

Root of Unity Exponential Sum Function related  
to Generalized Split-complex numbers  
<https://github.com/Nazgand/nazgandMathBook>

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**Abstract**

The goal of this paper is to analyze a class of functions which are equal to their own  $n$ th derivative.

## 1 Exponential Sum Definition

For the fundamental definition where  $n \in \mathbb{Z}^+ \wedge x \in \mathbb{C}$ , define:

$$\text{rues}_n(x) = \sum_{k=0}^{\infty} \frac{x^{nk}}{(nk)!} \quad (1.1)$$

## 2 Laplace inverse transform form

**Theorem 2.1.**

$$\text{rues}_n(t) = \mathcal{L}^{-1} \left\{ \frac{s^{n-1}}{s^n - 1} \right\} (t) \quad (2.1)$$

*Proof.* Equivalence is shown using the General derivative rule for Laplace transforms:

$$\mathcal{L} \left\{ f^{(n)}(t) \right\} (s) = s^n \mathcal{L} \{ f(t) \} (s) - \sum_{k=1}^n s^{n-k} f^{(k-1)}(0^+) \quad (2.2)$$

Substitute  $f$  with  $\text{rues}_n^{(k)}$  in (2.2), use the derivatives provided by (1.1) to simplify noting  $\text{mod}(-m, n) = \text{mod}(k-1, n)$  is the only surviving summand:

$$\mathcal{L} \left\{ \text{rues}_n^{(m)}(t) \right\} (s) = \mathcal{L} \left\{ \text{rues}_n^{(m+n)}(t) \right\} (s) = s^n \mathcal{L} \left\{ \text{rues}_n^{(m)}(t) \right\} (s) - s^{\text{mod}(m-1, n)} \quad (2.3)$$

Solve:

$$\mathcal{L} \left\{ \text{rues}_n^{(m)}(t) \right\} (s) = \frac{s^{\text{mod}(m-1, n)}}{s^n - 1} \quad (2.4)$$

□

### 3 As a sum of exponential functions

The reason this function is named *rues* is because it is a Root of Unity Exponential Sum function.

**Theorem 3.1.**

$$\text{rues}_n(x) = \frac{1}{n} \sum_{k=1}^n \exp\left(xe^{2ki\pi/n}\right) \quad (3.1)$$

*Proof.* Proof of equivalence to (1.1) via Taylor series:

$$\sum_{k=1}^n \exp\left(xe^{2ki\pi/n}\right) = \sum_{k=1}^n \sum_{j=0}^{\infty} e^{2jki\pi/n} \frac{x^j}{j!} = \sum_{j=0}^{\infty} \frac{x^j}{j!} \sum_{k=1}^n e^{2jki\pi/n} = \sum_{m=0}^{\infty} \frac{nx^{nm}}{(nm)!} \quad (3.2)$$

□

### 4 Real formulae derived from (3.1)

This section exploits this fact:

$$\exp(xe^{iy}) + \exp(xe^{-iy}) = 2e^{x \cos(y)} \cos(x \sin(y)) \quad (4.1)$$

Thus:

$$\text{rues}_n(x) = \frac{1}{n} \left( e^x + e^{-x} \frac{1 + \cos(\pi n)}{2} + 2 \sum_{k=1}^{\lceil n/2 \rceil - 1} e^{x \cos\left(\frac{2k\pi}{n}\right)} \cos\left(x \sin\left(\frac{2k\pi}{n}\right)\right) \right) \quad (4.2)$$

And:

$$\text{rues}_n\left(xe^{i\pi/n}\right) = \frac{1}{n} \left( e^{-x} \frac{1 - \cos(\pi n)}{2} + 2 \sum_{k=1}^{\lfloor n/2 \rfloor} e^{x \cos\left(\frac{2k-1}{n}\pi\right)} \cos\left(x \sin\left(\frac{2k-1}{n}\pi\right)\right) \right) \quad (4.3)$$

### 5 other

Notable values of  $n$ :

$$\text{rues}_1(x) = e^x \wedge \text{rues}_2(x) = \cosh(x) \wedge \text{rues}_4(x) = \cosh\left(\frac{x}{1+i}\right) \cosh\left(\frac{x}{1-i}\right) \quad (5.1)$$

Complex rotation property:

$$\text{rues}_n^{(k)}(x) = \text{rues}_n^{(k)}\left(x * e^{2i\pi/n}\right) e^{2ki\pi/n} \quad (5.2)$$

