Trigonometry

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Abstract

The goal of this paper is to review trigonometry.

1 Geometric Foundations

In [Elementary Geometry and Trigonometry.kig] by moving the angle control point to the second quadrant and comparing, we can see

$$\sin\left(\theta + \frac{\pi}{2}\right) = (\cos(\theta)), \cos\left(\theta + \frac{\pi}{2}\right) = -(\sin(\theta)) \tag{1.1}$$

Moreover, we see as $\theta \to 0$

$$\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 0 \tag{1.2}$$

Comparing with [PythagoreanTheorem.png], we find

$$1 = \sin(\theta)^2 + \cos(\theta)^2 \tag{1.3}$$

From [SumOfAnglesSinCos.kig], we get

$$\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta) \tag{1.4}$$

And

$$\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) \tag{1.5}$$

2 Derivatives

Theorem 2.1.

$$\frac{\partial}{\partial x}\sin(x) = \cos(x) \tag{2.1}$$

Proof.

$$\frac{\partial}{\partial x}\sin(x) = \lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h} \tag{2.2}$$

Use (1.4)

$$\frac{\partial}{\partial x}\sin(x) = \lim_{h \to 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h} \tag{2.3}$$

Split the limit:

$$\frac{\partial}{\partial x}\sin(x) = \lim_{h \to 0} \frac{\cos(x)\sin(h)}{h} + \lim_{h \to 0} \frac{\sin(x)(\cos(h) - 1)}{h} \tag{2.4}$$

Simplify

$$\frac{\partial}{\partial x}\sin(x) = \cos(x)\lim_{h \to 0} \frac{\sin(h)}{h} + \sin(x)\lim_{h \to 0} \frac{\cos(h) - 1}{h} \tag{2.5}$$

Use (1.2) on left limit and multiply right limit by $\frac{\cos(h)+1}{\cos(h)+1}$

$$\frac{\partial}{\partial x}\sin(x) = \cos(x) + \sin(x)\lim_{h \to 0} \frac{\cos(h)^2 - 1}{h(\cos(h) + 1)} \tag{2.6}$$

Use (1.3)

$$\frac{\partial}{\partial x}\sin(x) = \cos(x) + \sin(x)\lim_{h \to 0} \frac{\sin(h)^2}{h(\cos(h) + 1)} \tag{2.7}$$

Split the limit:

$$\frac{\partial}{\partial x}\sin(x) = \cos(x) + \sin(x)\lim_{a \to 0} \frac{\sin(a)}{h}\lim_{b \to 0}\sin(b)\lim_{c \to 0}\frac{1}{(\cos(c) + 1)} \tag{2.8}$$

Evaluate using (1.2) again

$$\frac{\partial}{\partial x}\sin(x) = \cos(x) + 1 * 0 * \frac{1}{2} = \cos(x) \tag{2.9}$$

□

Theorem 2.2.

$$\frac{\partial}{\partial x}\cos(x) = -\sin(x) \tag{2.10}$$

Proof. Use (1.1)

$$\frac{\partial}{\partial x}\cos(x) = \frac{\partial}{\partial x}\sin\left(x + \frac{\pi}{2}\right) \tag{2.11}$$

Use the chain rule:

$$\frac{\partial}{\partial x}\cos(x) = \left(\frac{\partial}{\partial z}\sin(z) : z \to x + \frac{\pi}{2}\right)\frac{\partial}{\partial x}\left(x + \frac{\pi}{2}\right) \tag{2.12}$$

Simplify

$$\frac{\partial}{\partial x}\cos(x) = \cos\left(x + \frac{\pi}{2}\right) \tag{2.13}$$

Use Use (1.1)

$$\frac{\partial}{\partial x}\cos(x) = -\sin(x) \tag{2.14}$$

$\square$

3 Taylor Series

Theorem 3.1.

$$\sin(x) = \sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{(2k+1)!} \tag{3.1}$$

Proof. The multiple derivatives cycle through 4 functions. Let $n \in \mathbb{Z}$:

$$\frac{\partial^{4n}}{\partial x^{4n}}\sin(x) = \sin(x) \tag{3.2}$$

$$\frac{\partial^{4n+1}}{\partial x^{4n+1}}\sin(x) = \cos(x) \tag{3.3}$$

$$\frac{\partial^{4n+2}}{\partial x^{4n+2}}\sin(x) = -\sin(x) \tag{3.4}$$

$$\frac{\partial^{4n+3}}{\partial x^{4n+3}}\sin(x) = -\cos(x) \tag{3.5}$$

The derivatives at 0 thus cycle through $0, 1, 0, -1$. Use [TaylorSeries(0.1)] and simplify out the zeroes. $\square$

Theorem 3.2.

$$\cos(x) = \sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}}{(2k)!} \tag{3.6}$$

Proof. The multiple derivatives cycle through 4 functions. Let $n \in \mathbb{Z}$:

$$\frac{\partial^{4n}}{\partial x^{4n}}\sin(x) = \cos(x) \tag{3.7}$$

$$\frac{\partial^{4n+1}}{\partial x^{4n+1}}\sin(x) = -\sin(x) \tag{3.8}$$

$$\frac{\partial^{4n+2}}{\partial x^{4n+2}}\sin(x) = -\cos(x) \tag{3.9}$$

$$\frac{\partial^{4n+3}}{\partial x^{4n+3}}\sin(x) = \sin(x) \tag{3.10}$$

The derivatives at 0 thus cycle through $1, 0, -1, 0$. Use [TaylorSeries(0.1)] and simplify out the zeroes. $\square$