The Riemann Hypothesis

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Abstract

The Riemann hypothesis is one of the most famous and important unsolved problems in mathematics. It is about the distribution of prime numbers and the properties of a complex function called the Riemann zeta function. The Riemann hypothesis states that all the non-trivial zeros of the zeta function have real part equal to $\frac{1}{2}$. In this document, we will introduce the zeta function, its functional equation, and some equivalent statements of the Riemann hypothesis.

1 The Riemann zeta function

The Riemann zeta function is defined as:

$$\begin{aligned} \zeta(s) &= \sum_{n=1}^{\infty} \frac{1}{n^s} \\ &= \prod_{p \text{ prime}} \frac{1}{1-p^{-s}}, \quad \text{for } \Re(s) > 1 \end{aligned} \tag{1}$$

In Equation 1, s is a complex variable and $\Re(s)$ is its real part. The zeta function can be extended to the whole complex plane (except for a simple pole at $s = 1$) by using a formula called the Euler product.

The zeta function satisfies a remarkable symmetry relation known as the functional equation, which relates the values of the zeta function at s and $1 - s$. The functional equation can be written as in Equation 2.

$$\xi(s) = \xi(1-s) \tag{2}$$

In Equation 2 $\xi(s)$ is a function derived from $\zeta(s)$ by multiplying it with some factors involving $\Gamma(s)$, the gamma function, and π^s, the power of pi. The exact form of $\xi(s)$ is not important for our purposes, but it can be found in any standard reference on the zeta function.

2 The Riemann hypothesis

A zero of the zeta function is a complex number s such that $\zeta(s) = 0$. The trivial zeros are the negative even integers, and they are easy to find. The non-trivial zeros are more mysterious, and their locations have deep implications for number theory.

The Riemann hypothesis states that all the non-trivial zeros of the zeta function have real part equal to $\frac{1}{2}$. In other words, if $\zeta(s) = 0$ and $s \neq -2n$ for any positive integer n, then $s = \frac{1}{2} + it$ for some real number t.

The Riemann hypothesis is equivalent to many other statements, such as the following:

- The error term in the prime number theorem is $O(\sqrt{x} \log x)$, where x is a positive real number and $\pi(x)$ is the number of primes less than or equal to x.
- The error term in the Dirichlet divisor problem is $O(x^{1/4+\epsilon})$, where x is a positive real number, $D(x)$ is the sum of the number of divisors of the integers up to x, and ϵ is any positive real number.
- The Mangoldt function $\Lambda(n)$, which is equal to $\log p$ if n is a power of a prime p and zero otherwise, satisfies the orthogonality relation:

$$\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} \overline{\left(\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{1-s}} \right)} = 0, \quad \text{for } \Re(s) > 1 \tag{3}$$

  In Equation 3, $\overline{z}$ denotes the complex conjugate of z.

These statements are just a few examples of the many consequences and connections of the Riemann hypothesis.

Tasks

1. Replicate at least 3 inline mathematical expressions
2. Replicate all the numbered display mode expressions
3. Replicate the references to the numbered equations