Course 1

Linear Algebra

Week 1 - Systems of linear Equations

Linear Algebra and Machine Learning

→ Linear Reggresion $\xrightarrow{\text{is}}$ Supervised Machine Learning : It means that we have some Inputs and the Output and we want to find the relation between Inputs and the Output.

System of sentences

Assume that we only have 1 dog and 1 cat and they're both of only 1 color.
We are given some information and our goal is to try to figure out the color of each of the animals.

1	2	3
The dog is black. The cat is orange.	The dog is black The dog is black.	The dog is black. The dog is white.
2 sentences and 2 pieces of information	2 sentences that they're exactly the same	the sentences contradict each other
Complete System	Redundant System	Contradictory System
⇓	⇓	⇓
Non-Singular System	Singular System	Singular System

In a nutshell (= summarize) :

- Non_Singular System : is a system that carries as many pieces of information as sentences. So It's the most informative system can be. So we can find a unique solution.

- Singular System : is less informative than a non_singular one. Because the system doesn't carry enough information, the system has infinitely many Solutions. / or no solution if the system is contradictory and singular.

What is a Linear Equation ?

visualized as lines

- Linear : A Linear Equation is a equation that can have as many variables as we want , and we can multiply the variable by scalars and then add them or subtract them and then add a constant.

Ex: $a+b=10$ / $2a+3b-15=0$ / $3.4a+8b-2.2c=122.5$

- Non_Linear : is a equation that can have squares , things like Sin / Cos , powers , multiply variables , divided variables, log and etc.

Ex: $a^2+b^2=10$ / $\sin(a)+b^3=15$ / $ab^2+\frac{b}{a}-\log(c)=4^9$

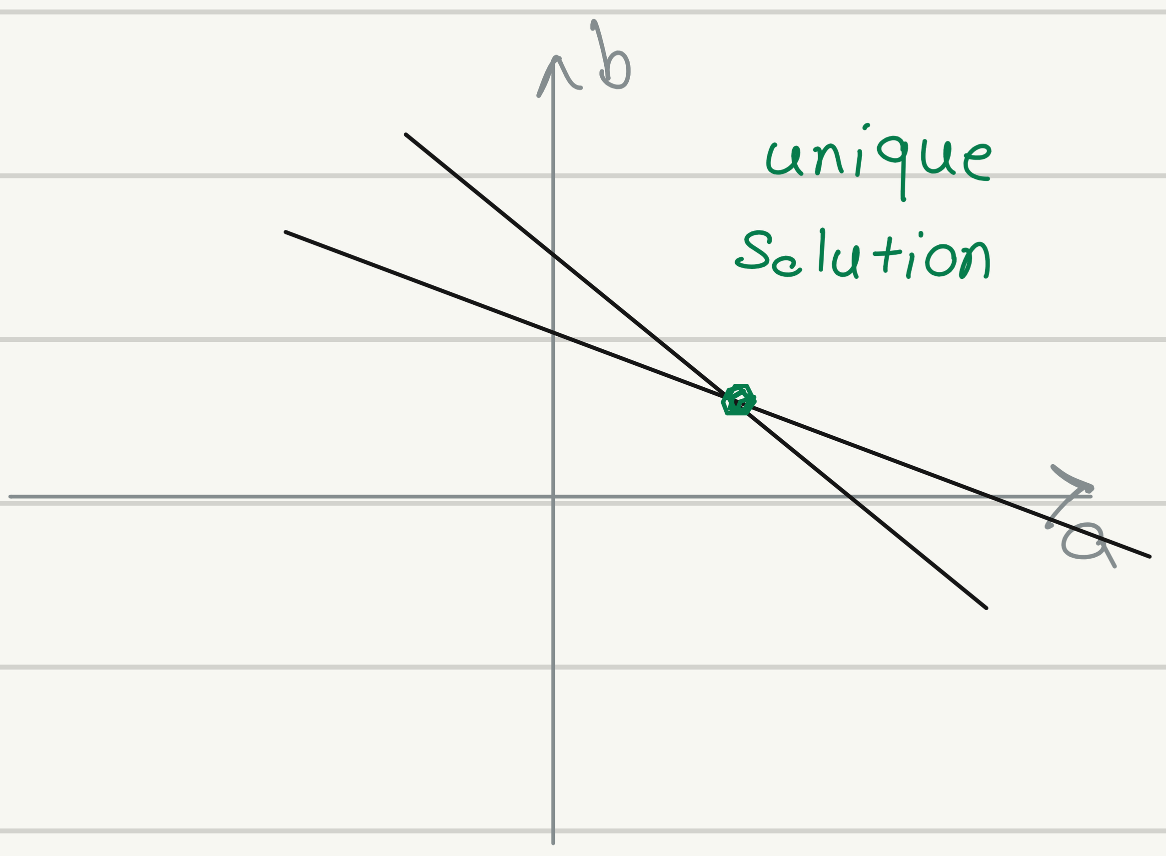
System of equations as lines

Example:

system 1

$a + b = 10$

$a + 2b = 12$



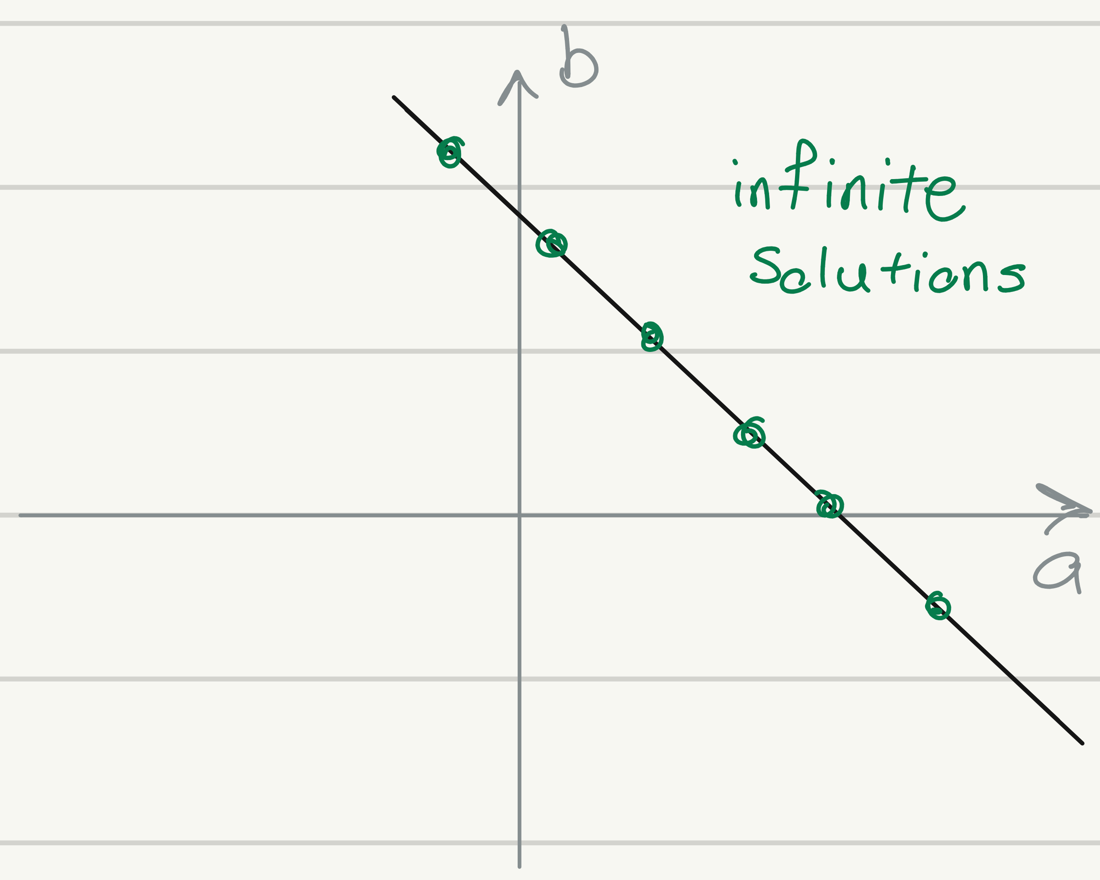
Complete

Non-Singular

system 2

$a + b = 10$

$2a + 2b = 20$



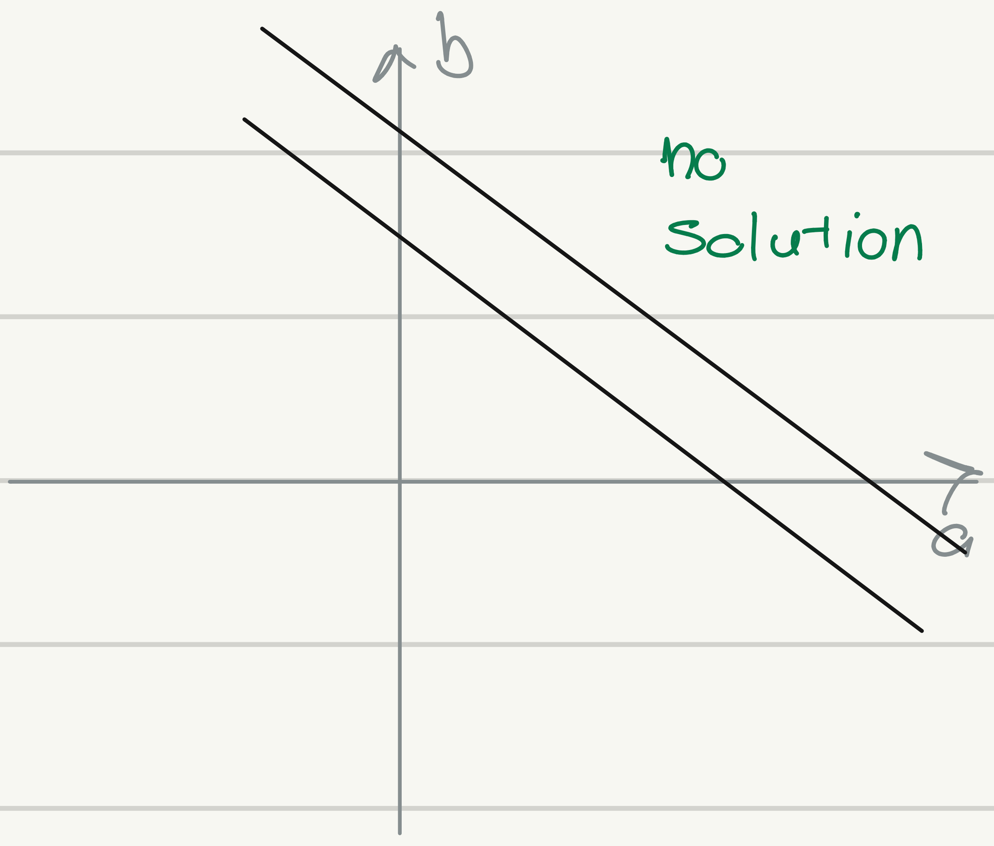
Redundant

Singular

system 3

$a + b = 10$

$2a + 2b = 24$



Contradictory

Singular

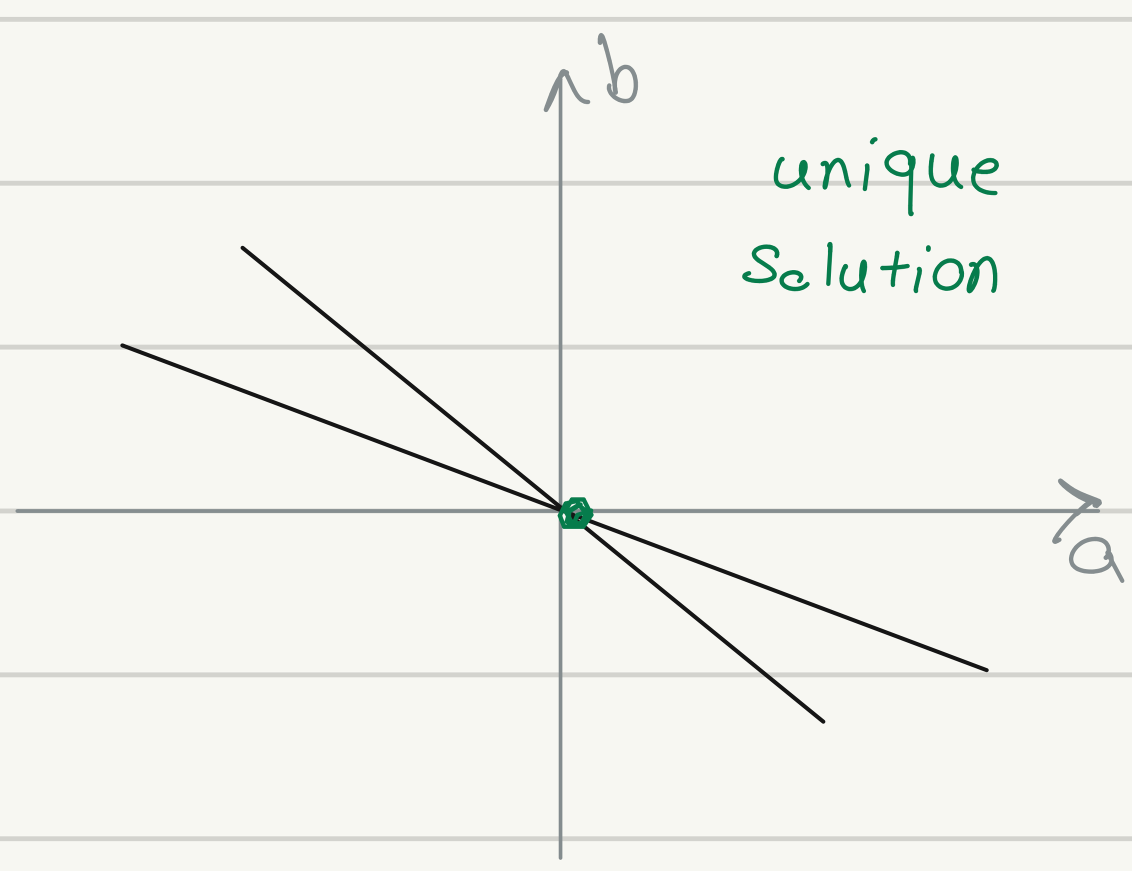
* Note that Constants don't matter for Singularity.

Changing Example:

system 1

$a + b = 0$

$a + 2b = 0$



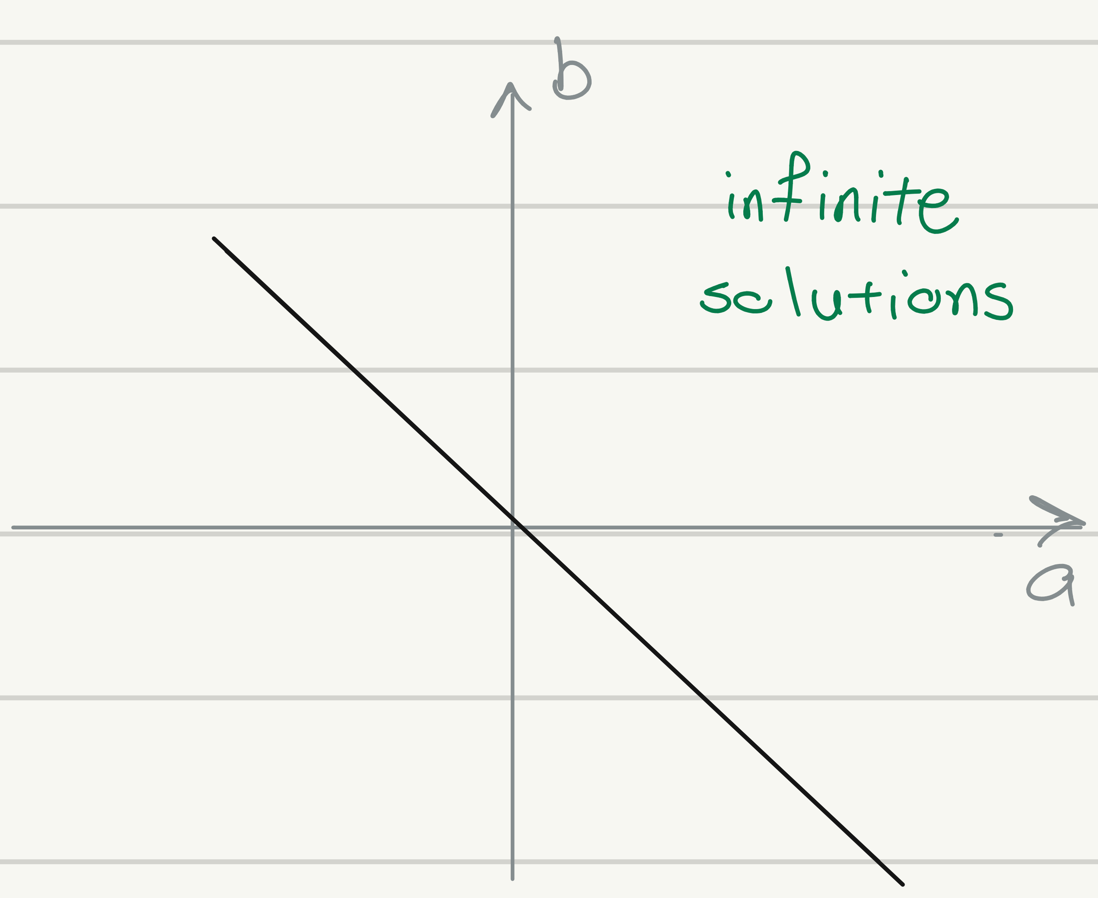
Complete

Non-Singular

system 2

$a + b = 0$

$2a + 2b = 0$



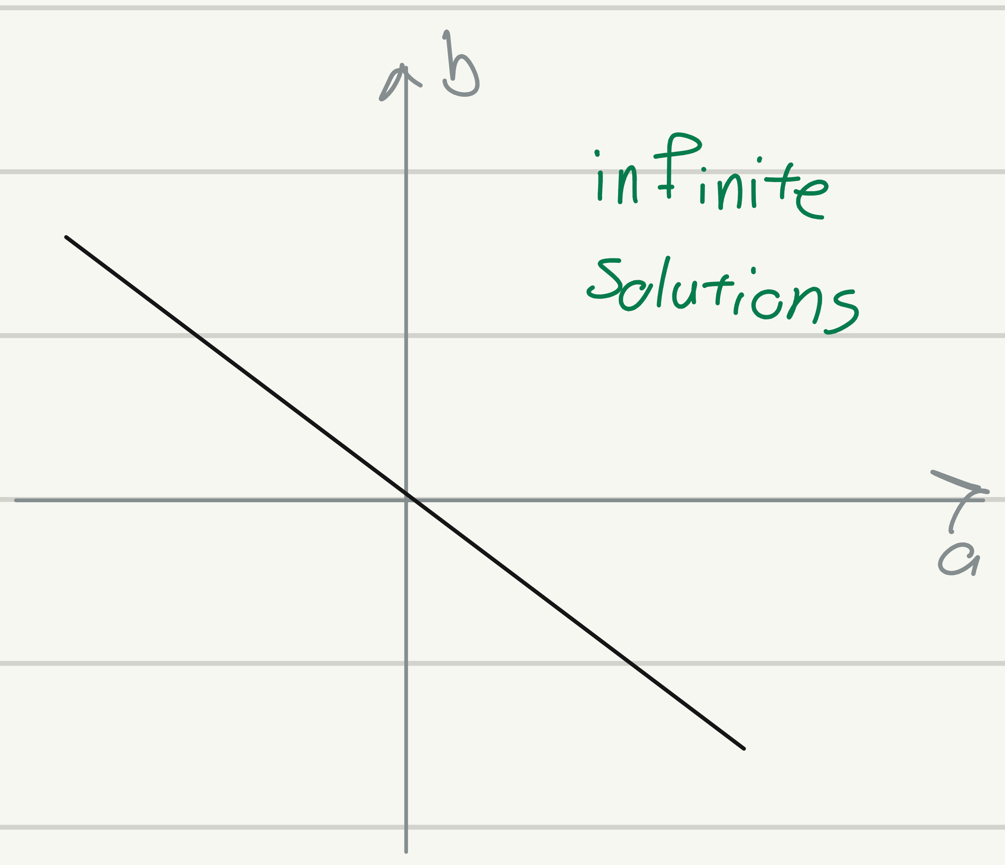
Redundant

Singular

system 3

$a + b = 0$

$2a + 2b = 0$



Reduntant

Singular

Constants matter for the reason of singularity : Redundant / contradictory / ...

Singular vs Non-Singular Matrices

- **Coefficient Matrix:** is a Matrix consisting of the coefficients of the variables in a set of linear equations.

Example:

system 1

$$\underbrace{\begin{matrix} a+b=0 \\ a+2b=0 \end{matrix}}_{\text{Non-singular system}} \qquad \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}}_{\text{Non-Singular matrix}}$$

unique solution

system 2

$$\underbrace{\begin{matrix} a+b=0 \\ 2a+2b=0 \end{matrix}}_{\text{singular system}} \qquad \underbrace{\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}}_{\text{Singular matrix}}$$

Infinity many solutions

Coefficient matrix

Linear Dependence and Independence

In above example:

system 1

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \times ?$$

system 2

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \times 2$$

* In system 2 we can reach the second row with multiplying first row to 2. So in system 2 rows are linearly dependent but in system 1 rows are linearly independent.

- **Linearly Dependent:** If between the rows of a matrix exist any relations.

The Determinant : a much faster way to tell if a matrix is Singular or Non - Singular.

- A Singular matrix:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow[\text{if}]{\text{singular}} [a \ \ b] * k = [c \ \ d]$$

$$\Downarrow$$

$$ak = c \quad \text{and} \quad bk = d$$

$$\Downarrow$$

$$\frac{c}{a} = k \quad \text{and} \quad \frac{d}{b} = k$$

$$\Downarrow$$

$$\frac{c}{a} = \frac{d}{b} \quad \text{So} \quad \underline{ad = bc}$$

$$\Downarrow \text{or}$$

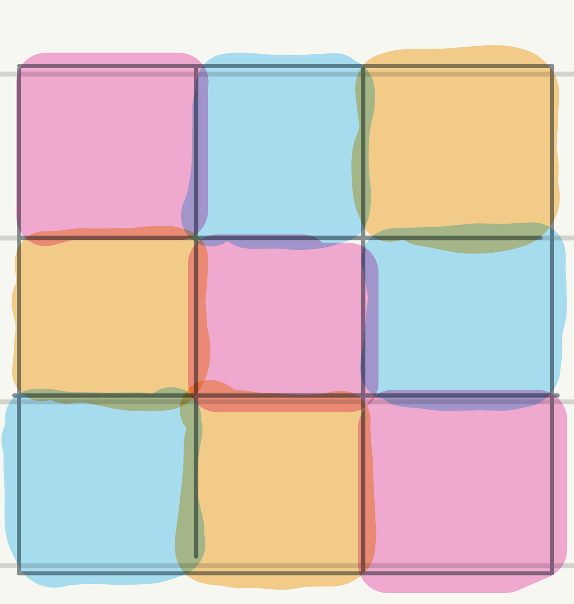
$$\text{Determinant} \longleftarrow ad - bc = 0$$

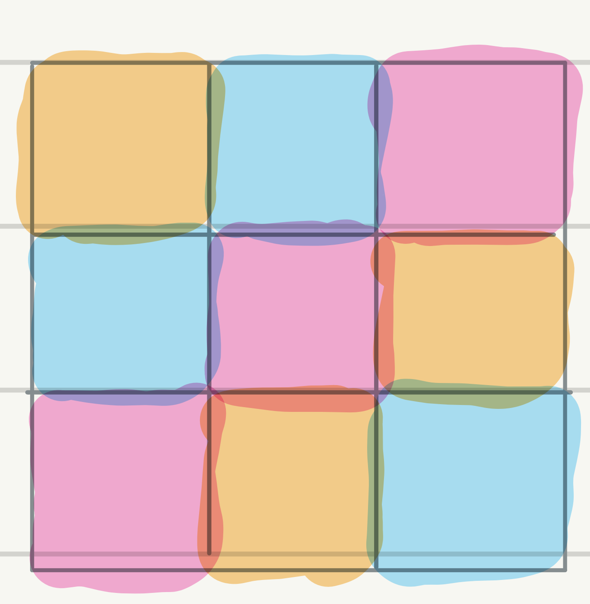
- If Determinant
  - is zero : The matrix is Singular
  - is not zero : The matrix is Non- Singular

- Determinant for 2x2 matrix: the product of the numbers in the main diagonal minus the product of the numbers in the antidiagonal.

