Argument Sum Rules From Homogeneous Linear Differential Equations Of Constant Coefficients Conjecture
https://github.com/Nazgand/nazgandMathBook

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Abstract

The goal of this paper is to conjecture a seemingly fundamental calculus fact.

1 Assumptions and definitions

A [homogeneous linear differential equation of constant coefficients] has the form

$$0 = \sum_{k=0}^{n} a_k \frac{\partial^k}{\partial z^k} f(z) = \sum_{k=0}^{n} a_k f^{(k)}(z) \tag{1.1}$$

where $\forall k, a_k \in \mathbb{C}, n \in \mathbb{Z}^+, a_n \neq 0$, and $f : \mathbb{C} \to \mathbb{C}$ is differentiable at least n times everywhere. Let $g_0(z), \ldots, g_{n-1}(z)$ be solutions that are a basis of the vector space of solutions of the [homogeneous linear differential equation of constant coefficients]. To be clear:

$$\text{Solutions} = \left\{ h(z) \mid 0 = \sum_{k=0}^{n} a_k h^{(k)}(z) \right\} = \left\{ \sum_{k=0}^{n-1} b_k g_k(z) \mid b_k \in \mathbb{C} \right\} \tag{1.2}$$

Let us define a column vector and clarify its transpose (a row vector):

$$v(z_0) = \begin{pmatrix} g_0(z_0) \\ \vdots \\ g_{n-1}(z_0) \end{pmatrix}, v(z_1)^T = \begin{pmatrix} g_0(z_1) & \ldots & g_{n-1}(z_1) \end{pmatrix} \tag{1.3}$$

2 Main conjecture with 2 arguments

I conjecture there exists a complex-valued constant n by n symmetric matrix A ($A = A^T$) such that

$$f(z_0 + z_1) = v(z_1)^T A v(z_0) = v(z_0)^T A v(z_1) \tag{2.1}$$

2.1 A reason A is symmetric

Suppose instead of a symmetric matrix A, we find a matrix B such that

$$f(z_0 + z_1) = v(z_1)^T B v(z_0) \tag{2.2}$$

Then take the transpose of the equation and substitute $z_0 \to z_1, z_1 \to z_0$, resulting in the following equation:

$$f(z_0 + z_1) = v(z_1)^T B^T v(z_0) \tag{2.3}$$

Average both equations:

$$f(z_0 + z_1) = v(z_1)^T \frac{B + B^T}{2} v(z_0) \tag{2.4}$$

Note that we can set $A = \frac{B+B^T}{2}$ because it is symmetric. □

3 Motivation

The first thing of note is

$$f(z) \in \text{Solutions} \tag{3.1}$$

The next thing of note is that offsetting the function's argument by a constant results in the same differential equation, with a solution of the same form.

$$\frac{\partial}{\partial z} z_0 = 0 \Rightarrow f(z + z_0) \in \text{Solutions} \tag{3.2}$$

One thing to note is that all constants are replaced with functions of the constant z_0. This allows us that change z_0 from a constant to a variable and still have working math when we consider all possible constants z_0.

$$\left[\frac{\partial}{\partial z} z_0 = 0 \land f(z+z_0) \in \text{Solutions}\right] \Rightarrow \exists b_{0,k} : \mathbb{C} \to \mathbb{C}, f(z+z_0) = \sum_{k=0}^{n-1} b_{0,k}(z_0) g_k(z) \tag{3.3}$$

Because we were careful about making all constants a function of the constant z_0, we can make z_0 a variable with the conclusion

$$\exists b_{0,k} : \mathbb{C} \to \mathbb{C}, f(z+z_0) = \sum_{k=0}^{n-1} b_{0,k}(z_0) g_k(z) \tag{3.4}$$

Follow the same logic with a constant named z_1:

$$\frac{\partial}{\partial z} z_1 = 0 \Rightarrow f(z+z_1) \in \text{Solutions} \tag{3.5}$$

$$\left[\frac{\partial}{\partial z} z_1 = 0 \land f(z+z_1) \in \text{Solutions}\right] \Rightarrow \exists b_{1,k} : \mathbb{C} \to \mathbb{C}, f(z+z_1) = \sum_{k=0}^{n-1} b_{1,k}(z_1) g_k(z) \tag{3.6}$$

$$\exists b_{1,k} : \mathbb{C} \to \mathbb{C}, f(z+z_1) = \sum_{k=0}^{n-1} b_{1,k}(z_1) g_k(z) \tag{3.7}$$

Now we substitute $z \to z_1$ in (3.4), and we substitute $z \to z_0$ in (3.7), and combine them.

$$\exists b_{0,k} : \mathbb{C} \to \mathbb{C}, \exists b_{1,k} : \mathbb{C} \to \mathbb{C}, f(z_0+z_1) = \sum_{k=0}^{n-1} b_{0,k}(z_0) g_k(z_1) = \sum_{k=0}^{n-1} b_{1,k}(z_1) g_k(z_0) \tag{3.8}$$

We now have 2 ways of viewing $f(z_0+z_1)$, and we could imagine a sum of a line from the front, a sum of a line from the right, and 'deduce' that the true form is probably the sum of a square from above, which leads to the matrix form conjectured, but this is slightly sketchy logic that needs more rigor.

4 LEAN 4 code

This conjecture has been formalized in LEAN 4:

https://github.com/Nazgand/NazgandLean4/blob/master/NazgandLean4/ArgumentSumConjecture.lean

For a given differential equation, if the conjecture is true for 1 basis, then it is true for all basises.

5 Conjecture generalized to positive integer amounts of arguments in addition to 2

The following statements use the same assumptions and definitions as the main conjecture.

5.1 1 argument

The following statement, which I call ArgSumCon(1), is known to be true and seems to have been proved by Euler, though I have not seen a rigorous formalized proof yet.

$$\exists c(k_0) \in \mathbb{C}, f(z_0) = \sum_{k_0=0}^{n-1} c(k_0) g_{k_0}(z_0) \tag{5.1}$$

5.2 2 arguments

The following statement, which I call ArgSumCon(2), is equivalent to the main conjecture, minus the symmetry requirement.

$$\exists c(k_0,k_1) \in \mathbb{C}, f(z_0+z_1) = \sum_{k_0=0}^{n-1} \sum_{k_1=0}^{n-1} c(k_0,k_1) g_{k_0}(z_0) g_{k_1}(z_1) \tag{5.2}$$

5.3 3 arguments

The following statement, I call ArgSumCon(3).

$$\exists c(k_0,k_1,k_2) \in \mathbb{C}, f(z_0+z_1+z_2) = \sum_{k_0=0}^{n-1} \sum_{k_1=0}^{n-1} \sum_{k_2=0}^{n-1} c(k_0,k_1,k_2) g_{k_0}(z_0) g_{k_1}(z_1) g_{k_2}(z_2) \tag{5.3}$$

5.4 m arguments for $m \in \mathbb{Z}^+$

The following statement, I call ArgSumCon(m).

$$\exists c(k_0,k_1,\ldots,k_{m-1}) \in \mathbb{C}, f\left(\sum_{k=0}^{n-1} z_k\right) = \sum_{k_0=0}^{n-1} \sum_{k_1=0}^{n-1} \cdots \sum_{k_{m-1}=0}^{n-1} c(k_0,k_1,\ldots,k_{m-1}) \prod_{j=0}^{n-1} g_{k_j}(z_j) \tag{5.4}$$

Note ArgSumCon($m+1$) $\Rightarrow$ ArgSumCon(m), as can be seen by replacing 1 of the variables with 0.