Degree 2 Differential Equation Argument Sum Rules
https://github.com/Nazgand/nazgandMathBook

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1 Assumptions and definitions

A degree 2 [homogeneous linear differential equation of constant coefficients] has the form

$$0=\sum_{k=0}^{2} a_k f^{(k)}(z) \tag{1.1}$$

where $\forall k, a_k \in \mathbb{C}, a_2 \neq 0$, and $f:\mathbb{C}\to\mathbb{C}$ is differentiable infinitely many times.

Let

$$\lambda_0=\frac{-a_1-\sqrt{a_1^2-4a_0a_2}}{2a_2}, \lambda_1=\frac{-a_1+\sqrt{a_1^2-4a_0a_2}}{2a_2} \tag{1.2}$$

2 Case $\lambda_0 \neq \lambda_1$

$$f(z)=c_0\exp(\lambda_0 z)+c_1\exp(\lambda_1 z), c_0=\frac{f^{(1)}(0)-f(0)\lambda_1}{\lambda_0-\lambda_1}, c_1=\frac{f^{(1)}(0)-f(0)\lambda_0}{\lambda_1-\lambda_0} \tag{2.1}$$

We can choose the basis of the set of solutions to be $\{\exp(\lambda_0 z), \exp(\lambda_1 z)\}$. Let

$$v_0(z)=\begin{pmatrix}\exp(\lambda_0 z)\\ \exp(\lambda_1 z)\end{pmatrix}, B_0=\begin{pmatrix}c_0 & 0\\ 0 & c_1\end{pmatrix}, B_1=\begin{pmatrix}c_0\lambda_0 & 0\\ 0 & c_1\lambda_1\end{pmatrix} \tag{2.2}$$

Then

$$\left(f(z_0+z_1)\right)=v_0(z_0)^\top B_0 v_0(z_1), \left(f^{(1)}(z_0+z_1)\right)=v_0(z_0)^\top B_1 v_0(z_1) \tag{2.3}$$

These facts are a bit too obvious. We want $\left\{f(z), f^{(1)}(z)\right\}$ to be a basis of the set of solutions to get an awesome equation. Thus we require $c_0 \neq 0, c_1 \neq 0$.

Let

$$M=\begin{pmatrix}c_0 & c_1\\ c_0\lambda_0 & c_1\lambda_1\end{pmatrix}, v_1(z)=\begin{pmatrix}f(z)\\ f^{(1)}(z)\end{pmatrix} \tag{2.4}$$

Then

$$Mv_0(z)=v_1(z), M^{-1}=\frac{\begin{pmatrix}-c_1\lambda_1 & c_1\\ c_0\lambda_0 & -c_0\end{pmatrix}}{c_0c_1(\lambda_0-\lambda_1)}, v_0(z)=M^{-1}v_1(z) \tag{2.5}$$

Thus, by substitution,

$$\left(f(z_0+z_1)\right)=\left(M^{-1}v_1(z_0)\right)^\top B_0M^{-1}v_1(z_1), \left(f^{(1)}(z_0+z_1)\right)=\left(M^{-1}v_1(z_0)\right)^\top B_1M^{-1}v_1(z_1) \tag{2.6}$$

Simplify:

$$\left(f(z_0+z_1)\right)=\frac{\begin{pmatrix}f(z_0)\\ f^{(1)}(z_0)\end{pmatrix}^\top\begin{pmatrix}c_0\lambda_0^2+c_1\lambda_1^2 & -c_0\lambda_0-c_1\lambda_1\\ -c_0\lambda_0-c_1\lambda_1 & c_0+c_1\end{pmatrix}\begin{pmatrix}f(z_1)\\ f^{(1)}(z_1)\end{pmatrix}}{c_0c_1(\lambda_0-\lambda_1)^2} \tag{2.7}$$

