

# Root of Unity Exponential Sum Function related to Generalized Split-complex numbers

<https://github.com/Nazgand/nazgandMathBook>

Mark Andrew Gerads: [Nazgand@Gmail.Com](mailto:Nazgand@Gmail.Com)

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## Abstract

The goal of this paper is to analyze a class of functions which are equal to their own  $n$ th derivative.

## 1 Exponential Sum Definition

For the fundamental definition where  $n \in \mathbb{Z}^+ \wedge x \in \mathbb{C}$ , define:

$$\text{rues}_n(x) = \sum_{k=0}^{\infty} \frac{x^{nk}}{(nk)!} \quad (1.1)$$

## 2 Laplace inverse transform form

**Theorem 2.1.**

$$\text{rues}_n(t) = \mathcal{L}^{-1} \left\{ \frac{s^{n-1}}{s^n - 1} \right\}(t) \quad (2.1)$$

*Proof.* Equivalence is shown using the General derivative rule for Laplace transforms:

$$\mathcal{L} \left\{ f^{(n)}(t) \right\}(s) = s^n \mathcal{L} \{ f(t) \}(s) - \sum_{k=1}^n s^{n-k} f^{(k-1)}(0^+) \quad (2.2)$$

Substitute  $f$  with  $\text{rues}_n^{(k)}$  in (2.2), use the derivatives provided by (1.1) to simplify noting  $\text{mod}(-m, n) = \text{mod}(k-1, n)$  is the only surviving summand:

$$\mathcal{L} \left\{ \text{rues}_n^{(m)}(t) \right\}(s) = \mathcal{L} \left\{ \text{rues}_n^{(m+n)}(t) \right\}(s) = s^n \mathcal{L} \left\{ \text{rues}_n^{(m)}(t) \right\}(s) - s^{\text{mod}(m-1, n)} \quad (2.3)$$

Solve:

$$\mathcal{L} \left\{ \text{rues}_n^{(m)}(t) \right\}(s) = \frac{s^{\text{mod}(m-1, n)}}{s^n - 1} \quad (2.4)$$

□

## 3 As a sum of exponential functions

The reason this function is named rues is because it is a Root of Unity Exponential Sum function.

**Theorem 3.1.**

$$\text{rues}_n(x) = \frac{1}{n} \sum_{k=1}^n \exp \left( x e^{2ki\pi/n} \right) \quad (3.1)$$

*Proof.* Proof of equivalence to (1.1) via Taylor series:

$$\sum_{k=1}^n \exp \left( x e^{2ki\pi/n} \right) = \sum_{k=1}^n \sum_{j=0}^{\infty} e^{2jki\pi/n} \frac{x^j}{j!} = \sum_{j=0}^{\infty} \frac{x^j}{j!} \sum_{k=1}^n e^{2jki\pi/n} = \sum_{m=0}^{\infty} \frac{nx^{nm}}{(nm)!} \quad (3.2)$$

□

## 4 Real formulae derived from (3.1)

This section exploits this fact:

$$\exp(xe^{iy}) + \exp(xe^{-iy}) = 2e^{x \cos(y)} \cos(x \sin(y)) \quad (4.1)$$

Thus:

$$\text{rues}_n(x) = \frac{1}{n} \left( e^x + e^{-x} \frac{1 + \cos(\pi n)}{2} + 2 \sum_{k=1}^{\lceil n/2 \rceil - 1} e^{x \cos(\frac{2k\pi}{n})} \cos\left(x \sin\left(\frac{2k\pi}{n}\right)\right) \right) \quad (4.2)$$

And:

$$\text{rues}_n(xe^{i\pi/n}) = \frac{1}{n} \left( e^{-x} \frac{1 - \cos(\pi n)}{2} + 2 \sum_{k=1}^{\lfloor n/2 \rfloor} e^{x \cos(\frac{2k-1}{n}\pi)} \cos\left(x \sin\left(\frac{2k-1}{n}\pi\right)\right) \right) \quad (4.3)$$

## 5 other

Notable values of  $n$ :

$$\text{rues}_1(x) = e^x \wedge \text{rues}_2(x) = \cosh(x) \wedge \text{rues}_4(x) = \cosh\left(\frac{x}{1+i}\right) \cosh\left(\frac{x}{1-i}\right) \quad (5.1)$$

Complex rotation property:

$$\text{rues}_n^{(k)}(x) = \text{rues}_n^{(k)}(x * e^{2i\pi/n}) e^{2ki\pi/n} \quad (5.2)$$

