Root Of Unity Exponential Sum Function Related To Generalized Split-complex Numbers

Mark Andrew Gerads <Nazgand@Gmail.Com>

2026-01-21 04:46:10-08:00

Abstract

The goal of this paper is to analyze a class of functions which are equal to their own nth derivative.

1 Exponential Sum Definition

For the fundamental definition where $n \in \mathbb{Z}_{>0} \wedge x \in \mathbb{C}$, define:

$$\mathrm{Rues}_n(x) = \sum_{k=0}^{\infty} \frac{x^{nk}}{(nk)!} \tag{1.1}$$

2 Laplace inverse transform form

Theorem 2.1.

$$\mathrm{Rues}_n(t) = \mathcal{L}^{-1}\left\{\frac{s^{n-1}}{s^n - 1}\right\}(t) \tag{2.1}$$

Proof. Equivalence is shown using the General derivative rule for Laplace transforms:

$$\mathcal{L}\left\{f^{(n)}(t)\right\}(s) = s^n \mathcal{L}\{f(t)\}(s) - \sum_{k=1}^{n} s^{n-k} f^{(k-1)}(0^+) \tag{2.2}$$

Substitute f with $\mathrm{Rues}_n^{(k)}$ in (2.2), use the derivatives provided by (1.1) to simplify noting $\mathrm{Mod}(-m, n) = \mathrm{Mod}(k-1, n)$ is the only surviving summand:

$$\mathcal{L}\left\{\mathrm{Rues}_n^{(m)}(t)\right\}(s) = \mathcal{L}\left\{\mathrm{Rues}_n^{(m+n)}(t)\right\}(s) = s^n \mathcal{L}\left\{\mathrm{Rues}_n^{(m)}(t)\right\}(s) - s^{\mathrm{Mod}(m-1,n)} \tag{2.3}$$

Solve:

$$\mathcal{L}\left\{\mathrm{Rues}_n^{(m)}(t)\right\}(s) = \frac{s^{\mathrm{Mod}(m-1,n)}}{s^n - 1} \tag{2.4}$$

□

3 As a sum of exponential functions

The reason this function is named Rues is because it is a Root of Unity Exponential Sum function.

Theorem 3.1.

$$\mathrm{Rues}_n(x) = \frac{1}{n}\sum_{k=1}^{n} \exp\left(x e^{2ki\pi/n}\right) \tag{3.1}$$

Proof. Proof of equivalence to (1.1) via Taylor series:

$$\sum_{k=1}^{n} \exp\left(x e^{2ki\pi/n}\right) = \sum_{k=1}^{n}\sum_{j=0}^{\infty} e^{2jki\pi/n}\frac{x^j}{j!} = \sum_{j=0}^{\infty}\frac{x^j}{j!}\sum_{k=1}^{n} e^{2jki\pi/n} = \sum_{m=0}^{\infty}\frac{n x^{nm}}{(nm)!} \tag{3.2}$$

□

4 Real formulae derived from (3.1)

This section exploits this fact:

$$\exp\left(x e^{iy}\right) + \exp\left(x e^{-iy}\right) = 2e^{x\cos(y)}\cos(x\sin(y)) \tag{4.1}$$

Thus:

$$\text{Rues}_n(x) = \frac{1}{n}\left(e^x + e^{-x}\frac{1+\cos(\pi n)}{2} + 2\sum_{k=1}^{\lceil n/2\rceil - 1} e^{x\cos\left(\frac{2k\pi}{n}\right)}\cos\left(x\sin\left(\frac{2k\pi}{n}\right)\right)\right) \tag{4.2}$$

And:

$$\text{Rues}_n\left(xe^{i\pi/n}\right) = \frac{1}{n}\left(e^{-x}\frac{1-\cos(\pi n)}{2} + 2\sum_{k=1}^{\lfloor n/2\rfloor} e^{x\cos\left(\frac{2k-1}{n}\pi\right)}\cos\left(x\sin\left(\frac{2k-1}{n}\pi\right)\right)\right) \tag{4.3}$$

5 other

Notable values of n:

$$\text{Rues}_1(x) = e^x \wedge \text{Rues}_2(x) = \cosh(x) \wedge \text{Rues}_4(x) = \cosh\left(\frac{x}{1+i}\right)\cosh\left(\frac{x}{1-i}\right) \tag{5.1}$$

