https://GitHub.Com/Nazgand/NazgandMathBook

Trigonometry

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1 Geometric Foundations

In [ElementaryGeometryAndTrigonometry.kig] by moving the angle control point to the second quadrant and comparing, we can see

$$\sin\left(\theta+\frac{\pi}{2}\right)=\cos(\theta)\wedge\cos\left(\theta+\frac{\pi}{2}\right)=-\sin(\theta) \tag{1.1}$$

Moreover, we see as $\theta\to 0$

$$\lim_{\theta\to 0}\frac{\sin(\theta)}{\theta}=1 \tag{1.2}$$

Using the Pythagorean Theorem, we find

$$1=\sin(\theta)^2+\cos(\theta)^2 \tag{1.3}$$

From [SumOfAnglesSinCos.kig], we get

$$\sin(\alpha+\beta)=\sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta) \tag{1.4}$$

And

$$\cos(\alpha+\beta)=\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta) \tag{1.5}$$

2 Derivatives

Theorem 2.1.

$$\frac{\partial}{\partial x}\sin(x)=\cos(x) \tag{2.1}$$

Proof.

$$\frac{\partial}{\partial x}\sin(x)=\lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h} \tag{2.2}$$

Use (1.4)

$$\frac{\partial}{\partial x}\sin(x)=\lim_{h\to 0}\frac{\sin(x)\cos(h)+\cos(x)\sin(h)-\sin(x)}{h} \tag{2.3}$$

Split the limit:

$$\frac{\partial}{\partial x}\sin(x)=\lim_{h\to 0}\frac{\cos(x)\sin(h)}{h}+\lim_{h\to 0}\frac{\sin(x)(\cos(h)-1)}{h} \tag{2.4}$$

Simplify

$$\frac{\partial}{\partial x}\sin(x)=\cos(x)\lim_{h\to 0}\frac{\sin(h)}{h}+\sin(x)\lim_{h\to 0}\frac{\cos(h)-1}{h} \tag{2.5}$$

Use (1.2) on left limit and multiply right limit by $\frac{\cos(h)+1}{\cos(h)+1}$

$$\frac{\partial}{\partial x}\sin(x)=\cos(x)+\sin(x)\lim_{h\to 0}\frac{\cos(h)^2-1}{h(\cos(h)+1)} \tag{2.6}$$

Use (1.3)

$$\frac{\partial}{\partial x}\sin(x)=\cos(x)+\sin(x)\lim_{h\to 0}\frac{\sin(h)^2}{h(\cos(h)+1)} \tag{2.7}$$

Split the limit:

$$\frac{\partial}{\partial x}\sin(x)=\cos(x)+\sin(x)\lim_{a\to 0}\frac{\sin(a)}{a}\lim_{b\to 0}\sin(b)\lim_{c\to 0}\frac{1}{(\cos(c)+1)} \tag{2.8}$$

Evaluate using (1.2) again

$$\frac{\partial}{\partial x}\sin(x)=\cos(x)+\sin(x)*1*0*\frac{1}{2}=\cos(x) \tag{2.9}$$

□

Theorem 2.2.

$$\frac{\partial}{\partial x}\cos(x)=-\sin(x) \tag{2.10}$$

Proof. Use (1.1)

$$\frac{\partial}{\partial x}\cos(x) = \frac{\partial}{\partial x}\sin\left(x + \frac{\pi}{2}\right) \tag{2.11}$$

Use the chain rule:

$$\frac{\partial}{\partial x}\cos(x) = \left(\frac{\partial}{\partial z}\sin(z)\Big|_{z\to x+\frac{\pi}{2}}\right)\frac{\partial}{\partial x}\left(x + \frac{\pi}{2}\right) \tag{2.12}$$

Simplify

$$\frac{\partial}{\partial x}\cos(x) = \cos\left(x + \frac{\pi}{2}\right) \tag{2.13}$$

Use Use (1.1)

$$\frac{\partial}{\partial x}\cos(x) = -\sin(x) \tag{2.14}$$

□

3 Taylor Series

Theorem 3.1.

$$\sin(x) = \sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{(2k+1)!} \tag{3.1}$$

Proof. The multiple derivatives cycle through 4 functions. Let $n \in \mathbb{Z}_{\geq 0}$:

$$\frac{\partial^{4n}}{\partial x^{4n}}\sin(x) = \sin(x) \tag{3.2}$$

$$\frac{\partial^{4n+1}}{\partial x^{4n+1}}\sin(x) = \cos(x) \tag{3.3}$$

$$\frac{\partial^{4n+2}}{\partial x^{4n+2}}\sin(x) = -\sin(x) \tag{3.4}$$

$$\frac{\partial^{4n+3}}{\partial x^{4n+3}}\sin(x) = -\cos(x) \tag{3.5}$$

The derivatives at 0 thus cycle through $0, 1, 0, -1$. Use [TaylorSeries(0.1)] and simplify out the zeroes. □

Theorem 3.2.

$$\cos(x) = \sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}}{(2k)!} \tag{3.6}$$

Proof. The multiple derivatives cycle through 4 functions. Let $n \in \mathbb{Z}_{\geq 0}$:

$$\frac{\partial^{4n}}{\partial x^{4n}}\cos(x) = \cos(x) \tag{3.7}$$

$$\frac{\partial^{4n+1}}{\partial x^{4n+1}}\cos(x) = -\sin(x) \tag{3.8}$$

$$\frac{\partial^{4n+2}}{\partial x^{4n+2}}\cos(x) = -\cos(x) \tag{3.9}$$

$$\frac{\partial^{4n+3}}{\partial x^{4n+3}}\cos(x) = \sin(x) \tag{3.10}$$

The derivatives at 0 thus cycle through $1, 0, -1, 0$. Use [TaylorSeries(0.1)] and simplify out the zeroes. □