https://GitHub.Com/Nazgand/NazgandMathBook

1 With Varying Base Digits

Mark Andrew Gerads <Nazgand@Gmail.Com>

2026-01-21 04:45:49-08:00

Let exist a sequence $a_k \in \mathbb{Z}, a_k > 1$. Then

$$1 = \sum_{k \in \mathbb{Z}_{>0}} \frac{a_k - 1}{\prod_{m=1}^{k} a_m} = \sum_{k=1}^{\infty} \frac{a_k - 1}{\prod_{m=1}^{k} a_m} \tag{0.1}$$

This can be seen via the limit $j \to \infty$ in the following equation.

$$1 - \prod_{m=1}^{j} a_m^{-1} = \sum_{k=1}^{j} \frac{a_k - 1}{\prod_{m=1}^{k} a_m} \tag{0.2}$$

Proof of previous equation by induction: For the base case, $j = 0$, we have $1 - 1 = 0$. For the inductive step, we can simply prove

$$\left(1 - \prod_{m=1}^{j+1} a_m^{-1}\right) - \left(1 - \prod_{m=1}^{j} a_m^{-1}\right) = \left(\sum_{k=1}^{j+1} \frac{a_k - 1}{\prod_{m=1}^{k} a_m}\right) - \left(\sum_{k=1}^{j} \frac{a_k - 1}{\prod_{m=1}^{k} a_m}\right) \tag{0.3}$$

Simplify both sides to get:

$$\prod_{m=1}^{j} a_m^{-1} - \prod_{m=1}^{j+1} a_m^{-1} = \sum_{k=j+1}^{j+1} \frac{a_k - 1}{\prod_{m=1}^{k} a_m} = \frac{a_{j+1} - 1}{\prod_{m=1}^{j+1} a_m} \tag{0.4}$$

Multiply both sides by $\prod_{m=1}^{j+1} a_m$:

$$a_{j+1} - 1 = a_{j+1} - 1 \tag{0.5}$$

$\square$