Завдання 1.

1. Для функції $y = f(x)$ вибрати відрізок інтерполяції та вузли по яких буде використуватись інтерполяція.

Вільними відрізок інтерполяції - [1, 4]. Оскільки надані нам точки буде знайти інтерполяційне поліном Ньютона, то вільними доводиться вузли з кроком 0. = 1. Знайдемо значення функції у вузлах.

In [1]:
```
import numpy as np

def f(x):
    return (7*np.log(x*x) - 2*x + x**2 + 1)

for i in range(1, 5):
    print("f(%1d)=%1.4f" % (i,round(f(i),4)))
```

Output:
```
f(1)= 0.1000
f(2)= 0.1972
f(3)= 0.7726
f(4)= 2.2189
```

Отже маємо:

x	1	2	3	4
y	1.3863	3.1972	6.7726	12.2189

2. Побудуємо таблицю скінченних різниць.

№	x_i	y_i	Δy_i	$\Delta^2 y_i$	$\Delta^3 y_i$
0	1	1.3863	1.8109	1.7545	0.3561
1	2	3.1972	3.5754	1.8709	
2	3	6.7726	5.4463		
3	4	12.2189			

3. Запишемо інтерполяційний многочлен Лагранжа.

Інтерполяційний многочлен Лагранжа має вигляд:

$$L_n(x) = \sum_{i=0}^{n} \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

У нашому випадку:

$$L_3(x) = \sum_{i=0}^{3} \prod_{\substack{j=0 \\ j \neq i}}^{3} \frac{x - x_j}{x_i - x_j} = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1 +$$

$$\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3 =$$

$$\frac{(x-2)(x-3)(x-4)}{(1-2)(1-3)(1-4)} \cdot 1.3863 + \frac{(x-1)(x-3)(x-4)}{(2-1)(2-3)(2-4)} \cdot 3.1972 + \frac{(x-1)(x-2)(x-4)}{(3-1)(3-2)(3-4)} \cdot 6.7726 + \frac{(x-1)(x-2)(x-3)}{(4-1)(4-2)(4-3)} \cdot 12.2189 = 0.0177x^3 + 0.776x^2 - 0.6468x + 1.2335$$

Оскільки маємо:

$$L_3(x) = 0.0177x^3 + 0.776x^2 - 0.6468x + 1.2335$$

Запишемо перший поліном Ньютона.

У випадку $n = 3$ він має вигляд:

$$P_3^1(x) = y_0 + \frac{\Delta y_0}{h}(x - x_0) + \frac{\Delta^2 y_0}{2h^2}(x - x_0)(x - x_1) + \frac{\Delta^3 y_0}{3h^3}(x - x_0)(x - x_1)(x - x_2)$$

Тоді:

$$P_3^1 = 1.3863 + \frac{1.8109}{1}(x-1) + \frac{1.7545}{2 \cdot 1^2}(x-1)(x-2) + \frac{0.3561}{3 \cdot 1^3}(x-1)(x-2)(x-3) = 0.0177x^3 + 0.776x^2 - 0.6468x + 1.2335$$

Запишемо другий поліном Ньютона.

У випадку $n = 3$ він має вигляд:

$$P_3^2(x) = y_3 + \frac{\Delta y_2}{h}(x - x_3) + \frac{\Delta^2 y_1}{2h^2}(x - x_3)(x - x_2) + \frac{\Delta^3 y_0}{3h^3}(x - x_3)(x - x_2)(x - x_1)$$

Тоді:

$$P_3^2 = 12.2189 + 5.4463(x-3) + 3.5754 \cdot \frac{(x-3)(x-2)}{2 \cdot 1^2} + 0.3561 \cdot \frac{(x-3)(x-2)(x-1)}{3 \cdot 1^3} = 0.0177x^3 + 0.776x^2 - 0.6468x + 1.2335$$

Це показ'ємо так, що не вибираю однаков вузли для всіх многочлена.

