n-sided Hyperbolic Squares
https://github.com/Nazgand/nazgandMathBook

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Abstract

The goal of this paper is to analyze the properties (such as edge length, incircle radius, circumcircle radius) of an n-sided hyperbolic square, meaning a regular polygon with n right angles. These polygons can tile the hyperbolic plane, which looks nice plotted with the Cartesian Hyperbolic Plane Metric.

1 Calculations

Let the Gaussian curvature of the plane be -1.

The regular polygon with n right angles can be partitioned into $2n$ right triangles with angles $\frac{\pi}{n}, \frac{\pi}{2}, \frac{\pi}{4}$.

The radius of the circumcircle is $\operatorname{arcosh}\left(\cot\left(\frac{\pi}{n}\right)\cot\left(\frac{\pi}{4}\right)\right) = \operatorname{arcosh}\left(\cot\left(\frac{\pi}{n}\right)\right)$.

The radius of the incircle is $\operatorname{arcosh}\left(\frac{\cos\left(\frac{\pi}{4}\right)}{\sin\left(\frac{\pi}{n}\right)}\right)$.

The length of an edge is $2\operatorname{arcosh}\left(\frac{\cos\left(\frac{\pi}{n}\right)}{\sin\left(\frac{\pi}{4}\right)}\right)$.

The length of the perimeter is $2n\operatorname{arcosh}\left(\frac{\cos\left(\frac{\pi}{n}\right)}{\sin\left(\frac{\pi}{4}\right)}\right)$.

The area is $2n\left(\pi - \frac{\pi}{n} - \frac{\pi}{2} - \frac{\pi}{4}\right) = 2\pi n\left(\frac{1}{4} - \frac{1}{n}\right) = \frac{n\pi}{2} - 2\pi$.

It should be noted that as $n \to \infty$, the edge length approaches $2\operatorname{arcosh}\left(\sqrt{2}\right) \approx 1.7627471740390860504652186499595846180\,6$, where both the incircle and circumcircle are horocycles, and above which any edge length is allowed.

n	radius of circumcircle
5	0.842482081462007459111380941149711215310
6	1.14621583478058884390039365567400771581
7	1.36005184973956765038539610902535387417
8	1.52857091948099816127245618479367339329
9	1.66893379511935191412136015769839107377
10	1.78982041007171516740591260177484529866
11	1.89630753915117229883786713543359764353
12	1.99165239104943682406899667528592695415
13	2.07808427065529112824674264723998571714
14	2.15720370625423547470560897610422250689
15	2.23020294245828385075785639634686399084

n	radius of incircle
5	0.626869662906177814144463376211936401478
6	0.881373587019543025232609324979792309028
7	1.07040486155894418433137235164029245455
8	1.22422622383903789500265495681793457016
9	1.35504851879687183155393515348937116243
10	1.46935174436818527325584431736164761679
11	1.57108858003724067935632301765558416239
12	1.66288589105862107565248503907940605953
13	1.74659470452044937381132825499312321725
14	1.82357635695908839705212496034624551058
15	1.89486539086573102025718355160194478600

n	length of an edge
5	1.06127506190503565203301891621357348581
6	1.31695789692481670862504634730796844403
7	1.44907472267758633503217314325772678247
8	1.52857091948099816127245618479367339329
9	1.58069813785651414369050575393040896418
10	1.61692166751188651380348666582235462921
11	1.64319223747457343054577404285923637279
12	1.66288589105862107565248503907940605953
13	1.67804671290710848454907520339943861187
14	1.68997616809339866171304910723600185759
15	1.69953703096172053759430146589327334224