Root of Unity Exponential Sum Function related to Generalized Split-complex numbers

https://github.com/Nazgand/nazgandMathBook

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Abstract

The goal of this paper is to analyze a class of functions which are equal to their own nth derivative.

1 Exponential Sum Definition

For the fundamental definition where $n \in \mathbb{Z}^+ \wedge x \in \mathbb{C}$, define:

$$\operatorname{rues}_{n}(x) = \sum_{k=0}^{\infty} \frac{x^{nk}}{(nk)!}$$
(1.1)

2 Laplace inverse transform form

Theorem 2.1.

$$\operatorname{rues}_{n}(t) = \mathcal{L}^{-1} \left\{ \frac{s^{n-1}}{s^{n} - 1} \right\} (t)$$
 (2.1)

Proof. Equivalence is shown using the General derivative rule for Laplace transforms:

$$\mathcal{L}\left\{f^{(n)}(t)\right\}(s) = s^n \mathcal{L}\left\{f(t)\right\}(s) - \sum_{k=1}^n s^{n-k} f^{(k-1)}(0^+)$$
(2.2)

Substitute f with rues_n^(k) in (2.2), use the derivatives provided by (1.1) to simplify noting mod (-m, n) = mod(k-1, n) is the only surviving summand:

$$\mathcal{L}\left\{\operatorname{rues}_{n}^{(m)}(t)\right\}(s) = \mathcal{L}\left\{\operatorname{rues}_{n}^{(m+n)}(t)\right\}(s) = s^{n}\mathcal{L}\left\{\operatorname{rues}_{n}^{(m)}(t)\right\}(s) - s^{\operatorname{mod}(m-1,n)}$$
(2.3)

Solve:

$$\mathcal{L}\left\{\operatorname{rues}_{n}^{(m)}(t)\right\}(s) = \frac{s^{\operatorname{mod}(m-1,n)}}{s^{n}-1}$$
(2.4)

3 As a sum of exponential functions

The reason this function is named rues is because it is a Root of Unity Exponential Sum function.

Theorem 3.1.

$$\operatorname{rues}_{n}(x) = \frac{1}{n} \sum_{k=1}^{n} \exp\left(xe^{2ki\pi/n}\right)$$
(3.1)

Proof. Proof of equivalence to (1.1) via Taylor series:

$$\sum_{k=1}^{n} \exp\left(xe^{2ki\pi/n}\right) = \sum_{k=1}^{n} \sum_{j=0}^{\infty} e^{2jki\pi/n} \frac{x^{j}}{j!} = \sum_{j=0}^{\infty} \frac{x^{j}}{j!} \sum_{k=1}^{n} e^{2jki\pi/n} = \sum_{m=0}^{\infty} \frac{nx^{nm}}{(nm)!}$$
(3.2)

4 Real formulae derived from (3.1)

This section exploits this fact:

$$\exp\left(xe^{iy}\right) + \exp\left(xe^{-iy}\right) = 2e^{x\cos(y)}\cos\left(x\sin(y)\right) \tag{4.1}$$

Thus.

$$\operatorname{rues}_{n}(x) = \frac{1}{n} \left(e^{x} + e^{-x} \frac{1 + \cos(\pi n)}{2} + 2 \sum_{k=1}^{\lceil n/2 \rceil - 1} e^{x \cos\left(\frac{2k\pi}{n}\right)} \cos\left(x \sin\left(\frac{2k\pi}{n}\right)\right) \right)$$
(4.2)

And:

$$\operatorname{rues}_{n}\left(xe^{i\pi/n}\right) = \frac{1}{n}\left(e^{-x}\frac{1-\cos\left(\pi n\right)}{2} + 2\sum_{k=1}^{\lfloor n/2\rfloor} e^{x\cos\left(\frac{2k-1}{n}\pi\right)}\cos\left(x\sin\left(\frac{2k-1}{n}\pi\right)\right)\right) \tag{4.3}$$

5 other

Notable values of n

$$\operatorname{rues}_{1}(x) = e^{x} \wedge \operatorname{rues}_{2}(x) = \cosh(x) \wedge \operatorname{rues}_{4}(x) = \cosh\left(\frac{x}{1+i}\right) \cosh\left(\frac{x}{1-i}\right)$$

$$(5.1)$$

Complex rotation property

$$\operatorname{rues}_{n}^{(k)}(x) = \operatorname{rues}_{n}^{(k)}\left(x * e^{2i\pi/n}\right) e^{2ki\pi/n}$$
(5.2)

