Differentiation

https://github.com/Nazgand/nazgandMathBook

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June 12, 2023

Abstract

The goal of this paper is to review differentiation.

1 Definition

$$\frac{\partial}{\partial x} f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{1.1}$$

2 Sum Rule

Theorem 2.1.

$$\frac{\partial}{\partial x}(f(x) + g(x)) = \left(\frac{\partial}{\partial x} f(x)\right) + \left(\frac{\partial}{\partial x} g(x)\right) \tag{2.1}$$

Proof. Use (1.1)

$$\frac{\partial}{\partial x}(f(x) + g(x)) = \lim_{h \to 0} \frac{(f(x+h) + g(x+h)) - (f(x) + g(x))}{h} \tag{2.2}$$

Rearrange

$$\frac{\partial}{\partial x}(f(x) + g(x)) = \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h}\right) \tag{2.3}$$

Split the limit

$$\frac{\partial}{\partial x}(f(x) + g(x)) = \left(\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}\right) + \left(\lim_{a \to 0} \frac{g(x+a) - g(x)}{a}\right) \tag{2.4}$$

Simplify with (1.1)

$$\frac{\partial}{\partial x}(f(x) + g(x)) = \left(\frac{\partial}{\partial x} f(x)\right) + \left(\frac{\partial}{\partial x} g(x)\right) \tag{2.5}$$

□

3 Constant Multiple Rule

Theorem 3.1.

$$\frac{\partial}{\partial x}(a * f(x)) = a * \frac{\partial}{\partial x}(f(x)) \tag{3.1}$$

Proof. Use (1.1)

$$\frac{\partial}{\partial x}(a * f(x)) = \lim_{h \to 0} \frac{a * f(x+h) - a * f(x)}{h} = a * \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = a * \frac{\partial}{\partial x}(f(x)) \tag{3.2}$$

□

4 Chain Rule

Theorem 4.1.

$$\frac{\partial}{\partial x} f(g(x)) = \left(\frac{\partial}{\partial g} f(g) : g \to g(x)\right) * \frac{\partial}{\partial x} g(x) \tag{4.1}$$

Proof. Start with (1.1)

$$\frac{\partial}{\partial x} f(g(x)) = \lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{h} \tag{4.2}$$

Multiply by $1 = \frac{g(x+h)-g(x)}{g(x+h)-g(x)}$:

$$\frac{\partial}{\partial x} f(g(x)) = \lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \frac{g(x+h) - g(x)}{h} \tag{4.3}$$

Split the limit:

$$\frac{\partial}{\partial x} f(g(x)) = \lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \lim_{a\to 0} \frac{g(x+a) - g(x)}{a} \tag{4.4}$$

Simplify (1.1)

$$\frac{\partial}{\partial x} f(g(x)) = \lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \frac{\partial}{\partial x} g(x) \tag{4.5}$$

Substitute $g(x+h) \to g + \epsilon, g(x) \to g$

$$\frac{\partial}{\partial x} f(g(x)) = \left(\lim_{\epsilon\to 0} \frac{f(g+\epsilon) - f(g)}{\epsilon} : g \to g(x) \right) \frac{\partial}{\partial x} g(x) \tag{4.6}$$

Simplify (1.1)

$$\frac{\partial}{\partial x} f(g(x)) = \left(\frac{\partial}{\partial g} f(g) : g \to g(x) \right) \frac{\partial}{\partial x} g(x) \tag{4.7}$$

□

5 Logarithmic Derivative

Theorem 5.1.

$$\frac{\partial}{\partial x} f(x) = f(x) \frac{\partial}{\partial x} \ln\left(f(x)\right) \tag{5.1}$$

Proof. Use the chain rule

$$\frac{\partial}{\partial x} \ln\left(f(x)\right) = \left(\frac{\partial}{\partial z} \ln\left(z\right) : z \to f(x) \right) * \frac{\partial}{\partial x} f(x) \tag{5.2}$$

Simplify. [Logarithms(1.15)]

$$\frac{\partial}{\partial x} \ln\left(f(x)\right) = \left(\frac{1}{z} : z \to f(x) \right) * \frac{\partial}{\partial x} f(x) = \frac{1}{f(x)} * \frac{\partial}{\partial x} f(x) \tag{5.3}$$

Multiply

$$f(x) \frac{\partial}{\partial x} \ln\left(f(x)\right) = \frac{\partial}{\partial x} f(x) \tag{5.4}$$

□

6 Product Rule

Theorem 6.1.

$$\frac{\partial}{\partial x} \prod_{k=1}^{n} f(x) = \left(\prod_{k=1}^{n} f(x) \right) \sum_{k=1}^{n} \frac{\frac{\partial}{\partial x} f(x)}{f(x)} \tag{6.1}$$

Proof. Use (5.1)

$$\frac{\partial}{\partial x} \prod_{k=1}^{n} f(x) = \left(\prod_{k=1}^{n} f(x) \right) \frac{\partial}{\partial x} \ln\left(\prod_{k=1}^{n} f(x) \right) \tag{6.2}$$

Expand the logarithms

$$\frac{\partial}{\partial x} \prod_{k=1}^{n} f(x) = \left(\prod_{k=1}^{n} f(x) \right) \frac{\partial}{\partial x} \left(\sum_{k=1}^{n} \ln\left(f(x)\right) \right) \tag{6.3}$$

Sum rule (2.1)

$$\frac{\partial}{\partial x} \prod_{k=1}^{n} f(x) = \left(\prod_{k=1}^{n} f(x) \right) \left(\sum_{k=1}^{n} \frac{\partial}{\partial x} \ln\left(f(x)\right) \right) \tag{6.4}$$

Simplify logarithmic derivative using (5.1)

$$\frac{\partial}{\partial x} \prod_{k=1}^{n} f(x) = \left(\prod_{k=1}^{n} f(x) \right) \sum_{k=1}^{n} \frac{\frac{\partial}{\partial x} f(x)}{f(x)} \tag{6.5}$$

□