Introduction to Statistical Decision Theory

Arthur Small

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Data-driven decision-making

Data-driven decision-making: Motivations

Humans make billions of decisions every day. Many of these decisions are not optimal. Many are based on unexamined replication of whatever procedures were used historically. Sub-optimal decisions create waste.

Explosion of access to data, analytic tools, & computing power opens new opportunities for decisions to be more data driven.

The general form of the problems we'll take up

- There is a decision-maker who must choose one option from a menu of options.
- The decision-maker has objectives.
  - These objectives can be quantified in the form of an *objective function* (equivalently, a *loss function*).
- The payoffs to different choices depend on the values of certain *state variables* or unobserved *parameters*. The payoff function expresses the costs and benefits the result from the decision-maker's chosen action.
  - In some applications, it makes sense to model the payoffs as depending on the realized value of a random state variable.
  - In other applications, it makes sense to model the payoff's as depending on the value of the unobserved parameter.
- The decision-maker is in general *uncertain* about the values of the

Building decision models

The decision problem:

Who is the decision-maker?

What decision does this agent confront?

What is the set of options (or potential actions) from which this decision-maker chooses?

What are the stakes of the decision? What real costs and benefits are realized from making better vs. poorer decisions?

The predictive tool:

What information does your chosen predictive tool provide? How will the decision maker use the information generated by this tool to make better decisions?

Formalism for statistical decision models

Actions

A decision-maker must choose from exactly one action a from an *action space* $\mathbb{A}$: $a \in \mathbb{A}$

Example

- The decsion-maker is a doctor.
- $\mathbb{A} = \{$"give medication", "don't give medication"$\}$.

In some applications, each menu option a will correspond to a bundle or vector of distinct actions. In these cases, the action space $\mathbb{A}$ will include an exhaustive list of all such bundles or vectors.

States

In our nomenclature, *states* refer to *observable data* about relevant conditions.

A state x will correspond to a single point in a *state space*.

A state space $\mathbb{X}$ describes the set of all possible values that x might take: $x \in \mathbb{X}$.

Example

A test can be performed to determine whether a patient has a certain medical condition.
Let x denote the test result, which may be positive ($x =$"Y") or negative ($x =$"N").
The state space is the set of all possible outcomes for this test:
$x \in \mathbb{X} = \{$"N", "Y"$\}$.

Note

Notationally, it will often be more convenient to use numeric values for

Random states

Before observing the actual realized value x of the system's state, the system's true state is unknown.

In a typical application, we will nonetheless have some idea about the likelihood that the system's state x will attain each of the possible values in the state space $\mathbb{X}$.

Formally, our beliefs about the system's state can be represented by a *probability distribution over the state space* $\mathbb{X}$.

Example

For $x \in \mathbb{X} = \{0, 1\}$, let $f(1) = \Pr\{x = 1\}$ and $f(0) = 1 - f(1) = \Pr\{x = 0\}$.

Random variables

Prior to knowing the realized value of the system's state, its true value is an uncertain *random variable*.

By convention, we will use capital letters to refer to random variables, and lower case letters to refer to corresponding realized values.

Here, X denotes a random variable that takes values from on the state space $\mathbb{X}$, and x denotes its realized value in $\mathbb{X}$.

Let $f(x|\theta)$ denote the

Random states

For $x \in \mathbb{X}$, let $f(x)$ define a probability distribution over $\mathbb{X}$.

Let $B \subset \mathbb{X}$. If $\mathbb{X}$ is discrete, then

$$\Pr\{x \in B\} = \sum_{x \in B} f(x)$$

If $\mathbb{X}$ is continuous, then

$$\Pr\{x \in B\} = \int_{x \in B} f(x)$$

More generally, if x can take any of a finite number of values $1, 2, \ldots, N$, then a probability distribution over the state space $\mathbb{X} = \{1, \ldots, N\}$ can be represented by a vector $\theta =< \theta_1, \theta_2, \ldots, \theta_N >$, where $\theta_n = \Pr\{x = n\}$.

Since θ represents a probability distribution, we must have that $\theta_n \geq 0$ for all n, and $\sum \theta_n = 1$.

Parameters

A *parameter* is a variable that describes the condition of the system, that is *not* directly observable. An *unobserved* parameter θ takes values on a *parameter space* $\nleq$: $\theta \in \nleq$.

Payoffs

After choosing an action, the decision-maker realizes a *payoff*. This payoff depends on the chosen action a. Depending on the situation being modeled, it may make sense to express payoffs either in terms of the realized value of the system's state, or on the value of the unobserved parameters.