Derivative sum rules:

$$e^{x} = \sum_{k=0}^{n-1} \operatorname{rues}_{n}^{(k)}(x) \wedge \operatorname{rues}_{n}(x) = \sum_{k=0}^{m-1} \operatorname{rues}_{nm}^{(kn)}(x)$$
(5.3)

$$\operatorname{rues}_{m}(x) = \frac{1}{m} \sum_{k=1}^{n} \sum_{j=1}^{m} \operatorname{rues}_{n}^{(k)} \left(x \exp\left(\frac{2i\pi j}{m}\right) \right)$$
(5.4)

Theorem 5.1 (Argument sum rule).

$$rues_{n}(x+y) = \sum_{k=0}^{n-1} rues_{n}^{(k)}(x) rues_{n}^{(n-k)}(y)$$
(5.5)

Proof.

$$e^{x+y} = \sum_{k=0}^{n-1} \sum_{j=0}^{n-1} \operatorname{rues}_{n}^{(k)}(x) \operatorname{rues}_{n}^{(j)}(y)$$
(5.6)

$$\operatorname{rues}_{n}(x+y) = \frac{1}{n} \sum_{l=1}^{n} \sum_{k=0}^{n-1} \sum_{i=0}^{n-1} \operatorname{rues}_{n}^{(k)} \left(x \exp\left(\frac{2i\pi l}{n}\right) \right) \operatorname{rues}_{n}^{(j)} \left(y \exp\left(\frac{2i\pi l}{n}\right) \right)$$

$$\operatorname{rues}_{n}(x+y) = \frac{1}{n} \sum_{l=1}^{n} \sum_{k=0}^{n-1} \sum_{j=0}^{n-1} \operatorname{rues}_{n}^{(k)}(x) \operatorname{rues}_{n}^{(j)}(y) \exp\left(\frac{-2(j+k)i\pi l}{n}\right)$$
(5.8)

The *l* index cancels out all the terms where $j + k \neq 0$.

Corollary 5.2.

$$\operatorname{rues}_{n}^{(m)}(x+y) = \sum_{k=0}^{n-1} \operatorname{rues}_{n}^{(m+k)}(x) \operatorname{rues}_{n}^{(n-k)}(y)$$
(5.9)

Proof. Differentiate (5.5)

Corollary 5.3.

$$\delta(\text{mod}(m,n)) = \sum_{k=0}^{n-1} \text{rues}_n^{(m+k)}(x) \, \text{rues}_n^{(n-k)}(-x)$$
(5.10)

Proof. Substituting x = -y in (5.9) produces the equations analogous to $\sin(x)^2 + \cos(x)^2 = 1$

6 Real and imaginary parts

Using (3.1)

$$\Re(\operatorname{rues}_n(z)) = \frac{1}{n} \sum_{k=1}^n \exp\left(ze^{2ki\pi/n}\right) \exp\left(ze^{2ki\pi/n}\right)$$
(6.1)

For $n \geq 2$, arbitrary complex numbers many be decomposed as $z = \Re(z) + e^{i\pi/n}$

7 Weierstrass product conjecture

This conjecture states all zeroes of rues_n^(m) not on the origin are of degree 1 and are located on the critical rays $e^{i\pi(1+2k)/n}$, $k \in \mathbb{Z}$. The absolute values of the zeroes are denoted

$$\operatorname{Ez}(n, m, 1) = \min \left\{ \operatorname{rues}_{n}^{(m)} \left(r * e^{i\pi/n} \right) \middle| r \in \mathbb{R}^{+} \right\}$$

$$(7.1)$$

$$\operatorname{Ez}(n, m, k+1) = \min \left\{ \operatorname{rues}_{n}^{(m)} \left(r * e^{i\pi/n} \right) \middle| r \in \mathbb{R}^{+}, r > \operatorname{Ez}(n, m, k) \right\}$$

$$(7.2)$$

Moreover, for n > 1,

$$\operatorname{rues}_{n}^{(m)}(z) = z^{\operatorname{mod}(-m,n)} \prod_{k=1}^{\infty} \left(1 + \left(\frac{z}{\operatorname{Ez}(n,m,k)} \right)^{n} \right)$$
(7.3)

This conjecture is easily verifiable for $(n, m) \in \{(2, 0), (2, 1), (4, 0)\}.$

Work has been done in German to show this works for rues_n (z).

