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Proofs of the Argument Sum Conjectures for Homogeneous Linear Differential Equations

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1 Introduction

We consider homogeneous linear differential equations with constant coefficients of the form:

$$0 = \sum_{k=0}^{n} a_k f^{(k)}(z), \quad a_k \in \mathbb{C}, \quad a_n \neq 0, \tag{1.1}$$

where $f^{(k)}(z)$ denotes the k-th derivative of $f(z)$.

The solution space of this equation is n-dimensional, and we let $\{g_0(z), g_1(z), \ldots, g_{n-1}(z)\}$ be a basis for this space.

Define the vector:

$$v(z) = \begin{bmatrix} g_0(z) \\ g_1(z) \\ \vdots \\ g_{n-1}(z) \end{bmatrix}. \tag{1.2}$$

The conjectures $\text{ArgSumCon}(m)$ for positive integers $m \in \mathbb{Z}_{>0}$ assert that for any solution f, there exist constants $c(k_0, \ldots, k_{m-1})$ such that:

$$f\left(\sum_{j=0}^{m-1} z_j\right) = \sum_{k_0=0}^{n-1} \cdots \sum_{k_{m-1}=0}^{n-1} c(k_0, \ldots, k_{m-1}) \prod_{j=0}^{m-1} g_{k_j}(z_j). \tag{1.3}$$

In this document, we prove $\text{ArgSumCon}(2)$ explicitly, generalize to $\text{ArgSumCon}(m)$ using induction, and justify the invertibility of any matrix inverted during the proofs.

2 Proof of ArgSumCon(2)

Theorem 2.1. For any solution f to the differential equation, there exists a unique symmetric matrix $A \in \mathbb{C}^{n \times n}$ such that:

$$f(z_0 + z_1) = v(z_1)^\intercal A v(z_0). \tag{2.1}$$

Proof. We proceed through a series of steps, using lemmas and claims to build the result.

Lemma 2.2. For any solution f and fixed $z_1 \in \mathbb{C}$, the function $h(z_0) := f(z_0 + z_1)$ is also a solution to the differential equation.

Proof. Since the differential equation has constant coefficients, compute the derivatives:

$$h^{(k)}(z_0) = f^{(k)}(z_0 + z_1). \tag{2.2}$$

Applying the differential operator:

$$\sum_{k=0}^{n} a_k h^{(k)}(z_0) = \sum_{k=0}^{n} a_k f^{(k)}(z_0 + z_1) = 0, \tag{2.3}$$

since f satisfies the original equation. Thus, $h(z_0)$ is a solution. □

Lemma 2.3. There exist infinitely differentiable functions $c_k : \mathbb{C} \to \mathbb{C}$ such that:

$$f(z_0 + z_1) = \sum_{k=0}^{n-1} c_k(z_1) g_k(z_0). \tag{2.4}$$

Proof. Fix z_1. Since $h(z_0) = f(z_0 + z_1)$ is a solution, and $\{g_0(z_0), \ldots, g_{n-1}(z_0)\}$ is a basis, we can express:

$$h(z_0) = \sum_{k=0}^{n-1} c_k(z_1) g_k(z_0), \tag{2.5}$$

where the coefficients $c_k(z_1)$ depend on z_1. To find these coefficients, consider the Wronskian matrix of the basis at $z_0 = 0$:

$$W(0) = \begin{bmatrix} g_0(0) & g_1(0) & \cdots & g_{n-1}(0) \\ g_0'(0) & g_1'(0) & \cdots & g_{n-1}'(0) \\ \vdots & \vdots & \ddots & \vdots \\ g_0^{(n-1)}(0) & g_1^{(n-1)}(0) & \cdots & g_{n-1}^{(n-1)}(0) \end{bmatrix}. \tag{2.6}$$

Since $\{g_0, \ldots, g_{n-1}\}$ are linearly independent solutions, $W(0)$ is invertible (justified in Section 4). Solve for the coefficients using initial conditions at $z_0 = 0$:

$$\begin{bmatrix} h(0) \\ h'(0) \\ \vdots \\ h^{(n-1)}(0) \end{bmatrix} = \begin{bmatrix} f(z_1) \\ f'(z_1) \\ \vdots \\ f^{(n-1)}(z_1) \end{bmatrix} = W(0) \begin{bmatrix} c_0(z_1) \\ c_1(z_1) \\ \vdots \\ c_{n-1}(z_1) \end{bmatrix}. \tag{2.7}$$

Thus:

$$\begin{bmatrix} c_0(z_1) \\ c_1(z_1) \\ \vdots \\ c_{n-1}(z_1) \end{bmatrix} = W(0)^{-1} \begin{bmatrix} f(z_1) \\ f'(z_1) \\ \vdots \\ f^{(n-1)}(z_1) \end{bmatrix}. \tag{2.8}$$

