Exponential Function

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2026-01-21 04:46:18-08:00

1 Power function definition

Definition 1.1. Here I formally define the power function.

$$\text{Pow}(x, y) = x^y \tag{1.1}$$

$$\text{Pow}(x, 0) = 1 \tag{1.2}$$

$$\exists \text{Pow}(x, y-1) \Rightarrow \text{Pow}(x, y) = \text{Pow}(x, y-1) * x \tag{1.3}$$

$$\exists \text{Pow}(x, b)^{-1} \Rightarrow \text{Pow}(x, -b) = \text{Pow}(x, b)^{-1} \tag{1.4}$$

$$[\exists \text{Pow}(x, a) \wedge \exists \text{Pow}(x, b)] \Rightarrow \text{Pow}(x, a+b) = \text{Pow}(x, a)\,\text{Pow}(x, b) \tag{1.5}$$

$$[x \in \mathbb{R}^+ \wedge y \in \mathbb{R}] \Rightarrow \exists \text{Pow}(x, y) \in \mathbb{R} \tag{1.6}$$

$$[x \in \mathbb{R}^+ \wedge y \in \mathbb{R}] \Rightarrow \exists \left(\frac{\partial}{\partial z} \text{Pow}(x, z) \Big|_{z \to y} \right) \tag{1.7}$$

2 Basic results and definition of e

Theorem 2.1. The derivative of an exponential function is a multiple of the same exponential function.

$$\frac{\partial}{\partial y} x^y = \left(\frac{\partial}{\partial z} x^z \Big|_{z \to 0} \right) x^y \tag{2.1}$$

Proof. Substitute $y \to z + a$ in the left side of the equation to get the right side

$$\frac{\partial}{\partial y} x^y = \left(\frac{\partial}{\partial z} x^{z+a} \Big|_{z \to (y-a)} \right) \tag{2.2}$$

Substitute $x^{z+a} \to x^z x^a$

$$\frac{\partial}{\partial y} x^y = \left(\frac{\partial}{\partial z} x^z x^a \Big|_{z \to (y-a)} \right) \tag{2.3}$$

Bring x^a out:

$$\frac{\partial}{\partial y} x^y = x^a \left(\frac{\partial}{\partial z} x^z \Big|_{z \to (y-a)} \right) \tag{2.4}$$

Substitute $a \to y$

$$\frac{\partial}{\partial y} x^y = x^y \left(\frac{\partial}{\partial z} x^z \Big|_{z \to 0} \right) \tag{2.5}$$

□

Definition 2.2. Define e to be the base of the exponential function which has a derivative of 1 at 0.

$$1 = \left(\frac{\partial}{\partial z} e^z \Big|_{z \to 0} \right) \tag{2.6}$$

Lemma 2.3.

$$\frac{\partial}{\partial y} e^y = e^y \tag{2.7}$$

Proof. Substitute $x \to e$ in (2.5) and simplify with (2.6). □

Corollary 2.4. The derivative of an exponential function is the natural logarithm of the base of the exponential function times the exponential function.

$$\frac{\partial}{\partial y} x^y = \ln(x) x^y \tag{2.8}$$

Proof.

$$\left(\frac{\partial}{\partial z}x^z\Big|_{z\to 0}\right) = \ln(x) \tag{2.9}$$

$$\frac{\partial}{\partial y}x^y = x^y\left(\frac{\partial}{\partial z}x^z\Big|_{z\to 0}\right) \tag{2.10}$$

Substitute $x \to e^{\ln(x)}$ in the left to get the right

$$\frac{\partial}{\partial y}x^y = \frac{\partial}{\partial y}e^{\ln(x)y} \tag{2.11}$$

Apply the chain rule:

$$\frac{\partial}{\partial y}x^y = \left(\frac{\partial}{\partial z}e^z\Big|_{z\to \ln(x)y}\right) * \frac{\partial}{\partial y}\ln(x)y \tag{2.12}$$

Simplify

$$\frac{\partial}{\partial y}x^y = \left(e^z\Big|_{z\to \ln(x)y}\right) * \ln(x) = e^{\ln(x)y} * \ln(x) = \ln(x)x^y \tag{2.13}$$

□

3 Exponential Function Derivative

Theorem 3.1.

$$e^x = \sum_{k=0}^{\infty}\frac{x^k}{k!} \tag{3.1}$$

Proof. We have a formula which is it's own derivative (2.7). Another formula which is it's own derivative is

$$\exp(x) = \sum_{k=0}^{\infty}\frac{x^k}{k!} = \frac{\partial}{\partial x}\exp(x) \tag{3.2}$$

The differential equation $f(x) = f'(x)$ has 1 degree of freedom which is filled by $f(0) = 1$ by both formulae. Thus both formulae express the same function; $e^x = \exp(x)$. □

4 Convergence of $\exp(x)$

Theorem 4.1. $\exp(x)$ converges for all $x \in \mathbb{C}$

Proof. By the triangle inequality, an upper bound and a lower bound exist for all complex numbers.

$$-\frac{|x|^k}{k!} \le \frac{x^k}{k!} \le \frac{|x|^k}{k!} \tag{4.1}$$

$$-\exp(|x|) \le \exp(x) \le \exp(|x|) \tag{4.2}$$

Thus convergence for $x \in \mathbb{R}^+$ implies convergence for $x \in \mathbb{C}$. Let $x \in \mathbb{R}^+$. Bound part of the sum by a geometric series:

$$\exp(x) = \sum_{k=0}^{n-1}\frac{x^k}{k!} + \sum_{k=n}^{\infty}\frac{x^k}{k!} < \sum_{k=0}^{n-1}\frac{x^k}{k!} + \sum_{k=n}^{\infty}\frac{x^k}{n^{k-n}(n)!} \tag{4.3}$$

