Introduction to Statistical Decision Theory

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Data-driven decision-making

Data-driven decision-making: Motivations

Humans make billions of decisions every day. Many of these decisions are not optimal. Many are based on unexamined replication of whatever procedures were used historically. Sub-optimal decisions create waste.

Explosion of access to data, analytic tools, & computing power opens new opportunities for decisions to be more data driven.

The general form of the problems we'll take up

- There is a decision-maker who must choose one option from a menu of options.
- The decision-maker has objectives.
  - These objectives can be quantified in the form of an *objective function* (equivalently, a *loss function*).
- The payoffs to different choices depend on the values of certain *state variables* or unobserved *parameters*. The payoff function expresses the costs and benefits the result from the decision-maker's chosen action.
  - In some applications, it makes sense to model the payoffs as depending on the realized value of a random state variable.
  - In other applications, it makes sense to model the payoff's as depending on the value of the unobserved parameter.
- The decision-maker is in general *uncertain* about the values of the

Building decision models

The decision problem:

Who is the decision-maker?

What decision does this agent confront?

What is the set of options (or potential actions) from which this decision-maker chooses?

What are the stakes of the decision? What real costs and benefits are realized from making better vs. poorer decisions?

The predictive tool:

What information does your chosen predictive tool provide? How will the decision maker use the information generated by this tool to make better decisions?

Formalism for statistical decision models

Actions

A decision-maker must choose from exactly one action a from an *action space* $\mathbb{A}$: $a \in \mathbb{A}$

Example

- The decsion-maker is a doctor.
- $\mathbb{A} = \{$"give medication", "don't give medication"$\}$.

In some applications, each menu option a will correspond to a bundle or vector of distinct actions. In these cases, the action space $\mathbb{A}$ will include an exhaustive list of all such bundles or vectors.

States

In our nomenclature, *states* refer to *observable data* about relevant conditions.

A state x will correspond to a single point in a *state space*.

A state space $\mathbb{X}$ describes the set of all possible values that x might take: $x \in \mathbb{X}$.

Example

A test can be performed to determine whether a patient has a certain medical condition.
Let x denote the test result, which may be positive ($x =$"Y") or negative ($x =$"N").
The state space is the set of all possible outcomes for this test:
$x \in \mathbb{X} = \{$"N", "Y"$\}$.

Note

Notationally, it will often be more convenient to use numeric values for

Random states

Before observing the actual realized value x of the system's state, the system's true state is unknown.

In a typical application, we will nonetheless have some idea about the likelihood that the system's state x will attain each of the possible values in the state space $\mathbb{X}$.

Formally, our beliefs about the system's state can be represented by a *probability distribution over the state space* $\mathbb{X}$.

Example

For $x \in \mathbb{X} = \{0, 1\}$, let $\theta = \Pr\{x = 1\}$ and $1 - \theta = \Pr\{x = 0\}$.

Random variables

Prior to knowing the realized value of the system's state, its true value is an uncertain *random variable*.

By convention, we will use capital letters to refer to random variables, and lower case letters to refer to corresponding realized values.

Here, X denotes a random variable that takes values from on the state space $\mathbb{X}$, and x denotes its realized value in $\mathbb{X}$.

Random states

More generally, if x can take any of a finite number of values $1, 2, \ldots, N$, then a probability distribution over the state space $\mathbb{X} = \{1, \ldots, N\}$ can be represented by a vector $\theta =< \theta_1, \theta_2, \ldots, \theta_N >$, where $\theta_n = \Pr\{x = n\}$.

Since θ represents a probability distribution, we must have that $\theta_n \geq 0$ for all n, and $\sum \theta_n = 1$.

Example

Parameters

A *parameter* is a variable that describes the condition of the system, that is *not* directly observable. An *unobserved* parameter θ takes values on a *parameter space* $\nleq$: $\theta \in \nleq$.

Payoffs

After choosing an action, the decision-maker realizes a *payoff*. This payoff depends on the action chosen and, typically, on the realized value of

Statistical decision theory

Formalism for statistical decision models

The payoff to a decision depends on the value of some unobserved parameter θ, which could be a vector. We have some system for generating a probability distribution $\pi(\theta)$ over θ. In Bayesian approaches, the probability distribution $\pi(\theta)$ used is the posterior distribution of θ given the observed data: $\pi(\theta|y)$. Other approaches exist for creating probability distributions over states

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Additional notes

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Background: This course centers on a semester-long project to build a working algorithmic decision tool. As used here, the term algorithmic decision tool means a computational system that empowers a user to make data-driven, near-optimal decisions in a specific domain area.

In the broadest terms, you can think of a decision tool as having two components, or modules:

A prediction component that generates information (possibly imperfect) about the state of the world; and An optimization component that identifies the best action to take, given the information provided by your prediction tool. In the first two lectures, we've introduced the basic concepts of statistical decision theory - a rigorous framework for taking optimal decisions under conditions of uncertainty. In the context of a bare-bones simple model, we've addressed how to formalize a decision problem in terms of uncertain random variables, the menu of possible actions, the potential payoffs from choosing alternative actions, and the decision-maker's ultimate