3. Використовуючи однакові поліноми обчислити значення функції в кількох нерузлових точках (на вибір) та порівняти їх значення функції в цих точках.

Вільними так нерузлові точки: 1.5, 2.5, 3.5.

In [2]:
```
def L(x):
    return 0.0177*x**3 + 0.776*x**2 - 0.6468*x + 1.2335

print("Table: (x, y)", sep=sep)
for i in range(1, 4):
    print("%1f" % (i), "(%g,%1f)" % (i, round(f(i), 3)))

Table: (x, y)                   sep=sep
1.000000 (1,0.100)
2.000000 (2,3.197)
3.000000 (3,6.773)
```

Output:
```
1.000000 (1,0.100)
2.000000 (2,3.197)
3.000000 (3,6.773)
```

4. Обрати три вузли з тих, які були використані в попередніх пунктах та побудувати за обраними вузлами інтерполяційний кубічний сплайн дефекту 1.

Вибираємо такі вузли: $x_1 = 3, x_2 = 4, x_3 = 6$

У нашому випадку кубічний сплайн дефекту 1 має вигляд:

$$g(x) = \begin{cases} a_1 + b_1(x-3) + c_1(x-3)^2 + d_1(x-3)^3, & x \in [3;4] \\ a_2 + b_2(x-4) + c_2(x-4)^2 + d_2(x-4)^3, & x \in [3;4] \end{cases}$$

Позначимо:

$$g_1(x) = a_1 + b_1(x-3) + c_1(x-3)^2 + d_1(x-3)^3$$

$$g_2(x) = a_2 + b_2(x-4) + c_2(x-4)^2 + d_2(x-4)^3$$

складати систему з 8-ми невідомих

З умови інтерполяції у вузлах $x_1 = 3$ $\quad (g_1(3) = f(3))$ маємо:

$$a_1 = b_1 + c_1 - d_1 = 3.1972 \quad (1)$$

З умови інтерполяції у вузлах $x_2 = 3$ $\quad (g_2(3) = f(3))$ а також $\quad g_1(3) = f(3)$ маємо:

$$a_1 = 6.7726 \quad (2)$$

$$a_2 - b_2 + c_2 - d_2 = 6.7726 \quad (3)$$

З умови інтерполяції у вузлах $x_3 = 6$ $\quad (g_2(6) = f(6))$ маємо:

$$a_2 = 12.2189 \quad (4)$$

Оскільки $g'(3) = g'(4) = 0$, маємо:

$$2c_1 + 6d_1(2-3) = 2c_1 - 6d_1 = 0 \quad (5)$$

$$2c_2 + 6d_2(4-4) = 2c_2 = 0 \quad (6)$$

Функції $g_1(x)$ та $g_2(x)$, а також їх похідні першого та другого порядку у точці $x = 3$ мають однакове значення, тому маємо ще три рівняння:

$$a_1 = a_2 - b_2 + c_2 - d_2 \quad (7)$$

$$b_1 = b_2 - 2c_2 + 3d_2 \quad (8)$$

$$2c_1 = 2c_2 - 6d_2 \quad (9)$$

сьоме рівняння витікає з другого та третього тому воно не записуємо. Тоді маємо:

$$\begin{cases} a_1 - b_1 + c_1 - d_1 = 3.1972 \\ a_1 = 6.7726 \\ a_2 - b_2 + c_2 - d_2 = 6.7726 \\ a_2 = 12.2189 \\ 2c_1 - 6d_1 = 0 \\ c_2 = 0 \\ -b_1 + 2c_1 - 3c_2 + 3d_2 = 0 \\ -2c_1 + 2c_2 - 6d_2 = 0 \end{cases}$$