Derivative sum rules:

$$e^x = \sum_{k=0}^{n-1} \text{rues}_n^{(k)}(x) \wedge \text{rues}_n(x) = \sum_{k=0}^{m-1} \text{rues}_{nm}^{(kn)}(x) \quad (5.3)$$

$$\text{rues}_m(x) = \frac{1}{m} \sum_{k=1}^n \sum_{j=1}^m \text{rues}_n^{(k)} \left( x \exp \left( \frac{2i\pi j}{m} \right) \right) \quad (5.4)$$

**Theorem 5.1** (Argument sum rule).

$$\text{rues}_n(x+y) = \sum_{k=0}^{n-1} \text{rues}_n^{(k)}(x) \text{rues}_n^{(n-k)}(y) \quad (5.5)$$

*Proof.*

$$e^{x+y} = \sum_{k=0}^{n-1} \sum_{j=0}^{n-1} \text{rues}_n^{(k)}(x) \text{rues}_n^{(j)}(y) \quad (5.6)$$

$$\text{rues}_n(x+y) = \frac{1}{n} \sum_{l=1}^n \sum_{k=0}^{n-1} \sum_{j=0}^{n-1} \text{rues}_n^{(k)} \left( x \exp \left( \frac{2i\pi l}{n} \right) \right) \text{rues}_n^{(j)} \left( y \exp \left( \frac{2i\pi l}{n} \right) \right) \quad (5.7)$$

$$\text{rues}_n(x+y) = \frac{1}{n} \sum_{l=1}^n \sum_{k=0}^{n-1} \sum_{j=0}^{n-1} \text{rues}_n^{(k)}(x) \text{rues}_n^{(j)}(y) \exp \left( \frac{-2(j+k)i\pi l}{n} \right) \quad (5.8)$$

The  $l$  index cancels out all the terms where  $j+k \neq 0$ .  $\square$

**Corollary 5.2.**

$$\text{rues}_n^{(m)}(x+y) = \sum_{k=0}^{n-1} \text{rues}_n^{(m+k)}(x) \text{rues}_n^{(n-k)}(y) \quad (5.9)$$

*Proof.* Differentiate (5.5)  $\square$

**Corollary 5.3.**

$$\delta(\text{mod}(m, n)) = \sum_{k=0}^{n-1} \text{rues}_n^{(m+k)}(x) \text{rues}_n^{(n-k)}(-x) \quad (5.10)$$

*Proof.* Substituting  $x = -y$  in (5.9) produces the equations analogous to  $\sin(x)^2 + \cos(x)^2 = 1$   $\square$

## 6 Real and imaginary parts

Using (3.1)

$$\Re(\text{rues}_n(z)) = \frac{1}{n} \sum_{k=1}^n \exp\left(ze^{2ki\pi/n}\right) \exp\left(ze^{2ki\pi/n}\right) \quad (6.1)$$

For  $n \geq 2$ , arbitrary complex numbers may be decomposed as  $z = \Re(z) + e^{i\pi/n}$

## 7 Weierstrass product conjecture

This conjecture states all zeroes of  $\text{rues}_n^{(m)}$  not on the origin are of degree 1 and are located on the critical rays  $e^{i\pi(1+2k)/n}, k \in \mathbb{Z}$ . The absolute values of the zeroes are denoted

$$\text{Ez}(n, m, 1) = \min \left\{ \text{rues}_n^{(m)} \left( r * e^{i\pi/n} \right) \middle| r \in \mathbb{R}^+ \right\} \quad (7.1)$$

$$\text{Ez}(n, m, k+1) = \min \left\{ \text{rues}_n^{(m)} \left( r * e^{i\pi/n} \right) \middle| r \in \mathbb{R}^+, r > \text{Ez}(n, m, k) \right\} \quad (7.2)$$

Moreover, for  $n > 1$ ,

$$\text{rues}_n^{(m)}(z) = z^{\text{mod}(-m, n)} \prod_{k=1}^{\infty} \left( 1 + \left( \frac{z}{\text{Ez}(n, m, k)} \right)^n \right) \quad (7.3)$$

This conjecture is easily verifiable for  $(n, m) \in \{(2, 0), (2, 1), (4, 0)\}$ .

Work has been done in German to show this works for  $\text{rues}_n(z)$ .