Since f is infinitely differentiable (as a solution to a linear ODE with constant coefficients), each $c_k(z_1)$ is infinitely differentiable. □

Theorem 2.4. Each coefficient $c_k(z_1)$ satisfies the differential equation:

$$\sum_{j=0}^{n} a_j c_k^{(j)}(z_1) = 0. \tag{2.9}$$

Proof. Differentiate $f(z_0 + z_1) = \sum_{k=0}^{n-1} c_k(z_1) g_k(z_0)$ with respect to z_1:

$$\frac{\partial^j}{\partial z_1^j} f(z_0 + z_1) = \sum_{k=0}^{n-1} c_k^{(j)}(z_1) g_k(z_0). \tag{2.10}$$

Since $f(z_0 + z_1)$ satisfies the differential equation in $z_0 + z_1$, apply the operator with respect to z_1 (noting $\frac{\partial}{\partial z_1} f = f'$):

$$0 = \sum_{j=0}^{n} a_j \frac{\partial^j}{\partial z_1^j} f(z_0 + z_1) = \sum_{j=0}^{n} a_j \sum_{k=0}^{n-1} c_k^{(j)}(z_1) g_k(z_0) = \sum_{k=0}^{n-1} \left(\sum_{j=0}^{n} a_j c_k^{(j)}(z_1) \right) g_k(z_0). \tag{2.11}$$

Since the $g_k(z_0)$ are linearly independent, each coefficient must vanish:

$$\sum_{j=0}^{n} a_j c_k^{(j)}(z_1) = 0, \tag{2.12}$$

proving that each $c_k(z_1)$ is a solution. □

Lemma 2.5. There exist constants $d_{km} \in \mathbb{C}$ such that:

$$c_k(z_1) = \sum_{m=0}^{n-1} d_{km} g_m(z_1). \tag{2.13}$$

Proof. Since $c_k(z_1)$ satisfies the differential equation and $\{g_0(z_1), \ldots, g_{n-1}(z_1)\}$ is a basis for the solution space, we can write:

$$c_k(z_1) = \sum_{m=0}^{n-1} d_{km} g_m(z_1), \tag{2.14}$$

where the d_{km} are constants because the differential equation has constant coefficients. $\square$

Now, substitute into the expression for f:

$$f(z_0 + z_1) = \sum_{k=0}^{n-1} c_k(z_1) g_k(z_0) = \sum_{k=0}^{n-1} \left(\sum_{m=0}^{n-1} d_{km} g_m(z_1) \right) g_k(z_0) = \sum_{k=0}^{n-1} \sum_{m=0}^{n-1} d_{km} g_m(z_1) g_k(z_0). \tag{2.15}$$

Define the matrix D with entries $D_{mk} = d_{km}$, so:

$$f(z_0 + z_1) = v(z_1)^\intercal D v(z_0). \tag{2.16}$$

Since $f(z_0 + z_1) = f(z_1 + z_0)$, we have:

$$v(z_1)^\intercal D v(z_0) = v(z_0)^\intercal D v(z_1) = v(z_1)^\intercal D^\intercal v(z_0). \tag{2.17}$$

This holds for all z_0, z_1, so $D = D^\intercal$. Set $A = D$, which is symmetric. Thus:

$$f(z_0 + z_1) = v(z_1)^\intercal A v(z_0). \tag{2.18}$$

For uniqueness, suppose there exist two symmetric matrices A and B such that $v(z_1)^\intercal A v(z_0) = v(z_1)^\intercal B v(z_0)$ for all z_0, z_1. Then:

$$v(z_1)^\intercal (A - B) v(z_0) = 0. \tag{2.19}$$

Since the $g_k(z)$ span the solution space, and thus $v(z_0), v(z_1)$ can take on independent values, $A - B = 0$, so $A = B$. Hence, A is unique. $\square$

3 Proof of ArgSumCon(m)

Theorem 3.1. For any positive integer m, ArgSumCon(m) holds: there exist constants $c(k_0, \ldots, k_{m-1})$ such that:

$$f\left(\sum_{j=0}^{m-1} z_j\right) = \sum_{k_0=0}^{n-1} \cdots \sum_{k_{m-1}=0}^{n-1} c(k_0, \ldots, k_{m-1}) \prod_{j=0}^{m-1} g_{k_j}(z_j). \tag{3.1}$$