 $\verb|https://math.stackexchange.com/questions/3221569/conjecture-all-complex-roots-of-sum-k-0-infty-fraczk-left results for the constant of the$

7.1 Bounds on Complex Zeroes rues₃

Lemma

$$\cos(a+bi) = \cos(a)\cosh(b) - i\sin(a)\sinh(b) \tag{7.4}$$

$$\left|\cos\left(a+bi\right)\right| \le \left|\cos\left(a\right)\cosh\left(b\right)\right| + \left|\sin\left(a\right)\sinh\left(b\right)\right| \tag{7.5}$$

$$|\cos(a+bi)| \le |\cosh(b)| + |\sinh(b)| = \cosh(|b|) + \sinh(|b|) = \exp(|b|)$$
 (7.6)

A bound which supports this conjecture is attainable from (4.2).

$$\operatorname{rues}_{3}(x) = \frac{1}{3} \left(e^{x} + 2e^{x \cos\left(\frac{2\pi}{3}\right)} \cos\left(x \sin\left(\frac{2\pi}{3}\right)\right) \right)$$
 (7.8)

$$0 = \operatorname{rues}_{3}(a+bi) = \frac{1}{3} \left(e^{(a+bi)} + 2e^{(a+bi)\cos\left(\frac{2\pi}{3}\right)}\cos\left((a+bi)\sin\left(\frac{2\pi}{3}\right)\right) \right)$$
 (7.9)

$$-e^{(a+bi)} = 2e^{(a+bi)\cos\left(\frac{2\pi}{3}\right)}\cos\left((a+bi)\sin\left(\frac{2\pi}{3}\right)\right)$$
(7.10)

Absolute value:

$$e^{a} = 2e^{a\cos\left(\frac{2\pi}{3}\right)} \left|\cos\left((a+bi)\sin\left(\frac{2\pi}{3}\right)\right)\right| \tag{7.11}$$

$$\exp\left(\frac{3a}{2}\right) = 2\left|\cos\left((a+bi)\sin\left(\frac{2\pi}{3}\right)\right)\right| \tag{7.12}$$

Use lemma (7.7):

$$\exp\left(\frac{3a}{2}\right) \le 2\exp\left(|b|\sin\left(\frac{2\pi}{3}\right)\right) \tag{7.13}$$

$$\frac{3a}{2} \le \ln 2 + |b| \frac{\sqrt{3}}{2} \tag{7.14}$$

The last inequality carves out a sector of the complex plane where zeroes are not allowed. The symmetric properties of rues₃ restrict zeroes to appearing near the conjectured region.

7.2 Non-Rigorous Bounds on Complex Zeroes rues_n, $n \ge 3$

A bound which supports this conjecture is attainable from (4.2).

$$\operatorname{rues}_{n}(x) = \frac{1}{n} \left(e^{x} + e^{-x} \frac{1 + \cos(\pi n)}{2} + 2 \sum_{k=1}^{\lceil n/2 \rceil - 1} e^{x \cos\left(\frac{2k\pi}{n}\right)} \cos\left(x \sin\left(\frac{2k\pi}{n}\right)\right) \right)$$
(7.15)

Note rues_n (x) has the same zeroes as rues_n (x) exp $\left(-x\cos\left(\frac{2\pi}{n}\right)\right)$, and most terms go to zero as $\Re(x)\to\infty$.

$$\operatorname{rues}_{n}(x) \exp\left(-x \cos\left(\frac{2\pi}{n}\right)\right) \approx \mathcal{O}\left(\frac{1}{n} \left(e^{x\left(1-\cos\left(\frac{2\pi}{n}\right)\right)} + 2\cos\left(x \sin\left(\frac{2\pi}{n}\right)\right)\right)\right)$$
(7.16)

Zeroes of $rues_n(x)$ with large real parts are approximately bound by this equation:

$$0 = e^{x\left(1-\cos\left(\frac{2\pi}{n}\right)\right)} + 2\cos\left(x\sin\left(\frac{2\pi}{n}\right)\right) \tag{7.17}$$

$$-e^{(a+b*i)\left(1-\cos\left(\frac{2\pi}{n}\right)\right)} = 2\cos\left((a+b*i)\sin\left(\frac{2\pi}{n}\right)\right) \tag{7.18}$$

