Complex Number Law Of Sines

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Abstract

The goal of this paper is to appreciate The Law of Sines rewritten in the form of complex numbers.

0.1 The obvious single variable case where $z \in \mathbb{C}, \neg[z \in \{0,1\}]$:

$$\frac{\sin(\operatorname{Im}(\ln(z)))}{|1-z|} = \frac{\sin(\operatorname{Im}(\ln(1-\bar{z})))}{|z|} = \sin(\operatorname{Im}(\ln(z) + \ln(1-\bar{z}))) \tag{0.1}$$

For $x \in \mathbb{R}$,

$$\sin(\operatorname{Im}(\ln(z) + \ln(1-\bar{z}))) = \sin\left(\operatorname{Im}\left(\ln\left(z - |z|^2\right)\right)\right) = \sin\left(\operatorname{Im}\left(\ln\left(z|z|^x - |z|^{2+x}\right)\right)\right) \tag{0.2}$$

A nice substitution is $x \to -2$:

$$\sin\left(\operatorname{Im}\left(\ln\left(z|z|^x - |z|^{2+x}\right)\right)\right) = \sin\left(\operatorname{Im}\left(\ln\left(z|z|^{-2} - 1\right)\right)\right) = \sin\left(\operatorname{Im}\left(\ln\left(\bar{z}^{-1} - 1\right)\right)\right) \tag{0.3}$$

0.2 The distinct 3 variable case where $\{z_0, z_1, z_2\} \subset \mathbb{C}, z_0 \neq z_1, z_1 \neq z_2, z_2 \neq z_0$:

$$\frac{\sin\left(\operatorname{Im}\left(\ln\left(\frac{z_2-z_0}{z_1-z_0}\right)\right)\right)}{|z_2-z_1|} = \frac{\sin\left(\operatorname{Im}\left(\ln\left(\frac{z_0-z_1}{z_2-z_1}\right)\right)\right)}{|z_0-z_2|} = \frac{\sin\left(\operatorname{Im}\left(\ln\left(\frac{z_1-z_2}{z_0-z_2}\right)\right)\right)}{|z_1-z_0|} \tag{0.4}$$

is a formula with a symmetry rotating of the 3 variables, that is, substituting $z_0 \to z_1, z_1 \to z_2, z_2 \to z_0$. Furthermore, swapping 2 variables negates the value, so

$$\left|\frac{\sin\left(\operatorname{Im}\left(\ln\left(\frac{z_2-z_0}{z_1-z_0}\right)\right)\right)}{z_2-z_1}\right| \tag{0.5}$$

is invariant under any permutation of the 3 variables.