Portfolio Optimisation using Mean and Minimum Variance Optimisation Models

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Introduction:

The purpose of this assignment is to analyze and compare the out-of-sample performance of a mean-variance optimization model using historical data to estimate input parameters. We evaluate this model against a naive portfolio strategy, which equally weights all available assets. The analysis involves selecting an appropriate data set, choosing estimation windows for calculating mean returns and covariances, and conducting portfolio optimizations. By examining these strategies over different time horizons, we aim to derive insights into their effectiveness in varying market conditions.

Choosing the Data Set

Data Set: Europe 6 Portfolios Formed on ME and Investment (Developed Markets)

Source: Ken French's Data Library

The table below displays the sample dataset for the average value-weighted monthly returns for six different investment portfolios: SMALL LoINV, ME1 INV2, SMALL HiINV, BIG LoINV, ME2 INV2, and BIG HiINV, spanning from July 1990 till March 2024. This dataset offers insights into the respective returns and risk profiles over time, facilitating a comparative analysis of investment strategies across different periods and market conditions.

Table 1. Dataset Sample

Date	SMALL LoINV	ME1 INV2	SMALL HiINV	BIG LoINV	ME2 INV2	BIG HiINV
199007	5.73	5.33	4.7	6.02	4.72	4.85
199008	-9.52	-10.14	-10.74	-8.94	-10.8	-11.04
199009	-9.43	-9.75	-11.06	-11.1	-11.46	-13.02
199010	4.05	6.24	4.64	7.11	7.31	7.99
199011	-2.21	-2.07	-2.11	0.37	0.26	0.92
199012	-0.26	0.29	-0.93	-1.43	-0.8	-1.37

Reason for Choosing 6 European Portfolio Data Set:

1. **Geographic Focus:** The data set focuses on developed European markets, which are crucial for understanding the dynamics of mature economies. European markets represent a significant portion of global financial markets, providing a relevant and stable basis for portfolio analysis.

2. **Portfolio Composition:** It includes six portfolios based on market equity (ME) and investment (INV), allowing a detailed analysis of different investment strategies. This segmentation helps explore how firm size and investment levels impact returns and risk, facilitating a nuanced understanding of market behavior.

3. **Historical Depth:** Covering the period from July 1990 to the March 2024, the data set offers a long-term perspective on market trends and cycles. This extensive historical span is essential for identifying long-term trends, cyclical behaviors, and the impact of major financial events on portfolio performance.

Additionally, we are using average value weighted monthly returns because using average value-weighted monthly returns ensures the portfolio analysis accurately reflects market performance by giving more weight to larger companies, aligning with realistic investment practices. This approach helps in managing risk and assessing performance effectively.

Choosing Estimation Windows (M):

The choice to compare three different estimation windows (4 years, 8 years, and 12 years) is to understand how different historical periods and major events influence the financial markets and estimation of mean returns and covariance matrices, and consequently, the portfolio optimization results. This will highlight the trade-offs between responsiveness to recent market conditions and long-term stability, providing insights into optimal strategies for different investment horizons and market conditions. Through this comparison we can uncover major events such as Dot-Com Bubble (2000-2002), Global Financial Crisis (2008), Brexit Referendum (2016), COVID-19 Pandemic (2020-present) etc. impact on the market (Appendix A.2).

Estimation Windows:

1. 4 Years (48 months):

A 4-year window captures recent market trends and conditions. It reflects the latest economic events and shifts, making it relevant for investors adapting to current market dynamics. This window allows for a quick response to market changes but may capture short-term volatility, necessitating frequent portfolio adjustments. It is suitable for tactical asset allocation, where recent performance trends are prioritized. The impact of the Covid-19 and the Brexit can be prominently seen in this window, influencing European market behavior significantly.

2. 8 Years (96 months):

An 8-year window balances medium-term trends with reduced short-term noise. It encompasses several market cycles, providing more stable estimates of mean returns and covariances. This period smooths out short-term volatility, offering a reliable basis for long-term investment decisions. This window captures the aftermath of the Global Financial Crisis and the ongoing repercussions of the Brexit Referendum (2016) providing insights into medium-term recovery patterns.

3. 12 Years (144 months):

A 12-year window covers long-term market trends, minimizing the impact of short-term fluctuations. This window provides a comprehensive view of market performance across various economic cycles, including booms and recessions. It supports a long-term investment perspective, emphasizing stability and consistency over extended periods. This window spans multiple significant events, including the Global Financial Crisis, European Debt Crisis, and the onset of the COVID-19 pandemic, offering a holistic view of long-term market dynamics.