Absolute value:

$$\exp\left(a\left(1-\cos\left(\frac{2\pi}{n}\right)\right)\right) = 2\left|\cos\left((a+b*i)\sin\left(\frac{2\pi}{n}\right)\right)\right| \tag{7.19}$$

Use lemma (7.7):

$$\exp\left(a\left(1-\cos\left(\frac{2\pi}{n}\right)\right)\right) \le 2\exp\left(|b|\sin\left(\frac{2\pi}{n}\right)\right) \tag{7.20}$$

Logarithm

$$a\left(1-\cos\left(\frac{2\pi}{n}\right)\right) \le \ln\left(2\right) + |b|\sin\left(\frac{2\pi}{n}\right)$$
 (7.21)

Equation form with slope:

$$a \le \frac{\ln(2)}{1 - \cos\left(\frac{2\pi}{n}\right)} + |b| \cot\left(\frac{\pi}{n}\right) \tag{7.22}$$

The last inequality carves out a sector of the complex plane where zeroes are not allowed. The symmetric properties of rues_n restrict zeroes to appearing near the conjectured region.

8 Generalized split-complex numbers

8.1 Introduction

Split complex numbers a ring extending the reals with an extra unit which is a square root of 1 which is not a complex number.

8.2 Generalization

This concept may be generalized to rings with nth roots of 1, and these rings are interrelated. Define the general split units as

$$gs(r_0) * gs(r_1) = gs(mod(r_0 + r_1, 1)) \land [gs(z) \in \mathbb{C} \Leftrightarrow z \in \mathbb{Z}]$$
(8.1)

The general split units commute with complex numbers:

$$c \in \mathbb{C} \Rightarrow c * gs(r_0) = gs(r_0) * c \tag{8.2}$$

Simple deduction from the Taylor series of the exponential function for $m \in \mathbb{Z}, c \in \mathbb{C}$.

$$\exp\left(c * \operatorname{gs}\left(\frac{m}{n}\right)\right) = \sum_{k=0}^{n-1} \operatorname{gs}\left(\frac{m}{n}\right)^k \operatorname{rues}_n^{(n-k)}(c) = \sum_{k=0}^{n-1} \operatorname{gs}\left(\frac{km}{n}\right) \operatorname{rues}_n^{(n-k)}(c)$$
(8.3)

The n-split-complex number ring is

$$\mathbf{Gs}_n = \left\{ \sum_{k=0}^{n-1} c_k * \operatorname{gs}\left(\frac{k}{n}\right) \middle| c_k \in \mathbb{C} \right\}$$
(8.4)

Sub-ring property:

$$m \in \mathbb{Z}^+ \Rightarrow \mathbf{Gs}_n \subseteq \mathbf{Gs}_{m*n}$$
 (8.5)

This self-isomorphism of $f: \mathbf{Gs}_n \to \mathbf{Gs}_n$ exists for $m \in \mathbb{Z} \wedge \gcd(j, n) = 1$.

$$f\left(\sum_{k=0}^{n-1} c_k * \operatorname{gs}\left(\frac{k}{n}\right)\right) = \sum_{k=0}^{n-1} c_k \exp\left(\frac{2kmi\pi}{n}\right) \operatorname{gs}\left(\frac{kj}{n}\right)$$
(8.6)

The above is a homomorphism for arbitrary $j, m \in \mathbb{Z}$. $f: \mathbf{Gs}_n \to \mathbf{Gs}_{n/\gcd(j,n)}$

$$f\left(\sum_{k=0}^{n-1} c_k * gs\left(\frac{k}{n}\right)\right) = \sum_{k=0}^{n-1} c_k \exp\left(\frac{2kmi\pi}{n}\right) gs\left(\frac{kj}{n}\right)$$
(8.7)

Co-set isomorphism property, $f: \mathbf{Gs}_{m_0n}/\mathbf{Gs}_{m_0} \to \mathbf{Gs}_{m_1n}/\mathbf{Gs}_{m_1}$ for $n, m_0, m_1 \in \mathbb{Z}^+$

$$f\left(\mathbf{G}\mathbf{s}_{m_0} * \sum_{k=0}^{n-1} c_k * gs\left(\frac{k}{m_0 n}\right)\right) = \mathbf{G}\mathbf{s}_{m_1} * \sum_{k=0}^{n-1} c_k * gs\left(\frac{k}{m_1 n}\right)$$
(8.8)

