L^p Planar Algebraic Geometry
https://github.com/Nazgand/nazgandMathBook

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Abstract

1 The L^p norm for $p \in \mathbb{R}, p \geq 1$

The distance between points (x_1, y_1) and (x_1, y_1) is:

$$\sqrt[p]{|x_1 - x_2|^p + |y_1 - y_2|^p} \tag{1}$$

This has the triangle inequality:

$$\sqrt[p]{|x_1 - x_2|^p + |y_1 - y_2|^p} + \sqrt[p]{|x_2 - x_3|^p + |y_2 - y_3|^p} \geq \sqrt[p]{|x_1 - x_3|^p + |y_1 - y_3|^p} \tag{2}$$

Euclidean translations and scalings of objects preserves shape in L^p space. Euclidean rotations do not generally preserve shape.

2 Lengths of curves

A curve defined by a function $y = f(x)$ between a and b has the length

$$\lim_{\Delta x \to 0^+} \sum_{k=0}^{\lfloor (b-a)/\Delta x \rfloor} \sqrt[p]{(\Delta x)^p + |f(a + (k+1)\Delta x) - f(a + k\Delta x)|^p} \tag{3}$$

where the limit exists. This is equivalent to

$$\int_a^b \sqrt[p]{1 + |f'(x)|^p}\, dx \tag{4}$$

In polar coordinates,

$$\int_a^b \sqrt[p]{|\cos(\theta) r'(\theta) - \sin(\theta) r(\theta)|^p + |\sin(\theta) r'(\theta) + \cos(\theta) r(\theta)|^p}\, d\theta \tag{5}$$

$$\int_a^b |r(\theta)| \sqrt[p]{\left|\cos(\theta)\frac{r'(\theta)}{r(\theta)} - \sin(\theta)\right|^p + \left|\sin(\theta)\frac{r'(\theta)}{r(\theta)} + \cos(\theta)\right|^p}\, d\theta \tag{6}$$

3 Area

If for all disjoint regions R_0, R_1, if $\text{Area}(R_0) + \text{Area}(R_1) = \text{Area}(R_0 \cup R_1)$ and for all R_2 which is a Euclidean translation of R_0, $\text{Area}(R_0) = \text{Area}(R_2)$, then Area needs to be proportional to Euclidean area. Thus Euclidean area is used.

4 Circles

In the L^p norm, a circle which is the set of points distance r_p from the point (x_0, y_0) is.

$$|x - x_0|^p + |y - y_0|^p = r_p^p \tag{7}$$

The shape of circles is scale invariant:

$$\left|\frac{x}{r_p}\right|^p + \left|\frac{y}{r_p}\right|^p = 1 \tag{8}$$

Thus this work concentrates the unit circle in the first quadrant $x, y \in \mathbb{R}^+$, the other quadrants are reflections and not needing absolute value signs is nice during calculation. Thus what will be used is:

$$x^p + y^p = 1 \tag{9}$$

An important point for symmetry: $x = y = 2^{(-1/p)}$.

Let τ_p be the circumference of a unit circle in L^p. $\theta_p(\theta)$ is the function that translate the Euclidean angle to the L^p angle.

$$\theta_p(\theta) = -\theta_p(-\theta) = \theta_p\left(\theta + \frac{\pi}{2}\right) - \frac{\tau_p}{4} \tag{10}$$

In the first quadrant of the unit circle,

$$1 \geq x^p \Rightarrow y = \sqrt[p]{1 - x^p} \tag{11}$$

Area is important. Ratios between area and distance keep constant as do angles for scaling centered at the origin and rotations in L^p space. For $p > 0$,

$$\int_0^1 (1 - x^p)^{1/p}\, dx = \frac{\Gamma\left(1 + \frac{1}{p}\right)^2}{\Gamma\left(1 + \frac{2}{p}\right)} \tag{12}$$

and polar coordinates with respect to the Euclidean angle and Euclidean norm can be found via substituting $x = r_p \cos(\theta), y = r_p \sin(\theta)$

$$1 = r_p^p(|\cos(\theta)|^p + |\sin(\theta)|^p) \wedge r_p = (|\cos(\theta)|^p + |\sin(\theta)|^p)^{-1/p} \tag{13}$$