Comparison of Portfolio Strategies

The analysis involves comparing three portfolio strategies: mean-variance optimization, minimum-variance optimization, and the naive portfolio.

*1. **Mean-Variance Optimization***

This strategy aims to minimize the portfolio risk(volatility) for a given level of target return by using historical mean returns and covariance matrices to find the best risk-return trade-off. This approach offers the potential for higher returns with managed risk, but it is sensitive to estimation errors in the mean returns and covariances.

For this analysis, we are considering 0.72 percent monthly return as target return after calculating average monthly returns of the portfolios which is 0.72173663 for our value weighted monthly returns data. (Appendix A.1)

Mathematical Formulation for Mean-Variance Portfolio Optimization:

In this section, we focus on three different portfolio strategies: Mean-Variance, Minimum-Variance, and Naive.

Objective: Minimize portfolio volatility subject to a target expected return.

Formulation:

Let:

- w be the vector of portfolio weights, where;

  $\sum_{i=1}^{N} wi = 1 \; and \; wi \geq 0$

- μ be the vector of expected returns
- Σ be the covariance matrix of returns
- R_p be the target portfolio return

Objective Function:

$$\min \sqrt{w^T \Sigma w}$$

Subject to:

$$w^T \mu = R_p$$

$$\sum_{i=1}^{N} w_i = 1$$

$$w_i \geq 0 \ \forall i$$

Return Analysis:

Table 2. Mean-Variance Optimisation Performance Over 3 Windows

Window	Strategy	Mean Returns	Covariance Matrix	Weights	Return	Std Dev
4	Mean-Variance	{'SMALL LoINV': 1.2358, 'ME1 INV2': 1.3919, 'SMALL HiINV': 1.049, 'BIG LoINV': 1.2029, 'ME2 INV2': 1.3652, 'BIG HiINV': 1.4617}	[[34.578, 34.3542, 39.8516, 30.2217, 30.2036, 33.8814], [34.3542, 34.9559, 40.4661, 30.3048, 30.5539, 34.8063], [39.8516, 40.4661, 48.6404, 33.7868, 34.6713, 40.8991], [30.2217, 30.3048, 33.7868, 29.2092, 28.3966, 30.9234], [30.2036, 30.5539, 34.6713, 28.3966, 29.1022, 32.0665], [33.8814, 34.8063, 40.8991, 30.9234, 32.0665, 39.1433]]	[0.0, 0.0, 1.0, 0.0, 0.0, 0.0]	1.049	6.9743
8	Mean-Variance	{'SMALL LoINV': 0.5962, 'ME1 INV2': 0.7941, 'SMALL HiINV': 0.6053, 'BIG LoINV': 0.6316, 'ME2 INV2': 0.8436, 'BIG HiINV': 0.8243}	[[30.5564, 29.4258, 32.5373, 26.4023, 24.8782, 27.8695], [29.4258, 29.1023, 32.0537, 25.7048, 24.5456, 27.6249], [32.5373, 32.0537, 36.8757, 27.5273, 26.7988, 31.0533], [26.4023, 25.7048, 27.5273, 25.2087, 23.091, 25.1341], [24.8782, 24.5456, 26.7988, 23.091, 22.7789, 24.96], [27.8695, 27.6249, 31.0533, 25.1341, 24.96, 29.7923]]	[0.0, 0.0, 0.0, 0.5748, 0.4252, 0.0]	0.7217	4.8718
12	Mean-Variance	{'SMALL LoINV': 0.6786, 'ME1 INV2': 0.8266, 'SMALL HiINV': 0.6508, 'BIG LoINV': 0.5894, 'ME2 INV2': 0.7633, 'BIG HiINV': 0.7261}	[[27.0464, 25.4563, 27.524, 23.2775, 22.4269, 24.2247], [25.4563, 24.6392, 26.6289, 22.1838, 21.5974, 23.5573], [27.524, 26.6289, 30.0396, 23.3439, 23.0911, 25.9282], [23.2775, 22.1838, 23.3439, 22.8372, 21.39, 22.4846], [22.4269, 21.5974, 23.0911, 21.39, 21.5659, 22.7586], [24.2247, 23.5573, 25.9282, 22.4846, 22.7586, 25.9883]]	[0.0, 0.0, 0.0, 0.2392, 0.7608, 0.0]	0.7217	4.6448

4-Year Window:

- **Weights and Performance:** All weight assigned to 'SMALL HiINV' with a return of 1.049 and a standard deviation of 6.9743, indicating high volatility.