Simplify

$$\sum_{k=n}^{\infty}\frac{x^k}{n^{k-n}(n)!} = \frac{n^n}{(n)!}\sum_{k=n}^{\infty}\left(\frac{x}{n}\right)^k = \frac{x^n}{(n)!}\sum_{m=0}^{\infty}\left(\frac{x}{n}\right)^m \tag{4.4}$$

Find where the bounding geometric series converges. [GeometricSeries(0.3)]

$$\frac{x}{n} < 1 \Rightarrow \sum_{m=0}^{\infty}\left(\frac{x}{n}\right)^m = \frac{1}{1-\frac{x}{n}} \tag{4.5}$$

Every specific x has an integer larger than it and is bounded by a circle of convergence from a corresponding geometric series. Let $n \to \infty$ and $\exp(x)$ converges for all $x \in \mathbb{C}$. □

5 Limit Form of E

Theorem 5.1.

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \lim_{n\to\infty}\left(\frac{1+n}{n}\right)^n \tag{5.1}$$

Proof. Proof from `https://mathcs.clarku.edu/~djoyce/ma122/elimit.pdf`. Note

$$\left[\{n,t\}\subset\mathbb{R}^+ \wedge 1\le t\le\frac{1+n}{n}\right]\Rightarrow 1\ge\frac{1}{t}\ge\frac{n}{1+n} \tag{5.2}$$

Integrate over the inequality:

$$\int_1^{\frac{1+n}{n}} 1\,\mathrm{d}t \ge \int_1^{\frac{1+n}{n}}\frac{1}{t}\,\mathrm{d}t \ge \int_1^{\frac{1+n}{n}}\frac{n}{1+n}\,\mathrm{d}t \tag{5.3}$$

Simplify using [Logarithms(1.16)]

$$\frac{1}{n}\ge\ln\left(\frac{1+n}{n}\right)\ge\frac{1}{n+1} \tag{5.4}$$

Apply the exponential function:

$$e^{\frac{1}{n}}\ge\frac{1+n}{n}\ge e^{\frac{1}{n+1}} \tag{5.5}$$

Raise to the power of n and $n+1$

$$e\ge\left(\frac{1+n}{n}\right)^n\wedge\left(\frac{1+n}{n}\right)^{n+1}\ge e \tag{5.6}$$

Divide

$$e\ge\left(\frac{1+n}{n}\right)^n\wedge\left(\frac{1+n}{n}\right)^{n}\ge \frac{en}{1+n} \tag{5.7}$$

Let $n\to\infty$ using the squeeze theorem, simplify.

$$e\ge\lim_{n\to\infty}\left(\frac{1+n}{n}\right)^n\ge e \tag{5.8}$$

□

6 Exponential Function Limit Form

Theorem 6.1.

$$e^x = \lim_{m\to\infty}\left(1+\frac{x}{m}\right)^m \tag{6.1}$$

Proof. Raise (5.1) to the power of x

$$e^x = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{xn} \tag{6.2}$$

For $x\in\mathbb{R}^+$, a substitution $n\to\frac{m}{x}$ can be made to obtain a limit known to exist.

$$e^x = \lim_{m\to\infty}\left(1+\frac{x}{m}\right)^m \tag{6.3}$$

The existence of the limit for $x\in\mathbb{R}^+$ extends analytically to $x\in\mathbb{C}$ because the new formula fulfills the differential equation $f(x)=f'(x)$ and has $f(0)=1$. Chain rule used:

$$\frac{\partial}{\partial x}\lim_{m\to\infty}\left(1+\frac{x}{m}\right)^m = \lim_{m\to\infty}\left(\frac{\partial}{\partial z}z^m\Big|_{z\to(1+\frac{x}{m})}\right)*\frac{\partial}{\partial x}\left(1+\frac{x}{m}\right) \tag{6.4}$$

Simplify:

$$\frac{\partial}{\partial x}\lim_{m\to\infty}\left(1+\frac{x}{m}\right)^m = \lim_{m\to\infty} m\left(1+\frac{x}{m}\right)^{m-1}*\frac{1}{m} \tag{6.5}$$

Split the limit

$$\frac{\partial}{\partial x}\lim_{m\to\infty}\left(1+\frac{x}{m}\right)^m = \lim_{m\to\infty}\left(1+\frac{x}{m}\right)^{m}*\left(1+\frac{x}{m}\right)^{-1} \tag{6.6}$$

$$\frac{\partial}{\partial x}\lim_{m\to\infty}\left(1+\frac{x}{m}\right)^m = \lim_{m\to\infty}\left(1+\frac{x}{m}\right)^{m}*\lim_{p\to\infty}\left(1+\frac{x}{p}\right)^{-1} \tag{6.7}$$

Simplify to see $f(x)=f'(x)$.

$$\frac{\partial}{\partial x}\lim_{m\to\infty}\left(1+\frac{x}{m}\right)^m = \lim_{m\to\infty}\left(1+\frac{x}{m}\right)^{m} \tag{6.8}$$

□

7 Euler's Identity

Theorem 7.1.

$$e^{ix} = \cos(x) + i\sin(x) \tag{7.1}$$

Proof.

$$\frac{\partial^n}{\partial x^n} e^{ix} = i^n e^{ix} \tag{7.2}$$

The derivatives at 0 thus cycle through $1, i, -1, -i$. Use [TaylorSeries(0.1)] and compare to [Trigonometry(3.1)] and [Trigonometry(3.6)] □

8 Bibliography

https://en.wikipedia.org/wiki/Exponential_function