In the first quadrant:

$$0 \leq \theta \leq \frac{\pi}{2} \Rightarrow r_p(\theta) = (\cos(\theta)^p + \sin(\theta)^p)^{-1/p} \tag{14}$$

$$\frac{r'(\theta)}{r(\theta)} = \frac{\tan(\theta)\cos(\theta)^p - \cot(\theta)\sin(\theta)^p}{\cos(\theta)^p + \sin(\theta)^p} \tag{15}$$

Useing (6),

$$\tau_p = 4\int_0^{\pi/2} \sqrt[p]{(\cos(\theta)^p + \sin(\theta)^p)\left(\left|\cos(\theta)\frac{\tan(\theta)\cos(\theta)^p - \cot(\theta)\sin(\theta)^p}{\cos(\theta)^p + \sin(\theta)^p} - \sin(\theta)\right|^p + \left|\sin(\theta)\frac{\tan(\theta)\cos(\theta)^p}{\cos(\theta)^p}\right.\right.} \tag{16}$$

An important function is a scaling function based on the Euclidean angle.

$$s_p(\theta) = \left(|\cos(\theta)|^p + |\sin(\theta)|^p\right)^{1/p} \tag{17}$$

In the L^p norm, a formula for arc length is

$$\int_a^b s_p\left(\arctan\left(\frac{\partial y}{\partial x}\right)\right)\sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2}\,dx \tag{18}$$

In the L^p norm, a formula for arc length in Euclidean polar coordinates is

$$\int_a^b s_p\left(\arctan\left(\frac{\sin(\theta)\frac{\partial r}{\partial \theta} + r\cos(\theta)}{\cos(\theta)\frac{\partial r}{\partial \theta} - r\sin(\theta)}\right)\right)\sqrt{r^2 + \left(\frac{\partial r}{\partial \theta}\right)^2}\,d\theta \tag{19}$$

The circumference can be calculated:

$$\frac{\partial y}{\partial x} = \frac{\partial}{\partial x}\sqrt[p]{1 - x^p} = \frac{1}{p}(1 - x^p)^{\frac{1}{p}-1} * -px^{p-1} = -(1 - x^p)^{\frac{1}{p}-1}x^{p-1} \tag{20}$$

$$\tau_p = 4\int_0^1 \sqrt[p]{1 + \left|(1 - x^p)^{\frac{1}{p}-1}x^{p-1}\right|^p}\,dx = 4\int_0^1 \sqrt[p]{1 + (1 - x^p)^{1-p}x^{p^2-p}}\,dx \tag{21}$$

$$\tau_p = 4\int_0^1 \frac{\sqrt[p]{(1 - x^p)\Big((1 - x^p)^{p-1} + x^{p^2-p}\Big)}}{1 - x^p}\,dx = \frac{4}{p}\int_0^1 \frac{\sqrt[p]{z(1 - z)\Big((1 - z)^{p-1} + z^{p-1}\Big)}}{(1 - z)z}\,dz \tag{22}$$

$$\tau_p = \frac{8}{p}\int_0^{1/2} \frac{\sqrt[p]{z(1 - z)\Big((1 - z)^{p-1} + z^{p-1}\Big)}}{(1 - z)z}\,dz \tag{23}$$

Case $p = 3$: Choose $(1 - z)^2 + z^2 = y$, $\frac{\partial y}{\partial z} = 4z - 2$, $z(1 - z) = \frac{1-y}{2}$

$$\tau_p = \frac{16\sqrt[3]{4}}{3}\int_{1/2}^1 y(y(1 - y))^{-2/3}\sqrt{2y - 1}\,dy \tag{24}$$

