# Differentiation

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## 1 Definition

$$\frac{\partial}{\partial x}f(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \tag{1.1}$$

## 2 Sum Rule

**Theorem 2.1.**

$$\frac{\partial}{\partial x}(f(x)+g(x))=\left(\frac{\partial}{\partial x}f(x)\right)+\left(\frac{\partial}{\partial x}g(x)\right) \tag{2.1}$$

*Proof.* Use (1.1)

$$\frac{\partial}{\partial x}(f(x)+g(x))=\lim_{h\to 0}\frac{(f(x+h)+g(x+h))-(f(x)+g(x))}{h} \tag{2.2}$$

Rearrange

$$\frac{\partial}{\partial x}(f(x)+g(x))=\lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}+\frac{g(x+h)-g(x)}{h}\right) \tag{2.3}$$

Split the limit

$$\frac{\partial}{\partial x}(f(x)+g(x))=\left(\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\right)+\left(\lim_{a\to 0}\frac{g(x+a)-g(x)}{a}\right) \tag{2.4}$$

Simplify with (1.1)

$$\frac{\partial}{\partial x}(f(x)+g(x))=\left(\frac{\partial}{\partial x}f(x)\right)+\left(\frac{\partial}{\partial x}g(x)\right) \tag{2.5}$$

□

## 3 Constant Multiple Rule

**Theorem 3.1.**

$$\frac{\partial}{\partial x}(a*f(x))=a*\frac{\partial}{\partial x}(f(x)) \tag{3.1}$$

*Proof.* Use (1.1)

$$\frac{\partial}{\partial x}(a*f(x))=\lim_{h\to 0}\frac{a*f(x+h)-a*f(x)}{h}=a*\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=a*\frac{\partial}{\partial x}(f(x)) \tag{3.2}$$

□

## 4 Chain Rule

**Theorem 4.1.**

$$\frac{\partial}{\partial x}f(g(x))=\left(\frac{\partial}{\partial g}f(g)\Big|_{g\to g(x)}\right)*\frac{\partial}{\partial x}g(x) \tag{4.1}$$

*Proof.* Start with (1.1)

$$\frac{\partial}{\partial x}f(g(x))=\lim_{h\to 0}\frac{f(g(x+h))-f(g(x))}{h} \tag{4.2}$$

Multiply by $1=\frac{g(x+h)-g(x)}{g(x+h)-g(x)}$:

$$\frac{\partial}{\partial x}f(g(x))=\lim_{h\to 0}\frac{f(g(x+h))-f(g(x))}{g(x+h)-g(x)}\frac{g(x+h)-g(x)}{h} \tag{4.3}$$

Split the limit:

$$\frac{\partial}{\partial x}f(g(x))=\lim_{h\to 0}\frac{f(g(x+h))-f(g(x))}{g(x+h)-g(x)}\lim_{a\to 0}\frac{g(x+a)-g(x)}{a} \tag{4.4}$$

Simplify (1.1)

$$\frac{\partial}{\partial x} f(g(x)) = \lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \frac{\partial}{\partial x} g(x) \tag{4.5}$$

Substitute $g(x+h) \to g + \epsilon, g(x) \to g$

$$\frac{\partial}{\partial x} f(g(x)) = \left( \lim_{\epsilon \to 0} \frac{f(g+\epsilon) - f(g)}{\epsilon} \Big|_{g \to g(x)} \right) \frac{\partial}{\partial x} g(x) \tag{4.6}$$

Simplify (1.1)

$$\frac{\partial}{\partial x} f(g(x)) = \left( \frac{\partial}{\partial g} f(g) \Big|_{g \to g(x)} \right) \frac{\partial}{\partial x} g(x) \tag{4.7}$$

□

# 5 Logarithmic Derivative

**Theorem 5.1.**

$$\frac{\partial}{\partial x} f(x) = f(x) \frac{\partial}{\partial x} \ln(f(x)) \tag{5.1}$$

*Proof.* Use the chain rule

$$\frac{\partial}{\partial x} \ln(f(x)) = \left( \frac{\partial}{\partial z} \ln(z) \Big|_{z \to f(x)} \right) * \frac{\partial}{\partial x} f(x) \tag{5.2}$$

Simplify. [Logarithms(1.16)]

$$\frac{\partial}{\partial x} \ln(f(x)) = \left( \frac{1}{z} \Big|_{z \to f(x)} \right) * \frac{\partial}{\partial x} f(x) = \frac{1}{f(x)} * \frac{\partial}{\partial x} f(x) \tag{5.3}$$

Multiply

$$f(x) \frac{\partial}{\partial x} \ln(f(x)) = \frac{\partial}{\partial x} f(x) \tag{5.4}$$

□

# 6 Product Rule

**Theorem 6.1.**

$$\frac{\partial}{\partial x} \prod_{k=1}^{n} f_k(x) = \left( \prod_{k=1}^{n} f_k(x) \right) \sum_{k=1}^{n} \frac{\frac{\partial}{\partial x} f_k(x)}{f_k(x)} \tag{6.1}$$

*Proof.* Use (5.1)

$$\frac{\partial}{\partial x} \prod_{k=1}^{n} f_k(x) = \left( \prod_{k=1}^{n} f_k(x) \right) \frac{\partial}{\partial x} \ln \left( \prod_{k=1}^{n} f_k(x) \right) \tag{6.2}$$

Expand the logarithms

$$\frac{\partial}{\partial x} \prod_{k=1}^{n} f_k(x) = \left( \prod_{k=1}^{n} f_k(x) \right) \frac{\partial}{\partial x} \left( \sum_{k=1}^{n} \ln(f_k(x)) \right) \tag{6.3}$$

Sum rule (2.1)

$$\frac{\partial}{\partial x} \prod_{k=1}^{n} f_k(x) = \left( \prod_{k=1}^{n} f_k(x) \right) \left( \sum_{k=1}^{n} \frac{\partial}{\partial x} \ln(f_k(x)) \right) \tag{6.4}$$

Simplify logarithmic derivative using (5.1)

$$\frac{\partial}{\partial x} \prod_{k=1}^{n} f_k(x) = \left( \prod_{k=1}^{n} f_k(x) \right) \sum_{k=1}^{n} \frac{\frac{\partial}{\partial x} f_k(x)}{f_k(x)} \tag{6.5}$$

□