Complex rotation property:

$$\text{Rues}_n^{(k)}(x) = \text{Rues}_n^{(k)}\left(x * e^{2i\pi/n}\right)e^{2ki\pi/n} \tag{5.2}$$

Derivative sum rules:

$$e^x = \sum_{k=0}^{n-1}\text{Rues}_n^{(k)}(x) \wedge \text{Rues}_n(x) = \sum_{k=0}^{m-1}\text{Rues}_{nm}^{(kn)}(x) \tag{5.3}$$

$$\text{Rues}_m(x) = \frac{1}{m}\sum_{k=1}^{n}\sum_{j=1}^{m}\text{Rues}_n^{(k)}\left(x\exp\left(\frac{2i\pi j}{m}\right)\right) \tag{5.4}$$

Theorem 5.1 (Argument sum rule).

$$\text{Rues}_n(x+y) = \sum_{k=0}^{n-1}\text{Rues}_n^{(k)}(x)\,\text{Rues}_n^{(n-k)}(y) \tag{5.5}$$

Proof.

$$e^{x+y} = \sum_{k=0}^{n-1}\sum_{j=0}^{n-1}\text{Rues}_n^{(k)}(x)\,\text{Rues}_n^{(j)}(y) \tag{5.6}$$

$$\text{Rues}_n(x+y) = \frac{1}{n}\sum_{l=1}^{n}\sum_{k=0}^{n-1}\sum_{j=0}^{n-1}\text{Rues}_n^{(k)}\left(x\exp\left(\frac{2i\pi l}{n}\right)\right)\text{Rues}_n^{(j)}\left(y\exp\left(\frac{2i\pi l}{n}\right)\right) \tag{5.7}$$

$$\text{Rues}_n(x+y) = \frac{1}{n}\sum_{l=1}^{n}\sum_{k=0}^{n-1}\sum_{j=0}^{n-1}\text{Rues}_n^{(k)}(x)\,\text{Rues}_n^{(j)}(y)\exp\left(\frac{-2(j+k)i\pi l}{n}\right) \tag{5.8}$$

The l index cancels out all the terms where $j+k\neq 0$. □

Corollary 5.2.

$$\text{Rues}_n^{(m)}(x+y) = \sum_{k=0}^{n-1}\text{Rues}_n^{(m+k)}(x)\,\text{Rues}_n^{(n-k)}(y) \tag{5.9}$$

Proof. Differentiate (5.5) □

Corollary 5.3.

$$\delta(\text{Mod}(m,n)) = \sum_{k=0}^{n-1}\text{Rues}_n^{(m+k)}(x)\,\text{Rues}_n^{(n-k)}(-x) \tag{5.10}$$

Proof. Substituting $y \to -x$ in (5.9) produces the equations analogous to $\sin(x)^2 + \cos(x)^2 = 1$ □

6 Real and imaginary parts

Using (3.1)

$$\text{Re}(\text{Rues}_n(z)) = \frac{1}{n}\sum_{k=1}^{n}\exp\left(ze^{2ki\pi/n}\right)\exp\left(ze^{2ki\pi/n}\right) \tag{6.1}$$

For $n\geq 2$, arbitrary complex numbers many be decomposed as $z = \text{Re}(z) + e^{i\pi/n}$

7 Weierstrass product conjecture

This conjecture states all zeroes of $\text{Rues}_n^{(m)}$ not on the origin are of degree 1 and are located on the critical rays $e^{i\pi(1+2k)/n}, k \in \mathbb{Z}$. The absolute values of the zeroes are denoted

$$\text{Ez}\,(n,m,1) = \min\left\{\text{Rues}_n^{(m)}\left(r * e^{i\pi/n}\right)\middle| r \in \mathbb{R}^+\right\} \tag{7.1}$$

$$\text{Ez}\,(n,m,k+1) = \min\left\{\text{Rues}_n^{(m)}\left(r * e^{i\pi/n}\right)\middle| r \in \mathbb{R}^+, r > \text{Ez}\,(n,m,k)\right\} \tag{7.2}$$

Moreover, for $n > 1$,

$$\text{Rues}_n^{(m)}(z) = z^{\text{Mod}\,(-m,n)} \prod_{k=1}^{\infty}\left(1+\left(\frac{z}{\text{Ez}\,(n,m,k)}\right)^n\right) \tag{7.3}$$

This conjecture is easily verifiable for $(n,m) \in \{(2,0),(2,1),(4,0)\}$.