Derivative sum rules:

$$e^x = \sum_{k=0}^{n-1} \text{rues}_n^{(k)}(x) \wedge \text{rues}_n(x) = \sum_{k=0}^{m-1} \text{rues}_{nm}^{(kn)}(x) \quad (5.3)$$

$$\text{rues}_m(x) = \frac{1}{m} \sum_{k=1}^n \sum_{j=1}^m \text{rues}_n^{(k)}\left(x \exp\left(\frac{2i\pi j}{m}\right)\right) \quad (5.4)$$

**Theorem 5.1** (Argument sum rule).

$$\text{rues}_n(x+y) = \sum_{k=0}^{n-1} \text{rues}_n^{(k)}(x) \text{rues}_n^{(n-k)}(y) \quad (5.5)$$

*Proof.*

$$e^{x+y} = \sum_{k=0}^{n-1} \sum_{j=0}^{n-1} \text{rues}_n^{(k)}(x) \text{rues}_n^{(j)}(y) \quad (5.6)$$

$$\text{rues}_n(x+y) = \frac{1}{n} \sum_{l=1}^n \sum_{k=0}^{n-1} \sum_{j=0}^{n-1} \text{rues}_n^{(k)}\left(x \exp\left(\frac{2i\pi l}{n}\right)\right) \text{rues}_n^{(j)}\left(y \exp\left(\frac{2i\pi l}{n}\right)\right) \quad (5.7)$$

$$\text{rues}_n(x+y) = \frac{1}{n} \sum_{l=1}^n \sum_{k=0}^{n-1} \sum_{j=0}^{n-1} \text{rues}_n^{(k)}(x) \text{rues}_n^{(j)}(y) \exp\left(\frac{-2(j+k)i\pi l}{n}\right) \quad (5.8)$$

The  $l$  index cancels out all the terms where  $j+k \neq 0$ . □

**Corollary 5.2.**

$$\text{rues}_n^{(m)}(x+y) = \sum_{k=0}^{n-1} \text{rues}_n^{(m+k)}(x) \text{rues}_n^{(n-k)}(y) \quad (5.9)$$

*Proof.* Differentiate (5.5) □

**Corollary 5.3.**

$$\delta(\text{mod}(m, n)) = \sum_{k=0}^{n-1} \text{rues}_n^{(m+k)}(x) \text{rues}_n^{(n-k)}(-x) \quad (5.10)$$

*Proof.* Substituting  $x = -y$  in (5.9) produces the equations analogous to  $\sin(x)^2 + \cos(x)^2 = 1$  □

## 6 Real and imaginary parts

Using (3.1)

$$\Re(\text{rues}_n(z)) = \frac{1}{n} \sum_{k=1}^n \exp\left(ze^{2ki\pi/n}\right) \exp\left(ze^{2ki\pi/n}\right) \quad (6.1)$$

For  $n \geq 2$ , arbitrary complex numbers many be decomposed as  $z = \Re(z) + e^{i\pi/n}$

## 7 Weierstrass product conjecture

This conjecture states all zeroes of  $\text{rues}_n^{(m)}$  not on the origin are of degree 1 and are located on the critical rays  $e^{i\pi(1+2k)/n}, k \in \mathbb{Z}$ . The absolute values of the zeroes are denoted

$$\text{Ez}(n, m, 1) = \min \left\{ \text{rues}_n^{(m)} \left( r * e^{i\pi/n} \right) \middle| r \in \mathbb{R}^+ \right\} \quad (7.1)$$

$$\text{Ez}(n, m, k+1) = \min \left\{ \text{rues}_n^{(m)} \left( r * e^{i\pi/n} \right) \middle| r \in \mathbb{R}^+, r > \text{Ez}(n, m, k) \right\} \quad (7.2)$$

Moreover, for  $n > 1$ ,

$$\text{rues}_n^{(m)}(z) = z^{\text{mod}(-m, n)} \prod_{k=1}^{\infty} \left( 1 + \left( \frac{z}{\text{Ez}(n, m, k)} \right)^n \right) \quad (7.3)$$

This conjecture is easily verifiable for  $(n, m) \in \{(2, 0), (2, 1), (4, 0)\}$ .

Work has been done in German to show this works for  $\text{rues}_n(z)$ .

<https://math.stackexchange.com/questions/3221569/conjecture-all-complex-roots-of-sum-k-0-infty-fraczk-leftnk>

### 7.1 Bounds on Complex Zeroes $\text{rues}_3$

Lemma

$$\cos(a + bi) = \cos(a) \cosh(b) - i \sin(a) \sinh(b) \quad (7.4)$$

$$|\cos(a + bi)| \leq |\cos(a) \cosh(b)| + |\sin(a) \sinh(b)| \quad (7.5)$$

$$|\cos(a + bi)| \leq |\cosh(b)| + |\sinh(b)| = \cosh(|b|) + \sinh(|b|) = \exp(|b|) \quad (7.6)$$

$$|\cos(a + bi)| \leq \exp(|b|) \quad (7.7)$$

A bound which supports this conjecture is attainable from (4.2).