розв'язання систему за допомогою бібліотеки numpy

In [3]:
```
a = np.array([1, -1, 1, -1, 0, 0, 0, 0],
             [1, 0, 0, 0, 0, 0, 0, 0],
             [0, 0, 0, 0, 1, -1, 1, -1],
             [0, 0, 0, 0, 1, 0, 0, 0],
             [0, 1, -2, -3, 0, 0, 0, 0],
             [0, 0, 0, 0, 0, 1, -3, 3],
             [0, 0, 0, 0, 0, 0, 2, -6],
             [0, 0, 0, 0, 0, 0, 0, 2])

b = np.array([3.1972,
              6.7726,
              6.7726,
              12.2189,
              0,
              0,
              0,
              0])

np.set_printoptions(linewidth=80, k=4)
```

Output:
```
array([ 4.7726,
        6.7726,
        1.4551,
        1.4551,
        0.0000,
        0.0000,
        0.0000,
        0.0000])
```

Отже, отримаємо

$$g(x) = \begin{cases} 4.7726 + 4.5591(x-3) + 1.4551(x-3)^2 + 0.0077(x-3)^3, & x \in [3;5] \\ 12.2189 + 3.0545(x-4) - 0.0077(x-4)^3, & x \in [3;4] \end{cases}$$

5. Побудувати графіки отриманих поліномів та графік функції $f(x)$

Нижче подані графіки: Графік функції $f(x)$ - зелений; Графік поліномів - фіолетовий; Графік кубічного сплайну дефекту 1 на відрізку [2;5] набуваючий значення, а на [3;6] - синім.