Work has been done in German to show this works for $\text{Rues}_n(z)$.

Mathematics Stack Exchange | All complex roots of $\sum_{k=0}^{\infty}\frac{z^k}{(nk)!}$ are real

7.1 Bounds on Complex Zeroes Rues_3

Lemma

$$\cos(a+bi) = \cos(a)\cosh(b) - i\sin(a)\sinh(b) \tag{7.4}$$

$$|\cos(a+bi)| \leq |\cos(a)\cosh(b)| + |\sin(a)\sinh(b)| \tag{7.5}$$

$$|\cos(a+bi)| \leq |\cosh(b)| + |\sinh(b)| = \cosh(|b|) + \sinh(|b|) = \exp(|b|) \tag{7.6}$$

$$|\cos(a+bi)| \leq \exp(|b|) \tag{7.7}$$

A bound which supports this conjecture is attainable from (4.2).

$$\text{Rues}_3(x) = \frac{1}{3}\left(e^x + 2e^{x\cos\left(\frac{2\pi}{3}\right)}\cos\left(x\sin\left(\frac{2\pi}{3}\right)\right)\right) \tag{7.8}$$

$$0 = \text{Rues}_3(a+bi) = \frac{1}{3}\left(e^{(a+bi)} + 2e^{(a+bi)\cos\left(\frac{2\pi}{3}\right)}\cos\left((a+bi)\sin\left(\frac{2\pi}{3}\right)\right)\right) \tag{7.9}$$

$$-e^{(a+bi)} = 2e^{(a+bi)\cos\left(\frac{2\pi}{3}\right)}\cos\left((a+bi)\sin\left(\frac{2\pi}{3}\right)\right) \tag{7.10}$$

Absolute value:

$$e^a = 2e^{a\cos\left(\frac{2\pi}{3}\right)}\left|\cos\left((a+bi)\sin\left(\frac{2\pi}{3}\right)\right)\right| \tag{7.11}$$

$$\exp\left(\frac{3a}{2}\right) = 2\left|\cos\left((a+bi)\sin\left(\frac{2\pi}{3}\right)\right)\right| \tag{7.12}$$

Use lemma (7.7):

$$\exp\left(\frac{3a}{2}\right) \leq 2\exp\left(|b|\sin\left(\frac{2\pi}{3}\right)\right) \tag{7.13}$$

$$\frac{3a}{2} \leq \ln 2 + |b|\frac{\sqrt{3}}{2} \tag{7.14}$$

The last inequality carves out a sector of the complex plane where zeroes are not allowed. The symmetric properties of Rues_3 restrict zeroes to appearing near the conjectured region.

7.2 Non-Rigorous Bounds on Complex Zeroes $\text{Rues}_n, n \geq 3$

A bound which supports this conjecture is attainable from (4.2).

$$\text{Rues}_n(x) = \frac{1}{n}\left(e^x + e^{-x}\frac{1+\cos(\pi n)}{2} + 2\sum_{k=1}^{\lceil n/2\rceil-1} e^{x\cos\left(\frac{2k\pi}{n}\right)}\cos\left(x\sin\left(\frac{2k\pi}{n}\right)\right)\right) \tag{7.15}$$

Note $\text{Rues}_n(x)$ has the same zeroes as $\text{Rues}_n(x)\exp\left(-x\cos\left(\frac{2\pi}{n}\right)\right)$, and most terms go to zero as $\text{Re}(x) \to \infty$.

$$\text{Rues}_n(x)\exp\left(-x\cos\left(\frac{2\pi}{n}\right)\right) \approx \mathcal{O}\left(\frac{1}{n}\left(e^{x\left(1-\cos\left(\frac{2\pi}{n}\right)\right)} + 2\cos\left(x\sin\left(\frac{2\pi}{n}\right)\right)\right)\right) \tag{7.16}$$

Zeroes of $\text{Rues}_n(x)$ with large real parts are approximately bound by this equation:

$$0 = e^{x\left(1-\cos\left(\frac{2\pi}{n}\right)\right)} + 2\cos\left(x\sin\left(\frac{2\pi}{n}\right)\right) \tag{7.17}$$

$$-e^{(a+b*i)\left(1-\cos\left(\frac{2\pi}{n}\right)\right)} = 2\cos\left((a+b*i)\sin\left(\frac{2\pi}{n}\right)\right) \tag{7.18}$$