The Determinant : 3x3 Matrix

Diagonals in 3x3 matrix





ADD + SUBTRACT = DETERMINANT

Ex:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$4 + 1 + 1 - (2 + 2 + 1) = 1$$

* Note : Whenever you have a matrix where anything underneath the diagonal is zero, the determinant is going to be the product of the elements in the main diagonal.

a	x	x
0	b	x
0	0	c

Determinant $= a * b * c$

Read Lab1 (Introduction to numpy arrays) and Lab2 (Solving linear systems - 2 variables).

Week 2 _ Solving Systems of Linear Equations

Manipulating Equations

(For non_Singular system)

- Swapping equations
- Adding "
- subtracting "
- Multiplying equations by a costant

Matrix Row Reduction

Matrix row reduction, also called Gaussian Elimination, consist of applying the exact same manipulations to the rows of a matrix.

Example :

original system	Intermediate System	Solved system
$5a + b = 17$	$a + 0.2b = 3.4$	$a = 3$
$4a + 3b = 6$	$b = 2$	$b = 2$

$$\begin{cases} 5a + b = 17 \\ 4a + 3b = 6 \end{cases} \longrightarrow \begin{cases} a + 0.2b = 3.4 \\ b = 2 \end{cases} \longrightarrow \begin{cases} a = 3 \\ b = 2 \end{cases}$$

original matrix	upper diagonal matrix	Diagonal matrix
$\begin{bmatrix} 5 & 1 \\ 4 & -3 \end{bmatrix}$	$\begin{bmatrix} 1 & 0.2 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} 5 & 1 \\ 4 & -3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0.2 \\ 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Coefficient matrices

row echelon form

reduced row echelon form

Row Echelon Form

Looks like:

$$\begin{bmatrix} 1 & * & * & * & * & * \\ 0 & 1 & * & * & * & * \\ 0 & 0 & 1 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- On the main diagonal, we have a bunch of <u>ones</u> followed by perhaps a bunch of <u>zeros</u>. (could have all <u>ones</u> or <u>zeros</u>).
- Below the diagonal, everything is a <u>zero</u>.
- To the right of the ones, any number is allowed.
- To the right of the zeros, everything must be zero.

2x2 → matrices

① $\begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix}$ two 1s in the diagonal

② $\begin{bmatrix} 1 & * \\ 0 & 0 \end{bmatrix}$ one 1 in the diagonal

③ $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ two 0s in the diagonal

Row Operations that preserve Singularity

The same manipulations that we use to solve systems of linear equations, can be used in matrices. These are called row operations in a matrix.

* A very important property that they have is that they preserve the Singularity of a matrix.

1) Switching Rows:

Ex: $\begin{bmatrix} 5 & 1 \\ 4 & 3 \end{bmatrix} \xrightarrow{\text{switch rows}} \begin{bmatrix} 4 & 3 \\ 5 & 1 \end{bmatrix}$

$\det = 15 - 4 = 11$ $\Downarrow$ non-singular

$\det = 4 - 15 = -11$ $\Downarrow$ non-singular

2) Multiplying a row by a (non-zero) Scalar:

Ex: $\begin{bmatrix} 5 & 1 \\ 4 & 3 \end{bmatrix} \rightarrow [5 \quad 1] * 10 = [50 \quad 10] \rightarrow \begin{bmatrix} 50 & 10 \\ 4 & 3 \end{bmatrix}$

$\det = 15 - 4 = 11$ $\Downarrow$ non-singular

$\det = 150 - 40 = 110$ $\Downarrow$ non-singular

3) Adding a row to another row:

Ex: $\begin{bmatrix} 5 & 1 \\ 4 & 3 \end{bmatrix} \rightarrow \begin{array}{r} [5 \quad 1] \\ + [4 \quad 3] \\ \hline [9 \quad 4] \end{array} \rightarrow \begin{bmatrix} 9 & 4 \\ 4 & 3 \end{bmatrix}$

$\det = 11$ $\Downarrow$ non-singular

$\det = 27 - 16 = 11$ $\Downarrow$ non-singular

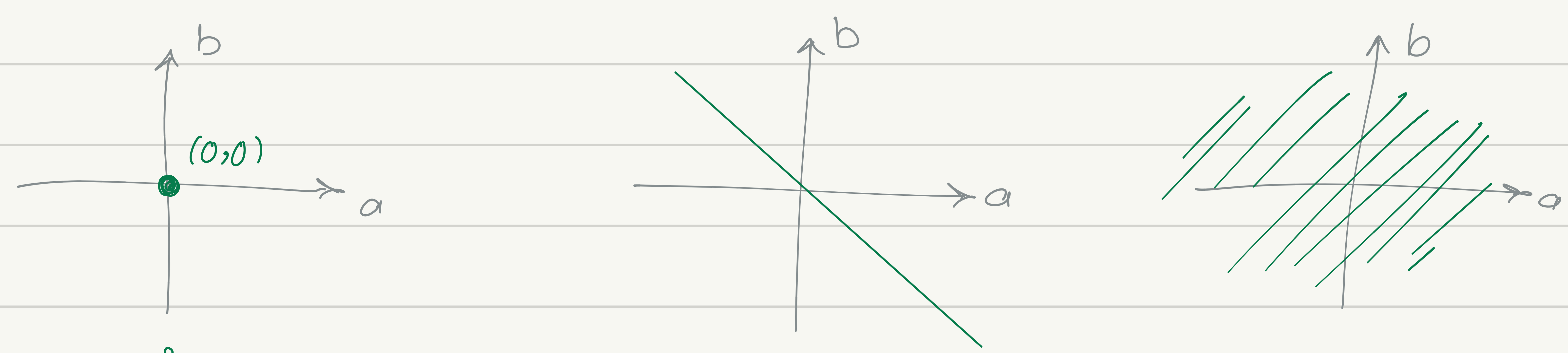
Rank of a Matrix → application = Image Compression

Example:

System 1	System 2	System 3
$a + b = 0$	$a + b = 0$	$0a + 0b = 0$
$a + 2b = 0$	$2a + 2b = 0$	$0a + 0b = 0$
↓	↓	↓
two equations	two equations	two equation
two pieces of info	one piece of info	zero piece of info
⇓	⇓	⇓
Rank = 2	Rank = 1	Rank = 0

Rank and Solutions to the system

System 1	System 2	System 3
$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
Rank = 2	Rank = 1	Rank = 0



Dimension of solution space	= 0	= 1	= 2
	* non_singular	Singular	Singular

So, for 2x2 Matrices:

$$\text{Rank} = 2 - (\text{Dimension of solution Space})$$

* A matrix is non-singular if and only if it has full rank. (rank = number of rows)

Rank of a Matrix : General Case

- Question : Is there an easier way to calculate the rank ?
- Answer : Yes! As before, it is the number of ones in the diagonal of the reduced row echelon form of the matrix.

Row Echelon Form

Ex 1:

original Matrix

$$\begin{bmatrix} 5 & 1 \\ 4 & -3 \end{bmatrix} \xrightarrow[\text{Divide each row by the leftmost coefficient}]{} \begin{bmatrix} 1 & 0.2 \\ 1 & -0.75 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0.2 \\ 0 & -0.95 \end{bmatrix}$$

$$\begin{array}{r} [1 \quad -0.75] \\ -\ [1 \quad 0.2] \\ \hline [0 \quad -0.95] \end{array}$$

Row Echelon Form

$$\begin{bmatrix} \underline{\underline{1}} & 0.2 \\ 0 & \underline{\underline{1}} \end{bmatrix}$$

Divide the second row by the left most non-zero Coefficient

* This matrix has $\underline{\underline{2}}$ ones in the diagonal of the row echelon form, so it has Rank = 2 .

Ex 2. system 1

2	*	*	*
0	1	*	*
0	0	3	*
0	0	0	-5

Rank = 4

system 2

3	*	*	*
0	0	1	*
0	0	0	0
0	0	0	0

Rank = 2

- Zero rows at the bottom
- Each row has a pivot (leftmost non-zero entry)
- Each pivot is to the right of the pivots one the rows above.
- Rank of the matrix is the number of pivots.

Reduced Row Echelon Form

Ex: original matrix → upper diagnol matrix → diagonal matrix

$$\begin{bmatrix} 5 & 1 \\ -4 & 3 \end{bmatrix} \longrightarrow \underbrace{\begin{bmatrix} 1 & 0.2 \\ 0 & 1 \end{bmatrix}}_{\text{Row echelon form}} \longrightarrow \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\text{Reduced row echelon form}}$$

How?

→ Answer:

$$\underbrace{\begin{bmatrix} 1 & 0.2 \\ 0 & 1 \end{bmatrix}}_{\text{row ehelon form}} \xrightarrow{1} \begin{array}{r} [0 \quad 1] \\ \times \quad 0.2 \\ \hline [0 \quad 0.2] \end{array} \xrightarrow{2} \begin{array}{r} [1 \quad 0.2] \\ - \; [0 \quad 0.2] \\ \hline [1 \quad 0] \end{array} \xrightarrow{3} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\text{reduced row echelon form}}$$

Reduced Row Echelon Form : In General

1)

1	0	0	0	0
0	1	0	0	0
0	0	1	0	0
0	0	0	1	0
0	0	0	0	1

Rank = 5

2)

1	*	0	0	*
0	0	1	0	*
0	0	0	1	*
0	0	0	0	0
0	0	0	0	0

Rank = 3

Rules:

- Is in row echelon form
- Each pivot is a 1
- Any number above a pivot is 0
- Rank of the matrix is the number of pivots

* A General method to go from a row echelon form matrix to a reduced row echelon form:

We can just divide each row by the leading coefficient.

Ex 1:

Row echelon form

3	*	*	*	*
0	0	2	*	*
0	0	0	-4	*
0	0	0	0	0
0	0	0	0	0

1	*	*	*	*
0	0	1	*	*
0	0	0	1	*
0	0	0	0	0
0	0	0	0	0

divide each row by the value of the pivot

Reduced row echelon form

1	*	0	0	*
0	0	1	0	*
0	0	0	1	*
0	0	0	0	0
0	0	0	0	0

Turn anything above a pivot to 0

Ex 2:

Row echelon form ... Reduced row echelon form

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Subtract 2 times the second row from the first one.

Add 5 times the thired row to the first one.

Subtract 4 times the third row from the second one.

The Gaussian Elimination Algorithm

- **Augmented Matrix**: is a matrix formed by combining the colums of two matrices to form a new matrix.

Ex :

$$2a - b + c = 1$$
$$2a + 2b + 4c = -2$$
$$4a + b = -1$$

$$\Longrightarrow \quad \underbrace{\begin{array}{c|ccc|c} R_1 & 2 & -1 & 1 & 1 \\ R_2 & 2 & 2 & 4 & -2 \\ R_3 & 4 & 1 & 0 & -1 \end{array}}_{\text{Coefficient matrix} \; + \; \text{Constant values} \; = \; \text{Augmented Matrix}}$$

Coefficient matrix

Constant values

+

Augmented Matrix

Let's Solve the equations :

① Pivoting : turn the pivot from first row into a 1.

$R_1 \leftarrow \frac{1}{2} R_1 \Longrightarrow R_1 \; [\,1 \quad -\frac{1}{2} \quad \frac{1}{2} \quad \frac{1}{2}\,]$ updated row 1

② Set all the values below the pivot to 0 : using row operations

1) $R_2 \leftarrow R_2 - 2\overset{\frown}{R_1}$ (updated row) $\Longrightarrow$

$$\begin{array}{rl} R_2 & [\,2 \quad 2 \quad 4 \quad -2\,] \\ -\; 2R_1 & [\,2 \quad -1 \quad 1 \quad 1\,] \\ \hline \text{new } R_2 & [\,0 \quad 3 \quad 3 \quad -3\,] \end{array}$$

updated row 2

2) $R_3 \leftarrow R_3 - 4R_1 \Longrightarrow$

$$\begin{array}{rl} R_3 & [\,4 \quad 1 \quad 0 \quad -1\,] \\ -\; 4R_1 & [\,4 \quad -2 \quad 2 \quad 2\,] \\ \hline \text{new } R_3 & [\,0 \quad 3 \quad -2 \quad -3\,] \end{array}$$

updated row 3

updated Matrix:

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}\left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 3 & 3 & -3 \\ 0 & 3 & -2 & 2 \end{array}\right]$$

(3) Pivoting: turn the pivot from second row into a $\underline{\underline{1}}$.

$R_2 \leftarrow \frac{1}{3} R_2 \implies R_2\ [0\ \ 1\ \ 1\ \ -1]$ updated row $\underline{\underline{2}}$

(4) Set all the values below the pivot to $\underline{\underline{0}}$: using row operations

$$R_3 \leftarrow R_3 - 3\overset{\text{updated}}{\widehat{R_2}} \implies \begin{array}{rl} & R_3\ [0\ \ 3\ \ -2\ \ 2] \\ - & 3R_2\ [0\ \ 3\ \ 3\ \ -3] \\ \hline & \text{new } R_3\ [0\ \ 0\ \ -5\ \ 5] \end{array}$$

updated row $\underline{\underline{3}}$

(5) Step 3 for the pivot of thired row.

$R_3 \leftarrow -\frac{1}{5} R_3 \implies R_3\ [0\ \ 0\ \ 1\ \ -1]$

Updated Matrix:

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}\left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$\underbrace{\hspace{6em}}_{\text{row echelon form}}$

- Back substitution: Use the information in the last column to solve the system of equations. How? We will start from the bottom row and work our way to the top. We will use the pivot from each row to cancel the values in the cells above it. This process actually looks very similar to creating the pivots in the first place.

⑥ Back substitution :

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

1) $R_2 \leftarrow R_2 - R_3 \Rightarrow$

$$\begin{array}{rl} & R_2 \, [0 \quad 1 \quad 1 \quad -1] \\ - & R_3 \, [0 \quad 0 \quad 1 \quad -1] \\ \hline & \text{new } R_2 \, [0 \quad 1 \quad 0 \quad 0] \end{array}$$

2) $R_1 \leftarrow R_1 - \frac{1}{2} R_3 \Rightarrow$

$$\begin{array}{rl} - & R_1 \, [1 \quad -\frac{1}{2} \quad \frac{1}{2} \quad \frac{1}{2}] \\ & \frac{1}{2} R_3 \, [0 \quad 0 \quad \frac{1}{2} \quad -\frac{1}{2}] \\ \hline & \text{new } R_1 \, [1 \quad -\frac{1}{2} \quad 0 \quad 1] \end{array}$$

3) for second column :

$R_1 \leftarrow R_1 + \frac{1}{2} R_2 \Rightarrow$

$$\begin{array}{rl} & R_1 \, [1 \quad -\frac{1}{2} \quad 0 \quad 1] \\ + & \frac{1}{2} R_2 \, [0 \quad \frac{1}{2} \quad 0 \quad 0] \\ \hline & \text{new } R_1 \, [1 \quad 0 \quad 0 \quad 1] \end{array}$$

The result :

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right] \Rightarrow \begin{array}{l} a = 1 \\ b = 0 \\ c = -1 \end{array}$$

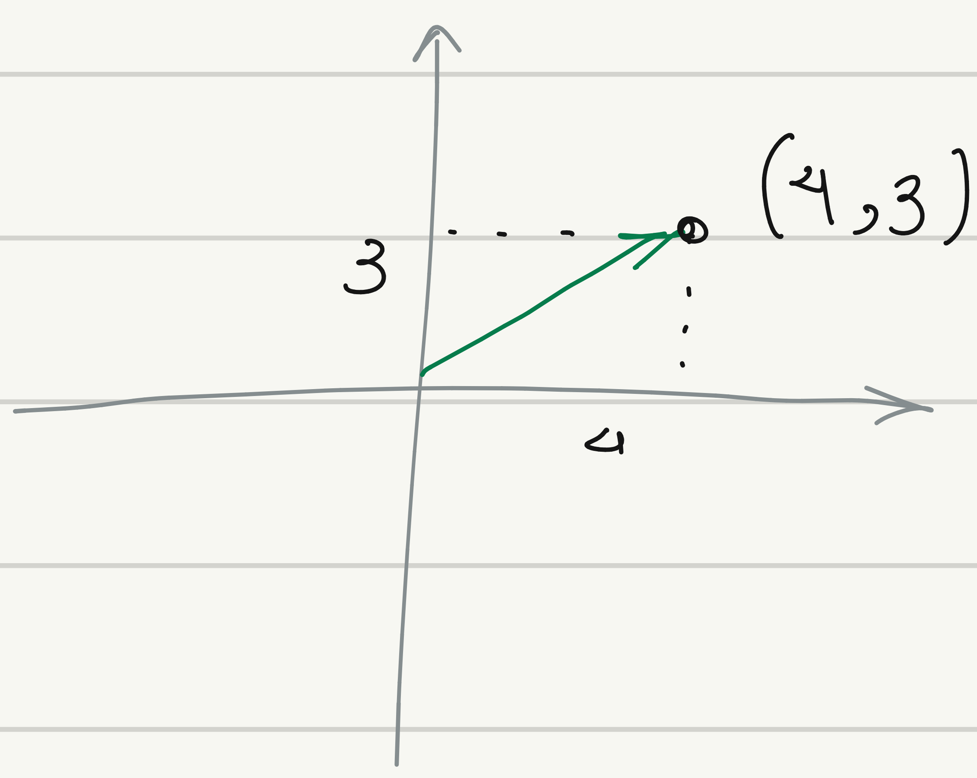
reduced row echelon form

solution to the system

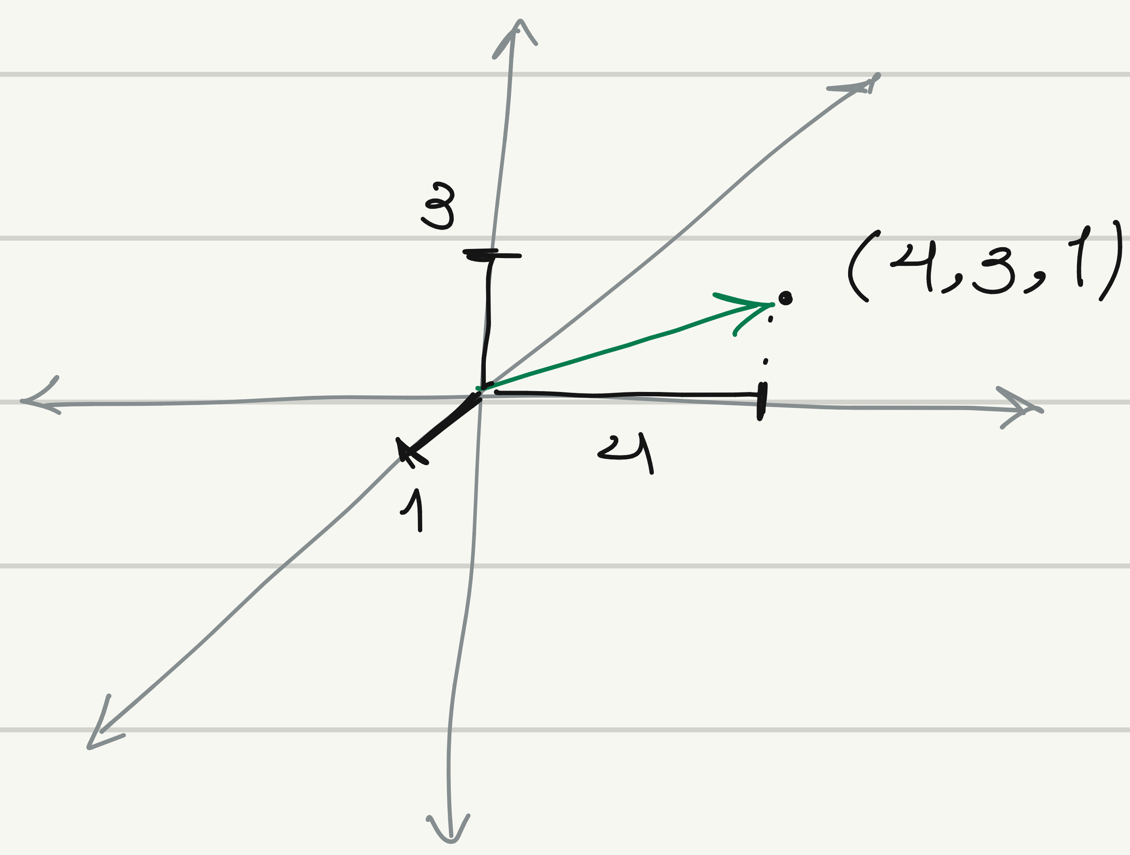
Identity Matrix : is the square matrix that has only 1s in the diagonal.

