n-sided Hyperbolic Squares
https://github.com/Nazgand/nazgandMathBook

Mark Andrew Gerads: Nazgand@Gmail.Com

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Abstract

The goal of this paper is to analyze the properties (such as edge length, incircle radius, circumcircle radius) of an n-sided hyperbolic square, meaning a regular polygon with n right angles. These polygons can tile the hyperbolic plane, which looks nice plotted with the Cartesian Hyperbolic Plane Metric.

1 Calculations

Let the Gaussian curvature of the plane be -1.

The regular polygon with n right angles can be partitioned into $2n$ right triangles with angles $\frac{\pi}{n}, \frac{\pi}{2}, \frac{\pi}{4}$.

The radius of the circumcircle is $\operatorname{arcosh}\left(\cot\left(\frac{\pi}{n}\right)\cot\left(\frac{\pi}{4}\right)\right) = \operatorname{arcosh}\left(\cot\left(\frac{\pi}{n}\right)\right)$.

The radius of the incircle is $\operatorname{arcosh}\left(\frac{\cos\left(\frac{\pi}{4}\right)}{\sin\left(\frac{\pi}{n}\right)}\right)$.

The length of an edge is $2\operatorname{arcosh}\left(\frac{\cos\left(\frac{\pi}{n}\right)}{\sin\left(\frac{\pi}{4}\right)}\right)$.

The length of the perimeter is $2n\operatorname{arcosh}\left(\frac{\cos\left(\frac{\pi}{n}\right)}{\sin\left(\frac{\pi}{4}\right)}\right)$.

The area is $2n\left(\pi - \frac{\pi}{n} - \frac{\pi}{2} - \frac{\pi}{4}\right) = 2\pi n\left(\frac{1}{4} - \frac{1}{n}\right) = \frac{n\pi}{2} - 2\pi$.

It should be noted that as $n \to \infty$, the edge length approaches $2\operatorname{arcosh}\left(\sqrt{2}\right) \approx 1.762747174039086050465218649959584618 06$, where both the incircle and circumcircle are horocycles, and above which any edge length is allowed.

n	radius of circumcircle
5	0.84248208146200745911138094114971121531 0
6	1.1462158347805888439003936556740077158 1
7	1.3600518497395676503853961090253538741 7
8	1.5285709194809981612724561847936733932 9
9	1.6689337951193519141213601576983910737 7
10	1.7898204100717151674059126017748452986 6
11	1.8963075391511722988378671354335976435 3
12	1.9916523910494368240689966752859269541 5
13	2.0780842706552911282467426472399857171 4
14	2.1572037062542354747056089761042225068 9
15	2.2302029424582838507578563963468639908 4

n	radius of incircle
5	0.6268696629061778141444633762119364014 78
6	0.8813735870195430252326093249797923090 28
7	1.0704048615589441843313723516402924545 5
8	1.2242262238390378950026549568179345701 6
9	1.3550485187968718315539351534893711624 3
10	1.4693517443681852732558443173616476167 9
11	1.5710885800372406793563230176555841623 9
12	1.6628858910586210756524850390794060595 3
13	1.7465947045204493738113282549931232172 5
14	1.8235763569590883970521249603462455105 8
15	1.8948653908657310202571835516019447860 0

n	length of an edge
5	1.06127506190503565203301891621357348581
6	1.31695789692481670862504634730796844403
7	1.44907472267758633503217314325772678247
8	1.52857091948099816127245618479367339329
9	1.58069813785651414369050575393040896418
10	1.61692166751188651380348666582235462921
11	1.64319223747457343054577404285923637279
12	1.66288589105862107565248503907940605953
13	1.67804671290710848454907520339943861187
14	1.68997616809339866171304910723600185759
15	1.69953703096172053759430146589327334224