Logarithms

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1 Logarithms

Definition 1.1. The logarithm functions are the inverse functions of the exponentiation functions:

$$[\{b, x\} \subset \mathbb{C} \land b \neq 0] \Rightarrow b^{\log_b(x)} = x \tag{1.1}$$

$$[\{b, x\} \subset \mathbb{R} \land b > 0] \Rightarrow \log_b(b^x) = x \tag{1.2}$$

$$\log_e(x) = \ln(x) \tag{1.3}$$

Theorem 1.2 (Power rule).

$$[\{b, x\} \subset \mathbb{R} \land b > 0] \Rightarrow \log_b(x^p) = p * \log_b(x) \tag{1.4}$$

Proof. Start with (1.1):

$$b^{\log_b(x)} = x \tag{1.5}$$

Raise both sides to the power of p:

$$b^{p*\log_b(x)} = x^p \tag{1.6}$$

Apply the $\log_b$ function to both sides, simplifying with (1.2):

$$p * \log_b(x) = \log_b(x^p) \tag{1.7}$$

□

Theorem 1.3 (Product sum rule).

$$\log_b(x * y) = \log_b(x) + \log_b(y) \tag{1.8}$$

Proof. Use (1.1)

$$\log_b(x * y) = \log_b\left(b^{\log_b(x)} * b^{\log_b(x)}\right) \tag{1.9}$$

Use the additive property of exponential functions

$$\log_b(x * y) = \log_b\left(b^{\log_b(x)+\log_b(x)}\right) \tag{1.10}$$

Use (1.1)

$$\log_b(x * y) = \log_b(x) + \log_b(x) \tag{1.11}$$

□

Theorem 1.4 (Change of base).

$$\log_b(x) = \frac{\log_k(x)}{\log_k(b)} \tag{1.12}$$

Proof. Start with (1.1):

$$b^{\log_b(x)} = x \tag{1.13}$$

Apply the $\log_k$ function to both sides:

$$\log_k\left(b^{\log_b(x)}\right) = \log_k(x) \tag{1.14}$$

$$\log_b(x)\log_k(b) = \log_k(x) \tag{1.15}$$

Divide both sides by $\log_k(b)$ □

Theorem 1.5 (Integral form).

$$z \in \mathbb{R}^+ \Rightarrow \ln(z) = \int_1^z \frac{1}{x}\, \mathrm{d}x \tag{1.16}$$

Proof. Proof from https://math.stackexchange.com/q/1341995. The derivative of the logarithm functions exist because the exponential functions have derivatives. https://en.wikipedia.org/wiki/Inverse_function_theorem

$$\frac{\partial}{\partial x} x = 1 \tag{1.17}$$

Substitute (1.1):

$$\frac{\partial}{\partial x} e^{\ln(x)} = 1 \tag{1.18}$$

The chain rule [Differentiation(4.1)]

$$\left(\frac{\partial}{\partial z} e^z \Big|_{z \to \ln(x)} \right) \frac{\partial}{\partial x} \ln(x) = 1 \tag{1.19}$$

The derivative of the natural exponential function is itself: [ExponentialFunction(2.7)]

$$\left(e^z \Big|_{z \to \ln(x)} \right) \frac{\partial}{\partial x} \ln(x) = 1 \tag{1.20}$$

Substitute:

$$e^{\ln(x)} \frac{\partial}{\partial x} \ln(x) = 1 \tag{1.21}$$

Simplify:

$$x \frac{\partial}{\partial x} \ln(x) = 1 \tag{1.22}$$

Divide:

$$\frac{\partial}{\partial x} \ln(x) = \frac{1}{x} \tag{1.23}$$

Integrate:

$$\ln(z) = \ln(z) - \ln(1) = \int_1^z \frac{1}{x} \, \mathrm{d}x \tag{1.24}$$

□

https://en.wikipedia.org/wiki/Logarithm