

# $L^p$ Planar Algebraic Geometry

<https://github.com/Nazgand/nazgandMathBook>

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## Abstract

### 1 The $L^p$ norm for $p \in \mathbb{R}, p \geq 1$

The distance between points  $(x_1, y_1)$  and  $(x_2, y_2)$  is:

$$\sqrt[p]{|x_1 - x_2|^p + |y_1 - y_2|^p} \quad (1)$$

This has the triangle inequality:

$$\sqrt[p]{|x_1 - x_2|^p + |y_1 - y_2|^p} + \sqrt[p]{|x_2 - x_3|^p + |y_2 - y_3|^p} \geq \sqrt[p]{|x_1 - x_3|^p + |y_1 - y_3|^p} \quad (2)$$

Euclidean translations and scalings of objects preserves shape in  $L^p$  space. Euclidean rotations do not generally preserve shape.

### 2 Lengths of curves

A curve defined by a function  $y = f(x)$  between  $a$  and  $b$  has the length

$$\lim_{\Delta x \rightarrow 0+} \sum_{k=0}^{\lfloor (b-a)/\Delta x \rfloor} \sqrt[p]{(\Delta x)^p + |f(a + (k+1)\Delta x) - f(a + k\Delta x)|^p} \quad (3)$$

where the limit exists. This is equivalent to

$$\int_a^b \sqrt[p]{1 + |f'(x)|^p} dx \quad (4)$$

In polar coordinates,

$$\int_a^b \sqrt[p]{|\cos(\theta)r'(\theta) - \sin(\theta)r(\theta)|^p + |\sin(\theta)r'(\theta) + \cos(\theta)r(\theta)|^p} d\theta \quad (5)$$

$$\int_a^b |r(\theta)| \sqrt[p]{\left| \cos(\theta) \frac{r'(\theta)}{r(\theta)} - \sin(\theta) \right|^p + \left| \sin(\theta) \frac{r'(\theta)}{r(\theta)} + \cos(\theta) \right|^p} d\theta \quad (6)$$

### 3 Area

If for all disjoint regions  $R_0, R_1$ , if  $\text{Area}(R_0) + \text{Area}(R_1) = \text{Area}(R_0 \cup R_1)$  and for all  $R_2$  which is a Euclidean translation of  $R_0$ ,  $\text{Area}(R_0) = \text{Area}(R_2)$ , then Area needs to be proportional to Euclidean area. Thus Euclidean area is used.

### 4 Circles

In the  $L^p$  norm, a circle which is the set of points distance  $r_p$  from the point  $(x_0, y_0)$  is.

$$|x - x_0|^p + |y - y_0|^p = r_p^p \quad (7)$$

The shape of circles is scale invariant:

$$\left| \frac{x}{r_p} \right|^p + \left| \frac{y}{r_p} \right|^p = 1 \quad (8)$$

Thus this work concentrates the unit circle in the first quadrant  $x, y \in \mathbb{R}^+$ , the other quadrants are reflections and not needing absolute value signs is nice during calculation. Thus what will be used is:

$$x^p + y^p = 1 \quad (9)$$

An important point for symmetry:  $x = y = 2^{(-1/p)}$ .

Let  $\tau_p$  be the circumference of a unit circle in  $L^p$ .  $\theta_p(\theta)$  is the function that translate the Euclidean angle to the  $L^p$  angle.

$$\theta_p(\theta) = -\theta_p(-\theta) = \theta_p\left(\theta + \frac{\pi}{2}\right) - \frac{\tau_p}{4} \quad (10)$$

In the first quadrant of the unit circle,

$$1 \geq x^p \Rightarrow y = \sqrt[p]{1 - x^p} \quad (11)$$

Area is important. Ratios between area and distance keep constant as do angles for scaling centered at the origin and rotations in  $L^p$  space. For  $p > 0$ ,

$$\int_0^1 (1 - x^p)^{1/p} dx = \frac{\Gamma\left(1 + \frac{1}{p}\right)^2}{\Gamma\left(1 + \frac{2}{p}\right)} \quad (12)$$

and polar coordinates with respect to the Euclidean angle and Euclidean norm can be found via substituting  $x = r_p \cos(\theta)$ ,  $y = r_p \sin(\theta)$

$$1 = r_p^p (|\cos(\theta)|^p + |\sin(\theta)|^p) \wedge r_p = (|\cos(\theta)|^p + |\sin(\theta)|^p)^{-1/p} \quad (13)$$

