

# Does the Series Converge?

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2026-01-21 04:46:17-08:00

## 1 Problem -2

Given [a function from [the natural numbers] to [the real numbers]] which is [inclusively bounded in magnitude by 1], i.e.

$$\exists f_1 : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{R}, [\forall k \in \mathbb{Z}_{\geq 0}, |f_1(k)| \leq 1] \quad (1.1)$$

and [a positive real number smaller than 1], i.e.

$$\exists r \in \mathbb{R}^+, r < 1 \quad (1.2)$$

, prove

$$\exists \left[ \lim_{m \rightarrow \infty} \sum_{k=0}^m f_1(k)r^k \right] \in \mathbb{R} \quad (1.3)$$

Solution: Note

$$\forall n \in \mathbb{Z}_{\geq 0}, \left[ \exists \left[ \lim_{m \rightarrow \infty} \sum_{k=0}^m f_1(k)r^k \right] \in \mathbb{R} \Leftrightarrow \exists \left[ \lim_{m \rightarrow \infty} \sum_{k=n}^m f_1(k)r^k \right] \in \mathbb{R} \right] \quad (1.4)$$

and

$$-\frac{r^n - r^{n+1}}{1-r} = -\sum_{k=n}^m r^k \leq \sum_{k=n}^m f_1(k)r^k \leq \sum_{k=n}^m r^k = \frac{r^n - r^{n+1}}{1-r} \quad (1.5)$$

First, let  $m \rightarrow \infty$ .

$$-\frac{r^n}{1-r} \leq \lim_{m \rightarrow \infty} \sum_{k=n}^m f_1(k)r^k \leq \frac{r^n}{1-r} \quad (1.6)$$

Then let  $n \rightarrow \infty$  to observe the Squeeze Theorem.

$$0 \leq \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} \sum_{k=n}^m f_1(k)r^k \leq 0 \quad (1.7)$$

## 2 Problem -1

Given [a function from [the natural numbers] to [the complex numbers]] which is [inclusively bounded in magnitude by 1], i.e.

$$\exists f_1 : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{C}, [\forall k \in \mathbb{Z}_{\geq 0}, |f_1(k)| \leq 1] \quad (2.1)$$

and [a complex number smaller than 1 in magnitude], i.e.

$$\exists r \in \mathbb{C}, |r| < 1 \quad (2.2)$$

, prove

$$\exists \left[ \lim_{m \rightarrow \infty} \sum_{k=0}^m f_1(k)r^k \right] \in \mathbb{C} \quad (2.3)$$

Without loss of generality, we can set  $r = |r|$ , as seen by the substitutions  $f_1(k) \rightarrow f_1(k)\frac{r^k}{|r|^k}$ ,  $r \rightarrow |r|$ . The case  $r = 0$  is trivial, so let  $r \in \mathbb{R}$ ,  $0 < r < 1$ . The relevant Cauchy series can be split into 2 Cauchy series for the imaginary and real parts; [the complex limit can be proved to exist] by [proving the existence of the {real, imaginary} parts]; thus, the problem is reduced to [Problem -2]

## 3 Problem 0

Given [a function from [the natural numbers] to [the complex numbers]] which is [exponentially bounded in magnitude], i.e.

$$\exists f_a : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{C}, \exists a \in \mathbb{R}, [\forall k \in \mathbb{Z}_{\geq 0}, a^{k+1} \geq |f_a(k)|] \quad (3.1)$$

, and

$$\exists t \in \mathbb{R}^+ \quad (3.2)$$

, prove:

$$\exists \left[ \lim_{m \rightarrow \infty} \sum_{k=0}^m \frac{f_a(k)}{(k!)^t} \right] \in \mathbb{C} \quad (3.3)$$

This sum 'obviously' converges via  $\left[ \forall b \in \mathbb{R}^+, \mathcal{O}\left((k!)^t\right) > \mathcal{O}(b^k) \right]$ ,  $\mathcal{O}(a^k) \geq \mathcal{O}(|f_a(k)|)$ . We must be more rigorous than that to build a solid foundation. A good first step is to notice

$$\forall K \in \mathbb{Z}_{\geq 0}, \left[ \left[ \exists \left[ \lim_{m \rightarrow \infty} \sum_{k=0}^m \frac{f_a(k)}{(k!)^t} \right] \in \mathbb{C} \right] \Leftrightarrow \left[ \exists \left[ \lim_{m \rightarrow \infty} \sum_{k=K}^m \frac{f_a(k)}{(k!)^t} \right] \in \mathbb{C} \right] \right] \quad (3.4)$$

and let  $K$  be sufficiently large so that  $\frac{((K+1)!)^t}{(K!)^t} > a$ , i.e.  $K > \sqrt[t]{a} - 1$ . Then this problem reduces to [Problem -1].

## 4 Problem 1

Given [a function from [the natural numbers] to [the complex numbers]] which is [exponentially bounded in magnitude], i.e.

$$\exists f_b : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{C}, \exists b \in \mathbb{R}, [\forall k \in \mathbb{Z}_{\geq 0}, b^{k+1} \geq |f_b(k)|] \quad (4.1)$$

, and

$$\exists t \in \mathbb{R}^+, \exists z \in \mathbb{C} \quad (4.2)$$

, prove:

$$\exists \left[ \lim_{m \rightarrow \infty} \sum_{k=0}^m \frac{f_b(k)z^k}{(k!)^t} \right] \in \mathbb{C} \quad (4.3)$$

Solution: Let  $f_a(k) = f_b(k)z^k, a \geq b|z|$ , and note

$$[\forall k \in \mathbb{Z}_{\geq 0}, b^{k+1} \geq |f_b(k)|] \Rightarrow [\forall k \in \mathbb{Z}_{\geq 0}, a^{k+1} \geq |f_a(k)|] \quad (4.4)$$

Thus [Problem 1] has been reduced to [Problem 0].