Complex Iterated Derivatives

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Abstract

The goal of this paper is to analyze the iterated derivative extended to a complex number of iterations.

1 Complex Iterated Derivative For Functions, In General

Let us have a function, $\mathbb{C}^2 \to \mathbb{C}$, $\operatorname{cidf}(m, z)$ such that $\operatorname{cidf}(0, z) = f(z)$ and $\frac{\partial^k}{\partial z^k} \operatorname{cidf}(m, z) = \operatorname{cidf}(m + k, z)$. Let $\operatorname{Cid}(f(z))$ be the set of functions with the properties of cidf, so $\operatorname{cidf}(m, z) \in \operatorname{Cid}(f(z))$.

We can deduce ∞ more functions with the same properties as cidf. If we have an arbitrary wave function, $w(m) = w(m + 1)$, then it produces another function with the properties of cidf:

$$\operatorname{cidf}(m + w(m) - w(0), z) \in \operatorname{Cid}(f(z)) \tag{1.1}$$

$$(1 + w(m) - w(0)) \operatorname{cidf}(m, z) \in \operatorname{Cid}(f(z)) \tag{1.2}$$

The previous statement can be simplified:

$$(w(m) - w(0)) \operatorname{cidf}(m, z) \in \operatorname{Cid}(0) \tag{1.3}$$

A simple thing to note is:

$$k \in \mathbb{Z}_{\geq 0} \Rightarrow \frac{\partial^k}{\partial z^k} \operatorname{cidf}(m, bz) = b^k \operatorname{cidf}(m + k, bz) \tag{1.4}$$

(1.4) can be restated as

$$b^m \operatorname{cidf}(m, bz) \in \operatorname{Cid}(f(bz)) \tag{1.5}$$

Note that [the weighted average of a set of functions all having the property of cidf] also has the properties of cidf. Let $\sum_{j\in\mathbb{Z}} c_j = 1$ and $\operatorname{cidf}_j(m, z) \in \operatorname{Cid}(f(z))$. Then

$$\sum_{j\in\mathbb{Z}} c_j \operatorname{cidf}_j(m, z) \in \operatorname{Cid}(f(z)) \tag{1.6}$$

If we has solutions for the complex iterated derivative of 2 function, we can add the solutions to get a solution to the complex iterated derivative of the sum of the functions:

$$[\forall k, \operatorname{cidf}_k(m, z) \in \operatorname{Cid}(f_k(z))] \Rightarrow \operatorname{cidf}_1(m, z) + \operatorname{cidf}_2(m, z) \in \operatorname{Cid}(f_1(z) + f_2(z)) \tag{1.7}$$

The complications of multiple solutions can be shifted by adding the set $\operatorname{Cid}(0)$ to a single solution:

$$\operatorname{Cid}(f(z)) = \{\operatorname{cidf}(m, z) + \operatorname{cidzero}(m, z) \mid \operatorname{cidzero}(m, z) \in \operatorname{Cid}(0)\} \tag{1.8}$$

I thus declare the shorthand notation, reminiscent of the constant of integration:

$$\frac{\partial^m}{\partial z^m} f(z) = \operatorname{cidf}(m, z) + \operatorname{Cid}(0) \tag{1.9}$$

Another useful property is

$$\operatorname{cidf}(m + k, z) \in \operatorname{Cid}(\operatorname{cidf}(k, z)) \tag{1.10}$$

2 Complex Iterated Derivative For Specific Functions

First, the simplest solution to the complex iterated derivative of the exponential function:

$$e^z \in \operatorname{Cid}(e^z) \tag{2.1}$$

Thus, for an arbitrary exponential function, using (1.4), we can say:

$$\ln(t)^m t^z \in \operatorname{Cid}(t^z) \tag{2.2}$$

Thus, with the linearity of the complex iterated derivative, we have a way to evaluate the complex iterated derivative of arbitrary linear combinations of exponential functions. Example, where the sums converge or are continued analytically:

$$\zeta(z) - 1 = \sum_{t=2}^{\infty} t^{-z} \Rightarrow \sum_{t=2}^{\infty} \ln\left(t^{-1}\right)^m t^{-z} \in \operatorname{Cid}(\zeta(z) - 1) \tag{2.3}$$

We might want to add 1 to get $\zeta(z)$, so next is $\operatorname{Cid}(1)$:

$$\frac{z^{-m}}{\Gamma(1-m)} \in \operatorname{Cid}(1) \tag{2.4}$$

For $k \in \mathbb{Z}_{>0}$ and an arbitrary constant c,

$$\frac{cz^{-k-m}}{\Gamma(1-k-m)} \in \operatorname{Cid}(0) \tag{2.5}$$

From (1.10) and $\operatorname{Cid}(1)$, the power functions are obtained:

$$\frac{z^{-m-k}}{\Gamma(1-m-k)} \in \operatorname{Cid}\left(\frac{z^{-k}}{\Gamma(1-k)}\right) \tag{2.6}$$

As can be seen by dividing by $\Gamma(1-k)$ and substituting $k \to -k$, in the specific case where the left side of the following equation is not an indeterminate form:

$$\frac{\Gamma(1+k)z^{k-m}}{\Gamma(1+k-m)} \in \operatorname{Cid}\left(z^k\right) \tag{2.7}$$

The linearity of complex iterated derivatives combines with the linearity of integrals so that for an arbitrary function g and arbitrary constants a, b, where the integral converges:

$$\int_a^b g(t) \ln(t)^m t^z \, \mathrm{d}t \in \operatorname{Cid}\left(\int_a^b g(t) t^z \, \mathrm{d}t\right) \tag{2.8}$$

Letting $g(t) = e^{-t}, a = 0, b = \infty$, the complex iterated derivative of the gamma function is found where $z \in \mathbb{C} \land -z \notin \mathbb{Z}_{>0}$:

$$\Gamma(z+1) = \int_0^\infty e^{-t} t^z \, \mathrm{d}t \Rightarrow \int_0^\infty e^{-t} \ln(t)^m t^z \, \mathrm{d}t \in \operatorname{Cid}(\Gamma(z+1)) \tag{2.9}$$

3 Relation to earlier work

Where $n \in \mathbb{Z}_{>0} \land z \in \mathbb{C}$:

$$\operatorname{cidrues}_n(0, z) = \operatorname{Rues}_n(z) = \sum_{k=0}^{\infty} \frac{z^{nk}}{(nk)!} = \frac{1}{n} \sum_{k=1}^{n} \exp\left(ze^{2ki\pi/n}\right) \tag{3.1}$$

The reason these functions are named $\operatorname{cidrues}_n$ is because it is an acronym for Complex Iterated Derivative Root of Unity Exponential Sum function.

Definition 3.1. Where $n \in \mathbb{Z}_{>0}, \{m, z\} \subset \mathbb{C}$:

$$\operatorname{cidrues}_n(m, z) = \frac{1}{n} \sum_{k=1}^{n} \exp\left(ze^{2ki\pi/n} + 2mki\pi/n\right) \in \operatorname{Cid}(\operatorname{Rues}_n(z)) \tag{3.2}$$

The functions are periodic:

$$\operatorname{cidrues}_n(m, x) = \operatorname{cidrues}_n(m+n, x) \tag{3.3}$$

Thus there exists some functions $c_{n,k}(m)$ relatively constant to z such that

$$\operatorname{cidrues}_n(m, z) = \sum_{k=0}^{n-1} c_{n,k}(m) \operatorname{cidrues}_n(k, z) \tag{3.4}$$