<https://math.stackexchange.com/questions/3221569/conjecture-all-complex-roots-of-sum-k-0>

### 7.1 Bounds on Complex Zeroes $\text{rues}_3$

Lemma

$$\cos(a + bi) = \cos(a) \cosh(b) - i \sin(a) \sinh(b) \quad (7.4)$$

$$|\cos(a + bi)| \leq |\cos(a) \cosh(b)| + |\sin(a) \sinh(b)| \quad (7.5)$$

$$|\cos(a + bi)| \leq |\cosh(b)| + |\sinh(b)| = \cosh(|b|) + \sinh(|b|) = \exp(|b|) \quad (7.6)$$

$$|\cos(a + bi)| \leq \exp(|b|) \quad (7.7)$$

A bound which supports this conjecture is attainable from (4.2).

$$\text{rues}_3(x) = \frac{1}{3} \left( e^x + 2e^{x \cos(\frac{2\pi}{3})} \cos \left( x \sin \left( \frac{2\pi}{3} \right) \right) \right) \quad (7.8)$$

$$0 = \text{rues}_3(a + bi) = \frac{1}{3} \left( e^{(a+bi)} + 2e^{(a+bi) \cos(\frac{2\pi}{3})} \cos \left( (a+bi) \sin \left( \frac{2\pi}{3} \right) \right) \right) \quad (7.9)$$

$$-e^{(a+bi)} = 2e^{(a+bi) \cos\left(\frac{2\pi}{3}\right)} \cos\left((a+bi) \sin\left(\frac{2\pi}{3}\right)\right) \quad (7.10)$$

Absolute value:

$$e^a = 2e^{a \cos\left(\frac{2\pi}{3}\right)} \left| \cos\left((a+bi) \sin\left(\frac{2\pi}{3}\right)\right) \right| \quad (7.11)$$

$$\exp\left(\frac{3a}{2}\right) = 2 \left| \cos\left((a+bi) \sin\left(\frac{2\pi}{3}\right)\right) \right| \quad (7.12)$$

Use lemma (7.7):

$$\exp\left(\frac{3a}{2}\right) \leq 2 \exp\left(|b| \sin\left(\frac{2\pi}{3}\right)\right) \quad (7.13)$$

$$\frac{3a}{2} \leq \ln 2 + |b| \frac{\sqrt{3}}{2} \quad (7.14)$$

The last inequality carves out a sector of the complex plane where zeroes are not allowed. The symmetric properties of  $\text{rues}_3$  restrict zeroes to appearing near the conjectured region.

## 7.2 Non-Rigorous Bounds on Complex Zeroes $\text{rues}_n, n \geq 3$

A bound which supports this conjecture is attainable from (4.2).

$$\text{rues}_n(x) = \frac{1}{n} \left( e^x + e^{-x} \frac{1 + \cos(\pi n)}{2} + 2 \sum_{k=1}^{\lceil n/2 \rceil - 1} e^{x \cos\left(\frac{2k\pi}{n}\right)} \cos\left(x \sin\left(\frac{2k\pi}{n}\right)\right) \right) \quad (7.15)$$

Note  $\text{rues}_n(x)$  has the same zeroes as  $\text{rues}_n(x) \exp(-x \cos(\frac{2\pi}{n}))$ , and most terms go to zero as  $\Re(x) \rightarrow \infty$ .

$$\text{rues}_n(x) \exp\left(-x \cos\left(\frac{2\pi}{n}\right)\right) \approx \mathcal{O}\left(\frac{1}{n} \left( e^{x(1 - \cos(\frac{2\pi}{n}))} + 2 \cos\left(x \sin\left(\frac{2\pi}{n}\right)\right) \right) \right) \quad (7.16)$$

Zeroes of  $\text{rues}_n(x)$  with large real parts are approximately bound by this equation:

$$0 = e^{x(1 - \cos(\frac{2\pi}{n}))} + 2 \cos\left(x \sin\left(\frac{2\pi}{n}\right)\right) \quad (7.17)$$

$$-e^{(a+bi)(1 - \cos(\frac{2\pi}{n}))} = 2 \cos\left((a+bi) \sin\left(\frac{2\pi}{n}\right)\right) \quad (7.18)$$

Absolute value:

$$\exp\left(a \left(1 - \cos\left(\frac{2\pi}{n}\right)\right)\right) = 2 \left| \cos\left((a+bi) \sin\left(\frac{2\pi}{n}\right)\right) \right| \quad (7.19)$$

