

CALCULUS AND PROBABILITY BASED STATISTICS

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1. FINITE STATISTICS

1.1. Introduction. The purpose of this text is to have statistics built on a solid foundation of probability and calculus.

Definition 1.1. A is an independently repeatable event.

Definition 1.2. $p \in [0, 1]$ is the probability that A will happen when A is given a chance to happen. For simplicity, we assume that p is the same every time A is given a chance to happen.

Definition 1.3. $n \in \mathbb{W}$ is the number of times A had a chance to happen.

Definition 1.4. $k \in \mathbb{W}$ is the number of times A actually happened.

Remark 1.5. $k \leq n$.

Definition 1.6. f_p is the probability distribution for p .

Remark 1.7. $1 = \int_0^1 f_p dp$

Remark 1.8. k/n is the expected probability of p and is where the maximum value of the function f_p is.

Definition 1.9. (a, b) is the confidence interval of p such that $b - a$ is minimized.

Definition 1.10. c is the desired certainty for the confidence interval of p .

Remark 1.11. $0 \leq c = \int_a^b f_p dp \leq 1$

Remark 1.12. $0 \leq a \leq b \leq 1$

Remark 1.13. The symmetry mentioned is where you swap p with $1 - p$ and k with $n - k$ and A with $\neg A$.

1.2. The Problem. n, k, c are known, and we must solve for a, b .

Remark 1.14. The probability of k events given n chances is $\binom{n}{k} p^k (1 - p)^{n-k}$, but we must scale it to satisfy 1.7:

$$(1.1) \quad f_p = \frac{\binom{n}{k} p^k (1 - p)^{n-k}}{\int_0^1 \binom{n}{k} p^k (1 - p)^{n-k} dp} = \frac{p^k (1 - p)^{n-k}}{\int_0^1 p^k (1 - p)^{n-k} dp} = (n + 1) \binom{n}{k} p^k (1 - p)^{n-k}$$

1.3. The 4 cases of finding the interval.

1.3.1. *Case* $n = k = 0$. This case is self-symmetrical.

Remark 1.15. We have

$$(1.2) \quad (a, b) = \left(\frac{1 - c}{2}, \frac{1 + c}{2} \right)$$

1.3.2. *Case* $n > k = 0$. This case is symmetrical to case 1.3.3.

Remark 1.16. We can solve

$$(1.3) \quad \int_0^b (n+1)(1-p)^n dp = c = 1 - (1-b)^{n+1}$$

Remark 1.17. We have

$$(1.4) \quad (a, b) = \left(0, 1 - (1-c)^{\frac{1}{n+1}}\right)$$

Example 1.18. Now we can analyze the old saying "Third time's the charm.". $n = 2$.

$$(1.5) \quad c = 0.5 \Rightarrow b \approx 0.2062994740, c = 0.95 \Rightarrow b \approx 0.6315968501$$

We can now see that if you fail at something twice, then success before the 8th try is worth approximately a coin flip.

Example 1.19. Suppose we were repeatedly trying a difficult task. How many times of trying without success must there be for us to conclude with $c = 0.99$ that $b \leq 0.01$?

$$(1.6) \quad n = \frac{2 \log(3) + \log(11)}{2 \log(2) - 2 \log(3) + 2 \log(5) - \log(11)} \approx 457.2105766$$

We can see "If at first you don't succeed, try, try again." is sensible for things that we want to be certain are improbable before giving up.

1.3.3. *Case* $n = k > 0$. This case is symmetrical to case 1.3.2.

Remark 1.20. We can solve

$$(1.7) \quad \int_a^1 (k+1)p^k dp = c = 1 - a^{k+1}$$

Remark 1.21. We have

$$(1.8) \quad (a, b) = \left((1-c)^{\frac{1}{k+1}}, 1\right)$$

1.3.4. *Case* $n > k > 0$. This case is self-symmetrical.

Remark 1.22. We desire to solve

$$(1.9) \quad \int_a^b (n+1) \binom{n}{k} p^k (1-p)^{n-k} dp = c = (n+1) \binom{n}{k} (B_b(k+1, -k+n+1) - B_a(k+1, -k+n+1))$$

where B is the incomplete beta function.

