# Does the Series Converge?
https://github.com/Nazgand/nazgandMathBook

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**Abstract**

The goal of this paper is to prove the convergence of certain series.

## 1 Problem -2

Given [a function from [the natural numbers] to [the real numbers]] which is [inclusively bounded in magnitude by 1], i.e.

$$\exists f_1 : \mathbb{Z}_{\geq 0} \to \mathbb{R}, [\forall k \in \mathbb{Z}_{\geq 0}, |f_1(k)| \leq 1] \tag{1.1}$$

and [a positive real number smaller than 1], i.e.

$$\exists r \in \mathbb{R}^+, r < 1 \tag{1.2}$$

, prove

$$\exists \left[\lim_{m\to\infty} \sum_{k=0}^{m} f_1(k) r^k\right] \in \mathbb{R} \tag{1.3}$$

Solution: Note

$$\forall n \in \mathbb{Z}_{\geq 0}, \left[\exists \left[\lim_{m\to\infty} \sum_{k=0}^{m} f_1(k) r^k\right] \in \mathbb{R} \Leftrightarrow \exists \left[\lim_{m\to\infty} \sum_{k=n}^{m} f_1(k) r^k\right] \in \mathbb{R}\right] \tag{1.4}$$

and

$$-\frac{r^n - r^{m+1}}{1-r} = -\sum_{k=n}^{m} r^k \leq \sum_{k=n}^{m} f_1(k) r^k \leq \sum_{k=n}^{m} r^k = \frac{r^n - r^{m+1}}{1-r} \tag{1.5}$$

First, let $m \to \infty$.

$$-\frac{r^n}{1-r} \leq \lim_{m\to\infty} \sum_{k=n}^{m} f_1(k) r^k \leq \frac{r^n}{1-r} \tag{1.6}$$

Then let $n \to \infty$ to observe the Squeeze Theorem.

$$0 \leq \lim_{n\to\infty} \lim_{m\to\infty} \sum_{k=n}^{m} f_1(k) r^k \leq 0 \tag{1.7}$$

## 2 Problem -1

Given [a function from [the natural numbers] to [the complex numbers]] which is [inclusively bounded in magnitude by 1], i.e.

$$\exists f_1 : \mathbb{Z}_{\geq 0} \to \mathbb{C}, [\forall k \in \mathbb{Z}_{\geq 0}, |f_1(k)| \leq 1] \tag{2.1}$$

and [a complex number smaller than 1 in magnitude], i.e.

$$\exists r \in \mathbb{C}, |r| < 1 \tag{2.2}$$

, prove

$$\exists \left[\lim_{m\to\infty} \sum_{k=0}^{m} f_1(k) r^k\right] \in \mathbb{C} \tag{2.3}$$

Without loss of generality, we can set $r = |r|$, as seen by the substitutions $f_1(k) \to f_1(k)\frac{r^k}{|r|^k}, r \to |r|$. The case $r = 0$ is trivial, so let $r \in \mathbb{R}, 0 < r < 1$. The relevant Cauchy series can be split into 2 Cauchy series for the imaginary and real parts; [the complex limit can be proved to exist] by [proving the existence of the {real, imaginary} parts]; thus, the problem is reduced to [Problem -2]

## 3 Problem 0

Given [a function from [the natural numbers] to [the complex numbers]] which is [exponentially bounded in magnitude], i.e.

$$\exists f_a : \mathbb{Z}_{\geq 0} \to \mathbb{C}, \exists a \in \mathbb{R}, \left[\forall k \in \mathbb{Z}_{\geq 0}, a^{k+1} \geq |f_a(k)|\right] \tag{3.1}$$

, and

$$\exists t \in \mathbb{R}^+ \tag{3.2}$$

, prove:

$$\exists \left[\lim_{m\to\infty} \sum_{k=0}^{m} \frac{f_a(k)}{(k!)^t}\right] \in \mathbb{C} \tag{3.3}$$

This sum 'obviously' converges via $\left[\forall b \in \mathbb{R}^+, \mathcal{O}\left((k!)^t\right) > \mathcal{O}\left(b^k\right)\right], \mathcal{O}\left(a^k\right) \geq \mathcal{O}(|f_a(k)|)$. We must be more rigorous than that to build a solid foundation. A good first step is to notice

$$\forall K \in \mathbb{Z}_{\geq 0}, \left[\left[\exists \left[\lim_{m\to\infty} \sum_{k=0}^{m} \frac{f_a(k)}{(k!)^t}\right] \in \mathbb{C}\right] \Leftrightarrow \left[\exists \left[\lim_{m\to\infty} \sum_{k=K}^{m} \frac{f_a(k)}{(k!)^t}\right] \in \mathbb{C}\right]\right] \tag{3.4}$$

and let $K$ be sufficiently large so that $\frac{((K+1)!)^t}{(K!)^t} > a$, i.e. $K > \sqrt[t]{a} - 1$. Then this problem reduces to [Problem -1].

## 4 Problem 1

Given [a function from [the natural numbers] to [the complex numbers]] which is [exponentially bounded in magnitude], i.e.

$$\exists f_b : \mathbb{Z}_{\geq 0} \to \mathbb{C}, \exists b \in \mathbb{R}, \left[\forall k \in \mathbb{Z}_{\geq 0}, b^{k+1} \geq |f_b(k)|\right] \tag{4.1}$$

, and

$$\exists t \in \mathbb{R}^+, \exists z \in \mathbb{C} \tag{4.2}$$

, prove:

$$\exists \left[\lim_{m\to\infty} \sum_{k=0}^{m} \frac{f_b(k) z^k}{(k!)^t}\right] \in \mathbb{C} \tag{4.3}$$

Solution: Let $f_a(k) = f_b(k) z^k, a \geq b|z|$, and note

$$\left[\forall k \in \mathbb{Z}_{\geq 0}, b^{k+1} \geq |f_b(k)|\right] \Rightarrow \left[\forall k \in \mathbb{Z}_{\geq 0}, a^{k+1} \geq |f_a(k)|\right] \tag{4.4}$$

Thus [Problem 1] has been reduced to [Problem 0].