$$\text{rues}_3(x) = \frac{1}{3} \left( e^x + 2e^{x \cos(\frac{2\pi}{3})} \cos \left( x \sin \left( \frac{2\pi}{3} \right) \right) \right) \quad (7.8)$$

$$0 = \text{rues}_3(a + bi) = \frac{1}{3} \left( e^{(a+bi)} + 2e^{(a+bi) \cos(\frac{2\pi}{3})} \cos \left( (a+bi) \sin \left( \frac{2\pi}{3} \right) \right) \right) \quad (7.9)$$

$$-e^{(a+bi)} = 2e^{(a+bi) \cos(\frac{2\pi}{3})} \cos \left( (a+bi) \sin \left( \frac{2\pi}{3} \right) \right) \quad (7.10)$$

Absolute value:

$$e^a = 2e^{a \cos(\frac{2\pi}{3})} \left| \cos \left( (a+bi) \sin \left( \frac{2\pi}{3} \right) \right) \right| \quad (7.11)$$

$$\exp \left( \frac{3a}{2} \right) = 2 \left| \cos \left( (a+bi) \sin \left( \frac{2\pi}{3} \right) \right) \right| \quad (7.12)$$

Use lemma (7.7):

$$\exp \left( \frac{3a}{2} \right) \leq 2 \exp \left( |b| \sin \left( \frac{2\pi}{3} \right) \right) \quad (7.13)$$

$$\frac{3a}{2} \leq \ln 2 + |b| \frac{\sqrt{3}}{2} \quad (7.14)$$

The last inequality carves out a sector of the complex plane where zeroes are not allowed. The symmetric properties of  $\text{rues}_3$  restrict zeroes to appearing near the conjectured region.

### 7.2 Non-Rigorous Bounds on Complex Zeroes $\text{rues}_n, n \geq 3$

A bound which supports this conjecture is attainable from (4.2).

$$\text{rues}_n(x) = \frac{1}{n} \left( e^x + e^{-x} \frac{1 + \cos(\pi n)}{2} + 2 \sum_{k=1}^{\lceil n/2 \rceil - 1} e^{x \cos(\frac{2k\pi}{n})} \cos \left( x \sin \left( \frac{2k\pi}{n} \right) \right) \right) \quad (7.15)$$

Note  $\text{rues}_n(x)$  has the same zeroes as  $\text{rues}_n(x) \exp(-x \cos(\frac{2\pi}{n}))$ , and most terms go to zero as  $\Re(x) \rightarrow \infty$ .

$$\text{rues}_n(x) \exp \left( -x \cos \left( \frac{2\pi}{n} \right) \right) \approx \mathcal{O} \left( \frac{1}{n} \left( e^{x(1 - \cos(\frac{2\pi}{n}))} + 2 \cos \left( x \sin \left( \frac{2\pi}{n} \right) \right) \right) \right) \quad (7.16)$$

Zeroes of  $\text{rues}_n(x)$  with large real parts are approximately bound by this equation:

$$0 = e^{x(1 - \cos(\frac{2\pi}{n}))} + 2 \cos \left( x \sin \left( \frac{2\pi}{n} \right) \right) \quad (7.17)$$

$$-e^{(a+b*i)(1 - \cos(\frac{2\pi}{n}))} = 2 \cos \left( (a+b*i) \sin \left( \frac{2\pi}{n} \right) \right) \quad (7.18)$$

Absolute value:

$$\exp \left( a \left( 1 - \cos \left( \frac{2\pi}{n} \right) \right) \right) = 2 \left| \cos \left( (a + b * i) \sin \left( \frac{2\pi}{n} \right) \right) \right| \quad (7.19)$$

Use lemma (7.7):

$$\exp \left( a \left( 1 - \cos \left( \frac{2\pi}{n} \right) \right) \right) \leq 2 \exp \left( |b| \sin \left( \frac{2\pi}{n} \right) \right) \quad (7.20)$$

Logarithm:

$$a \left( 1 - \cos \left( \frac{2\pi}{n} \right) \right) \leq \ln(2) + |b| \sin \left( \frac{2\pi}{n} \right) \quad (7.21)$$

Equation form with slope:

$$a \leq \frac{\ln(2)}{1 - \cos \left( \frac{2\pi}{n} \right)} + |b| \cot \left( \frac{\pi}{n} \right) \quad (7.22)$$

The last inequality carves out a sector of the complex plane where zeroes are not allowed. The symmetric properties of  $\text{rues}_n$  restrict zeroes to appearing near the conjectured region.

## 8 Generalized split-complex numbers

### 8.1 Introduction

Split complex numbers a ring extending the reals with an extra unit which is a square root of 1 which is not a complex number.