Remark 1.23. We can use the hints

$$(1.10) \quad a^k (1-a)^{n-k} = b^k (1-b)^{n-k}$$

Remark 1.24. By solving

$$(1.11) \quad \frac{\partial f_p}{\partial p} = 0$$

, we get

$$(1.12) \quad \lim_{c \rightarrow 0} (a, b) = \left(\frac{k}{n}, \frac{k}{n}\right), m = (n+1) \binom{n}{k} \left(\frac{k}{n}\right)^k \left(1 - \frac{k}{n}\right)^{n-k}$$

1.4. Table of small trial half probability intervals. Table with $c = \frac{1}{2}$:

	$k = 0$	$k = 1$	$k = 2$	$k = 3$	$k = 4$
$n = 0$	$a \approx 0.2500000000$ $b \approx 0.7500000000$				
$n = 1$	$a \approx 0.0000000000$ $b \approx 0.2928932188$	$a \approx 0.7071067812$ $b \approx 1.0000000000$			
$n = 2$	$a \approx 0.0000000000$ $b \approx 0.2062994740$	$a \approx 0.326351822$ $b \approx 0.673648178$	$a \approx 0.7937005260$ $b \approx 1.0000000000$		
$n = 3$	$a \approx 0.0000000000$ $b \approx 0.1591035847$	$a \approx 0.1966713241$ $b \approx 0.4921993002$	$a \approx 0.507800700$ $b \approx 0.803328676$	$a \approx 0.8408964153$ $b \approx 1.0000000000$	
$n = 4$	$a \approx 0.0000000000$ $b \approx 0.1294494367$	$a \approx 0.1387754454$ $b \approx 0.3894044243$	$a \approx 0.3594361648$ $b \approx 0.640563835$	$a \approx 0.610595575$ $b \approx 0.861224555$	$a \approx 0.8705505633$ $b \approx 1.0000000000$
$n = 5$	$a \approx 0.0000000000$ $b \approx 0.1091012819$	$a \approx 0.1065967785$ $b \approx 0.3226910840$	$a \approx 0.2770309506$ $b \approx 0.532125995$	$a \approx 0.4678740055$ $b \approx 0.722969050$	$a \approx 0.677308916$ $b \approx 0.893403222$
$n = 6$	$a \approx 0.0000000000$ $b \approx 0.09427633574$	$a \approx 0.0862905298$ $b \approx 0.2757178692$	$a \approx 0.2249291772$ $b \approx 0.4551185682$	$a \approx 0.3788484407$ $b \approx 0.621151559$	$a \approx 0.544881432$ $b \approx 0.775070823$
$n = 7$	$a \approx 0.0000000000$ $b \approx 0.08299595680$	$a \approx 0.07237321411$ $b \approx 0.2407853463$	$a \approx 0.1891248358$ $b \approx 0.3976355162$	$a \approx 0.3180913474$ $b \approx 0.544067630$	$a \approx 0.4559323705$ $b \approx 0.681908652$
$n = 8$	$a \approx 0.0000000000$ $b \approx 0.07412528771$	$a \approx 0.06226629927$ $b \approx 0.2137614499$	$a \approx 0.1630534052$ $b \approx 0.3530822174$	$a \approx 0.2740192600$ $b \approx 0.4838784786$	$a \approx 0.3919638426$ $b \approx 0.608036157$
	$k = 5$	$k = 6$	$k = 7$	$k = 8$	
$n = 5$	$a \approx 0.8908987181$ $b \approx 1.0000000000$				
$n = 6$	$a \approx 0.724282131$ $b \approx 0.913709470$	$a \approx 0.9057236643$ $b \approx 1.0000000000$			
$n = 7$	$a \approx 0.602364484$ $b \approx 0.810875164$	$a \approx 0.759214654$ $b \approx 0.927626786$	$a \approx 0.9170040432$ $b \approx 1.0000000000$		
$n = 8$	$a \approx 0.516121520$ $b \approx 0.725980740$	$a \approx 0.646917782$ $b \approx 0.836946595$	$a \approx 0.786238550$ $b \approx 0.937733701$	$a \approx 0.9258747123$ $b \approx 1.0000000000$	