Argument Sum Rules From Homogeneous Linear Differential Equations Of Constant Coefficients

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2026-01-21 04:45:57-08:00

Abstract

The goal of this paper is to describe a theorem (previously a conjecture).

1 Assumptions and definitions

A [homogeneous linear differential equation of constant coefficients] has the form

$$0 = \sum_{k=0}^{n} a_k \frac{\partial^k}{\partial z^k} f(z) = \sum_{k=0}^{n} a_k f^{(k)}(z) \tag{1.1}$$

where $\forall k, a_k \in \mathbb{C}$ (1.2), $n \in \mathbb{Z}_{\geq 0}$ (1.3), $a_n \neq 0$ (1.4). Let $f : \mathbb{C} \to \mathbb{C}$ (1.5) be differentiable at least n times everywhere.

Every [homogeneous linear differential equation of constant coefficients] has at least 1 basis for the vector space of its solutions.

Let $g_0(z), \ldots, g_{n-1}(z)$ be such a basis. To be clear:

$$\text{Solutions} = \left\{ h(z) \mid 0 = \sum_{k=0}^{n} a_k h^{(k)}(z) \right\} = \left\{ \sum_{k=0}^{n-1} b_k g_k(z) \mid b_k \in \mathbb{C} \right\} \tag{1.6}$$

Let us define a column vector and clarify its transpose (a row vector):

$$v(z_0) = \begin{pmatrix} g_0(z_0) \\ \vdots \\ g_{n-1}(z_0) \end{pmatrix}, v(z_1)^\intercal = \begin{pmatrix} g_0(z_1) & \ldots & g_{n-1}(z_1) \end{pmatrix} \tag{1.7}$$

2 Main theorem with 2 arguments

There exists a unique symmetric n by n matrix A ($A = A^\intercal$) such that

$$f(z_0 + z_1) = v(z_1)^\intercal A v(z_0) = v(z_0)^\intercal A v(z_1) \tag{2.1}$$

2.1 A reason A is symmetric

Suppose instead of a symmetric matrix A, we find a matrix B such that

$$f(z_0 + z_1) = v(z_1)^\intercal B v(z_0) \tag{2.2}$$

Then take the transpose of the equation and substitute $z_0 \to z_1, z_1 \to z_0$, resulting in the following equation:

$$f(z_0 + z_1) = v(z_1)^\intercal B^\intercal v(z_0) \tag{2.3}$$

Average both equations:

$$f(z_0 + z_1) = v(z_1)^\intercal \frac{B + B^\intercal}{2} v(z_0) \tag{2.4}$$

Note that we can set $A = \frac{B+B^\intercal}{2}$ because it is symmetric. □

3 Theorem generalized to positive integer amounts of arguments in addition to 2

The following statements use the same assumptions and definitions as the main theorem.

3.1 1 argument

The following statement, which I call ArgSumCon(1), is trivial by assumption the (1.6).

$$\exists! c(k_0) \in \mathbb{C}, f(z_0) = \sum_{k_0=0}^{n-1} c(k_0) g_{k_0}(z_0) \tag{3.1}$$

3.2 2 arguments

The following statement, which I call ArgSumCon(2), is equivalent to (2.1).

$$\exists! c(k_0, k_1) \in \mathbb{C}, f(z_0 + z_1) = \sum_{k_0=0}^{n-1} \sum_{k_1=0}^{n-1} c(k_0, k_1) g_{k_0}(z_0) g_{k_1}(z_1) \tag{3.2}$$

3.3 3 arguments

The following statement, I call ArgSumCon(3).

$$\exists! c(k_0, k_1, k_2) \in \mathbb{C}, f(z_0 + z_1 + z_2) = \sum_{k_0=0}^{n-1} \sum_{k_1=0}^{n-1} \sum_{k_2=0}^{n-1} c(k_0, k_1, k_2) g_{k_0}(z_0) g_{k_1}(z_1) g_{k_2}(z_2) \tag{3.3}$$

3.4 m arguments for $m \in \mathbb{Z}_{>0}$

The following statement, I call ArgSumCon(m).

$$\exists! c(k_0, k_1, \ldots, k_{m-1}) \in \mathbb{C}, f\left(\sum_{j=0}^{m-1} z_j\right) = \sum_{k_0=0}^{n-1} \sum_{k_1=0}^{n-1} \cdots \sum_{k_{m-1}=0}^{n-1} c(k_0, k_1, \ldots, k_{m-1}) \prod_{j=0}^{m-1} g_{k_j}(z_j) \tag{3.4}$$

Note ArgSumCon$(m+1) \Rightarrow$ ArgSumCon(m), as can be seen by replacing 1 of the variables with 0.

4 Lean 4 code

ArgSumCon(m) has been formally proved in and verified by Lean 4:

https://GitHub.Com/Nazgand/NazgandLean4/blob/master/NazgandLean4/HomogeneousLinearDifferentialEquationsOfConstantCoefficients/ArgumentSumRule.lean

Also verified by Lean 4: the relevant tensor is unique and symmetric.