### 8.2 Generalization

This concept may be generalized to rings with  $n$ th roots of 1, and these rings are interrelated. Define the general split units as

$$\text{gs}(r_0) * \text{gs}(r_1) = \text{gs}(\text{mod}(r_0 + r_1, 1)) \wedge [\text{gs}(z) \in \mathbb{C} \Leftrightarrow z \in \mathbb{Z}] \quad (8.1)$$

The general split units commute with complex numbers:

$$c \in \mathbb{C} \Rightarrow c * \text{gs}(r_0) = \text{gs}(r_0) * c \quad (8.2)$$

Simple deduction from the Taylor series of the exponential function for  $m \in \mathbb{Z}, c \in \mathbb{C}$ .

$$\exp \left( c * \text{gs} \left( \frac{m}{n} \right) \right) = \sum_{k=0}^{n-1} \text{gs} \left( \frac{m}{n} \right)^k \text{rues}_n^{(n-k)}(c) = \sum_{k=0}^{n-1} \text{gs} \left( \frac{km}{n} \right) \text{rues}_n^{(n-k)}(c) \quad (8.3)$$

The  $n$ -split-complex number ring is

$$\mathbf{Gs}_n = \left\{ \sum_{k=0}^{n-1} c_k * \text{gs} \left( \frac{k}{n} \right) \middle| c_k \in \mathbb{C} \right\} \quad (8.4)$$

Sub-ring property:

$$m \in \mathbb{Z}^+ \Rightarrow \mathbf{Gs}_n \subseteq \mathbf{Gs}_{m*n} \quad (8.5)$$

This self-isomorphism of  $f : \mathbf{Gs}_n \rightarrow \mathbf{Gs}_n$  exists for  $m \in \mathbb{Z} \wedge \text{gcd}(j, n) = 1$ .

$$f \left( \sum_{k=0}^{n-1} c_k * \text{gs} \left( \frac{k}{n} \right) \right) = \sum_{k=0}^{n-1} c_k \exp \left( \frac{2kmi\pi}{n} \right) \text{gs} \left( \frac{kj}{n} \right) \quad (8.6)$$

The above is a homomorphism for arbitrary  $j, m \in \mathbb{Z}$ .  $f : \mathbf{Gs}_n \rightarrow \mathbf{Gs}_{n/\text{gcd}(j, n)}$ .

$$f \left( \sum_{k=0}^{n-1} c_k * \text{gs} \left( \frac{k}{n} \right) \right) = \sum_{k=0}^{n-1} c_k \exp \left( \frac{2kmi\pi}{n} \right) \text{gs} \left( \frac{kj}{n} \right) \quad (8.7)$$

Co-set isomorphism property,  $f : \mathbf{Gs}_{m_0n}/\mathbf{Gs}_{m_0} \rightarrow \mathbf{Gs}_{m_1n}/\mathbf{Gs}_{m_1}$  for  $n, m_0, m_1 \in \mathbb{Z}^+$

$$f \left( \mathbf{Gs}_{m_0} * \sum_{k=0}^{n-1} c_k * \text{gs} \left( \frac{k}{m_0n} \right) \right) = \mathbf{Gs}_{m_1} * \sum_{k=0}^{n-1} c_k * \text{gs} \left( \frac{k}{m_1n} \right) \quad (8.8)$$

Idempotent elements: Easily found:

$$\left( \sum_{k=0}^{n-1} \frac{1}{n} \text{gs} \left( \frac{k}{n} \right) \right)^2 = \sum_{k=0}^{n-1} \frac{1}{n} \text{gs} \left( \frac{k}{n} \right) \quad (8.9)$$

Because homomorphisms map idempotent elements to idempotent elements, using (8.7) produces a set of idempotent elements

$$\left\{ \sum_{k=0}^{n-1} \frac{1}{n} \exp \left( \frac{2kmi\pi}{n} \right) \text{gs} \left( \frac{kj}{n} \right) \middle| m, j \in \mathbb{Z} \right\} \quad (8.10)$$

The set of idempotent elements is closed under multiplication in abelian groups.

$$[n \in \mathbb{Z}^+ \wedge (F_1 F_2)^n = (F_2 F_1)^n \wedge F_1^2 = F_1 \wedge F_2^2 = F_2] \Rightarrow ((F_1 F_2)^n)^2 = (F_1 F_2)^n \quad (8.11)$$

.

$$[F_1 F_2 = F_2 F_1 \wedge F_1^2 = F_1 \wedge F_2^2 = F_2] \Rightarrow (F_1 + F_2 - 2F_1 F_2)^2 = (F_1 + F_2 - 2F_1 F_2) \quad (8.12)$$

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