Proof. We prove this by induction on m.
Base Case ($m = 1$): For any solution $f(z_0)$, since $\{g_0(z_0), \ldots, g_{n-1}(z_0)\}$ is a basis:

$$f(z_0) = \sum_{k_0=0}^{n-1} c(k_0) g_{k_0}(z_0), \tag{3.2}$$

where the $c(k_0)$ are constants. Thus, ArgSumCon(1) holds.
Inductive Step: Assume ArgSumCon(m) holds for some $m \geq 1$. We show it holds for $m + 1$. Consider:

$$f\left(\sum_{j=0}^{m} z_j\right). \tag{3.3}$$

Define $u = \sum_{j=0}^{m-1} z_j$, so:

$$f\left(\sum_{j=0}^{m} z_j\right) = f(u + z_m). \tag{3.4}$$

By ArgSumCon(2) (Theorem 1), there exists a symmetric matrix A such that:

$$f(u + z_m) = v(z_m)^\intercal A v(u). \tag{3.5}$$

Now, $v(u) = \begin{bmatrix} g_0(u) \\ \vdots \\ g_{n-1}(u) \end{bmatrix}$, where $u = \sum_{j=0}^{m-1} z_j$. Since each $g_l(u)$ is a solution, by the inductive hypothesis:

$$g_l(u) = g_l\left(\sum_{j=0}^{m-1} z_j\right) = \sum_{k_0=0}^{n-1} \cdots \sum_{k_{m-1}=0}^{n-1} d_l(k_0, \ldots, k_{m-1}) \prod_{j=0}^{m-1} g_{k_j}(z_j), \tag{3.6}$$

for some constants $d_l(k_0, \ldots, k_{m-1})$. Substitute into the expression:

$$f(u + z_m) = v(z_m)^\intercal A v(u) = \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} A_{kl} g_k(z_m) g_l(u) \tag{3.7}$$

$$= \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} A_{kl} g_k(z_m) \left[\sum_{k_0=0}^{n-1} \cdots \sum_{k_{m-1}=0}^{n-1} d_l(k_0, \ldots, k_{m-1}) \prod_{j=0}^{m-1} g_{k_j}(z_j)\right]. \tag{3.8}$$

Rearrange the summations:

$$f\left(\sum_{j=0}^{m} z_j\right) = \sum_{k_0=0}^{n-1} \cdots \sum_{k_{m-1}=0}^{n-1} \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} A_{kl} d_l(k_0, \ldots, k_{m-1}) g_k(z_m) \prod_{j=0}^{m-1} g_{k_j}(z_j). \tag{3.9}$$

Let $k_m = k$, and define:

$$c(k_0, \ldots, k_m) = \sum_{l=0}^{n-1} A_{kl_m} d_l(k_0, \ldots, k_{m-1}), \tag{3.10}$$

which are constants. Then:

$$f\left(\sum_{j=0}^{m} z_j\right) = \sum_{k_0=0}^{n-1} \cdots \sum_{k_{m-1}=0}^{n-1} \sum_{k_m=0}^{n-1} \left(\sum_{l=0}^{n-1} A_{k_m l} d_l(k_0, \ldots, k_{m-1})\right) g_{k_m}(z_m) \prod_{j=0}^{m-1} g_{k_j}(z_j) \tag{3.11}$$

$$= \sum_{k_0=0}^{n-1} \cdots \sum_{k_m=0}^{n-1} c(k_0, \ldots, k_m) \prod_{j=0}^{m} g_{k_j}(z_j), \tag{3.12}$$

proving ArgSumCon$(m+1)$. By induction, ArgSumCon(m) holds for all $m \geq 1$. □

4 Justification of Matrix Invertibility

In the proof of ArgSumCon(2), we inverted the Wronskian matrix $W(0)$. Here, we justify its invertibility.

Lemma 4.1. The Wronskian matrix $W(z)$ of the basis $\{g_0(z), g_1(z), \ldots, g_{n-1}(z)\}$ is invertible for some $z \in \mathbb{C}$, including $z = 0$.

Proof. The Wronskian matrix at z is:

$$W(z) = \begin{bmatrix} g_0(z) & g_1(z) & \cdots & g_{n-1}(z) \\ g_0'(z) & g_1'(z) & \cdots & g_{n-1}'(z) \\ \vdots & \vdots & \ddots & \vdots \\ g_0^{(n-1)}(z) & g_1^{(n-1)}(z) & \cdots & g_{n-1}^{(n-1)}(z) \end{bmatrix}. \tag{4.1}$$

The determinant $\det W(z)$ is the Wronskian of the functions $g_0, \ldots, g_{n-1}$. Since these are linearly independent solutions to an n-th order linear differential equation, $\det W(z) \neq 0$ for some z. For constant-coefficient equations, if the characteristic equation has distinct roots, the solutions (e.g., $e^{\lambda_i z}$) ensure $\det W(z) \neq 0$ for all z. Even with repeated roots, using solutions like $z^k e^{\lambda z}$, the Wronskian is non-zero at $z = 0$. Thus, $W(0)$ is invertible. □

This completes the justification of all matrix inversions in the proofs.