Use lemma (7.7):

$$\exp\left(a \left(1 - \cos\left(\frac{2\pi}{n}\right)\right)\right) \leq 2 \exp\left(|b| \sin\left(\frac{2\pi}{n}\right)\right) \quad (7.20)$$

Logarithm:

$$a \left( 1 - \cos \left( \frac{2\pi}{n} \right) \right) \leq \ln(2) + |b| \sin \left( \frac{2\pi}{n} \right) \quad (7.21)$$

Equation form with slope:

$$a \leq \frac{\ln(2)}{1 - \cos \left( \frac{2\pi}{n} \right)} + |b| \cot \left( \frac{\pi}{n} \right) \quad (7.22)$$

The last inequality carves out a sector of the complex plane where zeroes are not allowed. The symmetric properties of  $\text{rues}_n$  restrict zeroes to appearing near the conjectured region.

## 8 Riemann Liouville Operator

TODO

## 9 Generalized split-complex numbers

### 9.1 Introduction

Split complex numbers a ring extending the reals with an extra unit which is a square root of 1 which is not a complex number.

### 9.2 Generalization

This concept may be generalized to rings with  $n$ th roots of 1, and these rings are interrelated. Define the general split units as

$$\text{gs}(r_0) * \text{gs}(r_1) = \text{gs}(\text{mod}(r_0 + r_1, 1)) \wedge [\text{gs}(z) \in \mathbb{C} \Leftrightarrow z \in \mathbb{Z}] \quad (9.1)$$

The general split units commute with complex numbers:

$$c \in \mathbb{C} \Rightarrow c * \text{gs}(r_0) = \text{gs}(r_0) * c \quad (9.2)$$

Simple deduction from the Taylor series of the exponential function for  $m \in \mathbb{Z}, c \in \mathbb{C}$ .

$$\exp \left( c * \text{gs} \left( \frac{m}{n} \right) \right) = \sum_{k=0}^{n-1} \text{gs} \left( \frac{m}{n} \right)^k \text{rues}_n^{(n-k)}(c) = \sum_{k=0}^{n-1} \text{gs} \left( \frac{km}{n} \right) \text{rues}_n^{(n-k)}(c) \quad (9.3)$$

The  $n$ -split-complex number ring is

$$\mathbf{Gs}_n = \left\{ \sum_{k=0}^{n-1} c_k * \text{gs} \left( \frac{k}{n} \right) \mid c_k \in \mathbb{C} \right\} \quad (9.4)$$

Sub-ring property:

$$m \in \mathbb{Z}^+ \Rightarrow \mathbf{Gs}_n \subseteq \mathbf{Gs}_{m*n} \quad (9.5)$$

This self-isomorphism of  $f : \mathbf{Gs}_n \rightarrow \mathbf{Gs}_n$  exists for  $m \in \mathbb{Z} \wedge \gcd(j, n) = 1$ .

$$f\left(\sum_{k=0}^{n-1} c_k * \text{gs}\left(\frac{k}{n}\right)\right) = \sum_{k=0}^{n-1} c_k \exp\left(\frac{2kmi\pi}{n}\right) \text{gs}\left(\frac{kj}{n}\right) \quad (9.6)$$

The above is a homomorphism for arbitrary  $j, m \in \mathbb{Z}$ .  $f : \mathbf{Gs}_n \rightarrow \mathbf{Gs}_{n/\gcd(j, n)}$ .

$$f\left(\sum_{k=0}^{n-1} c_k * \text{gs}\left(\frac{k}{n}\right)\right) = \sum_{k=0}^{n-1} c_k \exp\left(\frac{2kmi\pi}{n}\right) \text{gs}\left(\frac{kj}{n}\right) \quad (9.7)$$

Co-set isomorphism property,  $f : \mathbf{Gs}_{m_0 n} / \mathbf{Gs}_{m_0} \rightarrow \mathbf{Gs}_{m_1 n} / \mathbf{Gs}_{m_1}$  for  $n, m_0, m_1 \in \mathbb{Z}^+$

$$f\left(\mathbf{Gs}_{m_0} * \sum_{k=0}^{n-1} c_k * \text{gs}\left(\frac{k}{m_0 n}\right)\right) = \mathbf{Gs}_{m_1} * \sum_{k=0}^{n-1} c_k * \text{gs}\left(\frac{k}{m_1 n}\right) \quad (9.8)$$