$$\left(f^{(1)}(z_0+z_1)\right)=\frac{\begin{pmatrix}f(z_0)\\ f^{(1)}(z_0)\end{pmatrix}^\top\begin{pmatrix}\lambda_0\lambda_1(c_0\lambda_0+c_1\lambda_1) & -\lambda_0\lambda_1(c_0+c_1)\\ -\lambda_0\lambda_1(c_0+c_1) & c_0\lambda_1+c_1\lambda_0\end{pmatrix}\begin{pmatrix}f(z_1)\\ f^{(1)}(z_1)\end{pmatrix}}{c_0c_1(\lambda_0-\lambda_1)^2} \tag{2.8}$$

3 Case $\lambda_0 = \lambda_1 = \lambda$

$$f(z) = c_0 \exp(\lambda z) + c_1 z \exp(\lambda z), c_0 = f(0), c_1 = f^{(1)}(0) - f(0)\lambda \tag{3.1}$$

We can choose the basis of the set of solutions to be $\{\exp(\lambda z), z\exp(\lambda z)\}$. Let

$$v_0(z) = \begin{pmatrix} \exp(\lambda z) \\ z\exp(\lambda z) \end{pmatrix}, B_0 = \begin{pmatrix} c_0 & c_1 \\ c_1 & 0 \end{pmatrix}, B_1 = \begin{pmatrix} c_1 + c_0\lambda & c_1\lambda \\ c_1\lambda & 0 \end{pmatrix} \tag{3.2}$$

Then

$$\big(f(z_0 + z_1)\big) = v_0(z_0)^\top B_0 v_0(z_1), \big(f^{(1)}(z_0 + z_1)\big) = v_0(z_0)^\top B_1 v_0(z_1) \tag{3.3}$$

These facts are a bit too obvious. We want $\left\{f(z), f^{(1)}(z)\right\}$ to be a basis of the set of solutions to get an awesome equation. Thus we require $c_1 \neq 0$.

Let

$$M = \begin{pmatrix} c_0 & c_1 \\ c_1 + c_0\lambda & c_1\lambda \end{pmatrix}, v_1(z) = \begin{pmatrix} f(z) \\ f^{(1)}(z) \end{pmatrix} \tag{3.4}$$

Then

$$Mv_0(z) = v_1(z), M^{-1} = \frac{\begin{pmatrix} -c_1\lambda & c_1 \\ c_1 + c_0\lambda & -c_0 \end{pmatrix}}{c_1^2}, v_0(z) = M^{-1}v_1(z) \tag{3.5}$$

Thus, by substitution,

$$\big(f(z_0 + z_1)\big) = \big(M^{-1}v_1(z_0)\big)^\top B_0 M^{-1} v_1(z_1), \big(f^{(1)}(z_0 + z_1)\big) = \big(M^{-1}v_1(z_0)\big)^\top B_1 M^{-1} v_1(z_1) \tag{3.6}$$

Simplify:

$$\big(f(z_0 + z_1)\big) = \frac{\begin{pmatrix} f(z_0) \\ f^{(1)}(z_0) \end{pmatrix}^\top \begin{pmatrix} -\lambda(2c_1 + c_0\lambda) & c_1 + c_0\lambda \\ c_1 + c_0\lambda & -c_0 \end{pmatrix} \begin{pmatrix} f(z_1) \\ f^{(1)}(z_1) \end{pmatrix}}{c_1^2} \tag{3.7}$$

$$\big(f^{(1)}(z_0 + z_1)\big) = \frac{\begin{pmatrix} f(z_0) \\ f^{(1)}(z_0) \end{pmatrix}^\top \begin{pmatrix} -\lambda^2(c_1 + c_0\lambda) & c_0\lambda^2 \\ c_0\lambda^2 & c_1 - c_0\lambda \end{pmatrix} \begin{pmatrix} f(z_1) \\ f^{(1)}(z_1) \end{pmatrix}}{c_1^2} \tag{3.8}$$