Payoffs as a function of action and state

Let $u(a, x)$ denote the payoffs from a given action a and state x.

$$E[u(a, X)|\theta] = \int_{\mathbb{X}} p(x)u(a, x)dx$$

Then define the expected payoff $U(a, \theta)$ as a function of θ in these terms:

$$U(a, \theta) = E[u(a, X)|\theta] = \int_{\mathbb{X}} p(x)u(a, x)dx$$

More generally in terms of utility:
$$ u()

Statistical decision theory

Formalism for statistical decision models

The payoff to a decision depends on the value of some unobserved parameter θ, which could be a vector. We have some system for generating a probability distribution $\pi(\theta)$ over θ. In Bayesian approaches, the probability distribution $\pi(\theta)$ used is the posterior distribution of θ given the observed data: $\pi(\theta|y)$. Other approaches exist for creating probability distributions over states

Review

Big picture: How to make optimal choices under statistical uncertainty?

- A decision-maker (a.k.a. *actor*) chooses one option from a menu of possible *actions*.
- Payoffs from that choice depend on the action chosen, and on the *true* value of the state of nature.
    - Payoffs may equivalently be represented as *losses* — negative payoffs relative to some baseline.
- The true value of the state of nature is *uncertain*. Hence the decision-maker confronts a problem of *decision-making under uncertainty*.
    - Each possible action thus maps to a *lottery* over uncertain payoffs.

Uncertainty

- Typically, the decision-maker will posses some information about the likelihood of different states of nature.
- This information can be represented formally by treating the state of nature as a random variable drawn from a known distribution.

Optimal choice

- Given a complete enumeration of
  - the menu of possible actions,
  - the set of possible states of nature,
  - a probability distribution over the set of states of nature, representing the likelihood of states, and – a *payoff function* that gives the payoffs (or losses) from each possible combination of action $\times$ state, in some units of measure.

  then each possible action maps to a *lottery* over payoffs (losses).

Optimization: Defining objectives

- Finally, given a lottery over payoffs, the actor chooses an *optimal* action guided by a *decision-making principle*:
  - *Example:* Maximize expected payoffs.
  - *Example:* Maximize expected utility of payoffs.
  - *Example:* 'Minimax': Choose the action to minimize possible loss, irrespective of probabilities.
- The decision-making principle encodes the actor's attitude towards risk – the willingness to accept losses in some uncertain states of the world, in exchange for acheiving gains in other states of the world.
  - The study of attitudes towards risk and loss is huge topic in economics, finance, and psychology. We will cover it only glancingly.
  - For your applications, just at minimum be aware that the actor's optimum choices may not be driven by goal to maximize expected gains (= minimize expected losses).

Integrating predictive and inferential tools into the decision calculus

In general, your predictive model serves to *reduce uncertainty* over future values of the states of nature.

They *sharpen the probability distribution* over states of nature.

(In terms of probability theory: they involve a *change of measure*.)

The actor can now choose an optimal action based not on her *prior* (or *naive*) beliefs, but on her *posterior* beliefs, conditioned on the current data.

Payoff functions: Define in terms of x (state) or θ (parameter value)?

Depending on your appplication, it may make sense to model payoffs (or losses) as either a function of observed realized state $x \in \mathbb{X}$, e.g., x denotes realized temperature:

$$L_0 = L_0(a; x)$$

or in terms of the value of an unobserved parameter, e.g., θ denotes mean temperature:

$$L_1 = L_1(a; \theta)$$

Parameter-dependent payoffs as reduced form of state-dependent payoffs

In many cases, you can represent the θ formulation as the *reduced form* of the x formulation, e.g.

$$L_1(a;\theta) = E[L_0(a;x)|\theta]$$

Additional notes

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Background: This course centers on a semester-long project to build a working algorithmic decision tool. As used here, the term algorithmic decision tool means a computational system that empowers a user to make data-driven, near-optimal decisions in a specific domain area.

In the broadest terms, you can think of a decision tool as having two components, or modules:

A prediction component that generates information (possibly imperfect) about the state of the world; and An optimization component that identifies the best action to take, given the information provided by your prediction tool. In the first two lectures, we've introduced the basic concepts of statistical decision theory - a rigorous framework for taking optimal decisions under conditions of uncertainty. In the context of a bare-bones simple model, we've addressed how to formalize a decision problem in terms of uncertain random variables, the menu of possible actions, the potential payoffs from choosing alternative actions, and the decision-maker's ultimate