Idempotent elements: Easily found:

$$\left(\sum_{k=0}^{n-1} \frac{1}{n} \operatorname{gs}\left(\frac{k}{n}\right)\right)^2 = \sum_{k=0}^{n-1} \frac{1}{n} \operatorname{gs}\left(\frac{k}{n}\right) \tag{8.9}$$

Because homomorphisms map idempotent elements to idempotent elements, using (8.7) produces a set of idempotent elements

$$\left\{ \sum_{k=0}^{n-1} \frac{1}{n} \exp\left(\frac{2kmi\pi}{n}\right) \operatorname{gs}\left(\frac{kj}{n}\right) \middle| m, j \in \mathbb{Z} \right\}$$
(8.10)

The set of idempotent elements is closed under multiplication in abelian groups

$$[n \in \mathbb{Z}^+ \wedge (F_1 F_2)^n = (F_2 F_1)^n \wedge F_1^2 = F_1 \wedge F_2^2 = F_2] \Rightarrow ((F_1 F_2)^n)^2 = (F_1 F_2)^n$$
(8.11)

 $[F_1F_2 = F_2F_1 \wedge F_1^2 = F_1 \wedge F_2^2 = F_2] \Rightarrow (F_1 + F_2 - 2F_1F_2)^2 = (F_1 + F_2 - 2F_1F_2)$ (8.12)

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8.3 Possible applications

http://www.math.usm.edu/lee/quantum.html Traditional Split-complex numbers, a sub-ring of Gs₂, are used in quantum mechanics, thus General Split-complex numbers likely also have application.

9 More general rings

9.1 Extending the reals

The goal here is to extend the real numbers, \mathbb{R} , using an abelian multiplicative group, G, such that exponential functions still work. The extension is the abelian ring $\left\{\sum_{g\in G}g*x_g \middle| x_g\in \mathbb{R}\right\}$, denoted $\mathbb{R}[G]$.

Defining

$$\exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} \tag{9.1}$$

, an immediate consequence for all rings is

$$x_1 * x_2 = x_2 * x_1 \Rightarrow \exp(x_1) * \exp(x_2) = \exp(x_1 + x_2)$$
 (9.2)

. The ring $\mathbb{R}[G]$ is abelian, thus $x_1, x_2 \in \mathbb{R}[G] \Rightarrow \exp(x_1) * \exp(x_2) = \exp(x_1 + x_2)$. The elements of finite order may be simplified:

$$\left[1 = g^n \wedge n \in \mathbb{Z}^+ \wedge g \in G \wedge x \in \mathbb{R}\right] \Rightarrow \exp\left(g * x\right) = \sum_{k=0}^{n-1} g^k * \operatorname{rues}_n^{(n-k)}(x) \tag{9.3}$$

Every homomorphism, f(a*b) = f(a)*f(b), $f: G_1 \to G_2$ may be extended to $f: \mathbb{R}[G_1] \to \mathbb{R}[G_2]$, f(a+b) = f(a) + f(b), f(a*b) = f(a)*f(b), f(a*b) = f(a)*f(b).

By the fundamental theorem of finite abelian groups, every finite abelian group is the direct sum of cyclic groups. Every cyclic group is isomorphic to a subgroup of $(\mathbb{R}/\mathbb{Z},+)$. Thus every finite abelian group is isomorphic to a subgroup of the ring generated by $(\mathbb{R}/\mathbb{Z})^{\aleph_0},+$.

9.2 Extending other rings

The goal here is to extend an abelian ring, R_a , with an exponential function, $x_1, x_2 \in R_a \Rightarrow f(x_1 + x_2) = f(x_1) * f(x_2)$, using an abelian multiplicative group, G. $f(0) = f(0)^2$, i.e. f(0) is idempotent. $\{f(x) * a | a^2 = a \land a \in R_a\}$ are all valid exponential functions.

10 Bibliography

https://math.stackexchange.com/questions/1542147/find-the-complex-or-real-roots-of-e-frac3-x22-cos-left-frac-shttps://en.wikipedia.org/wiki/Split-complex_number