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## 1 Assumptions and definitions

A degree 2 [homogeneous linear differential equation of constant coefficients] has the form

$$0 = \sum_{k=0}^2 a_k f^{(k)}(z) \quad (1.1)$$

where  $\forall k, a_k \in \mathbb{C}, a_2 \neq 0$ , and  $f : \mathbb{C} \rightarrow \mathbb{C}$  is differentiable infinitely many times.

Let

$$\lambda_0 = \frac{-a_1 - \sqrt{a_1^2 - 4a_0a_2}}{2a_2}, \lambda_1 = \frac{-a_1 + \sqrt{a_1^2 - 4a_0a_2}}{2a_2} \quad (1.2)$$

## 2 Case $\lambda_0 \neq \lambda_1$

$$f(z) = c_0 \exp(\lambda_0 z) + c_1 \exp(\lambda_1 z), c_0 = \frac{f^{(1)}(0) - f(0)\lambda_1}{\lambda_0 - \lambda_1}, c_1 = \frac{f^{(1)}(0) - f(0)\lambda_0}{\lambda_1 - \lambda_0} \quad (2.1)$$

We can choose the basis of the set of solutions to be  $\{\exp(\lambda_0 z), \exp(\lambda_1 z)\}$ . Let

$$v_0(z) = \begin{pmatrix} \exp(\lambda_0 z) \\ \exp(\lambda_1 z) \end{pmatrix}, B_0 = \begin{pmatrix} c_0 & 0 \\ 0 & c_1 \end{pmatrix}, B_1 = \begin{pmatrix} c_0\lambda_0 & 0 \\ 0 & c_1\lambda_1 \end{pmatrix} \quad (2.2)$$

Then

$$(f(z_0 + z_1)) = v_0(z_0)^T B_0 v_0(z_1), (f^{(1)}(z_0 + z_1)) = v_0(z_0)^T B_1 v_0(z_1) \quad (2.3)$$

These facts are a bit too obvious. We want  $\{f(z), f^{(1)}(z)\}$  to be a basis of the set of solutions to get an awesome equation. Thus we require  $c_0 \neq 0, c_1 \neq 0$ .

Let

$$M = \begin{pmatrix} c_0 & c_1 \\ c_0\lambda_0 & c_1\lambda_1 \end{pmatrix}, v_1(z) = \begin{pmatrix} f(z) \\ f^{(1)}(z) \end{pmatrix} \quad (2.4)$$

Then

$$Mv_0(z) = v_1(z), M^{-1} = \frac{\begin{pmatrix} -c_1\lambda_1 & c_1 \\ c_0\lambda_0 & -c_0 \end{pmatrix}}{c_0c_1(\lambda_0 - \lambda_1)}, v_0(z) = M^{-1}v_1(z) \quad (2.5)$$

Thus, by substitution,

$$(f(z_0 + z_1)) = (M^{-1}v_1(z_0))^T B_0 M^{-1} v_1(z_1), (f^{(1)}(z_0 + z_1)) = (M^{-1}v_1(z_0))^T B_1 M^{-1} v_1(z_1) \quad (2.6)$$

Simplify:

$$(f(z_0 + z_1)) = \frac{\begin{pmatrix} f(z_0) \\ f^{(1)}(z_0) \end{pmatrix}^T \begin{pmatrix} c_0\lambda_0^2 + c_1\lambda_1^2 & -c_0\lambda_0 - c_1\lambda_1 \\ -c_0\lambda_0 - c_1\lambda_1 & c_0 + c_1 \end{pmatrix} \begin{pmatrix} f(z_1) \\ f^{(1)}(z_1) \end{pmatrix}}{c_0c_1(\lambda_0 - \lambda_1)^2} \quad (2.7)$$

$$(f^{(1)}(z_0 + z_1)) = \frac{\begin{pmatrix} f(z_0) \\ f^{(1)}(z_0) \end{pmatrix}^T \begin{pmatrix} \lambda_0\lambda_1(c_0\lambda_0 + c_1\lambda_1) & -\lambda_0\lambda_1(c_0 + c_1) \\ -\lambda_0\lambda_1(c_0 + c_1) & c_0\lambda_1 + c_1\lambda_0 \end{pmatrix} \begin{pmatrix} f(z_1) \\ f^{(1)}(z_1) \end{pmatrix}}{c_0c_1(\lambda_0 - \lambda_1)^2} \quad (2.8)$$

## 3 Case $\lambda_0 = \lambda_1 = \lambda$

$$f(z) = c_0 \exp(\lambda z) + c_1 z \exp(\lambda z), c_0 = f(0), c_1 = f^{(1)}(0) - f(0)\lambda \quad (3.1)$$

We can choose the basis of the set of solutions to be  $\{\exp(\lambda z), z \exp(\lambda z)\}$ . Let

$$v_0(z) = \begin{pmatrix} \exp(\lambda z) \\ z \exp(\lambda z) \end{pmatrix}, B_0 = \begin{pmatrix} c_0 & c_1 \\ c_1 & 0 \end{pmatrix}, B_1 = \begin{pmatrix} c_1 + c_0\lambda & c_1\lambda \\ c_1\lambda & 0 \end{pmatrix} \quad (3.2)$$

Then

$$(f(z_0 + z_1)) = v_0(z_0)^\top B_0 v_0(z_1), (f^{(1)}(z_0 + z_1)) = v_0(z_0)^\top B_1 v_0(z_1) \quad (3.3)$$

These facts are a bit too obvious. We want  $\{f(z), f^{(1)}(z)\}$  to be a basis of the set of solutions to get an awesome equation. Thus we require  $c_1 \neq 0$ .

Let

$$M = \begin{pmatrix} c_0 & c_1 \\ c_1 + c_0\lambda & c_1\lambda \end{pmatrix}, v_1(z) = \begin{pmatrix} f(z) \\ f^{(1)}(z) \end{pmatrix} \quad (3.4)$$

Then

$$Mv_0(z) = v_1(z), M^{-1} = \frac{\begin{pmatrix} -c_1\lambda & c_1 \\ c_1 + c_0\lambda & -c_0 \end{pmatrix}}{c_1^2}, v_0(z) = M^{-1}v_1(z) \quad (3.5)$$

Thus, by substitution,

$$(f(z_0 + z_1)) = (M^{-1}v_1(z_0))^\top B_0 M^{-1}v_1(z_1), (f^{(1)}(z_0 + z_1)) = (M^{-1}v_1(z_0))^\top B_1 M^{-1}v_1(z_1) \quad (3.6)$$

Simplify:

$$(f(z_0 + z_1)) = \frac{\begin{pmatrix} f(z_0) \\ f^{(1)}(z_0) \end{pmatrix}^\top \begin{pmatrix} -\lambda(2c_1 + c_0\lambda) & c_1 + c_0\lambda \\ c_1 + c_0\lambda & -c_0 \end{pmatrix} \begin{pmatrix} f(z_1) \\ f^{(1)}(z_1) \end{pmatrix}}{c_1^2} \quad (3.7)$$

$$(f^{(1)}(z_0 + z_1)) = \frac{\begin{pmatrix} f(z_0) \\ f^{(1)}(z_0) \end{pmatrix}^\top \begin{pmatrix} -\lambda^2(c_1 + c_0\lambda) & c_0\lambda^2 \\ c_0\lambda^2 & c_1 - c_0\lambda \end{pmatrix} \begin{pmatrix} f(z_1) \\ f^{(1)}(z_1) \end{pmatrix}}{c_1^2} \quad (3.8)$$