Week 3 _ Vectors and Linear Transformations

Vectors

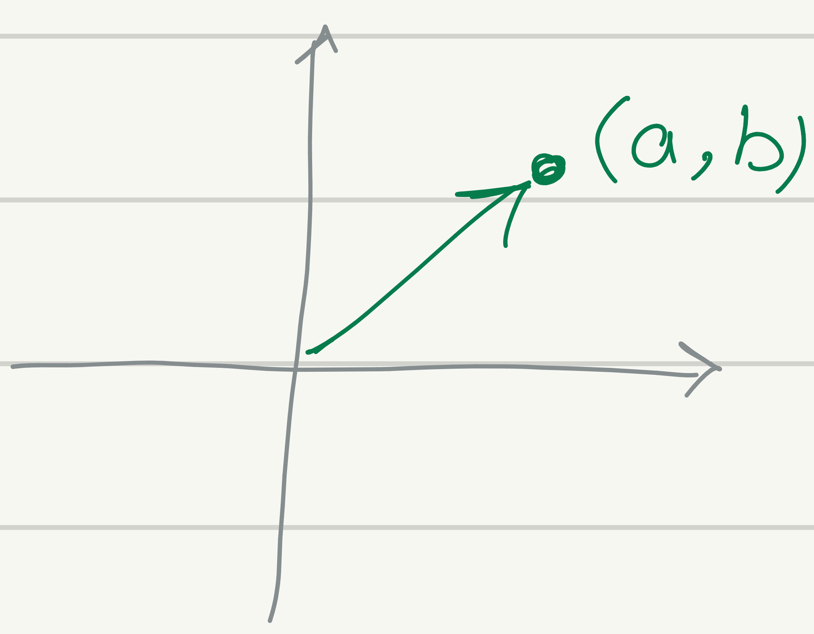


2D



3D

Norms

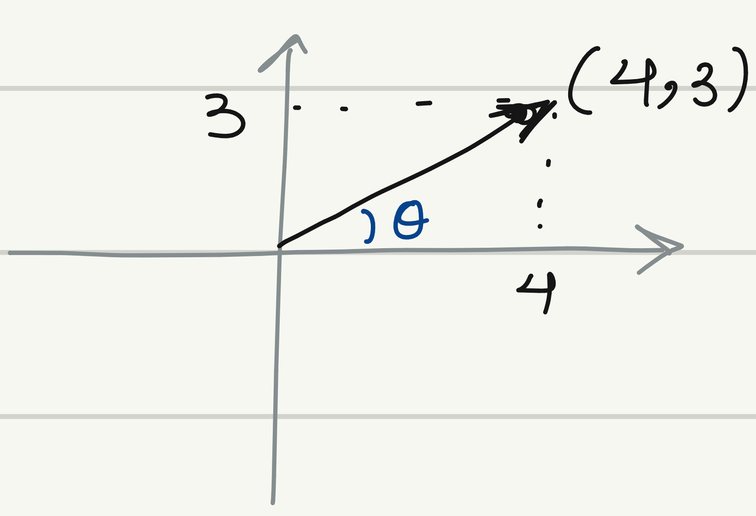


$$L1\_norm = ||(a,b)||_1 = |a| + |b|$$

$$L_2\_norm = ||(a,b)||_2 = \sqrt{a^2+b^2}$$

Example :

- Norm of a vector



$$\sqrt{4^2+3^2} = \sqrt{25} = 5$$

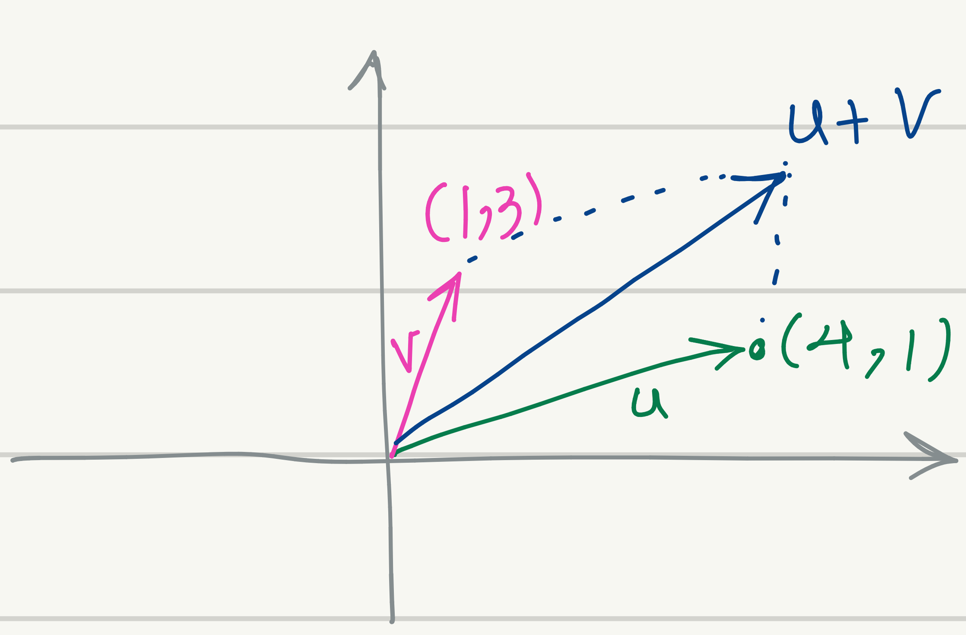
- Direction of a vector

$$\tan(\theta) = \frac{3}{4}$$

$$\theta = \text{Arctan}\left(\frac{3}{4}\right) = 0.64 = 36.87^{\circ}$$

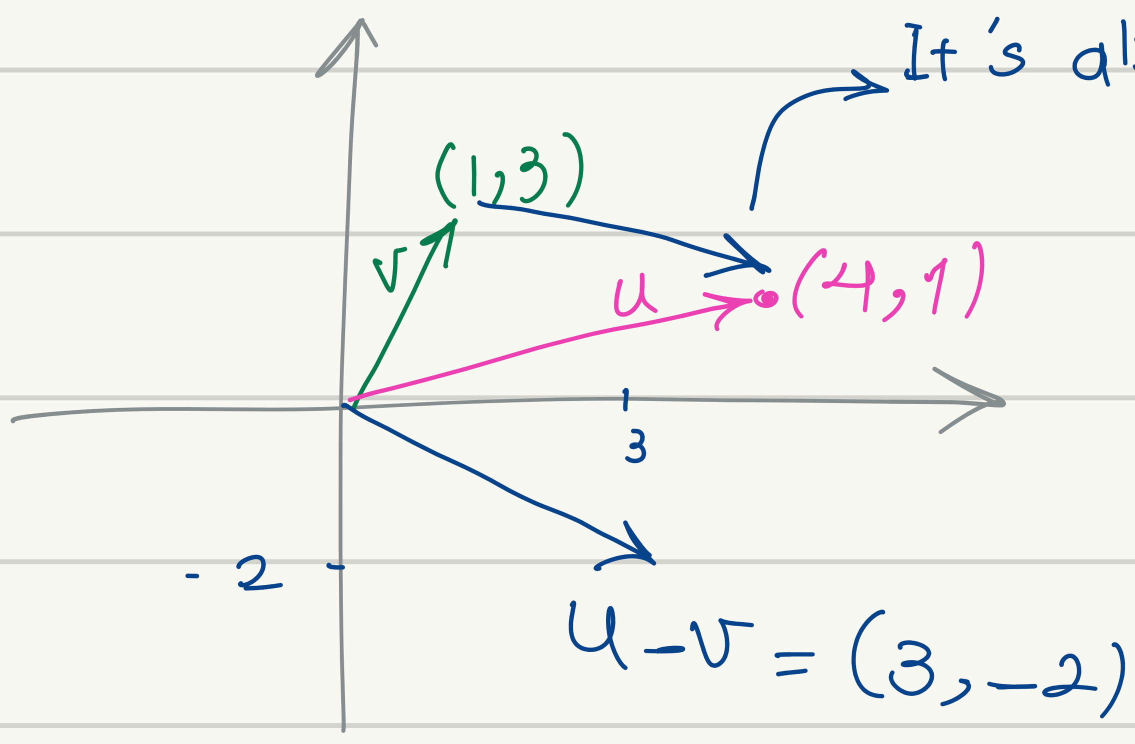
Vector Operations

1) Sum of vectors :



$$u+v = (4+1\ ,\ 3+1) = (5,4)$$

2) Difference of Vectors :



$u - v = (4-1,\ 1-3) = (3, -2)$

• Genera Definition •

$x = (x_1\ \ x_2\ \dots\ x_n)$, $y = (y_1\ \ y_2\ \dots\ y_n)$

Sum

Sum component by component

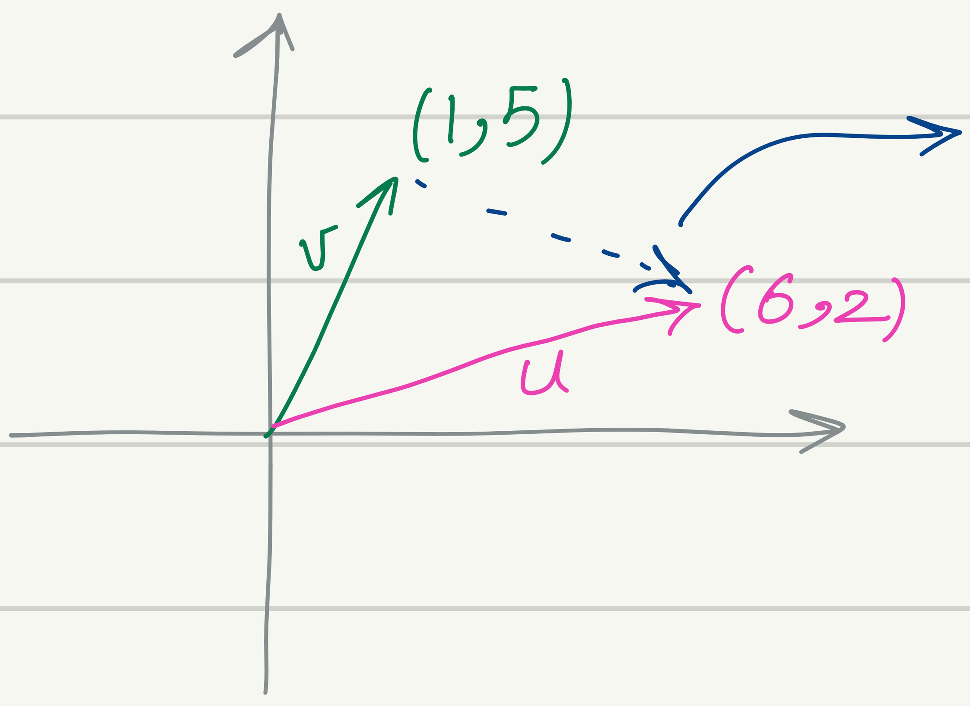
$x+y = (x_1+y_1\ \ x_2+y_2\ \dots\ x_n+y_n)$

Difference

subtract component by component

$x-y = (x_1-y_1\ \ x_2-y_2\ \dots\ x_n-y_n)$

Distance : The difference between two vectors is helpful to tell how far apart two vectors are from each other.



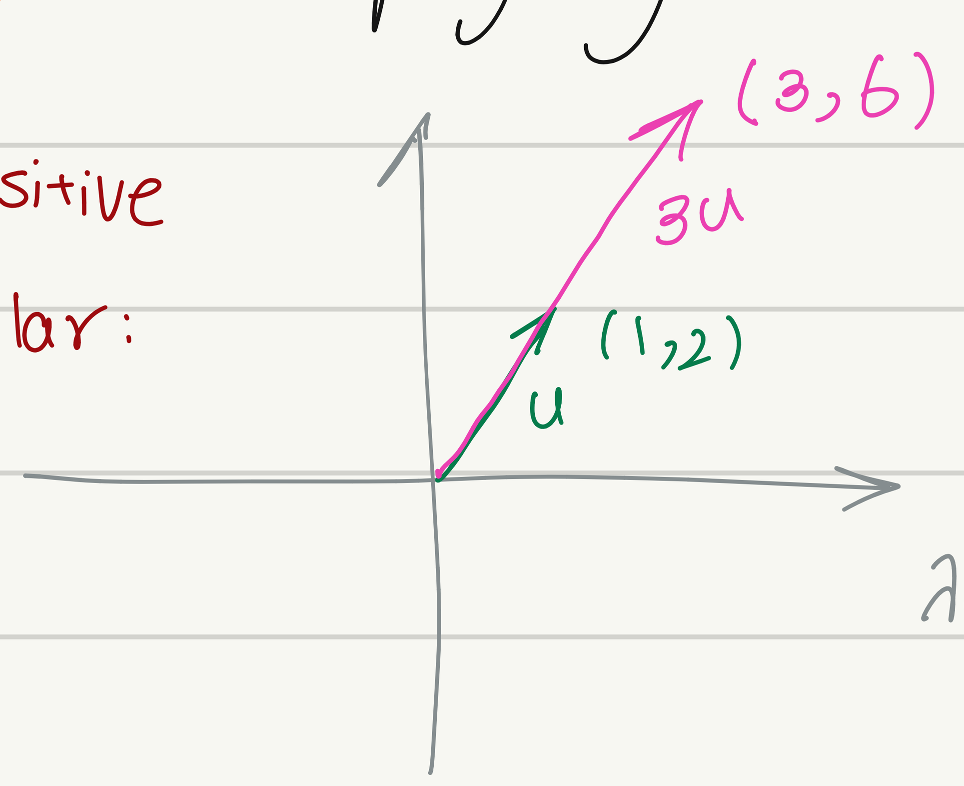
$u - v = (5, -3)$

$L1\_distance = |u-v|_1 = |5| + |3| = 8$

$L2\_distance = |u-v|_2 = \sqrt{5^2 + 3^2} = 5.83$

3) Multiplying a vector by a scalar :

• positive scalar :

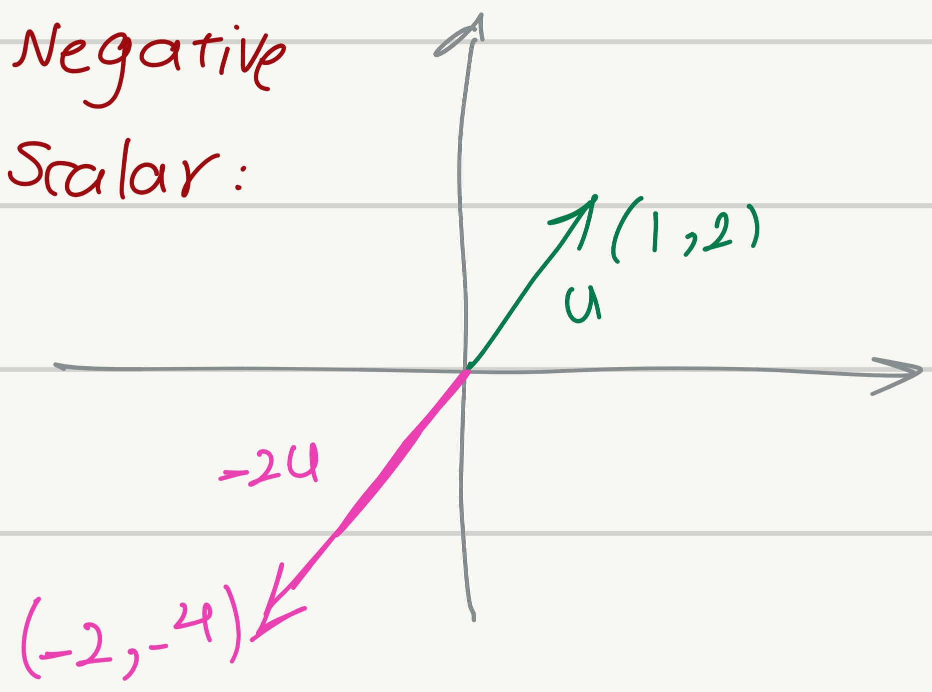


$u = (1,2)$

$\lambda = 3$

$\lambda u = (3,6)$

• Negative Scalar :



$u = (1,2)$

$\lambda = -2$

$\lambda u = (-2, -4)$

• General Def : $x = (x_1\ \ x_2\ \dots\ x_n) \longrightarrow \lambda x = (\lambda x_1\ \ \lambda x_2\ \dots\ \lambda x_n)$

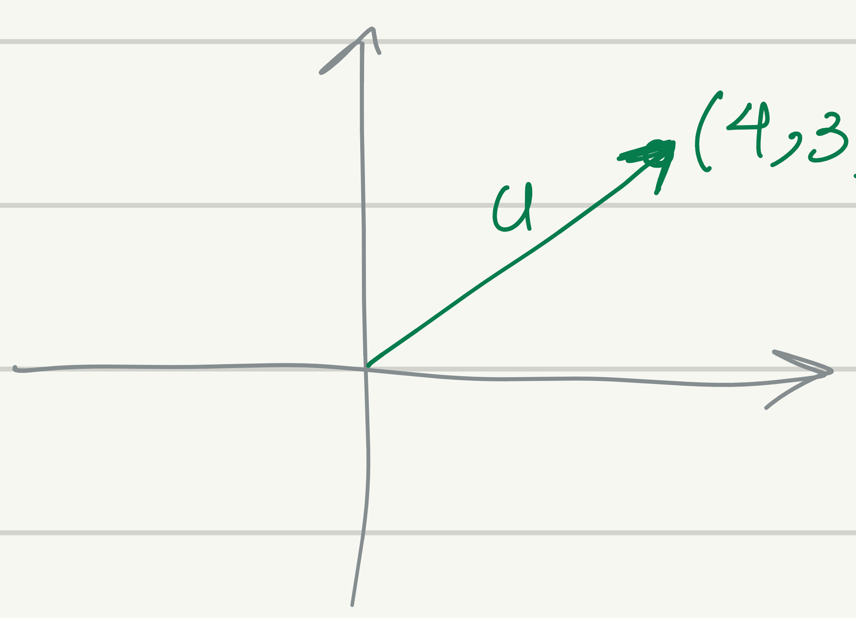
The Dot Product : Sum of each corresponding pair of entries.

$$\begin{bmatrix} 2 & 4 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix} = 28$$

How?

$$2 \cdot 3 + 4 \cdot 5 + 1 \cdot 2 = 28$$

Norm of a Vector using dot product :



$$L_2\text{-norm} = \sqrt{4^2 + 3^2} = \sqrt{25} = 5$$

dot product : $\begin{bmatrix} 4 & 3 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 3 \end{bmatrix} = (4 \times 4) + (3 \times 3) = 25$

So : $L_2\text{-norm} = \sqrt{\text{dot product}(u,u)} = \|u\|_2 = \sqrt{\langle u,u \rangle}$ → anothe notation for dot product

Vector Transpose

Transpose : Convert columns to rows

$$\begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}^T = \begin{bmatrix} 2 & 4 & 1 \end{bmatrix}$$

and vice versa

$$\begin{bmatrix} 2 & 4 & 1 \end{bmatrix}^T = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$$

Matrix Transpose

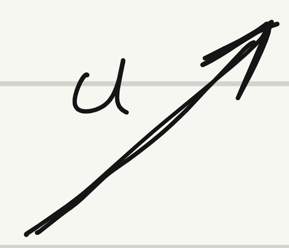
$$\begin{bmatrix} 2 & 5 \\ 4 & 7 \\ 1 & 3 \end{bmatrix}^T = \begin{bmatrix} 2 & 4 & 1 \\ 5 & 7 & 3 \end{bmatrix}$$

Columns → Rows

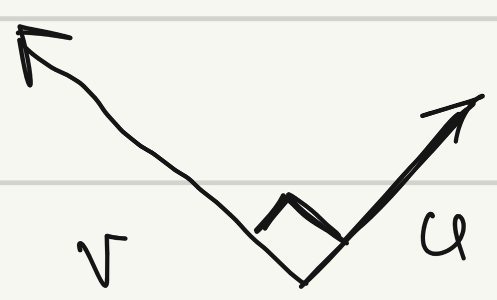
→ General Def : $x = (x_1 \; x_2 \dots x_n)$, $y = (y_1 \; y_2 \dots y_n)$

$$x \circ y = \langle x, y \rangle = (x_1 \times y_1) + (x_2 \times y_2) + \dots + (x_n \times y_n)$$

* The dot Product *

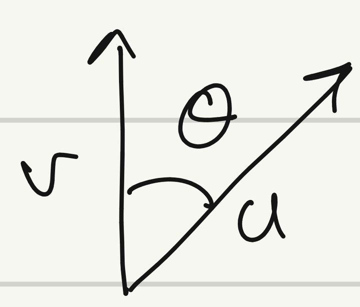
1) 

$\langle u, u \rangle = \|u\|^2 = \|u\| \cdot \|u\|$

2) 

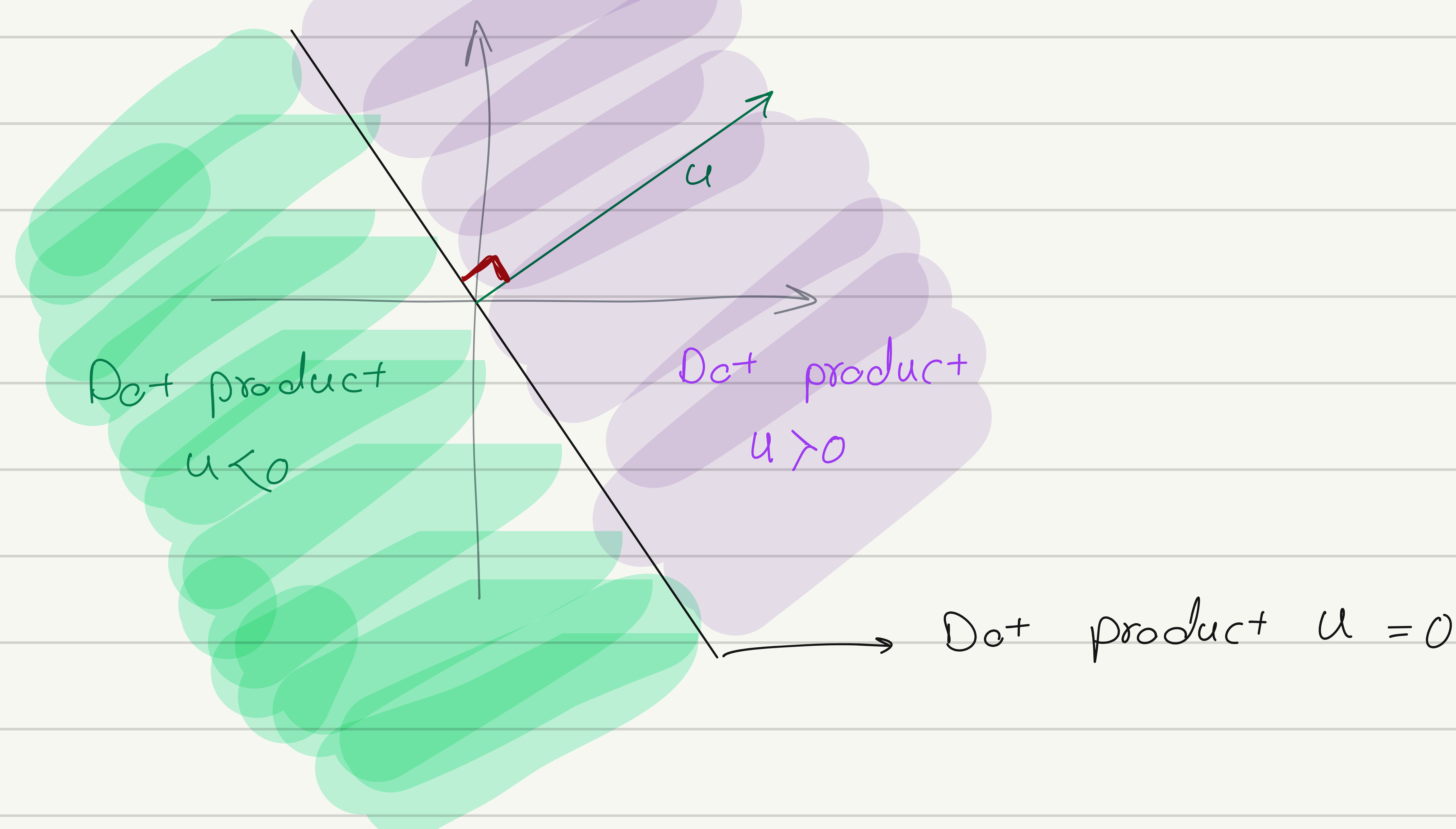
$\langle u, v \rangle = 0$

* orthogonal vectors have dot product 0 *

3) 

$\langle u, v \rangle = \|u\| \cdot \|v\| \cdot \cos(\theta)$

Geometric dot product



Equations as dot product

$a + b + c = 10$ $\quad$ $a + 2b + c = 15$ $\quad$ $a + b + 2c = 12$

$$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 10 \qquad \begin{bmatrix} 1 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 15 \qquad \begin{bmatrix} 1 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 12$$

