Alternative series test
https://github.com/Nazgand/nazgandMathBook

Mark Andrew Gerads: Nazgand@Gmail.Com

June 12, 2023

1 Alternating series with monotone decreasing absolute value

Theorem 1.1. *Let a sequence a_n exist such that $|a_n| \geq |a_{n+1}|$, $a_{2n} \geq 0, a_{2n+1} \leq 0$, and $\lim\limits_{n\to\infty} a_n = 0$. Then the series $\sum_{n=0}^{\infty} a_n$ converges.*

Proof.

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{2k} a_n + \sum_{n=2k+1}^{\infty} a_n = \sum_{n=0}^{2k} a_n + \sum_{m=0}^{\infty} (a_{2m+1} + a_{2m+2}) \tag{1.1}$$

Because each pair $(a_{2m+1} + a_{2m+2})$ is negative, $\sum_{m=0}^{\infty} (a_{2m+1} + a_{2m+2})$ is negative, and thus $\sum_{n=0}^{2k} a_n \geq \sum_{n=0}^{\infty} a_n$. Likewise,

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{2k+1} a_n + \sum_{n=2k+2}^{\infty} a_n = \sum_{n=0}^{2k+1} a_n + \sum_{n=m}^{\infty} (a_{2m+2} + a_{2m+3}) \tag{1.2}$$

Because each pair $(a_{2m+2} + a_{2m+3})$ is positive, $\sum_{m=0}^{\infty} (a_{2m+2} + a_{2m+3})$ is positive, and thus $\sum_{n=0}^{2k+1} a_n \leq \sum_{n=0}^{\infty} a_n$. Thus

$$\sum_{n=0}^{2k+1} a_n \leq \sum_{n=0}^{\infty} a_n \leq \sum_{n=0}^{2k} a_n \tag{1.3}$$

The right and left sides differ only by a_{2k} and $\lim\limits_{k\to\infty} a_{2k} = 0$. Thus by the squeeze theorem, $\sum_{n=0}^{\infty} a_n$ converges. □

2 $\int_0^{\infty} \sin(xf(x))\partial x$ converges for positive monotone increasing unbounded continuous $f(x)$

Definition 2.1. *Let a speed function $f : \mathbb{R}^+ \to \mathbb{R}^+$ exist such that*

$$\left[\{a, b\} \subseteq \mathbb{R}^+ \wedge a < b\right] \Rightarrow f(a) < f(b) \tag{2.1}$$

As $f(x) \to \infty$, the period of $\sin(xf(x))$ approaches zero, making the integral converge. This is valid by using the Alternating Series test on the positive and negative chunks of the integral which decrease in area monotonically.

https://en.wikipedia.org/wiki/Alternating_series_test