In the first quadrant:

$$0 \leq \theta \leq \frac{\pi}{2} \Rightarrow r_p(\theta) = (\cos(\theta)^p + \sin(\theta)^p)^{-1/p} \quad (14)$$

$$\frac{r'(\theta)}{r(\theta)} = \frac{\tan(\theta) \cos(\theta)^p - \cot(\theta) \sin(\theta)^p}{\cos(\theta)^p + \sin(\theta)^p} \quad (15)$$

Using (6),

$$\tau_p = 4 \int_0^{\pi/2} \sqrt[p]{(\cos(\theta)^p + \sin(\theta)^p) \left( \left| \cos(\theta) \frac{\tan(\theta) \cos(\theta)^p - \cot(\theta) \sin(\theta)^p}{\cos(\theta)^p + \sin(\theta)^p} - \sin(\theta) \right|^p + \left| \sin(\theta) \frac{\tan(\theta) \cos(\theta)^p - \cot(\theta) \sin(\theta)^p}{\cos(\theta)^p + \sin(\theta)^p} + \cos(\theta) \right|^p \right)} d\theta \quad (16)$$

An important function is a scaling function based on the Euclidean angle.

$$s_p(\theta) = (|\cos(\theta)|^p + |\sin(\theta)|^p)^{1/p} \quad (17)$$

In the  $L^p$  norm, a formula for arc length is

$$\int_a^b s_p \left( \arctan \left( \frac{\partial y}{\partial x} \right) \right) \sqrt{1 + \left( \frac{\partial y}{\partial x} \right)^2} dx \quad (18)$$

In the  $L^p$  norm, a formula for arc length in Euclidean polar coordinates is

$$\int_a^b s_p \left( \arctan \left( \frac{\sin(\theta) \frac{\partial r}{\partial \theta} + r \cos(\theta)}{\cos(\theta) \frac{\partial r}{\partial \theta} - r \sin(\theta)} \right) \right) \sqrt{r^2 + \left( \frac{\partial r}{\partial \theta} \right)^2} d\theta \quad (19)$$

The circumference can be calculated:

$$\frac{\partial y}{\partial x} = \frac{\partial}{\partial x} \sqrt[p]{1 - x^p} = \frac{1}{p} (1 - x^p)^{\frac{1}{p}-1} * -px^{p-1} = -(1 - x^p)^{\frac{1}{p}-1} x^{p-1} \quad (20)$$

$$\tau_p = 4 \int_0^1 \sqrt[p]{1 + \left| (1 - x^p)^{\frac{1}{p}-1} x^{p-1} \right|^p} dx = 4 \int_0^1 \sqrt[p]{1 + (1 - x^p)^{1-p} x^{p^2-p}} dx \quad (21)$$

$$\tau_p = 4 \int_0^1 \frac{\sqrt[p]{(1 - x^p) \left( (1 - x^p)^{p-1} + x^{p^2-p} \right)}}{1 - x^p} dx = \frac{4}{p} \int_0^1 \frac{\sqrt[p]{z(1 - z) \left( (1 - z)^{p-1} + z^{p-1} \right)}}{(1 - z)z} dz \quad (22)$$

$$\tau_p = \frac{8}{p} \int_0^{1/2} \frac{\sqrt[p]{z(1 - z) \left( (1 - z)^{p-1} + z^{p-1} \right)}}{(1 - z)z} dz \quad (23)$$

Case  $p = 3$ : Choose  $(1 - z)^2 + z^2 = y$ ,  $\frac{\partial y}{\partial z} = 4z - 2$ ,  $z(1 - z) = \frac{1-y}{2}$

$$\tau_p = \frac{16 \sqrt[3]{4}}{3} \int_{1/2}^1 y(y(1 - y))^{-2/3} \sqrt{2y - 1} dy \quad (24)$$