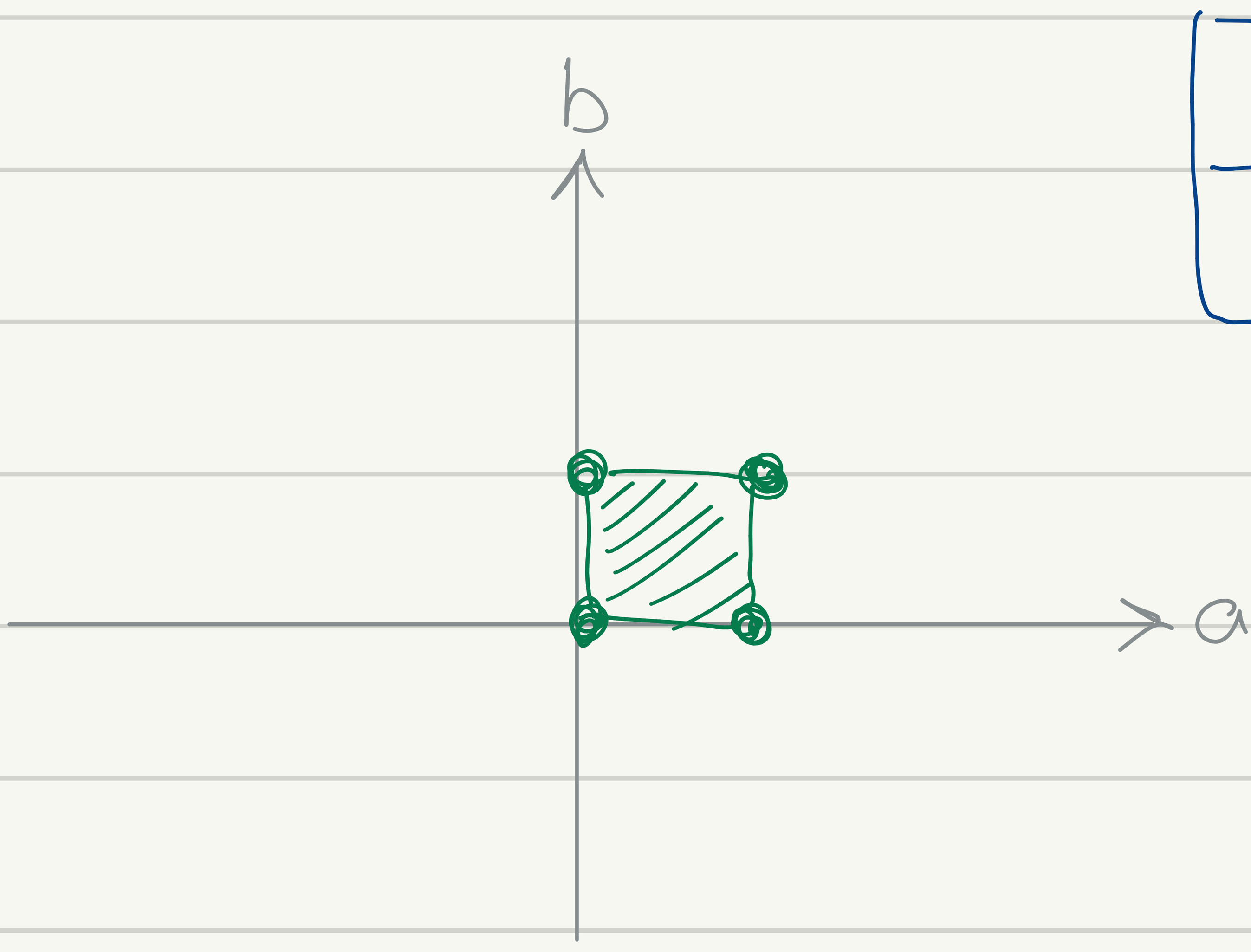
matrix product

→ Merging:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 10 \\ 15 \\ 12 \end{bmatrix}$$

** A columns = lenght of vector **

Matrices as Linear Transformations



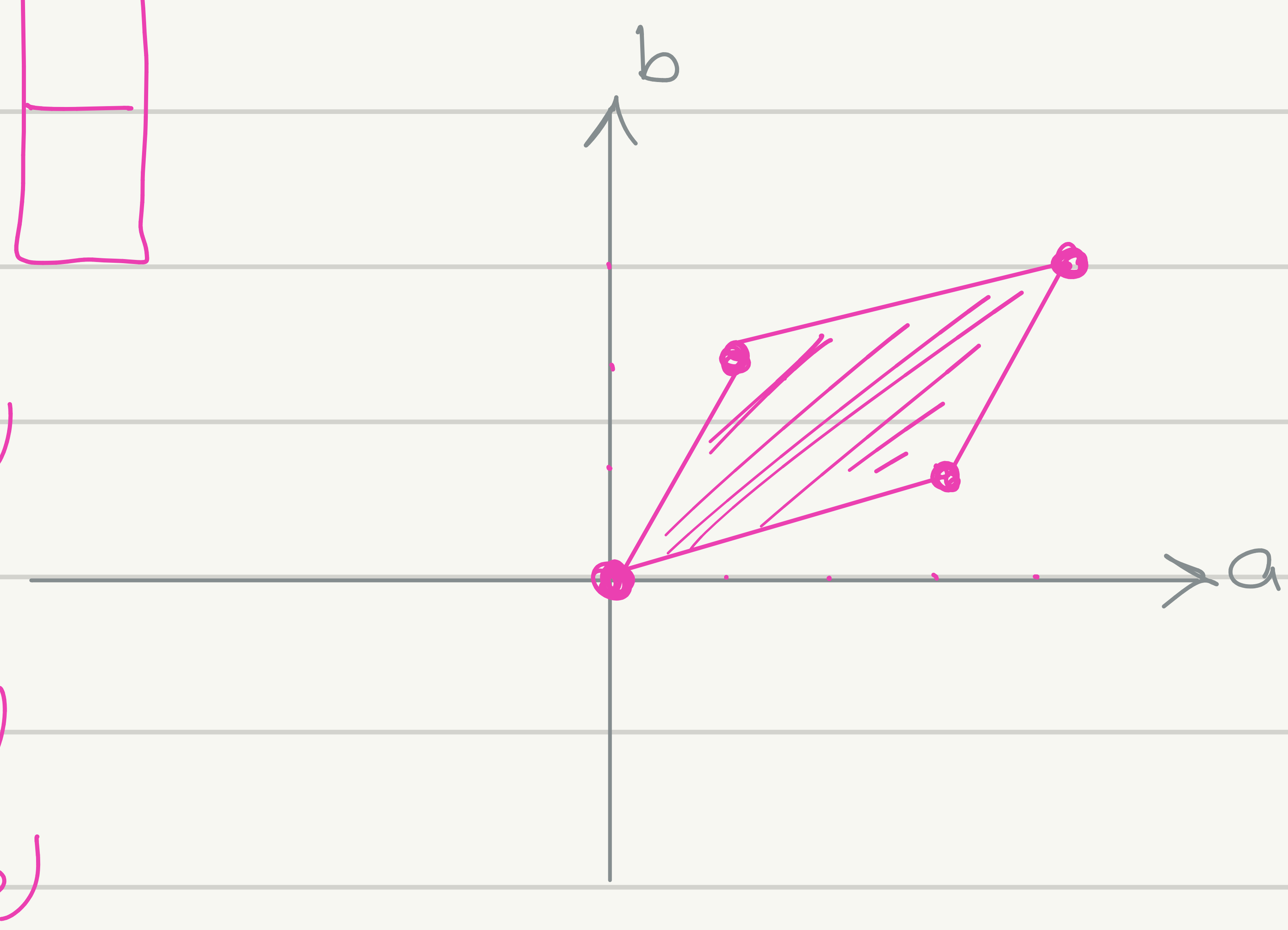
$$\begin{bmatrix}3 & 1\\ 1 & 2\end{bmatrix}\begin{bmatrix}\ \\ \ \end{bmatrix} = \begin{bmatrix}\ \\ \ \end{bmatrix}$$

$(0,0) \longrightarrow (0,0)$

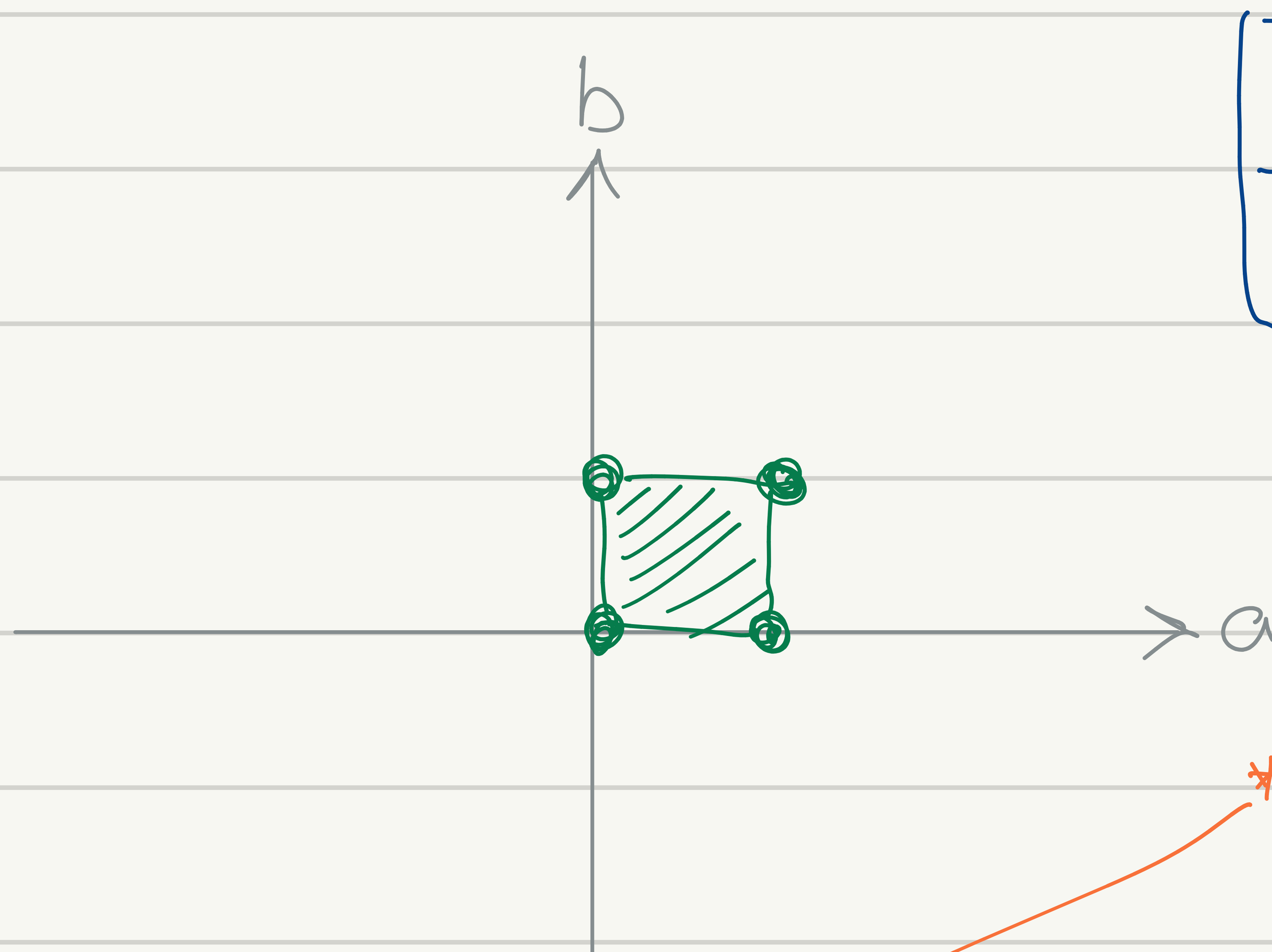
$(0,1) \longrightarrow (1,2)$

$(1,0) \longrightarrow (3,1)$

$(1,1) \longrightarrow (4,3)$



Linear Transformations as Matrices



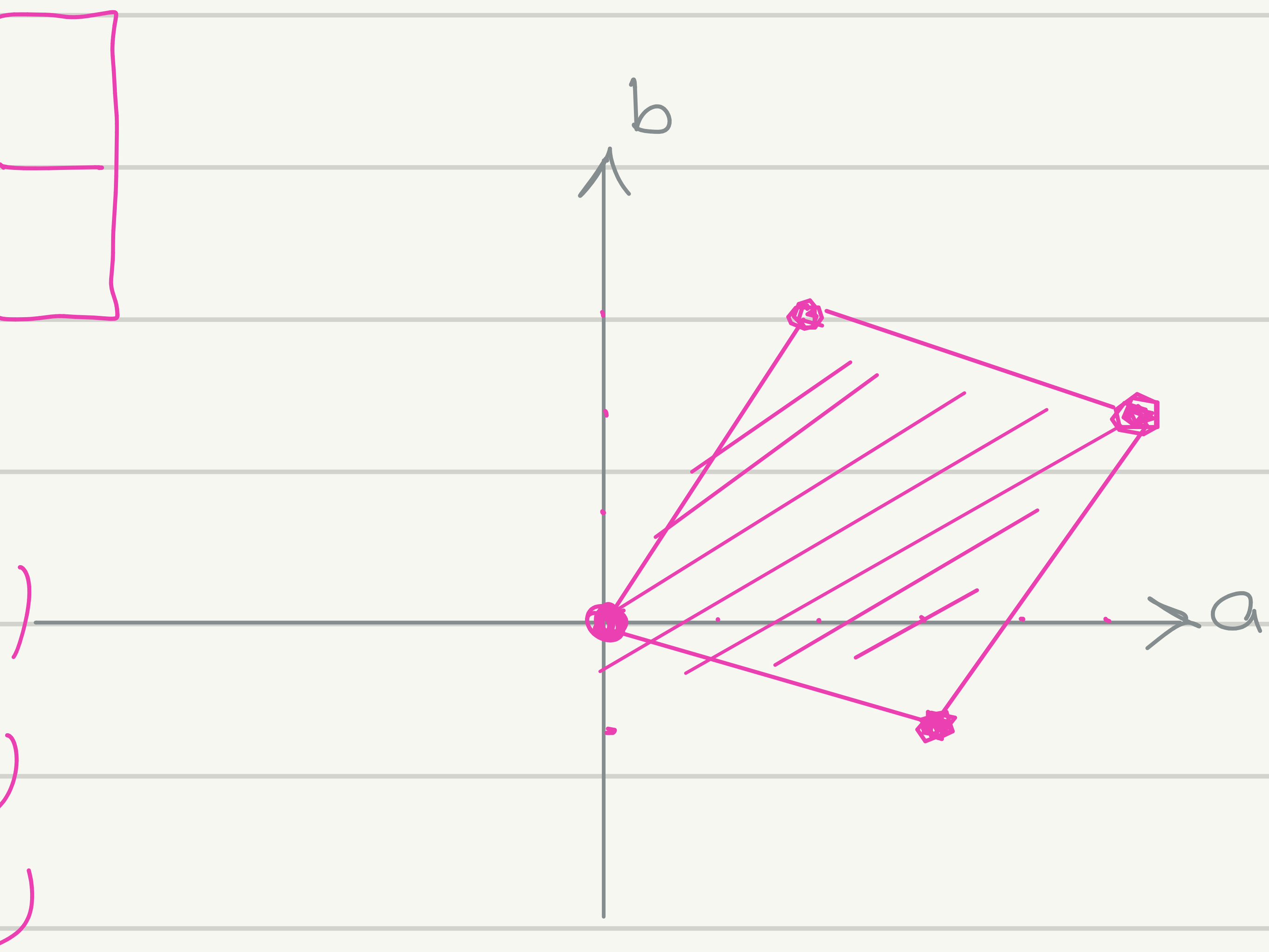
$$\begin{bmatrix}? & ?\\ ? & ?\end{bmatrix}\begin{bmatrix}\ \\ \ \end{bmatrix} = \begin{bmatrix}\ \\ \ \end{bmatrix}$$

$(0,0) \longrightarrow (0,0)$

$*(1,0) \longrightarrow (3,-1)$

$*(0,1) \longrightarrow (2,3)$

$(1,1) \longrightarrow (5,2)$



check these So: $\begin{bmatrix}3 & 2\\ -1 & 3\end{bmatrix}$

Multiplying Matrices

→ we can show multiplication of matrices with combining linear transformations.

$$\begin{bmatrix}2 & -1\\ 0 & 2\end{bmatrix}\cdot\begin{bmatrix}3 & 1\\ 1 & 2\end{bmatrix} = \begin{array}{c|c|c|c} 5 & [2\ \ -1]\begin{bmatrix}3\\1\end{bmatrix} & [2\ \ -1]\begin{bmatrix}1\\2\end{bmatrix} & 0 \\ 2 & [0\ \ 2]\begin{bmatrix}3\\1\end{bmatrix} & [0\ \ 2]\begin{bmatrix}1\\2\end{bmatrix} & 4 \end{array}$$

Dimension of Matrices

$$\begin{bmatrix} 3 & 1 & 4 \\ 2 & -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 3 & 0 & 1 & -2 \\ 1 & 5 & 2 & 0 \\ -2 & 1 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 9 & 21 & -6 \\ 1 & -3 & 8 & -4 \end{bmatrix}$$

2×3 3×4

must be equal

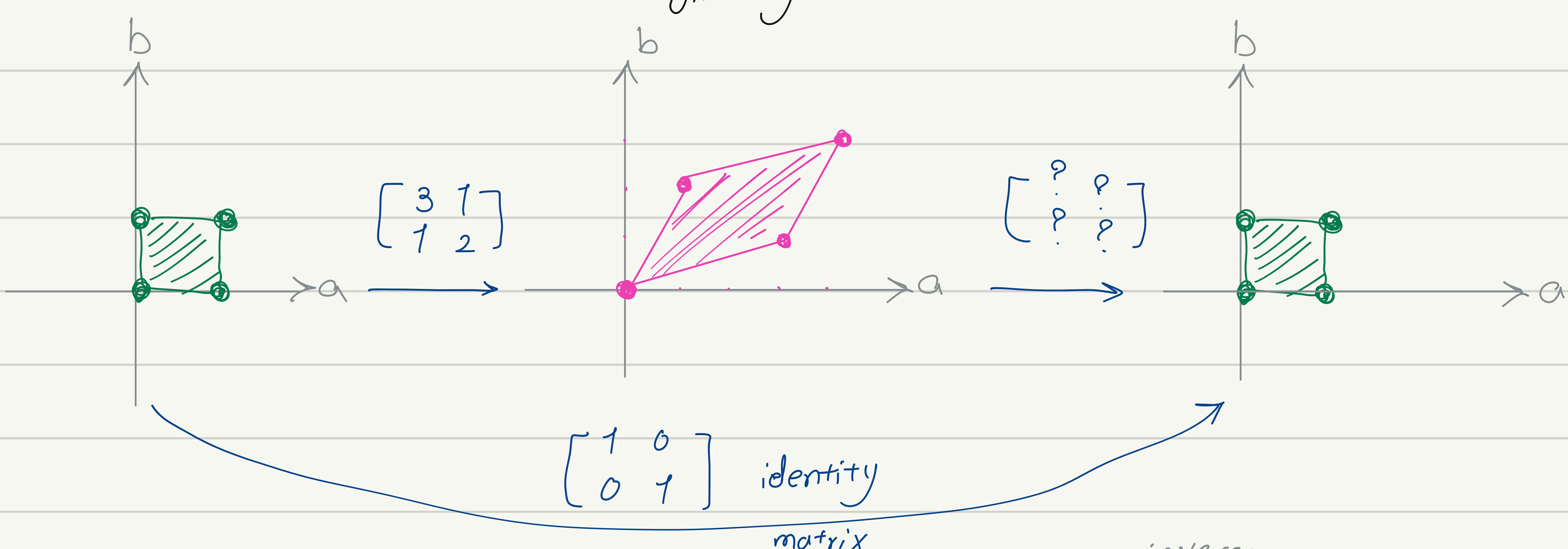
output will be 2×4

- **Identity Matrix:** is the matrix that when multiplied by any other matrix, It gives the same matrix.
  (1s in the diagonal, 0s every where else.)

Matrix Inverses:

is the matrix which the product of the matrices is the identity matrix.

- In Linear Transformations, the inverse matrix is the one that undoes the job of the original matrix, namely the one that returns the plane to where it was at the beginning.



$$\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 2/5 & -1/5 \\ -1/5 & 3/5 \end{bmatrix}$$

inverse

solving 4 linear equations

Which matrices have inverse ?