Choose $z = \frac{1}{1+e^{-x}}$, $\frac{\partial x}{\partial z} = \frac{1}{z(1-z)}$

$$\tau_p = \frac{8}{p}\int_0^\infty \sqrt[p]{\frac{e^x + e^{px}}{(1 + e^x)^{p+1}}}\,dx \tag{25}$$

Choose $z = e^x$, $\frac{\partial x}{\partial z} = \frac{1}{z}$

$$\tau_p = \frac{8}{p} \int_1^\infty \frac{1}{z(1+z)} \sqrt[p]{\frac{z+z^p}{1+z}}\, dz \tag{26}$$

Choose $x = 1 + z^{p-1}$, $(x-1)^{1/(p-1)} = z$, $\frac{\partial x}{\partial z} = \frac{1}{z}$

$$\tau_p = \frac{8}{p} \int_1^\infty \frac{1}{z(1+z)} \sqrt[p]{\frac{z+z^p}{1+z}}\, dz \tag{27}$$

Some exact values:

$$\tau_3 = \frac{2}{3}\Gamma\left(\frac{1}{3}\right)\left(\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{7}{12}\right)} + \frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{13}{12}\right)}\right), \tau_2 = 2\pi, \tau_1 = 8 \tag{28}$$

5 Regular polygons and circles around circles

Given a point at Euclidean angle θ a function is desired which gives the angle for the point distance $0 \le d \le 2$ counter-clockwise from the point at angle θ, and nesting of this function is desired to construct regular polygons.

$$\theta \le \operatorname{dcc}(\theta, d, p, 1) \le \theta + \pi \tag{29}$$

$$\begin{aligned}
(\Delta x) = &\quad r_p(\operatorname{dcc}(\theta, d, p, 1)) \cos(\operatorname{dcc}(\theta, d, p, 1)) - r_p(\theta)\cos(\theta) && (30)\\
(\Delta y) = &\quad r_p(\operatorname{dcc}(\theta, d, p, 1)) \sin(\operatorname{dcc}(\theta, d, p, 1)) - r_p(\theta)\sin(\theta) && (31)\\
d = &\quad \sqrt[p]{(\Delta x)^p + (\Delta y)^p} && (32)\\
\operatorname{dcc}(\theta, d, p, 0) &= \theta && (33)\\
\operatorname{dcc}(\theta, d, p, n) &= \theta - \operatorname{mod}\left(\theta, \frac{\pi}{2}\right) + \operatorname{dcc}\left(\operatorname{mod}\left(\theta, \frac{\pi}{2}\right), d, p, n\right) && (34)\\
&= \operatorname{dcc}(\operatorname{dcc}(\theta, d, p, n-m), d, p, m) && (35)\\
&= -\operatorname{dcc}(-\theta, d, p, -n) && (36)
\end{aligned}$$

Interestingly, regular hexagons have a constant side regardless of angle and p.

$$\begin{aligned}
\operatorname{dcc}(\theta, 1, p, 6) &= \theta + 2\pi && (37)\\
\operatorname{dcc}(\theta, 2, p, 2) &= \theta + 2\pi && (38)
\end{aligned}$$

6 Parabola for $p > 1$

Each point on a parabola has the same distance from the focus as from the directrix. With a directrix of $y = 0$ and a focus at $(0, 2a), a \in \mathbb{R}^+$, we have $f(x) = f(-x), f(0) = a, f(2a) = 2a$. Also,

$$|x|^p + |f(x) - 2a|^p = |f(x)|^p \tag{39}$$

$x \geq 0$ WLOG. 2 cases exist. $f(x) \geq 2a$:

$$x^p + (f(x) - 2a)^p = f(x)^p \tag{40}$$

and $f(x) \leq 2a$:

$$x^p + (2a - f(x))^p = f(x)^p \tag{41}$$

Both cases are equivalent when $2|p$.