Choose  $z = \frac{1}{1+e^{-x}}$ ,  $\frac{\partial x}{\partial z} = \frac{1}{z(1-z)}$

$$\tau_p = \frac{8}{p} \int_0^\infty \sqrt[p]{\frac{e^x + e^{px}}{(1 + e^x)^{p+1}}} dx \quad (25)$$

Choose  $z = e^x$ ,  $\frac{\partial x}{\partial z} = \frac{1}{z}$

$$\tau_p = \frac{8}{p} \int_1^\infty \frac{1}{z(1 + z)} \sqrt[p]{\frac{z + z^p}{1 + z}} dz \quad (26)$$

Choose  $x = 1 + z^{p-1}$ ,  $(x-1)^{1/(p-1)} = z$ ,  $\frac{\partial x}{\partial z} = \frac{1}{z}$

$$\tau_p = \frac{8}{p} \int_1^\infty \frac{1}{z(1+z)} \sqrt[p]{\frac{z+z^p}{1+z}} dz \quad (27)$$

Some exact values:

$$\tau_3 = \frac{2}{3} \Gamma\left(\frac{1}{3}\right) \left( \frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{7}{12})} + \frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{13}{12})} \right), \tau_2 = 2\pi, \tau_1 = 8 \quad (28)$$

## 5 Regular polygons and circles around circles

Given a point at Euclidean angle  $\theta$  a function is desired which gives the angle for the point distance  $0 \leq d \leq 2$  counter-clockwise from the point at angle  $\theta$ , and nesting of this function is desired to construct regular polygons.

$$\theta \leq \text{dcc}(\theta, d, p, 1) \leq \theta + \pi \quad (29)$$

$$(\Delta x) = r_p(\text{dcc}(\theta, d, p, 1)) \cos(\text{dcc}(\theta, d, p, 1)) - r_p(\theta) \cos(\theta) \quad (30)$$

$$(\Delta y) = r_p(\text{dcc}(\theta, d, p, 1)) \sin(\text{dcc}(\theta, d, p, 1)) - r_p(\theta) \sin(\theta) \quad (31)$$

$$d = \sqrt[p]{(\Delta x)^p + (\Delta y)^p} \quad (32)$$

$$\text{dcc}(\theta, d, p, 0) = \theta \quad (33)$$

$$\text{dcc}(\theta, d, p, n) = \theta - \text{mod}\left(\theta, \frac{\pi}{2}\right) + \text{dcc}\left(\text{mod}\left(\theta, \frac{\pi}{2}\right), d, p, n\right) \quad (34)$$

$$= \text{dcc}(\text{dcc}(\theta, d, p, n-m), d, p, m) \quad (35)$$

$$= -\text{dcc}(-\theta, d, p, -n) \quad (36)$$

Interestingly, regular hexagons have a constant side regardless of angle and  $p$ .

$$\text{dcc}(\theta, 1, p, 6) = \theta + 2\pi \quad (37)$$

$$\text{dcc}(\theta, 2, p, 2) = \theta + 2\pi \quad (38)$$

## 6 Parabola for $p > 1$

Each point on a parabola has the same distance from the focus as from the directrix. With a directrix of  $y = 0$  and a focus at  $(0, 2a)$ ,  $a \in \mathbb{R}^+$ , we have  $f(x) = f(-x)$ ,  $f(0) = a$ ,  $f(2a) = 2a$ . Also,

$$|x|^p + |f(x) - 2a|^p = |f(x)|^p \quad (39)$$

$x \geq 0$  WLOG. 2 cases exist.  $f(x) \geq 2a$ :

$$x^p + (f(x) - 2a)^p = f(x)^p \quad (40)$$

and  $f(x) \leq 2a$ :

$$x^p + (2a - f(x))^p = f(x)^p \quad (41)$$

Both cases are equivalent when  $2|p$ .