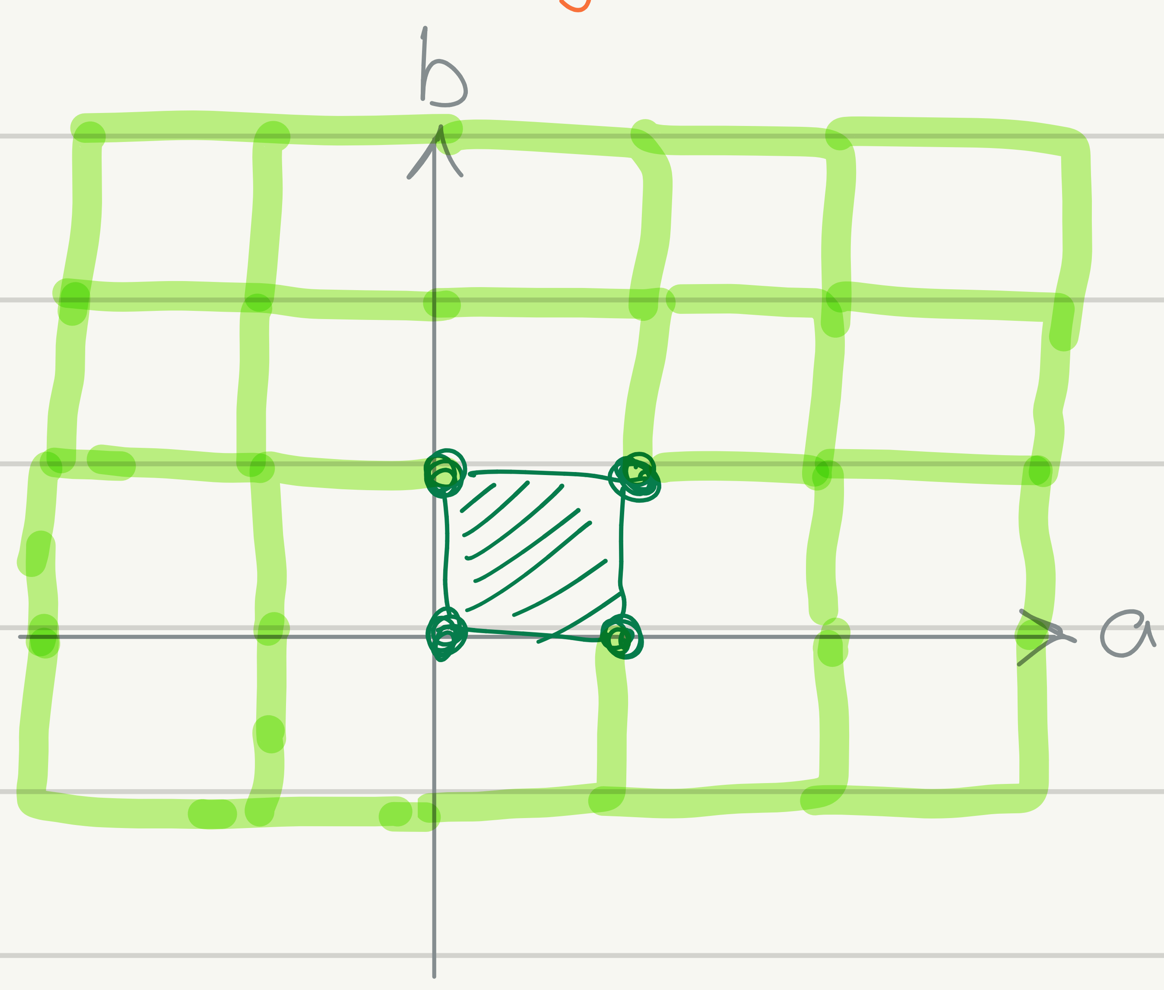
Non-Singular matrices always have inverse, So we call them Invertible matrice. (Det $\neq 0$)

Singular matrices never have inverse, So we call them Non-Invertible. (Det $= 0$)

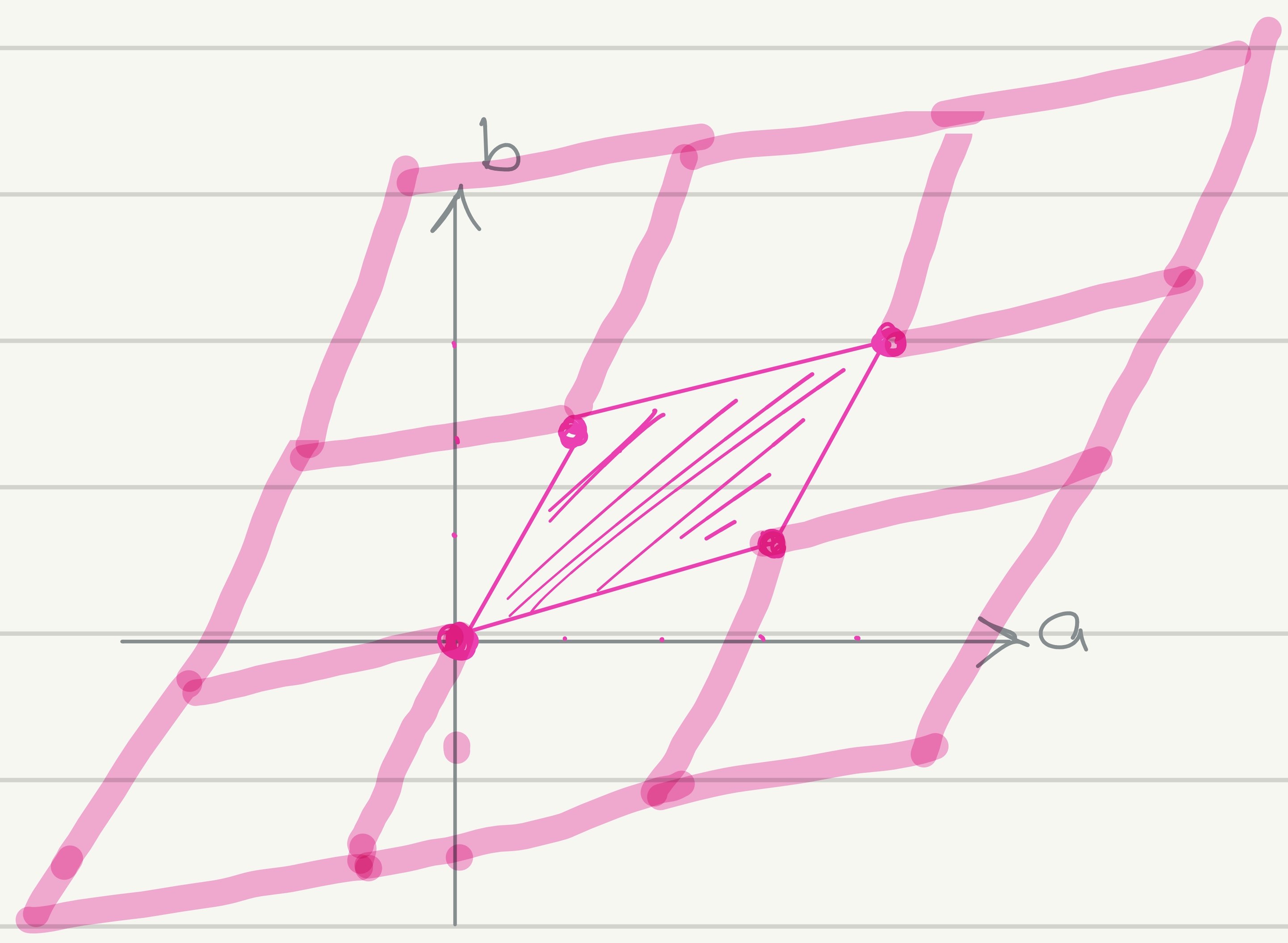
Week 4 – Determinants and Eigenvectors

Singularity and rank of linear transformations

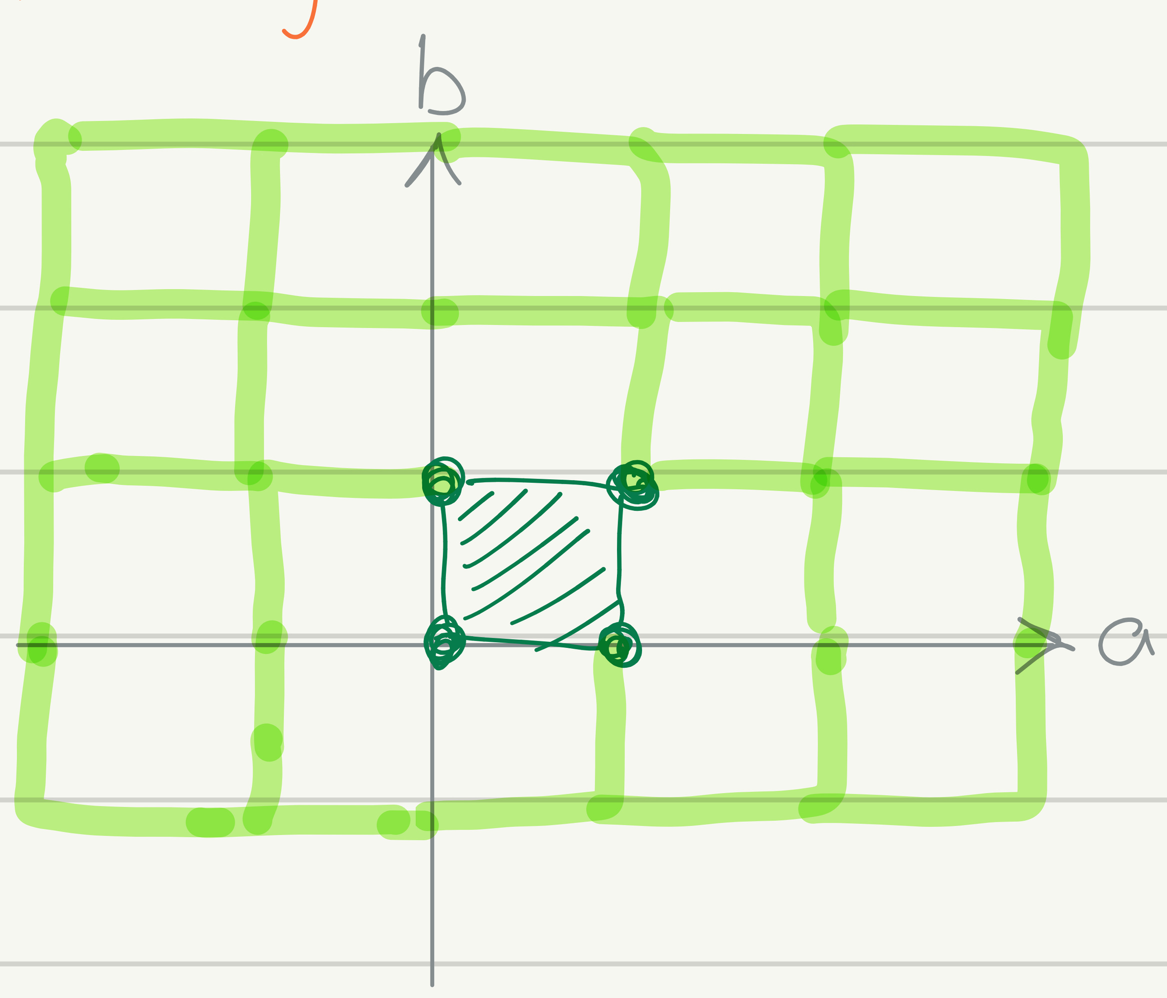
→ Non-singular transformation :



$$\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$$



→ Singular transformation :



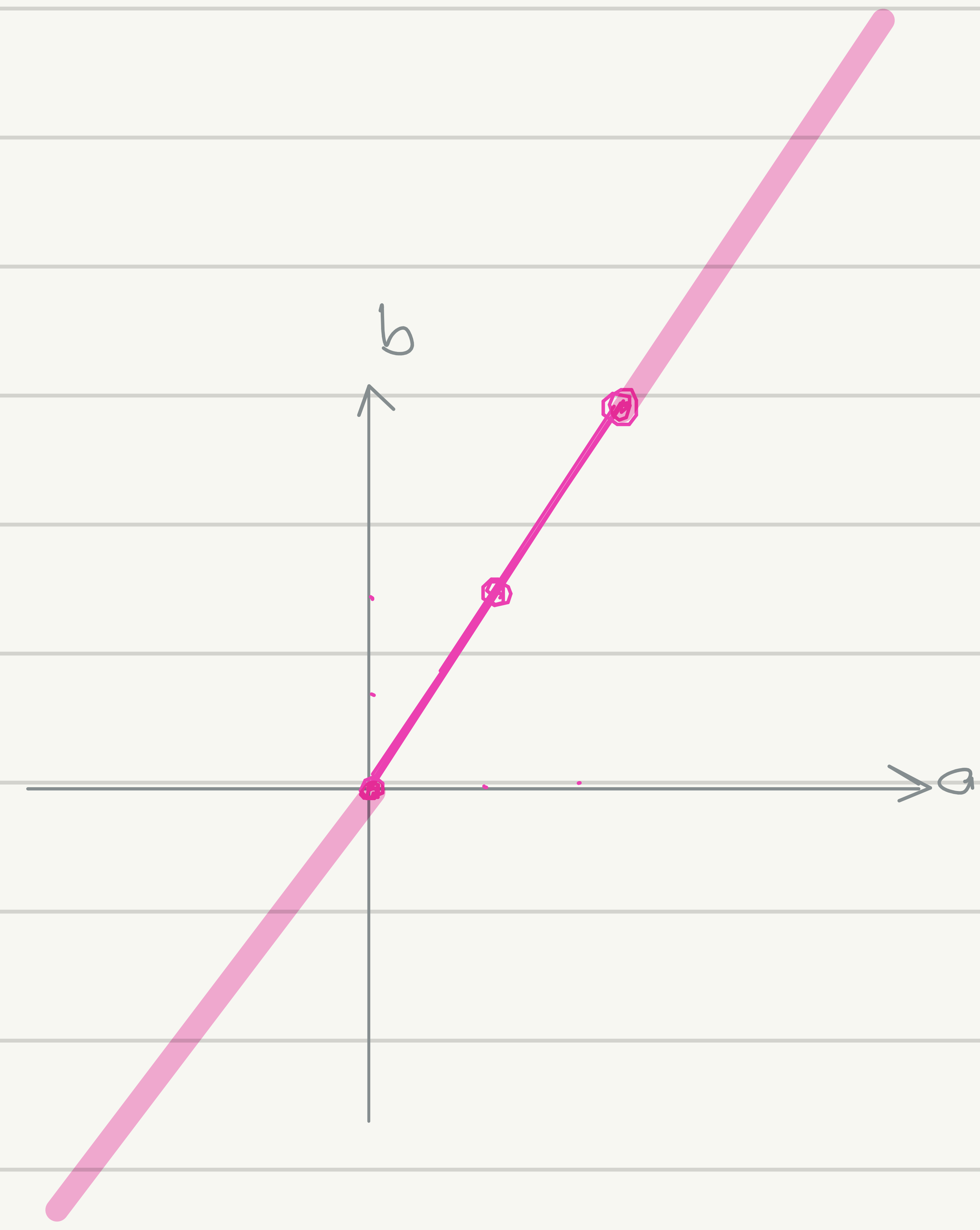
$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} \, \\ \, \end{bmatrix} = \begin{bmatrix} \, \\ \, \end{bmatrix}$$

(0,0) → (0,0)

(1,0) → (1,2)

(0,1) → (1,2)

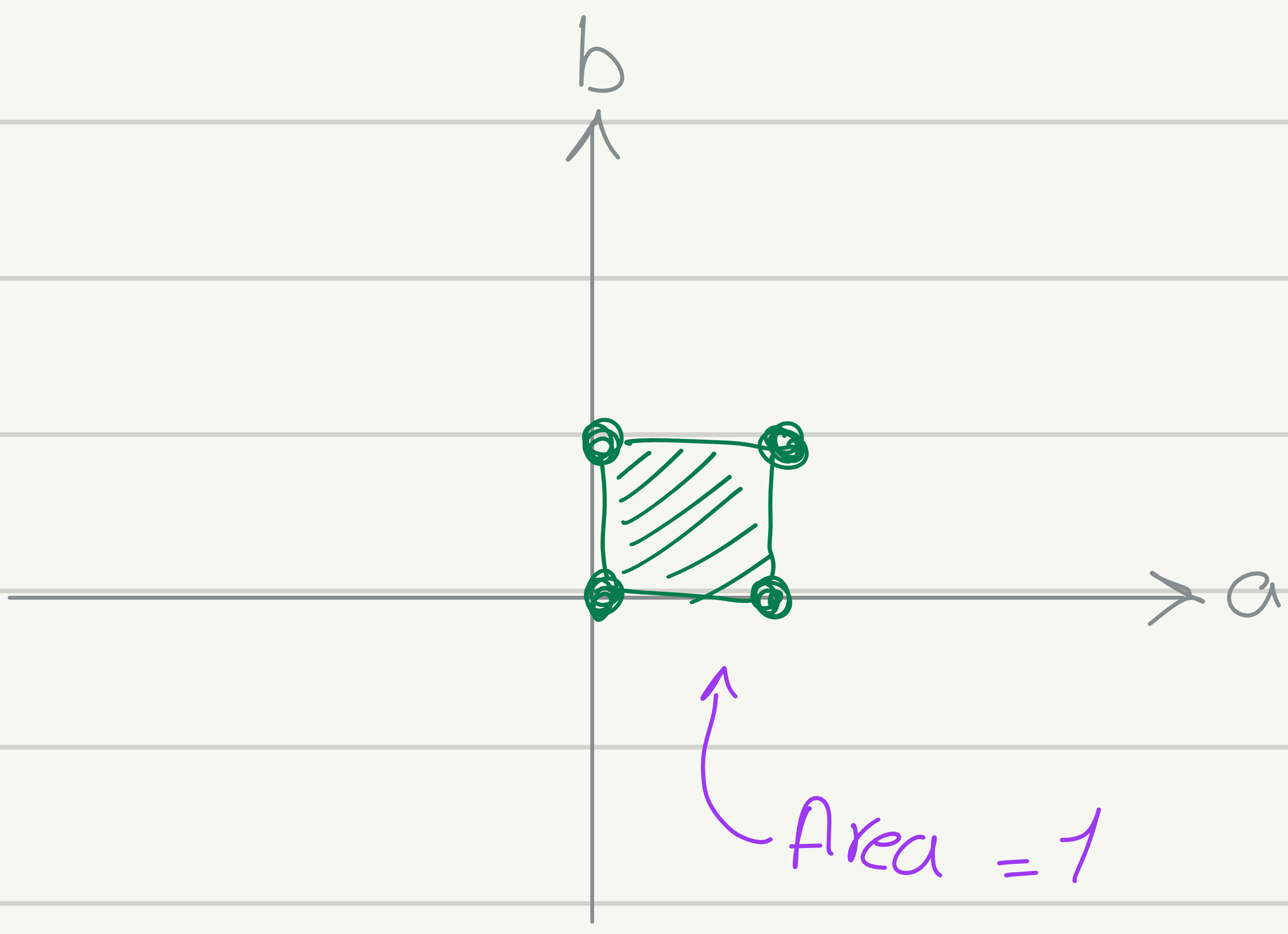
(1,1) → (2,4)



⇒ When a matrix is singular, the transformation plane is not covered entirely.

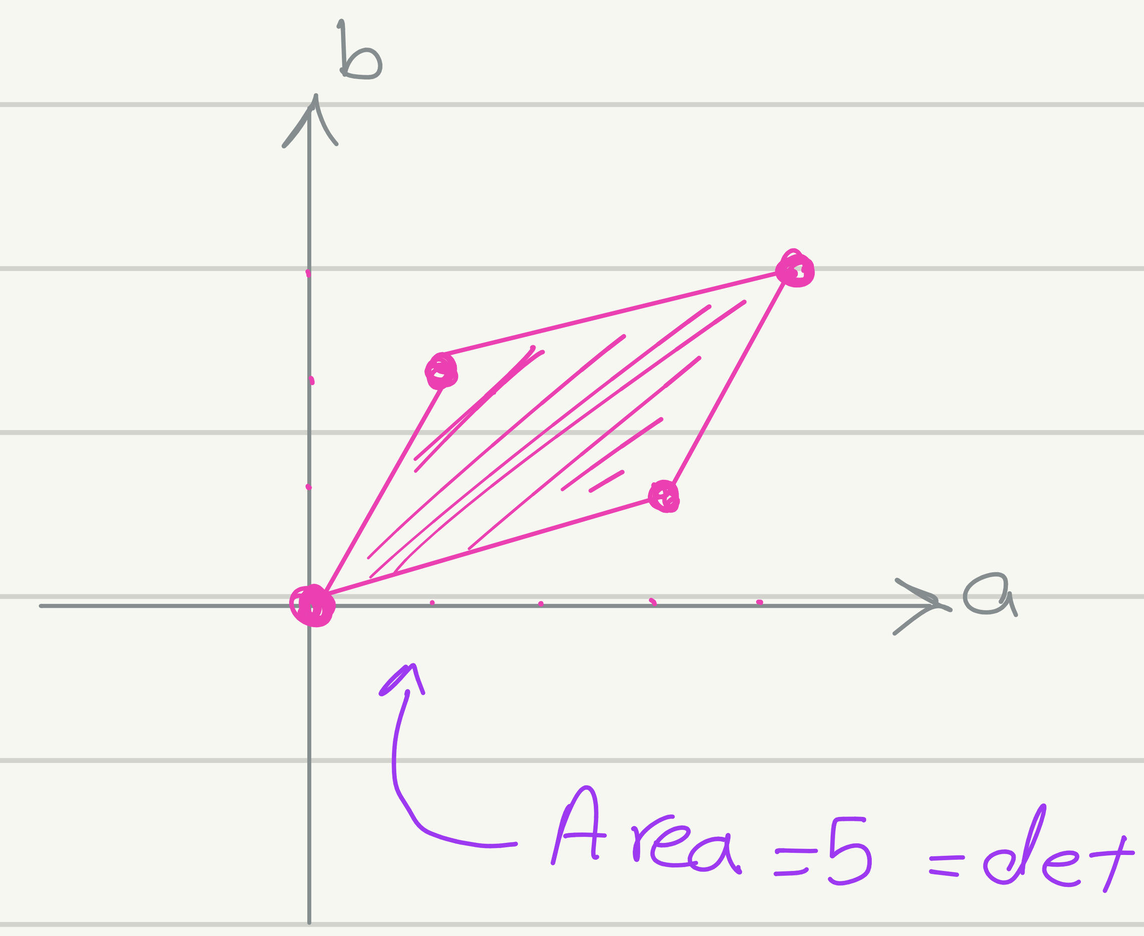
* Rank of linear transformation : the dimension of the image of the linear transformation.

