The Riemann Hypothesis

200042137

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Abstract

The Riemann hypothesis is one of the most famous and important unsolved problems in mathematics. It is about the distribution of prime numbers and the properties of a complex function called the Riemann zeta function. The Riemann hypothesis states that all the non-trivial zeros of the zeta function have real part equal to $\frac{1}{2}$. In this document, we will introduce the zeta function, its functional equation, and some equivalent statements of the Riemann hypothesis.

The Riemann zeta function is defined as:

$$\begin{aligned} \zeta(s) &= \sum_{n=1}^{\infty} \frac{1}{n^s} \\ &= \prod_{p \text{ prime}} \frac{1}{1-p^{-s}}, \quad \text{for } \Re(s) > 1 \end{aligned} \tag{1}$$

In Equation Equation 1, s is a complex variable

The zeta function satisfies a remarkable symmetry relation known as the functional equation, which relates the values of the zeta function at s and 1 s. The functional equation can be written as in Equation 2

$$\xi(s) = \xi(1-s) \tag{2}$$

The error term in the prime number theorem is $O\left(\sqrt{x} logx\right)$ where x is a positive $O\left(x^{1=4}\right)$

$$\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} \overline{\left(\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{1-s}} = 0\right)} = 0, \quad \text{for } \Re(s) > 1 \tag{3}$$

In Equation 3, z denotes the complex conjugate of z.