Absolute value:

$$\exp\left(a\left(1-\cos\left(\frac{2\pi}{n}\right)\right)\right) = 2\left|\cos\left((a+b*i)\sin\left(\frac{2\pi}{n}\right)\right)\right| \tag{7.19}$$

Use lemma (7.7):

$$\exp\left(a\left(1-\cos\left(\frac{2\pi}{n}\right)\right)\right) \leq 2\exp\left(|b|\sin\left(\frac{2\pi}{n}\right)\right) \tag{7.20}$$

Logarithm:

$$a\left(1-\cos\left(\frac{2\pi}{n}\right)\right) \leq \ln(2) + |b|\sin\left(\frac{2\pi}{n}\right) \tag{7.21}$$

Equation form with slope:

$$a \leq \frac{\ln(2)}{1-\cos\left(\frac{2\pi}{n}\right)} + |b|\cot\left(\frac{\pi}{n}\right) \tag{7.22}$$

The last inequality carves out a sector of the complex plane where zeroes are not allowed. The symmetric properties of Rues_n restrict zeroes to appearing near the conjectured region.

8 Generalized split-complex numbers

8.1 Introduction

Split complex numbers a ring extending the reals with an extra unit which is a square root of 1 which is not a complex number.

8.2 Generalization

This concept may be generalized to rings with nth roots of 1, and these rings are interrelated. Define the general split units as

$$\text{gs}(r_0) * \text{gs}(r_1) = \text{gs}(\text{Mod}(r_0+r_1, 1)) \wedge [\text{gs}(z) \in \mathbb{C} \Leftrightarrow z \in \mathbb{Z}] \tag{8.1}$$

The general split units commute with complex numbers:

$$c \in \mathbb{C} \Rightarrow c * \text{gs}(r_0) = \text{gs}(r_0) * c \tag{8.2}$$

Simple deduction from the Taylor series of the exponential function for $m \in \mathbb{Z}, c \in \mathbb{C}$.

$$\exp\left(c * \text{gs}\left(\frac{m}{n}\right)\right) = \sum_{k=0}^{n-1} \text{gs}\left(\frac{m}{n}\right)^k \text{Rues}_n^{(n-k)}(c) = \sum_{k=0}^{n-1} \text{gs}\left(\frac{km}{n}\right) \text{Rues}_n^{(n-k)}(c) \tag{8.3}$$

The n-split-complex number ring is

$$\mathbf{Gs}_n = \left\{\sum_{k=0}^{n-1} c_k * \text{gs}\left(\frac{k}{n}\right) \middle| c_k \in \mathbb{C}\right\} \tag{8.4}$$

Sub-ring property:

$$m \in \mathbb{Z}_{>0} \Rightarrow \mathbf{Gs}_n \subseteq \mathbf{Gs}_{m*n} \tag{8.5}$$

This self-isomorphism of $f : \mathbf{Gs}_n \to \mathbf{Gs}_n$ exists for $m \in \mathbb{Z} \wedge \gcd(j,n) = 1$.

$$f\left(\sum_{k=0}^{n-1} c_k * \text{gs}\left(\frac{k}{n}\right)\right) = \sum_{k=0}^{n-1} c_k \exp\left(\frac{2kmi\pi}{n}\right) \text{gs}\left(\frac{kj}{n}\right) \tag{8.6}$$

The above is a homomorphism for arbitrary $j, m \in \mathbb{Z}$. $f : \mathbf{Gs}_n \to \mathbf{Gs}_{n/\gcd(j,n)}$.

$$f\left(\sum_{k=0}^{n-1} c_k * \text{gs}\left(\frac{k}{n}\right)\right) = \sum_{k=0}^{n-1} c_k \exp\left(\frac{2kmi\pi}{n}\right) \text{gs}\left(\frac{kj}{n}\right) \tag{8.7}$$

Co-set isomorphism property, $f : \mathbf{Gs}_{m_0 n}/\mathbf{Gs}_{m_0} \to \mathbf{Gs}_{m_1 n}/\mathbf{Gs}_{m_1}$ for $n, m_0, m_1 \in \mathbb{Z}_{>0}$

$$f\left(\mathbf{Gs}_{m_0} * \sum_{k=0}^{n-1} c_k * \text{gs}\left(\frac{k}{m_0 n}\right)\right) = \mathbf{Gs}_{m_1} * \sum_{k=0}^{n-1} c_k * \text{gs}\left(\frac{k}{m_1 n}\right) \tag{8.8}$$

