

# Argument Sum Rules From Homogeneous Linear Differential Equations Of Constant Coefficients Conjecture

<https://github.com/Nazgand/nazgandMathBook>

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## Abstract

The goal of this paper is to conjecture a seemingly fundamental calculus fact.

A [homogeneous linear differential equation of constant coefficients] has the form

$$0 = \sum_{k=0}^n a_k \frac{\partial^k}{\partial z^k} f(z) = \sum_{k=0}^n a_k f^{(k)}(z) \quad (0.1)$$

where  $\forall k, a_k \in \mathbb{C}, a_n \neq 0$ , and  $f : \mathbb{C} \rightarrow \mathbb{C}$  is differentiable everywhere. Let  $g_0(z), \dots, g_{n-1}(z)$  be solutions that span the vector space of solutions of the [homogeneous linear differential equation of constant coefficients]. To be clear:

$$\left\{ h(z) \mid 0 = \sum_{k=0}^n a_k h^{(k)}(z) \right\} = \left\{ \sum_{k=0}^{n-1} b_k g_k(z) \mid b_k \in \mathbb{C} \right\} \quad (0.2)$$

Let us define a column vector and clarify its transpose (a row vector):

$$v(z_0) = \begin{pmatrix} g_0(z_0) \\ \vdots \\ g_{n-1}(z_0) \end{pmatrix}, v(z_1)^T = (g_0(z_1) \quad \dots \quad g_{n-1}(z_1)) \quad (0.3)$$

I conjecture there exists a complex-valued constant  $n$  by  $n$  symmetric matrix  $A$  ( $A = A^T$ ) such that

$$f(z_0 + z_1) = v(z_1)^T A v(z_0) = v(z_0)^T A v(z_1) \quad (0.4)$$

## 1 A reason $A$ is symmetric

Suppose instead of a symmetric matrix  $A$ , we find a matrix  $B$  such that

$$f(z_0 + z_1) = v(z_1)^T B v(z_0) \quad (1.1)$$

Then take the transpose of the equation and substitute  $z_0 \rightarrow z_1, z_1 \rightarrow z_0$ , resulting in the following equation:

$$f(z_0 + z_1) = v(z_1)^T B^T v(z_0) \quad (1.2)$$

Average both equations:

$$f(z_0 + z_1) = v(z_1)^T \frac{B + B^T}{2} v(z_0) \quad (1.3)$$

Note that we can set  $A = \frac{B+B^T}{2}$  because it is symmetric. □