Proof of $x^{\frac{1}{x}} \leq e^{\frac{1}{e}}$ for $x > 0$

Gemini 2.0 Flash Experimental

December 27, 2024

Abstract

We provide a complete proof that for any positive real number x, the inequality $x^{\frac{1}{x}} \leq e^{\frac{1}{e}}$ holds. This is achieved by analyzing the natural logarithm of the function $g(x) = x^{\frac{1}{x}}$, demonstrating that its maximum occurs at $x = e$.

1 Introduction

The inequality $x^{\frac{1}{x}} \leq e^{\frac{1}{e}}$ for $x > 0$ is a well-known result. Our approach involves examining the function $g(x) = x^{\frac{1}{x}}$. Since the natural logarithm is a strictly increasing function, finding the maximum of $\ln(g(x))$ is equivalent to finding the maximum of $g(x)$. This simplification makes the analysis considerably easier.

2 Proof

Let $g(x) = x^{\frac{1}{x}}$ for $x > 0$. We define $f(x) = \ln(g(x))$:

$$f(x) = \ln(x^{\frac{1}{x}}) = \frac{1}{x}\ln x = \frac{\ln x}{x}$$

Our goal is to find the maximum of $f(x)$. Since the natural logarithm is a monotonically increasing function, the x value at which $f(x)$ attains its maximum will be the same x value at which $g(x)$ attains its maximum.

To find the maximum of $f(x)$, we compute its first derivative:

$$f'(x) = \frac{\frac{1}{x} \cdot x - \ln x \cdot 1}{x^2} = \frac{1 - \ln x}{x^2}$$

We find the critical points by setting $f'(x) = 0$:

$$\frac{1 - \ln x}{x^2} = 0$$

Since $x^2 > 0$ for $x > 0$, this is equivalent to:

$$1 - \ln x = 0$$

$$\ln x = 1$$

$$x = e$$

Now, we use the second derivative test to determine if this critical point corresponds to a maximum or minimum. We calculate the second derivative of $f(x)$:

$$f''(x) = \frac{-\frac{1}{x} \cdot x^2 - (1 - \ln x) \cdot 2x}{x^4} = \frac{-x - 2x + 2x \ln x}{x^4} = \frac{-3x + 2x \ln x}{x^4} = \frac{-3 + 2 \ln x}{x^3}$$

We evaluate the second derivative at $x = e$:

$$f''(e) = \frac{-3 + 2 \ln e}{e^3} = \frac{-3 + 2}{e^3} = \frac{-1}{e^3}$$

Since $f''(e) < 0$, the function $f(x)$ has a local maximum at $x = e$.

Because $f'(x) > 0$ for $0 < x < e$ and $f'(x) < 0$ for $x > e$, $f(x)$ is increasing on $(0, e)$ and decreasing on (e, ∞). Thus, the local maximum at $x = e$ is a global maximum.

The maximum value of $f(x)$ is:

$$f(e) = \frac{\ln e}{e} = \frac{1}{e}$$

Since the maximum of $f(x) = \ln(g(x))$ occurs at $x = e$, the maximum of $g(x) = x^{\frac{1}{x}}$ also occurs at $x = e$. The maximum value of $g(x)$ is:

$$g(e) = e^{\frac{1}{e}}$$

Therefore, for all $x > 0$:

$$g(x) = x^{\frac{1}{x}} \leq e^{\frac{1}{e}}$$

This completes the proof.