Idempotent elements: Easily found:

$$\left(\sum_{k=0}^{n-1} \frac{1}{n} \text{gs}\left(\frac{k}{n}\right)\right)^2 = \sum_{k=0}^{n-1} \frac{1}{n} \text{gs}\left(\frac{k}{n}\right) \tag{8.9}$$

Because homomorphisms map idempotent elements to idempotent elements, using (8.7) produces a set of idempotent elements

$$\left\{\sum_{k=0}^{n-1}\frac{1}{n}\exp\left(\frac{2kmi\pi}{n}\right)\mathrm{gs}\left(\frac{kj}{n}\right)\middle|m,j\in\mathbb{Z}\right\} \tag{8.10}$$

The set of idempotent elements is closed under multiplication in abelian groups.

$$\left[n\in\mathbb{Z}_{>0}\wedge F_1F_2=F_2F_1\wedge F_1^2=F_1\wedge F_2^2=F_2\right]\Rightarrow\left((F_1F_2)^n\right)^2=(F_1F_2)^n \tag{8.11}$$

.

$$\left[F_1F_2=F_2F_1\wedge F_1^2=F_1\wedge F_2^2=F_2\right]\Rightarrow(F_1+F_2-2F_1F_2)^2=(F_1+F_2-2F_1F_2) \tag{8.12}$$

.

8.3 Possible applications

http://www.math.usm.edu/lee/quantum.html Traditional Split-complex numbers, a sub-ring of **Gs**$_2$, are used in quantum mechanics, thus General Split-complex numbers likely also have application.

9 More general rings

9.1 Extending the reals

The goal here is to extend the real numbers, $\mathbb{R}$, using an abelian multiplicative group, G, such that exponential functions still work. The extension is the abelian ring $\left\{\sum_{g\in G}g*x_g\middle|x_g\in\mathbb{R}\right\}$, denoted $\mathbb{R}[G]$.

Defining

$$\exp(x)=\sum_{k=0}^{\infty}\frac{x^k}{k!} \tag{9.1}$$

, an immediate consequence for all rings is

$$x_1*x_2=x_2*x_1\Rightarrow\exp(x_1)*\exp(x_2)=\exp(x_1+x_2) \tag{9.2}$$

. The ring $\mathbb{R}[G]$ is abelian, thus $x_1,x_2\in\mathbb{R}[G]\Rightarrow\exp(x_1)*\exp(x_2)=\exp(x_1+x_2)$. The elements of finite order may be simplified:

$$[1=g^n\wedge n\in\mathbb{Z}_{>0}\wedge g\in G\wedge x\in\mathbb{R}]\Rightarrow\exp(g*x)=\sum_{k=0}^{n-1}g^k*\mathrm{Rues}_n^{(n-k)}(x) \tag{9.3}$$

Every homomorphism, $f(a*b)=f(a)*f(b)$, $f:G_1\to G_2$ may be extended to $f:\mathbb{R}[G_1]\to\mathbb{R}[G_2]$, $f(a+b)=f(a)+f(b)$, $f(a*b)=f(a)*f(b)$, $f(\exp(a))=\exp(f(a))$.

By the fundamental theorem of finite abelian groups, every finite abelian group is the direct sum of cyclic groups. Every cyclic group is isomorphic to a subgroup of $(\mathbb{R}/\mathbb{Z},+)$. Thus every finite abelian group is isomorphic to a subgroup of the ring generated by $\left((\mathbb{R}/\mathbb{Z})^{\beth_0},+\right)$.

9.2 Extending other rings

The goal here is to extend an abelian ring, R_a, with an exponential function, $x_1,x_2\in R_a\Rightarrow f(x_1+x_2)=f(x_1)*f(x_2)$, using an abelian multiplicative group, G. $f(0)=f(0)^2$, i.e. $f(0)$ is idempotent. $\{f(x)*a|a^2=a\wedge a\in R_a\}$ are all valid exponential functions.

10 Bibliography

Mathematics Stack Exchange|Find the complex (or real) roots of $e^{\frac{3x}{2}}+2\cos\left(\frac{\sqrt{3}x}{2}\right)$

https://en.wikipedia.org/wiki/Split-complex_number