Argument Sum Rules From Homogeneous Linear Differential Equations Of Constant Coefficients Conjecture
https://github.com/Nazgand/nazgandMathBook

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Abstract

The goal of this paper is to conjecture a seemingly fundamental calculus fact.

A [homogeneous linear differential equation of constant coefficients] has the form

$$0 = \sum_{k=0}^{n} a_k \frac{\partial^k}{\partial z^k} f(z) = \sum_{k=0}^{n} a_k f^{(k)}(z) \tag{0.1}$$

where $a_k \in \mathbb{C}, f : \mathbb{C} \to \mathbb{C}$. Let $f_1(z), \ldots, f_n(z)$ be solutions that span the vector space of solutions of the [homogeneous linear differential equation of constant coefficients]. To be clear:

$$\left\{ f(z) \mid 0 = \sum_{k=0}^{n} a_k f^{(k)}(z) \right\} = \left\{ \sum_{k=1}^{n} b_k f_k(z) \mid b_k \in \mathbb{C} \right\} \tag{0.2}$$

Let us define a column vector and clarify its transpose (a row vector):

$$v(z_0) = \begin{pmatrix} f_1(z_0) \\ \vdots \\ f_n(z_0) \end{pmatrix}, v(z_1)^T = \begin{pmatrix} f_1(z_1) & \ldots & f_n(z_1) \end{pmatrix} \tag{0.3}$$

I conjecture there exists a complex-valued constant n by n symmetric matrix A ($A = A^T$) such that

$$f(z_0 + z_1) = v(z_1)^T A v(z_0) = v(z_0)^T A v(z_1) \tag{0.4}$$

1 A reason A is symmetric

Suppose instead of a symmetric matrix A, we find a matrix B such that

$$f(z_0 + z_1) = v(z_1)^T B v(z_0) \tag{1.1}$$

Then take the transpose of the equation and substitute $z_0 \to z_1, z_1 \to z_0$, resulting in the following equation:

$$f(z_0 + z_1) = v(z_1)^T B^T v(z_0) \tag{1.2}$$

Average both equations:

$$f(z_0 + z_1) = v(z_1)^T \frac{B + B^T}{2} v(z_0) \tag{1.3}$$

Note that we can set $A = \frac{B+B^T}{2}$ because it is symmetric. $\square$