Determinant as an area



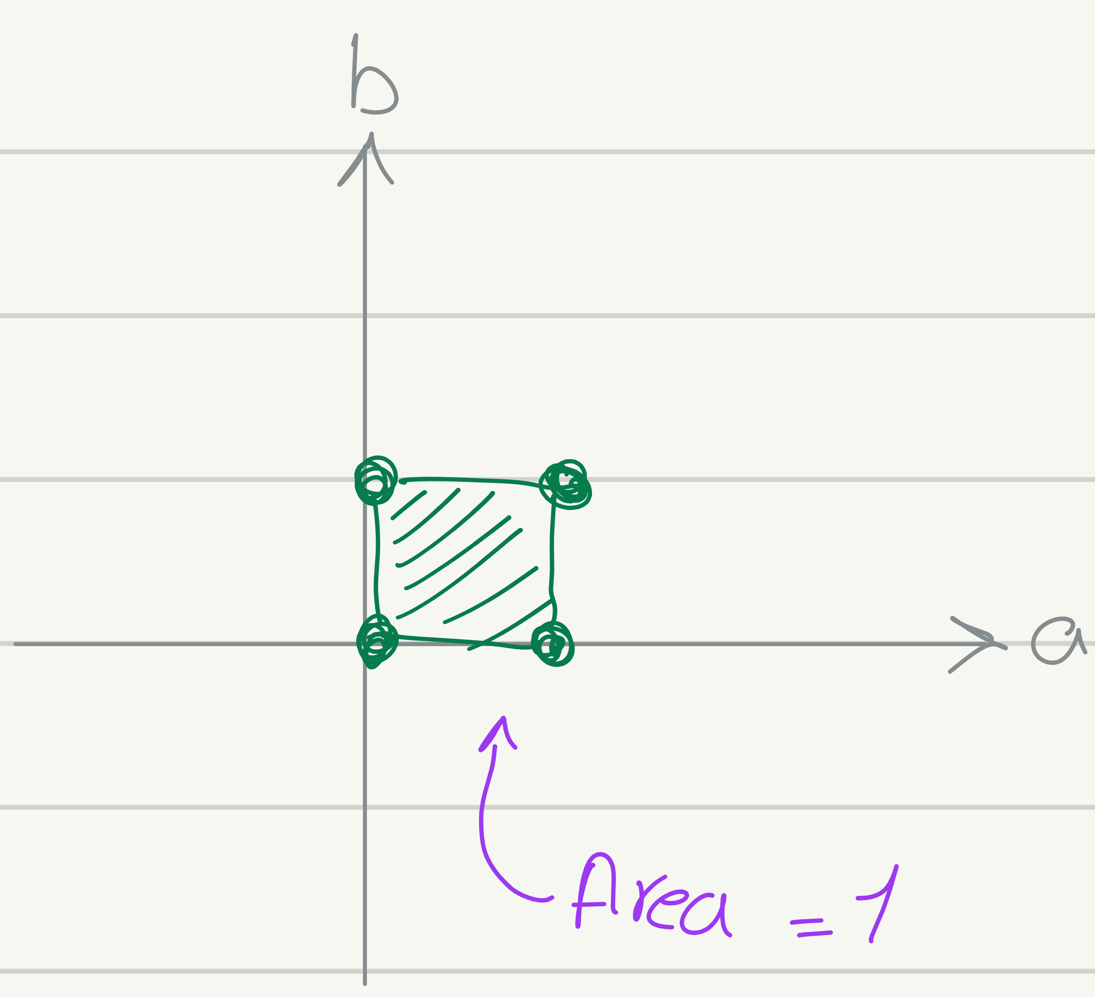
$$\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$$

$$Det = 6 - 1 = 5$$



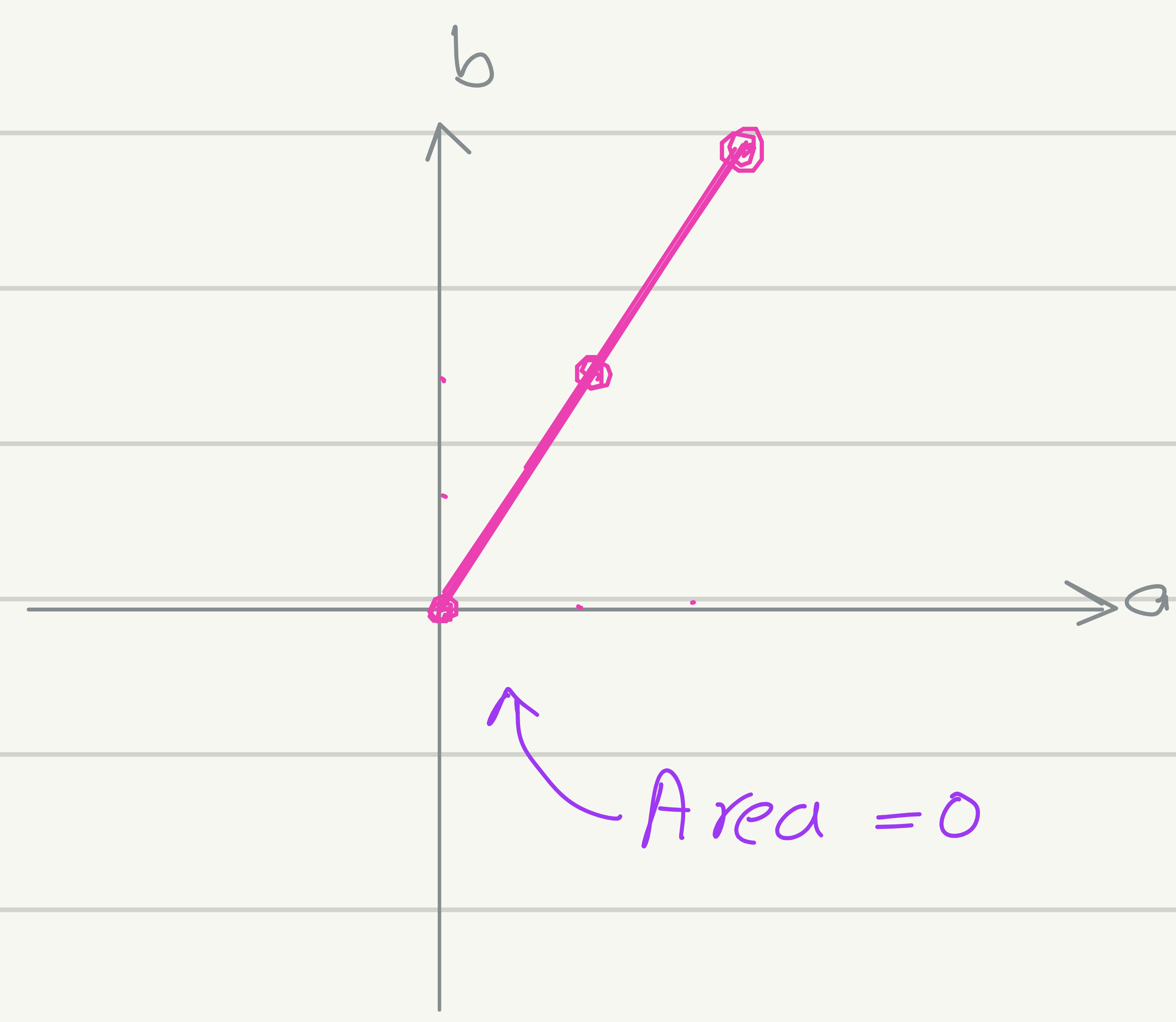
* The determinant of a matrix is the area of the image of the fundamental basis formed by the unit square on the left.

What if the matrix is singular?!

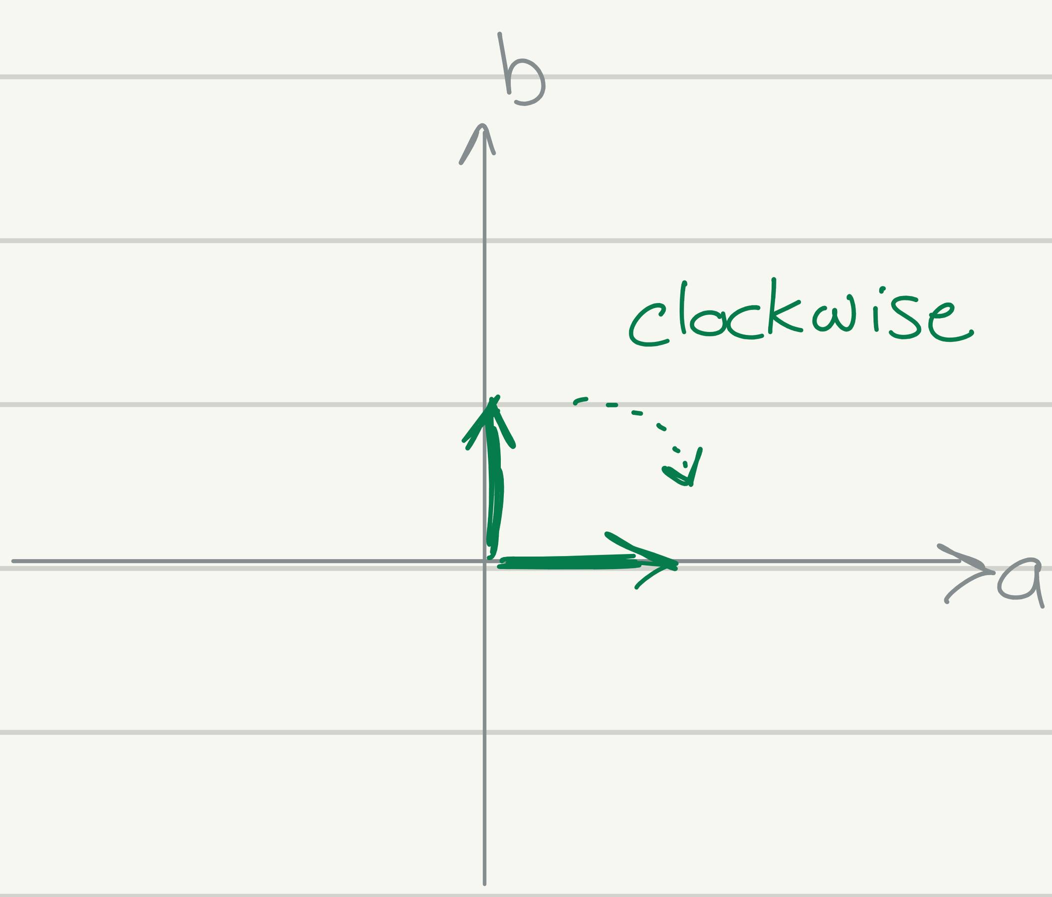


$$\begin{array}{cc} a & b \\ \left[\begin{array}{c} 1 \\ 2 \end{array}\right. & \left.\begin{array}{c} 1 \\ 2 \end{array}\right] \end{array}$$

$$Det = 2 - 2 = 0$$



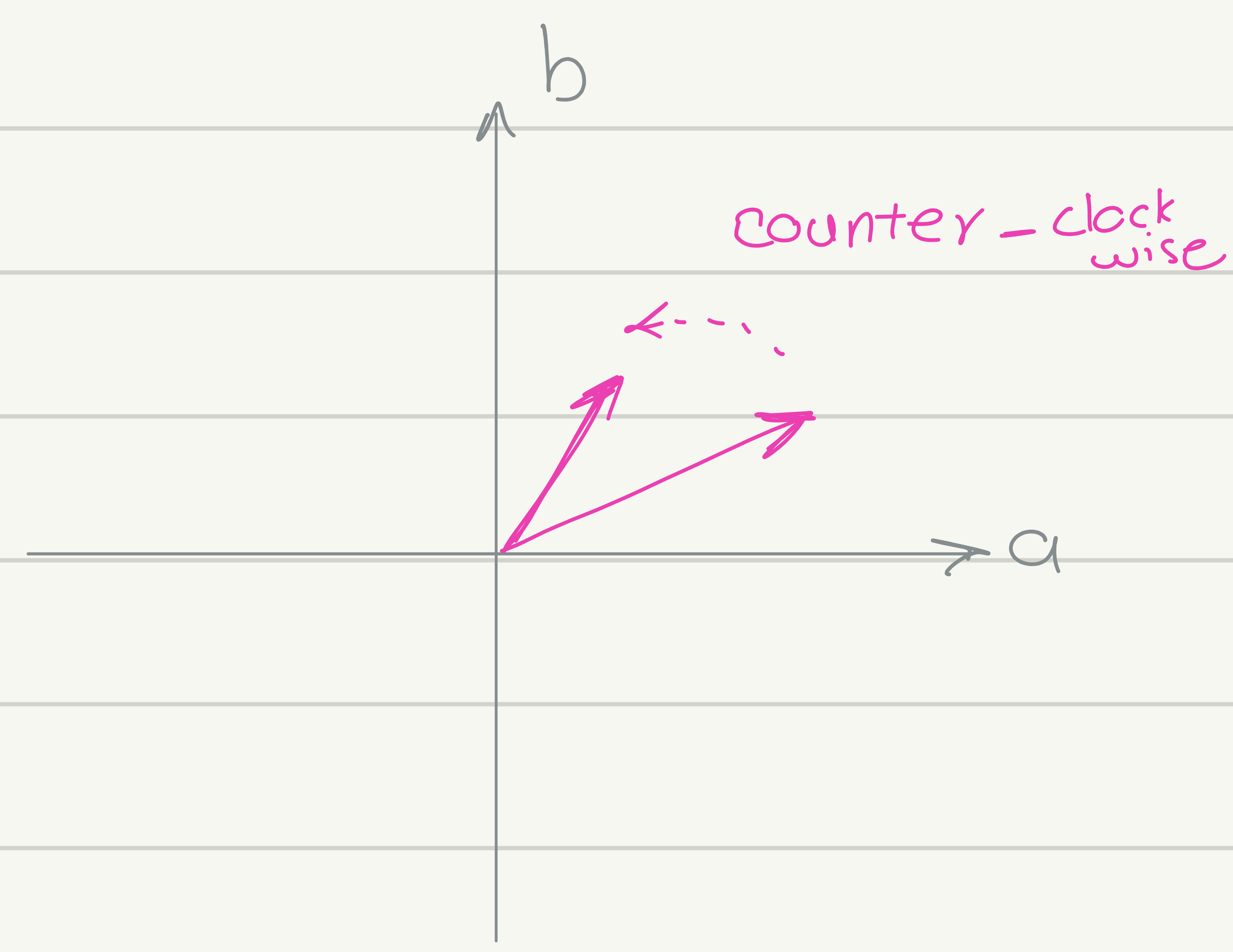
Negative determinants?



$$\begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$$

$$Det = -5$$

negative



Determinant of a product

$$\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ -3 & 3 \end{bmatrix}$$

det = 5   det = 3   det = 15

$$* \quad \det(AB) = \det(A) \cdot \det(B) \quad *$$

Determinant of inverse

$$\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 0.4 & -0.2 \\ -0.2 & 0.6 \end{bmatrix}$$

$$\underbrace{\text{det} = 5 \qquad \text{det} = 0.2}_{5^{-1} = 0.2}$$

$$* \quad \det(A^{-1}) = \frac{1}{\det(A)} \quad *$$

A is a invertible matrix

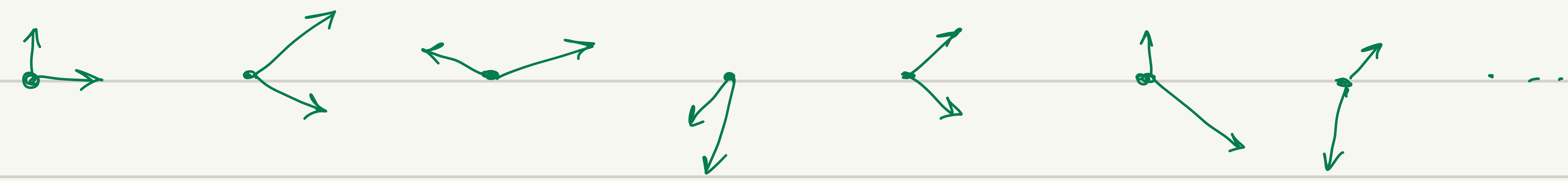
Determinant of the identity Matrix

$$* \quad \det(I) = 1 \quad *$$

Bases (plural of 'Basis)

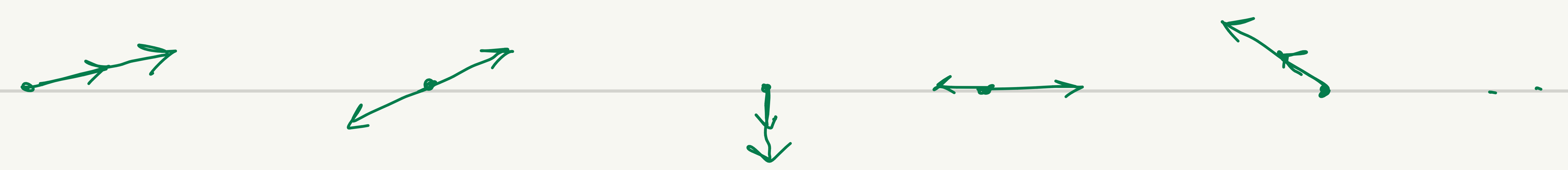
- **Basis** : The main property of a basis is that every point in the space can be expressed as a linear combination of elements in the basis.

Examples: Bases



- **Non_Basis** : Anything that comprises two vectors that go in the same direction and they could be opposite. (As long as they belong to the same line, the two vectors don't form a basis.)

Examples: Not Bases

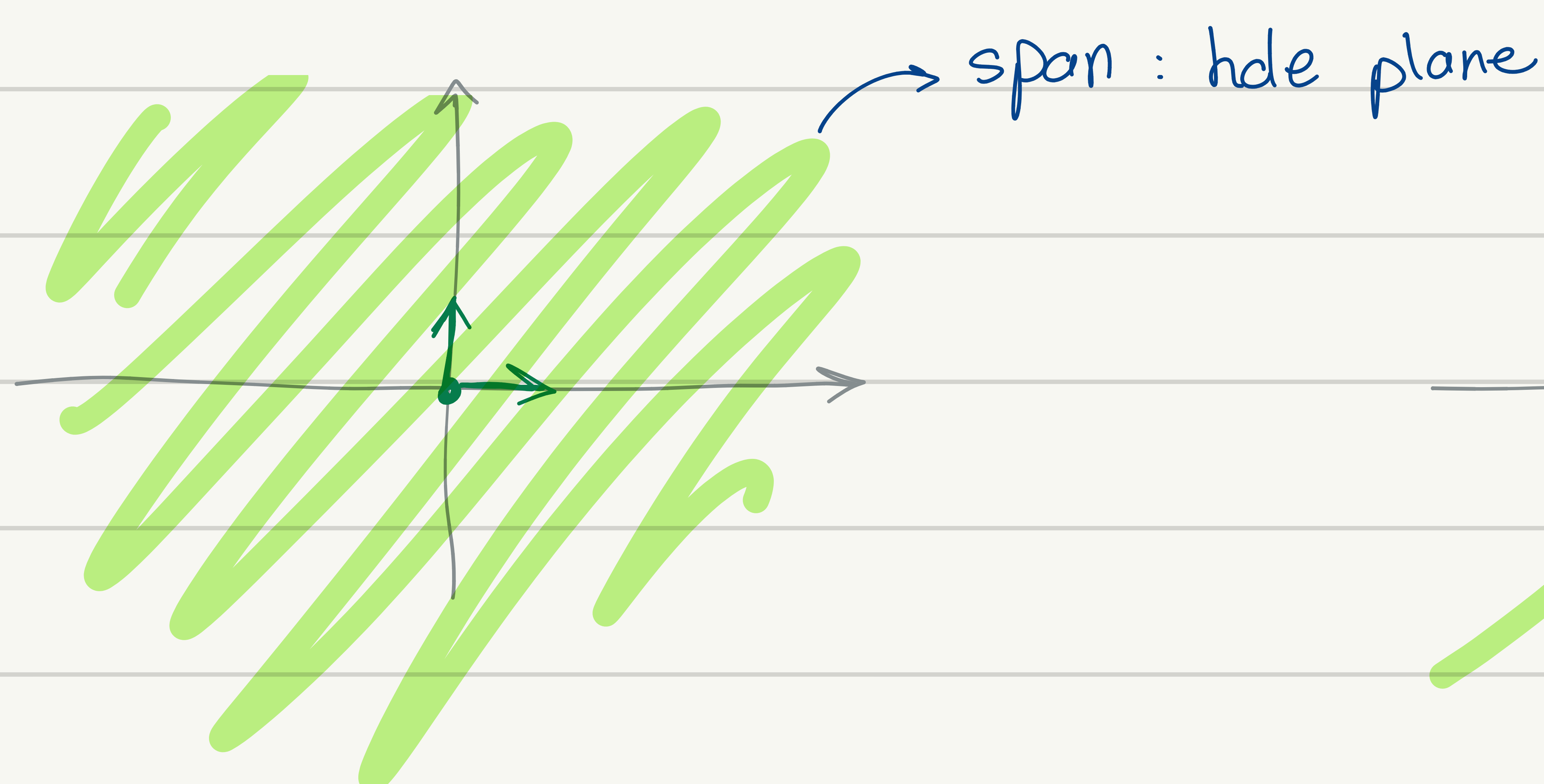


Span

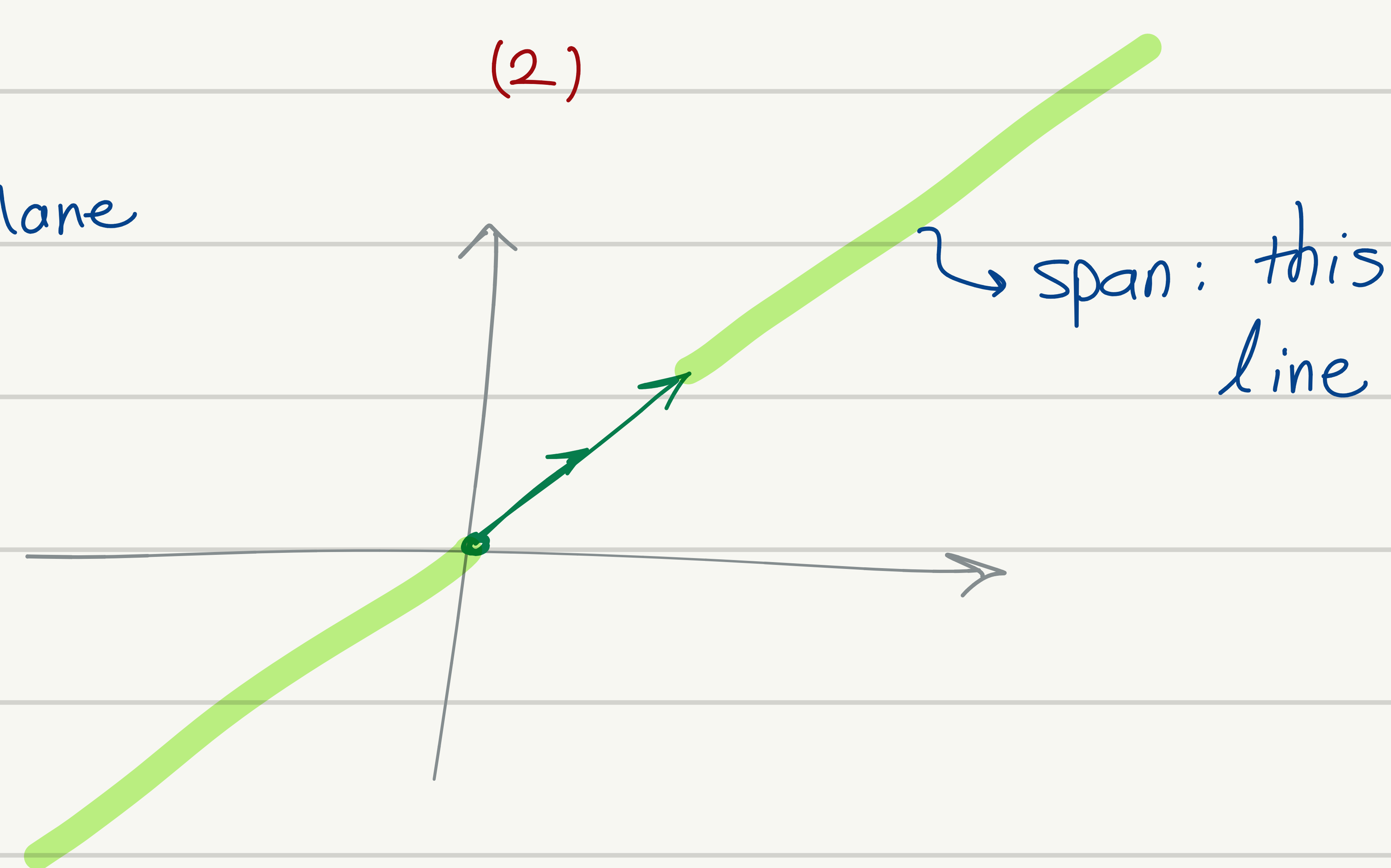
The span of a set of vectors is simply the set of points that can be reached by walking in the direction of these vectors in any combination.

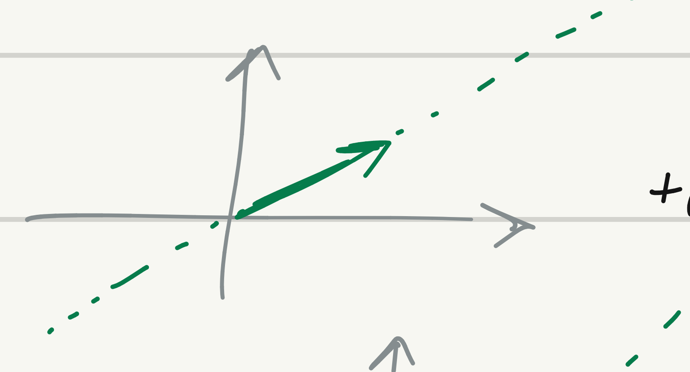
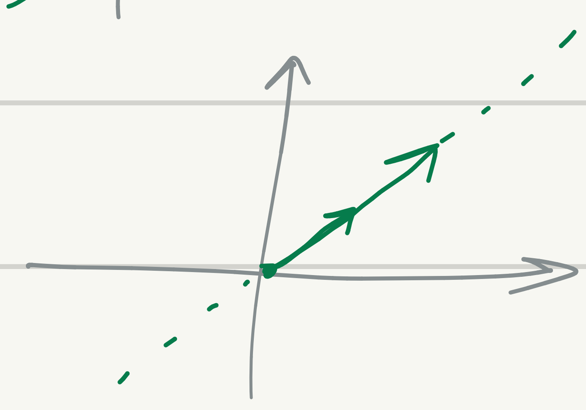
Example:

(1)



(2)

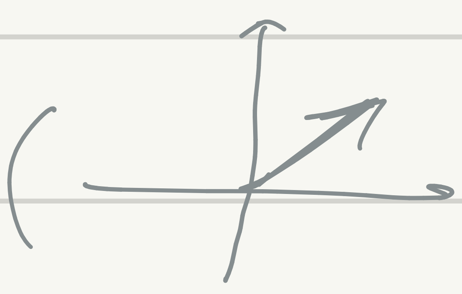


* A basis is a minimal spanning set. So  this vector forms a basis of the line. But these 2 vectors  don't, because they are too many.