In [4]:
```
plot_f(a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w,x,y,z,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27,28,29,30,31,32,33,34,35,36,37,38,39,40,41,42,43,44,45,46,47,48,49,50,51,52,53,54,55,56,57,58,59,60,61,62,63,64,65,66,67,68,69,70,71,72,73,74,75,76,77,78,79,80,81,82,83,84,85,86,87,88,89,90,91,92,93,94,95,96,97,98,99,100,101,102,103,104,105,106,107,108,109,110,111,112,113,114,115,116,117,118,119,120,121,122,123,124,125,126,127,128,129,130,131,132,133,134,135,136,137,138,139,140,141,142,143,144,145,146,147,148,149,150,151,152,153,154,155,156,157,158,159,160,161,162,163,164,165,166,167,168,169,170,171,172,173,174,175,176,177,178,179,180,181,182,183,184,185,186,187,188,189,190,191,192,193,194,195,196,197,198,199,200,201,202,203,204,205,206,207,208,209,210,211,212,213,214,215,216,217,218,219,220,221,222,223,224,225,226,227,228,229,230,231,232,233,234,235,236,237,238,239,240,241,242,243,244,245,246,247,248,249,250,251,252,253,254,255,256,257,258,259,260,261,262,263,264,265,266,267,268,269,270,271,272,273,274,275,276,277,278,279,280,281,282,283,284,285,286,287,288,289,290,291,292,293,294,295,296,297,298,299,300,301,302,303,304,305,306,307,308,309,310,311,312,313,314,315,316,317,318,319,320,321,322,323,324,325,326,327,328,329,330,331,332,333,334,335,336,337,338,339,340,341,342,343,344,345,346,347,348,349,350,351,352,353,354,355,356,357,358,359,360,361,362,363,364,365,366,367,368,369,370,371,372,373,374,375,376,377,378,379,380,381,382,383,384,385,386,387,388,389,390,391,392,393,394,395,396,397,398,399,400,401,402,403,404,405,406,407,408,409,410,411,412,413,414,415,416,417,418,419,420,421,422,423,424,425,426,427,428,429,430,431,432,433,434,435,436,437,438,439,440,441,442,443,444,445,446,447,448,449,450,451,452,453,454,455,456,457,458,459,460,461,462,463,464,465,466,467,468,469,470,471,472,473,474,475,476,477,478,479,480,481,482,483,484,485,486,487,488,489,490,491,492,493,494,495,496,497,498,499,500,501,502,503,504,505,506,507,508,509,510,511,512,513,514,515,516,517,518,519,520,521,522,523,524,525,526,527,528,529,530,531,532,533,534,535,536,537,538,539,540,541,542,543,544,545,546,547,548,549,550,551,552,553,554,555,556,557,558,559,560,561,562,563,564,565,566,567,568,569,570,571,572,573,574,575,576,577,578,579,580,581,582,583,584,585,586,587,588,589,590,591,592,593,594,595,596,597,598,599,600,601,602,603,604,605,606,607,608,609,610,611,612,613,614,615,616,617,618,619,620,621,622,623,624,625,626,627,628,629,630,631,632,633,634,635,636,637,638,639,640,641,642,643,644,645,646,647,648,649,650,651,652,653,654,655,656,657,658,659,660,661,662,663,664,665,666,667,668,669,670,671,672,673,674,675,676,677,678,679,680,681,682,683,684,685,686,687,688,689,690,691,692,693,694,695,696,697,698,699,700,701,702,703,704,705,706,707,708,709,710,711,712,713,714,715,716,717,718,719,720,721,722,723,724,725,726,727,728,729,730,731,732,733,734,735,736,737,738,739,740,741,742,743,744,745,746,747,748,749,750,751,752,753,754,755,756,757,758,759,760,761,762,763,764,765,766,767,768,769,770,771,772,773,774,775,776,777,778,779,780,781,782,783,784,785,786,787,788,789,790,791,792,793,794,795,796,797,798,799,800,801,802,803,804,805,806,807,808,809,810,811,812,813,814,815,816,817,818,819,820,821,822,823,824,825,826,827,828,829,830,831,832,833,834,835,836,837,838,839,840,841,842,843,844,845,846,847,848,849,850,851,852,853,854,855,856,857,858,859,860,861,862,863,864,865,866,867,868,869,870,871,872,873,874,875,876,877,878,879,880,881,882,883,884,885,886,887,888,889,890,891,892,893,894,895,896,897,898,899,900,901,902,903,904,905,906,907,908,909,910,911,912,913,914,915,916,917,918,919,920,921,922,923,924,925,926,927,928,929,930,931,932,933,934,935,936,937,938,939,940,941,942,943,944,945,946,947,948,949,950,951,952,953,954,955,956,957,958,959,960,961,962,963,964,965,966,967,968,969,970,971,972,973,974,975,976,977,978,979,980,981,982,983,984,985,986,987,988,989,990,991,992,993,994,995,996,997,998,999,1000,1001,1002,1003,1004,1005,1006,1007,1008,1009,1010,1011,1012,1013,1014,1015,1016,1017,1018,1019,1020,1021,1022,1023,1024,1025,1026,1027,1028,1029,1030,1031,1032,1033,1034,1035,1036,1037,1038,1039,1040,1041,1042,1043,1044,1045,1046,1047,1048,1049,1050,1051,1052,1053,1054,1055,1056,1057,1058,1059,1060,1061,1062,1063,1064,1065,1066,1067,1068,1069,1070,1071,1072,1073,1074,1075,1076,1077,1078,1079,1080,1081,1082,1083,1084,1085,1086,1087,1088,1089,1090,1091,1092,1093,1094,1095,1096,1097,1098,1099,1100,1101,1102,1103,1104,1105,1106,1107,1108,1109,1110,1111,1112,1113,1114,1115,1116,1117,1118,1119,1120,1121,1122,1123,1124,1125,1126,1127,1128,1129,1130,1131,1132,1133,1134,1135,1136,1137,1138,1139,1140,1141,1142,1143,1144,1145,1146,1147,1148,1149,1150,1151,1152,1153,1154,1155,1156,1157,1158,1159,1160,1161,1162,1163,1164,1165,1166,1167,1168,1169,1170,1171,1172,1173,1174,1175,1176,1177,1178,1179,1180,1181,1182,1183,1184,1185,1186,1187,1188,1189,1190,1191,1192,1193,1194,1195,1196,1197,1198,1199,1200,1201,1202,1203,1204,1205,1206,1207,1208,1209,121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