Kettenbruch Continued Fractions
https://github.com/Nazgand/nazgandMathBook

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Abstract

The goal of this paper is to have fun reviewing Kettenbruch notation for continued fractions.

1 Introduction

The 2 defining equations are:

$$\operatorname*{K}_{k=n}^{n} \frac{a_k}{b_k} = \frac{a_n}{b_n} \tag{1.1}$$

$$\operatorname*{K}_{k=n}^{m} \frac{a_k}{b_k} = \frac{a_n}{b_n + \operatorname*{K}_{k=n+1}^{m} \frac{a_k}{b_k}} \tag{1.2}$$

The most interesting continued fractions are infinite continued fractions, like so, yet careful consideration must be taken in this case, e.g. because the limit might not exist:

$$\operatorname*{K}_{k=1}^{\infty} \frac{a_k}{b_k} = \lim_{m\to\infty} \operatorname*{K}_{k=1}^{m} \frac{a_k}{b_k} = \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \ddots}}} \tag{1.3}$$

2 Examples

One of the simplest cases to consider is $a_k = a \in \mathbb{C}, b_k = b \in \mathbb{C}, b \neq 0$.

$$x = \operatorname*{K}_{k=1}^{\infty} \frac{a}{b} = \frac{a}{b + \operatorname*{K}_{k=1}^{\infty} \frac{a}{b}} \Rightarrow x^2 + bx - a = 0 \tag{2.1}$$

Here we run into a problem. The limit exists, but there are 2 values it might be. The simplest way to get around this problem is to choose $\{a, b\} \subset \mathbb{R}^+$ such that 1 of the solutions is not in $\mathbb{R}^+$, thus proving the limit is the other solution which would be in $\mathbb{R}^+$. E.g.:

$$x = \operatorname*{K}_{k=1}^{\infty} \frac{1}{1} \Rightarrow x^2 + x - 1 = 0 \Rightarrow x = \frac{-1 \pm \sqrt{5}}{2} \Rightarrow \operatorname*{K}_{k=1}^{\infty} \frac{1}{1} = \frac{-1+\sqrt{5}}{2} \tag{2.2}$$

A continued fraction I discovered without proof is:

$$x + \operatorname*{K}_{k=1}^{\infty} \frac{k}{x+k} = \frac{1}{e\gamma(x+1,1)} = \frac{1}{\int_0^1 t^x e^{1-t} dt} \tag{2.3}$$

where γ is the lower incomplete gamma function.