* Number of elements in the basis is the dimension.

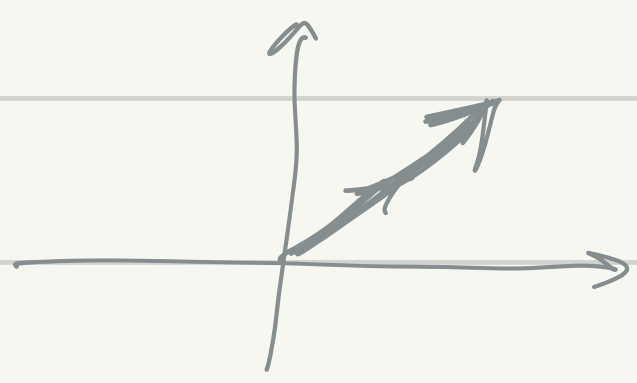
Linearly Independent and Linearly Dependent vectors

- **linearly independent**: A group of vectors is said to be linearly independent if none of the vectors in the group can be obtained as a linear combination of the others.

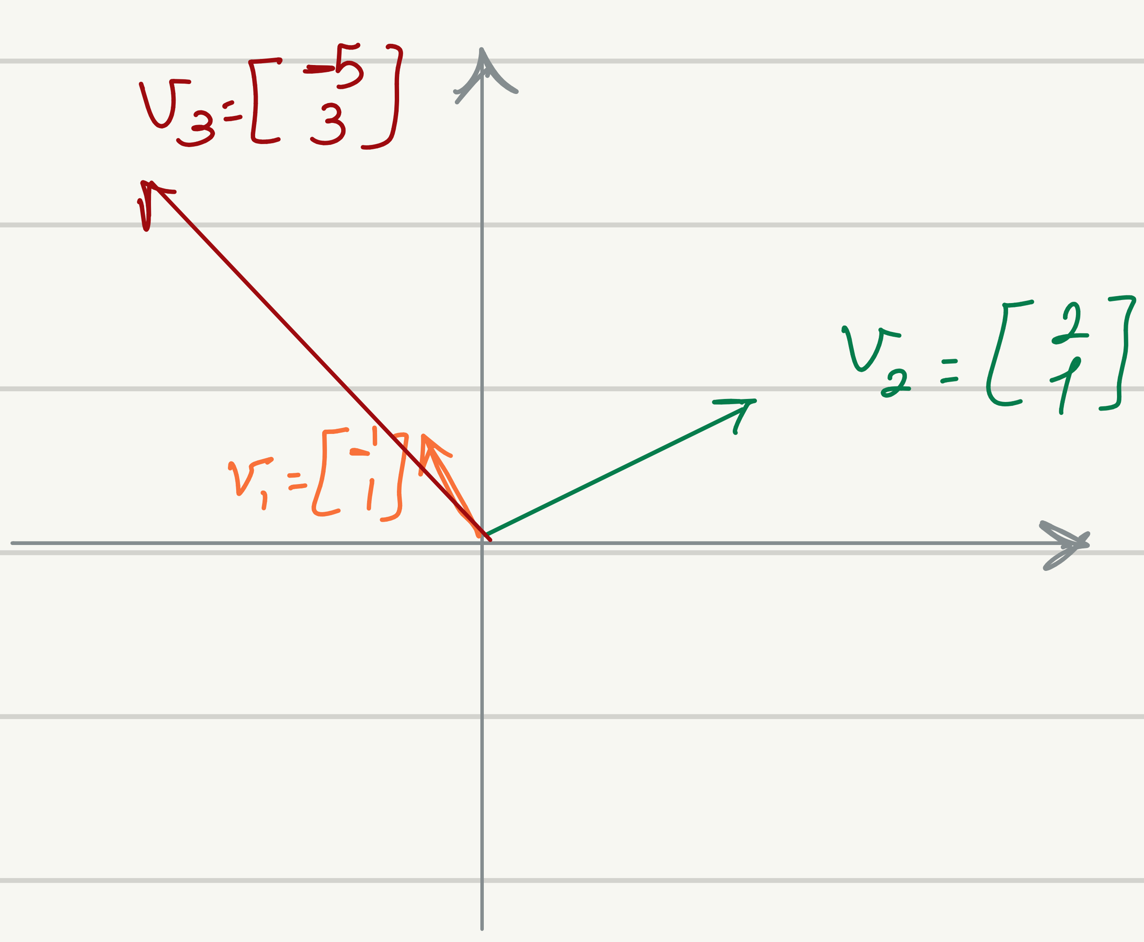
→ Just one vector in the plane (), it is always going to be linearly independent.

- **linearly dependent**: Since one vector can be obtained as a linear combination of the others, this set of vectors is called linearly dependent.

→ Like:



→ Let's see how to check for linear dependence:



$$\alpha V_1 + \beta V_2 = V_3$$

$$\alpha \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ 3 \end{bmatrix}$$

$$\Downarrow$$

$$\left.\begin{aligned} -\alpha + 2\beta &= -5 \\ \alpha + \beta &= 3 \end{aligned}\right\} \rightarrow \begin{aligned} \beta &= \frac{-2}{3} \\ \alpha &= 11/3 \end{aligned}$$

* V_3 is a linear combination of V_1 and V_2, So this set is linearly dependent.

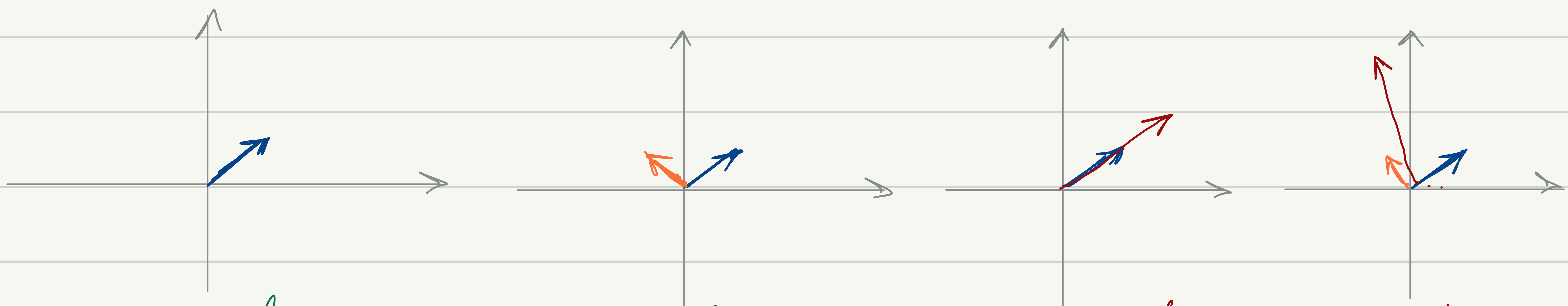
If the system has no solution, the set is linearly independent.

Basis : a formal definition

A Basis is a set of vectors that :

- Spans a vector space.
- Is linearly independent.

Ex :

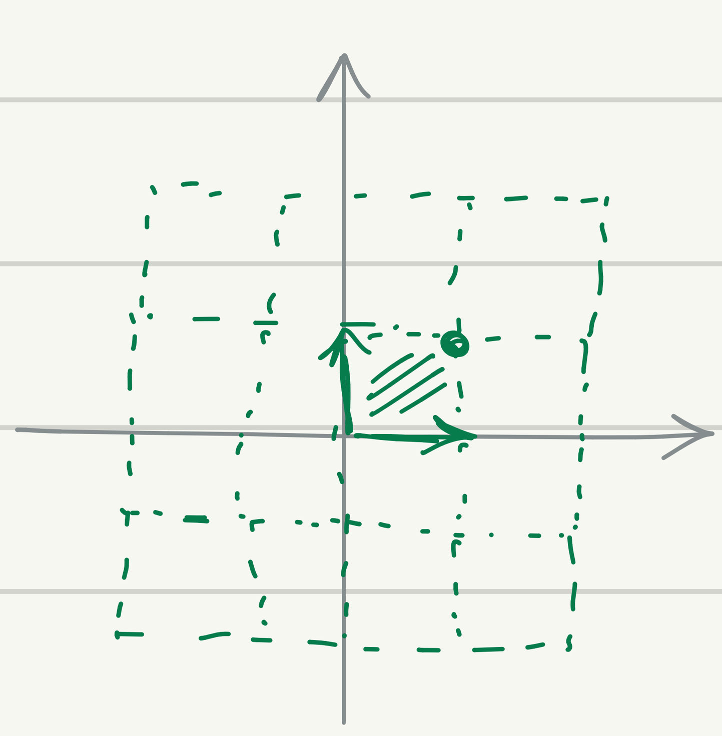


spans a line	spans the plane	Spans a line	spans the plane
linearly independent	linearly independent	linearly dependent	linearly dependent
is a basis	is a basis	Not a basis	Not a basis

** Not all sets of N vectors are a basis for N-dimensional space.

Eigen basis : If a set of eigenvectors of T forms a basis of the domain of T, then this basis is called an eigenbasis.

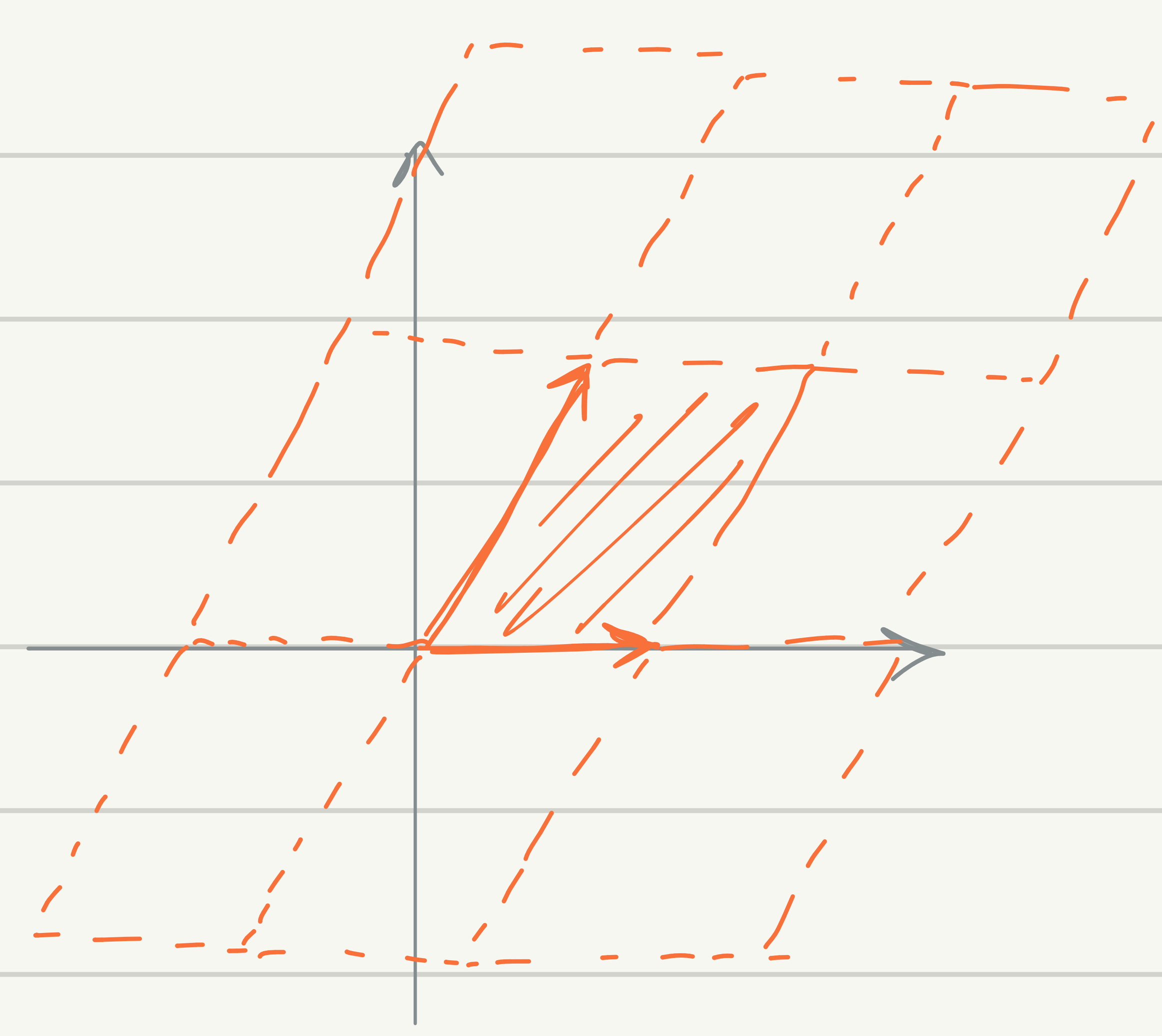
- Basis



$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$(1,0) \rightarrow (2,0)$

$(0,1) \rightarrow (1,3)$

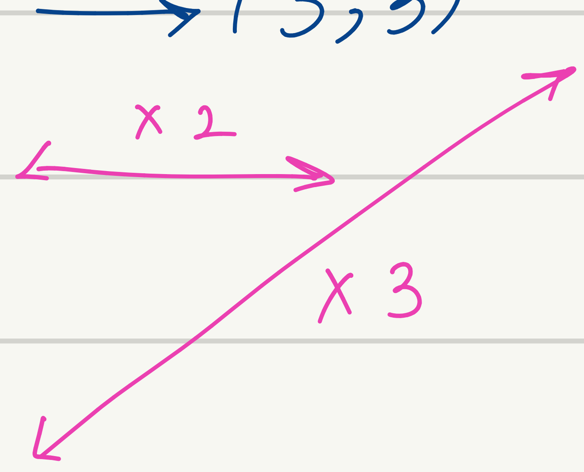


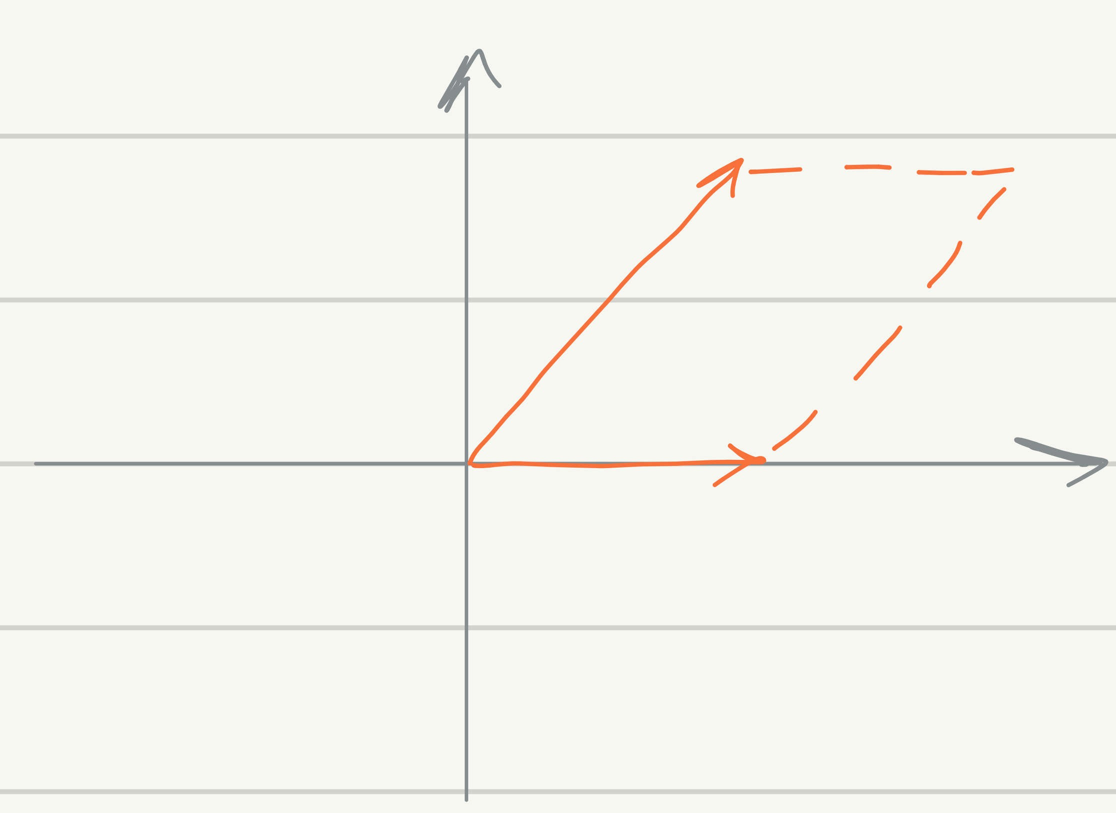
- Eigenbasis

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

$(1,0) \longrightarrow (2,0)$

$(1,1) \longrightarrow (3,3)$





Eigenvectors and Eigenvalues

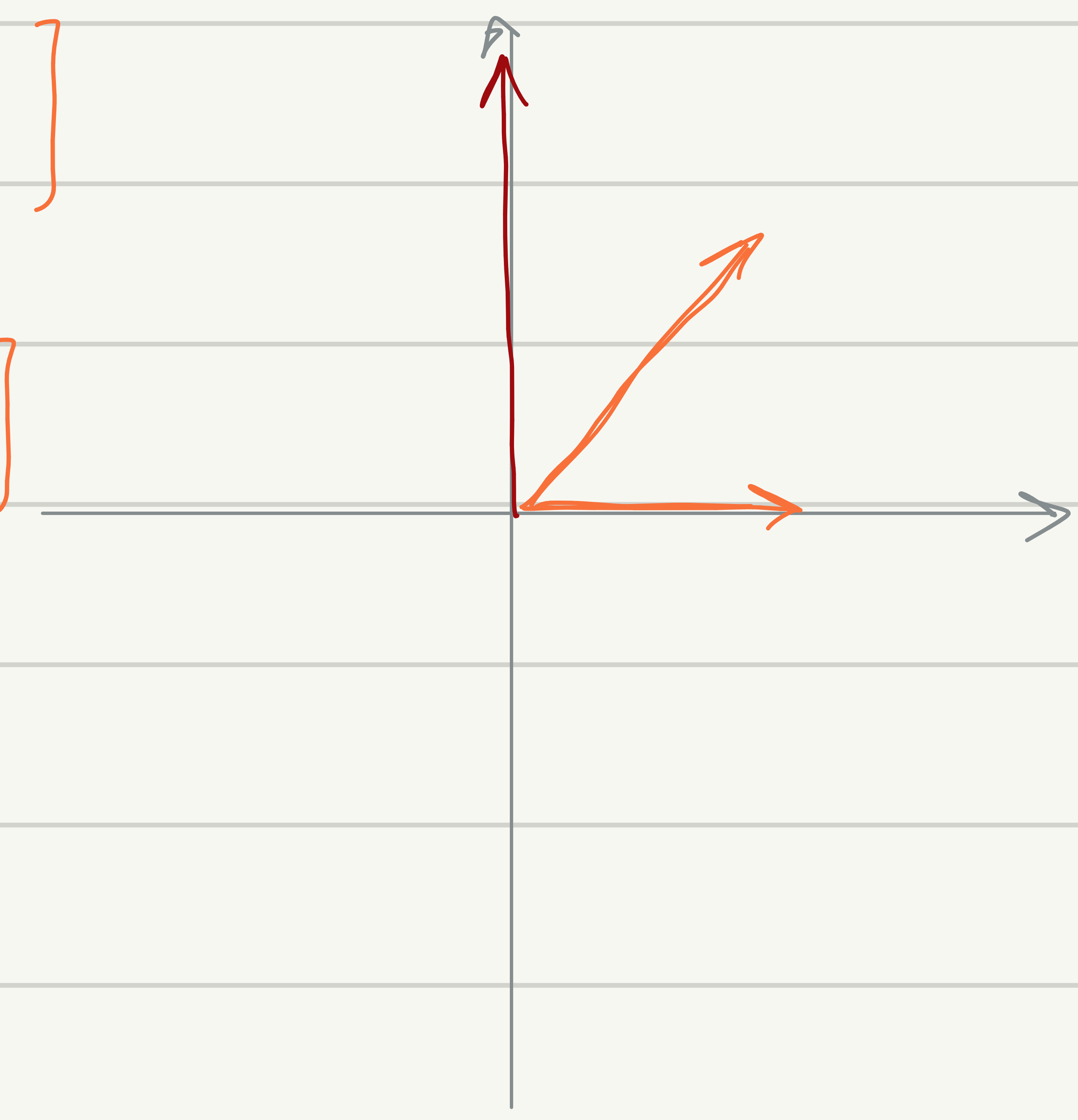
eigenvectors eigenvalue

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$* \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$$

$$\neq \lambda\begin{bmatrix} -1 \\ 2 \end{bmatrix}$$



So:

Matrix Mul More work

Scalar Mul less work

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightsquigarrow A v_1 = \lambda_1 v_1$$

8 multiplications 2 muls

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 1 \end{bmatrix} \rightsquigarrow A v_2 = \lambda_2 v_2$$

Eigenvectors

Eigenvalues

* Change $\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$ to Eigenvectors and Eigenvalues:

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}\left[-3\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right] = -6\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 6\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$$

In summary:
- $Av = \lambda v$ for each eigenvector / eigenvalue
- Eigevectors: Direction of strech
- Eigenvalues: how much strech
- Eigenbasis: the set of a matrix's eigenvectors, can be arranged as a matrix with one eigenvector in each column
- Save work and characterize a transformation.