Idempotent elements: Easily found:

$$\left(\sum_{k=0}^{n-1} \frac{1}{n} \text{gs}\left(\frac{k}{n}\right)\right)^2 = \sum_{k=0}^{n-1} \frac{1}{n} \text{gs}\left(\frac{k}{n}\right) \quad (9.9)$$

Because homomorphisms map idempotent elements to idempotent elements, using (9.7) produces a set of idempotent elements

$$\left\{ \sum_{k=0}^{n-1} \frac{1}{n} \exp\left(\frac{2kmi\pi}{n}\right) \text{gs}\left(\frac{kj}{n}\right) \middle| m, j \in \mathbb{Z} \right\} \quad (9.10)$$

The set of idempotent elements is closed under multiplication in abelian groups.

$$[n \in \mathbb{Z}^+ \wedge (F_1 F_2)^n = (F_2 F_1)^n \wedge F_1^2 = F_1 \wedge F_2^2 = F_2] \Rightarrow ((F_1 F_2)^n)^2 = (F_1 F_2)^n \quad (9.11)$$

.

$$[F_1 F_2 = F_2 F_1 \wedge F_1^2 = F_1 \wedge F_2^2 = F_2] \Rightarrow (F_1 + F_2 - 2F_1 F_2)^2 = (F_1 + F_2 - 2F_1 F_2) \quad (9.12)$$

.

### 9.3 Possible applications

<http://www.math.usm.edu/lee/quantum.html> Traditional Split-complex numbers, a sub-ring of  $\mathbf{Gs}_2$ , are used in quantum mechanics, thus General Split-complex numbers likely also have application.

## 10 More general rings

### 10.1 Extending the reals

The goal here is to extend the real numbers,  $\mathbb{R}$ , using an abelian multiplicative group,  $G$ , such that exponential functions still work. The extension is the abelian ring  $\left\{ \sum_{g \in G} g * x_g \mid x_g \in \mathbb{R} \right\}$ , denoted  $\mathbb{R}[G]$ .

Defining

$$\exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} \quad (10.1)$$

, an immediate consequence for all rings is

$$x_1 * x_2 = x_2 * x_1 \Rightarrow \exp(x_1) * \exp(x_2) = \exp(x_1 + x_2) \quad (10.2)$$

. The ring  $\mathbb{R}[G]$  is abelian, thus  $x_1, x_2 \in \mathbb{R}[G] \Rightarrow \exp(x_1) * \exp(x_2) = \exp(x_1 + x_2)$ . The elements of finite order may be simplified:

$$[1 = g^n \wedge n \in \mathbb{Z}^+ \wedge g \in G \wedge x \in \mathbb{R}] \Rightarrow \exp(g * x) = \sum_{k=0}^{n-1} g^k * \text{rues}_n^{(n-k)}(x) \quad (10.3)$$

Every homomorphism,  $f(a * b) = f(a) * f(b)$ ,  $f : G_1 \rightarrow G_2$  may be extended to  $f : \mathbb{R}[G_1] \rightarrow \mathbb{R}[G_2]$ ,  $f(a + b) = f(a) + f(b)$ ,  $f(a * b) = f(a) * f(b)$ ,  $f(\exp(a)) = \exp(f(a))$ .

By the fundamental theorem of finite abelian groups, every finite abelian group is the direct sum of cyclic groups. Every cyclic group is isomorphic to a subgroup of  $(\mathbb{R}/\mathbb{Z}, +)$ . Thus every finite abelian group is isomorphic to a subgroup of the ring generated by  $\left( (\mathbb{R}/\mathbb{Z})^{\aleph_0}, + \right)$ .

### 10.2 Extending other rings

The goal here is to extend an abelian ring,  $R_a$ , with an exponential function,  $x_1, x_2 \in R_a \Rightarrow f(x_1 + x_2) = f(x_1) * f(x_2)$ , using an abelian multiplicative group,  $G$ .  $f(0) = f(0)^2$ , i.e.  $f(0)$  is idempotent.  $\{f(x) * a \mid a^2 = a \wedge a \in R_a\}$  are all valid exponential functions.

## 11 Bibliography

<https://math.stackexchange.com/questions/1542147/find-the-complex-or-real-roots-of-e-frac3->  
[https://en.wikipedia.org/wiki/Split-complex\\_number](https://en.wikipedia.org/wiki/Split-complex_number)