* Finding Eigenvalues : 2 x 2 matrix *

if λ is an eigenvalue : $\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

for infinitely many (x,y)

$$\Bigg(\underbrace{\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}}_{\begin{bmatrix} 2-\lambda & 1 \\ 0 & 3-\lambda \end{bmatrix}}\Bigg)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\rightarrow$ Has infinitely many solutions

$$\det \begin{bmatrix} 2-\lambda & 1 \\ 0 & 3-\lambda \end{bmatrix} = 0$$

characteristic polynomial $(2-\lambda)(3-\lambda) - 1 = 0$ $\begin{cases} \lambda = 2 \\ \lambda = 3 \end{cases}$

* Finding Eigenvectors : 2x2 matrix *

Eigenvalues : $\lambda = 2$, $\lambda = 3$

Solve the equations:

Eigenvectors

$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 2\begin{bmatrix} x \\ y \end{bmatrix}$ $\quad$ $2x + y = 2x$, $0x + 3y = 2y$ $\quad$ $x = 1$, $y = 0$ $\quad$ $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 3\begin{bmatrix} x \\ y \end{bmatrix}$ $\quad$ $2x + y = 3x$, $0x + 3y = 3y$ $\quad$ $x = 1$, $y = 1$ $\quad$ $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

* Finding Eigen values : 3x3 matrix *

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & -3 \\ -1 & -3 & 0 \end{bmatrix} \qquad \lambda I = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

characteristic polynomial: $\det(A - \lambda I) = 0$

$$\det \begin{bmatrix} 2-\lambda & 1 & -1 \\ 1 & -\lambda & -3 \\ -1 & -3 & -\lambda \end{bmatrix} = 0 \longrightarrow (2-\lambda)\lambda^2 + 3 + 3 - 9(2-\lambda) + \lambda + \lambda =$$

$$\lambda^3 + 2\lambda^2 + 11\lambda - 12 = 0$$

$$-(\lambda+3)(\lambda-1)(\lambda-4) = 0 \quad \begin{cases} \lambda = -3 \\ \lambda = 1 \\ \lambda = 4 \end{cases}$$

* Finding Eigen vectors : 3x3 matrix *

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & -3 \\ -1 & -3 & 0 \end{bmatrix}$$

Eigenvalues: $\lambda = -3$, $\lambda = 1$, $\lambda = 4$

For $\lambda = 4$: $Av = \lambda v$

$$\begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & -3 \\ -1 & -3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 4 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \Longrightarrow \begin{aligned} 2x_1 + x_2 - x_3 &= 4x_1 \\ x_1 - 3x_2 &= 4x_2 \\ -x_1 - 3x_2 &= 4x_3 \end{aligned}$$

Solving

Eigenvector $= \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

Do the same thing for -3, 1 :

$\lambda_1 = 4$, $\lambda_2 = 1$, $\lambda_3 = -3$

Eigenvectors: $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

** Note : Eigenvalues $\longrightarrow$ determinant $\longrightarrow$ just square matrix

Number of Eigenvectors

	2x2 matrix	3x3 matrix
Eigenvalues	λ_1, λ_2	$\lambda_1, \lambda_2, \lambda_3$

2x2 matrix

if $\lambda_1 \neq \lambda_2 \rightarrow$ 2 eigenvectors (2 different directions)

if $\underbrace{\lambda_1 = \lambda_2}$ → 1 eigenvector (1 direction)
→ 2 eigenvectors (2 different dir)

Repeated Eigenvalues

3x3 matrix

if $\lambda_1 \neq \lambda_2 \neq \lambda_3 \rightarrow$ 3 eigenvectors (3 different dir)

if $\lambda_1 = \lambda_2 \neq \lambda_3$ → 2 eigenvectors (2 different dir)
→ 3 eigenvectors (3 different dir)

if $\lambda_1 = \lambda_2 = \lambda_3$ → 1 eigenvector (1 direction)
→ 2 eigenvector (2 different dir)
→ 3 eigenvector (3 different dir)

Dimensionality Reduction

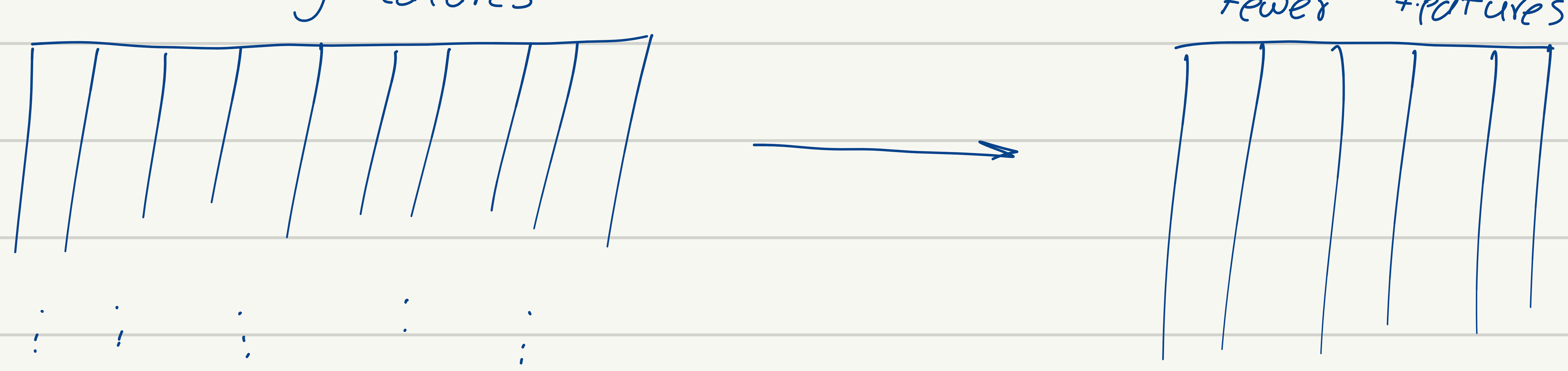
* Advantages *

→ Reduce dimensions (# of columns) of dataset
→ preserve as much information as possible

Leads to smaller dataset
Easier to visualize

Large table
Many Features

Smaller table
Fewer Features



Projections: The idea behind dimensionality reduction is to move your data points into a vector space with fewer dimensions. This is call a projection.

Ex:

x	y
1.0	1.0
1.2	1.6
-0.5	0.2
-1.3	-0.6

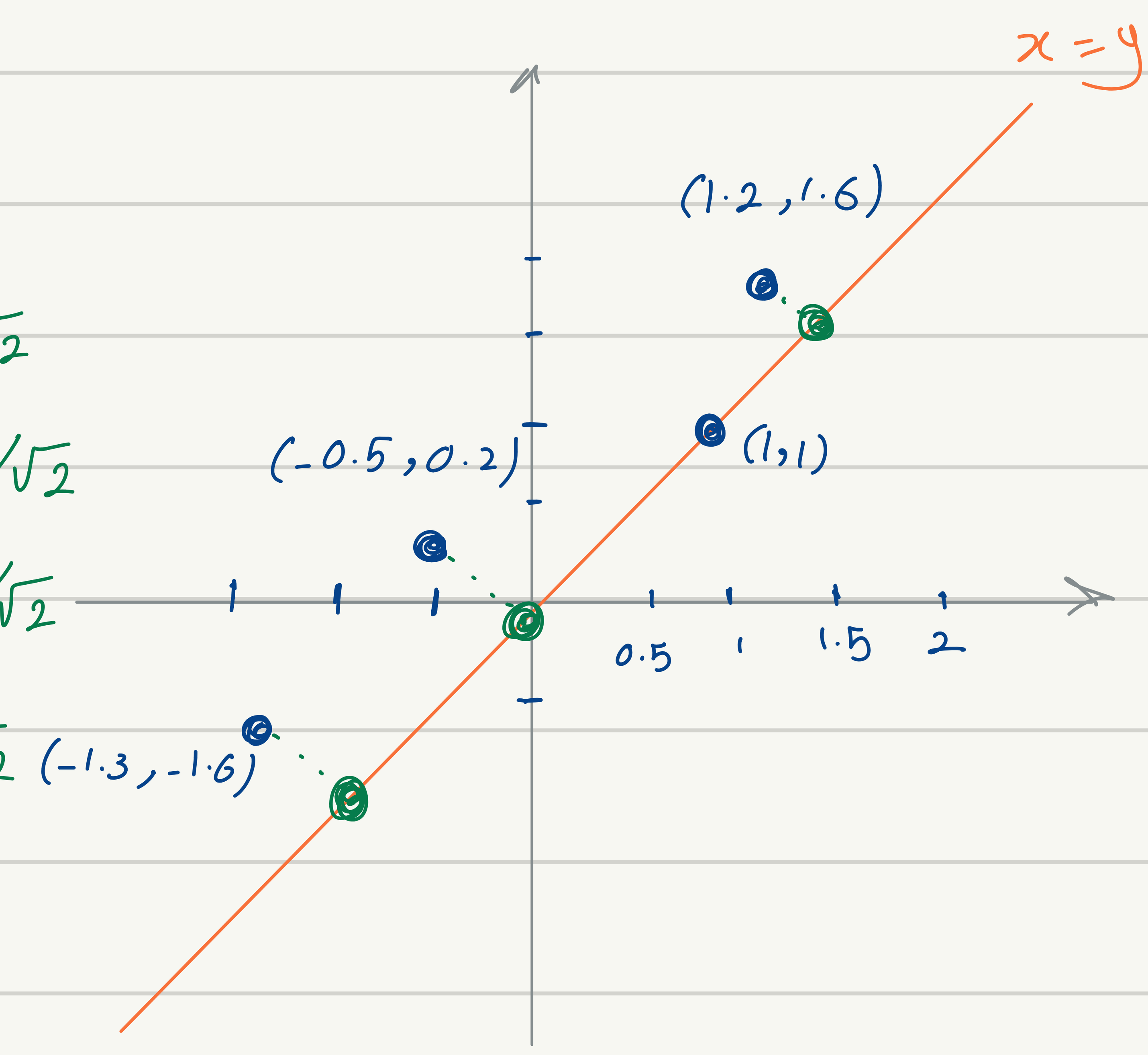
$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{1}{\sqrt{2}}$$

$(1+1)/\sqrt{2}$

$= (1.2+1.6)/\sqrt{2}$

$(-0.5+0.2)/\sqrt{2}$

$(-1.3-1.6)/\sqrt{2}$



* To project a matrix A onto a vector v

1) →

$$\underset{r \times 1}{A_p} = \underset{r \times c}{A} \cdot \underset{c \times 1}{\frac{v}{\|v\|_2}} \rightsquigarrow \text{norm}$$

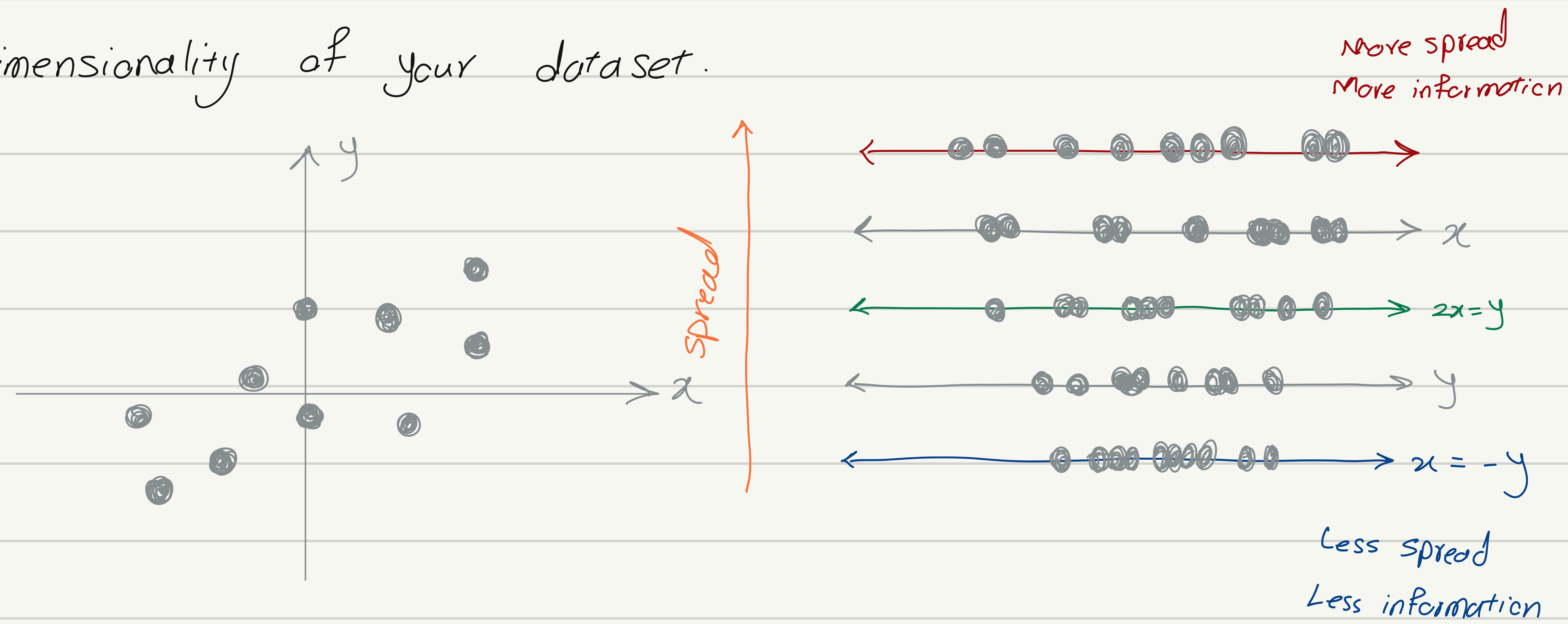
2) →

$$\underset{r \times 2}{A_p} = \underset{r \times c}{A} \cdot \underset{c \times 2}{\overbrace{\begin{bmatrix} \frac{v_1}{\|v_1\|_2} & \frac{v_2}{\|v_2\|_2} \end{bmatrix}}^{v}}$$

$$\Rightarrow \underset{r \times 2}{A_p} = \underset{r \times c}{A} \; \underset{c \times 2}{V}$$

Principal Component Analysis (PCA)

Goal: The goal of PCA will be to find the projection that preserve the max possible spread in your data, even as you reduce the dimensionality of your dataset.



* Benefits of Dimensionality Reduction *

- Easier data set to manage
- PCA reduces dimensions while minimizing information loss
- Simpler visualization

Mean: The average of the data

$$\text{Mean}(x) = \frac{1}{n}\sum_{i=1}^{n} x_i$$

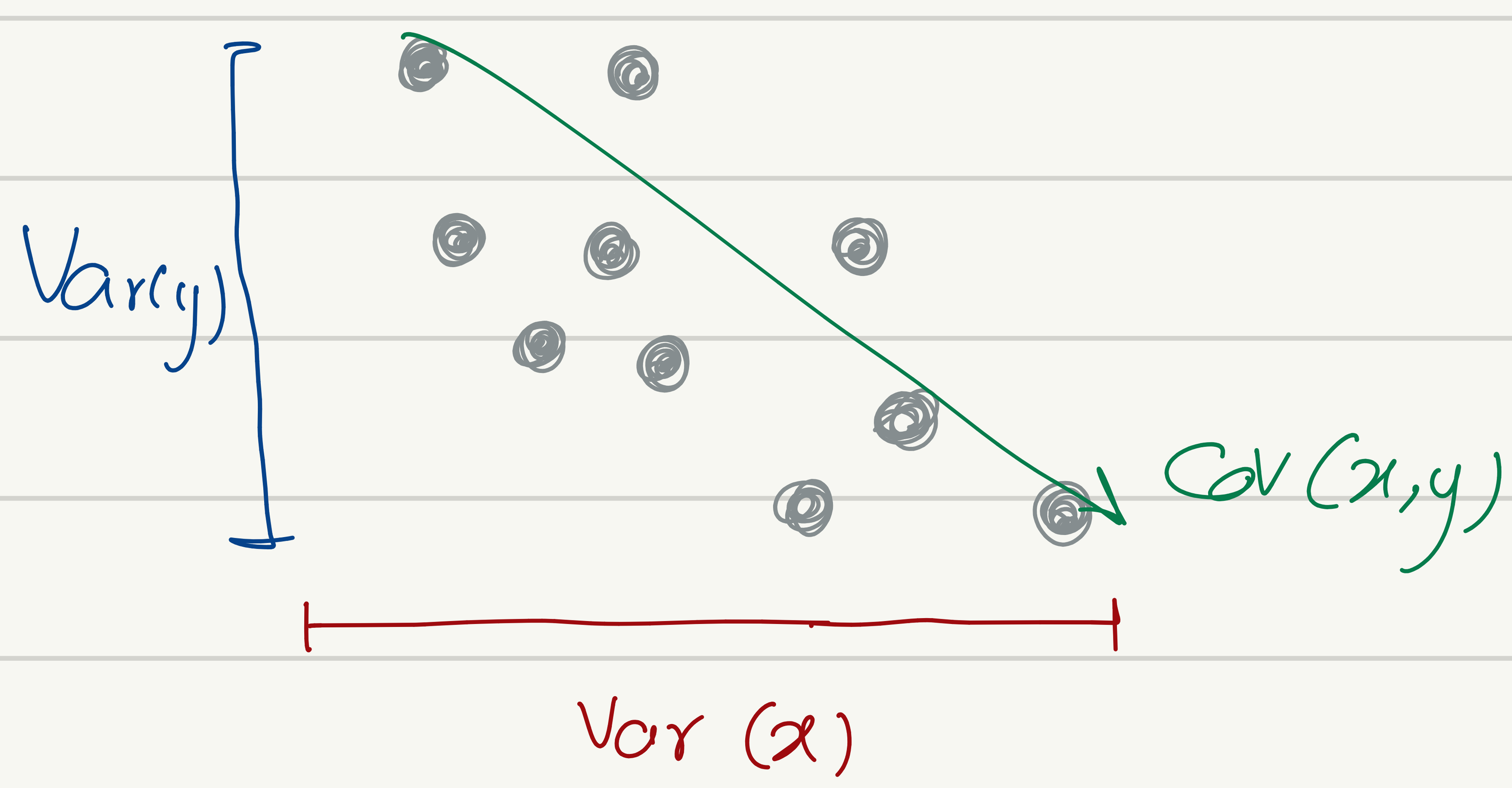
Variance: The average squard distance from the mean (how spread out our data is)

$$\text{Var}(x) = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \mu_x)^2$$

Covariance: The direction of the relationship between two variables

$$\text{Cov}(x,y) = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \mu_x)(y_i - \mu_y)$$

Covariance Matrix : is a square matrix giving the covariance between each pair of elements of a given random vector.



$$C = \begin{bmatrix} Var(x) & Cov(x,y) \\ Cov(x,y) & Var(y) \end{bmatrix}$$

* $Cov(x,x) = Var(x)$

→ **Calculating Covariance Matrix:**

Same shape

$$A = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} \qquad \text{mean matrix} \leftarrow \mu = \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \end{bmatrix}$$

observation's matrix

transport

$$C = \frac{1}{n-1} (A - \mu)^T (A - \mu)$$

$$C = \frac{1}{n-1} \begin{bmatrix} x_1 - \mu_x & x_2 - \mu_x & \cdots \\ y_1 - \mu_y & y_2 - \mu_y & \cdots \end{bmatrix} \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix}$$

covariance matrix

2×2 ← $\qquad 2 \times n \qquad n \times 2$

After multiplying : $C = \begin{bmatrix} Var(x) & Cov(x,y) \\ Cov(x,y) & Var(y) \end{bmatrix}$

Matrix Formula

$$A = \begin{bmatrix} x_1 & y_1 & z_1 \\ \vdots & \vdots & \vdots \\ x_n & y_n & z_n \end{bmatrix} \qquad C = \frac{1}{n-1}(A-\mu)^T(A-\mu)$$

1. Arrange data with a difference feature in each column
2. Calculate column averages
3. Subtract each average from their respective column to generate $A-\mu$
4. $\frac{1}{n-1}(A-\mu)^T(A-\mu)$ gives the covariance matrix C

PCA

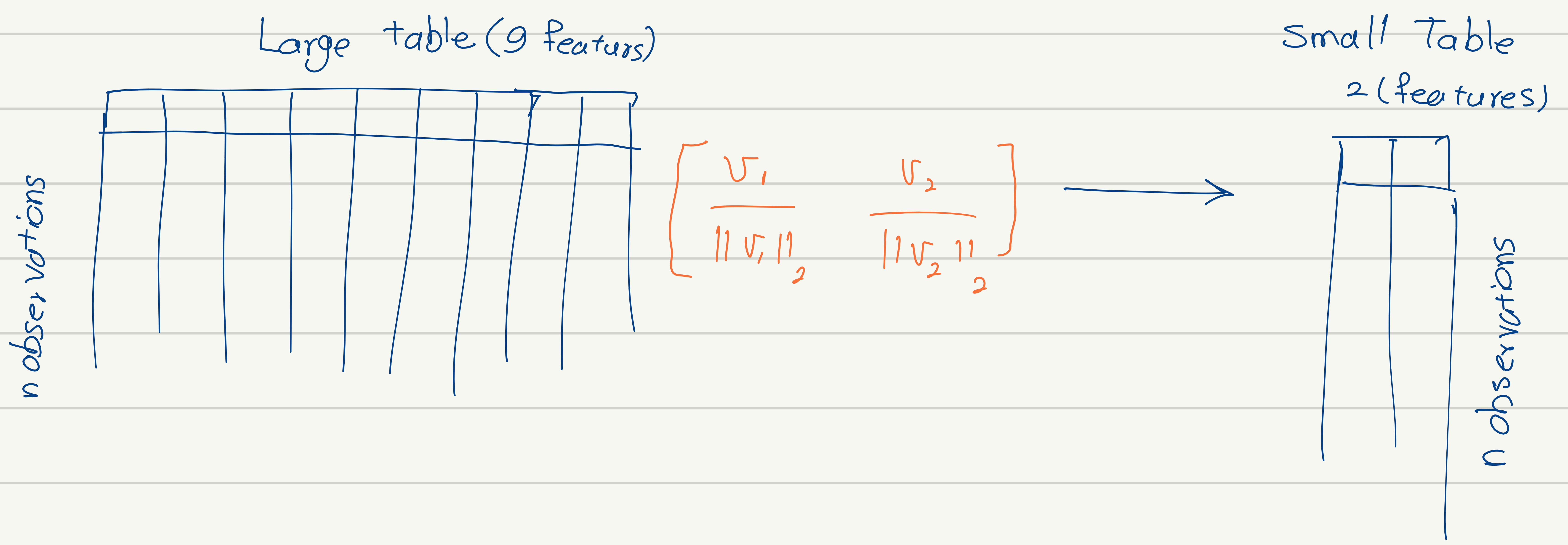
Projections

Eigenvectors / Eigenvalues — PCA

Covariance matrix

1. First we should calculate covariance matrix
2. Then we should find the eigenvalues and eigenvectors and Sort them according to eigenvalue, from the biggest to smallest
   (because for projection, we need the eigenvector with the biggest eigenvalue)
3. Imagine that you want to reduce your data to 2 variables, so you should select 2 biggest eigenvalue and their associated eigenvectors.

4. project data:



* PCA Mathematical Formulation *

we have n observations of 5 variables $(x_1, x_2, x_3, x_4, x_5)$

Goal : Reduce to 2 variables

1) Create Matrix :

$$X = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{15} \\ x_{21} & x_{22} & \cdots & x_{25} \\ \vdots & \vdots & & \\ x_{n1} & x_{n2} & \cdots & x_{n5} \end{bmatrix} \updownarrow \text{ n observations}$$

2) Center the data :

$$X - \mu = \begin{bmatrix} x_{11} - \mu_1 & x_{12} - \mu_2 & \cdots & x_{15} - \mu_5 \\ x_{21} - \mu_1 & x_{22} - \mu_2 & \cdots & x_{25} - \mu_5 \\ \vdots & \vdots & & \vdots \\ x_{n1} - \mu_1 & x_{n2} - \mu_2 & & x_{n5} - \mu_5 \end{bmatrix}$$

3) Calculate Covariance Matrix :

$$C = \frac{1}{n-1} (X - \mu)^T (X - \mu)$$

4) Calculate Eigenvectors and Eigenvalues :

big

λ_1 v_1

λ_2 v_2

λ_3 v_3

λ_4 v_4

small λ_5 v_5

5) Create projection Matrix :

$$V = \begin{bmatrix} \frac{v_1}{\|v_1\|_2} & \frac{v_2}{\|v_2\|_2} \end{bmatrix}$$

6) Project Centered data :

$$X_{PCA} = (X - \mu) \cdot V$$