- **Reasoning:** High allocation to small-cap stocks suggests they had strong short-term performance, with higher risk assumed for higher returns.

8-Year Window:

- **Weights and Performance:** Diversified weights, significant allocations to 'BIG LoINV' and 'ME2 INV2', with a return of 0.7217 and a standard deviation of 4.8718.
- **Reasoning:** Balance between stable large-cap and mid-cap stocks captures moderate returns with lower risk, reflecting stabilized market conditions over a mid-term period.

12-Year Window:

- **Weights and Performance:** Higher allocations to 'ME2 INV2' and 'BIG LoINV', with a return of 0.7217 and a standard deviation of 4.6448, indicating stability.
- **Reasoning**: Long-term horizons favor diversified, stable portfolios with reduced volatility, reflecting smoother market fluctuations over time.

2. Minimum-Variance Optimization:

This strategy focuses on minimizing portfolio risk (volatility) without considering expected returns. By concentrating solely on reducing the overall portfolio variance, this strategy provides lower risk and more stability. However, the downside is that it may lead to lower expected returns compared to the mean-variance strategy.

Mathematical Formulation for Minimum-Variance Portfolio Optimization:

Objective: Minimize the portfolio's risk (variance) without a constraint on the expected return.

Formulation:

Objective function:

$$\min \sqrt{w^T \Sigma w}$$

Subject to:

$$\sum_{i=1}^{N} w_i = 1$$

$$w_i \geq 0 \; \forall i$$

Return Analysis:

Table 3. Minimum-Variance Optimisation Performance Over 3 Windows

Window	Strategy	Mean Returns	Covariance Matrix	Weights	Return	Std Dev
4	Minimum-Variance	{'SMALL LoINV': 1.2358, 'ME1 INV2': 1.3919, 'SMALL HiINV': 1.049, 'BIG LoINV': 1.2029, 'ME2 INV2': 1.3652, 'BIG HiINV': 1.4617}	[[34.578, 34.3542, 39.8516, 30.2217, 30.2036, 33.8814], [34.3542, 34.9559, 40.4661, 30.3048, 30.5539, 34.8063], [39.8516, 40.4661, 48.6404, 33.7868, 34.6713, 40.8991], [30.2217, 30.3048, 33.7868, 29.2092, 28.3966, 30.9234], [30.2036, 30.5539, 34.6713, 28.3966, 29.1022, 32.0665], [33.8814, 34.8063, 40.8991, 30.9234, 32.0665, 39.1433]]	[0.0, 0.0, 0.0, 0.4648, 0.5352, 0.0]	1.2898	5.3642
8	Minimum-Variance	{'SMALL LoINV': 0.5962, 'ME1 INV2': 0.7941, 'SMALL HiINV': 0.6053, 'BIG LoINV': 0.6316, 'ME2 INV2': 0.8436, 'BIG HiINV': 0.8243}	[[30.5564, 29.4258, 32.5373, 26.4023, 24.8782, 27.8695], [29.4258, 29.1023, 32.0537, 25.7048, 24.5456, 27.6249], [32.5373, 32.0537, 36.8757, 27.5273, 26.7988, 31.0533], [26.4023, 25.7048, 27.5273, 25.2087, 23.091, 25.1341], [24.8782, 24.5456, 26.7988, 23.091, 22.7789, 24.96], [27.8695, 27.6249, 31.0533, 25.1341, 24.96, 29.7923]]	[0.0, 0.0, 0.0, 0.0, 1.0, 0.0]	0.8436	4.7727
12	Minimum-Variance	{'SMALL LoINV': 0.6786, 'ME1 INV2': 0.8266, 'SMALL HiINV': 0.6508, 'BIG LoINV': 0.5894, 'ME2 INV2': 0.7633, 'BIG HiINV': 0.7261}	[[27.0464, 25.4563, 27.524, 23.2775, 22.4269, 24.2247], [25.4563, 24.6392, 26.6289, 22.1838, 21.5974, 23.5573], [27.524, 26.6289, 30.0396, 23.3439, 23.0911, 25.9282], [23.2775, 22.1838, 23.3439, 22.8372, 21.39, 22.4846], [22.4269, 21.5974, 23.0911, 21.39, 21.5659, 22.7586], [24.2247, 23.5573, 25.9282, 22.4846, 22.7586, 25.9883]]	[0.0, 0.0, 0.0, 0.1083, 0.8917, 0.0]	0.7445	4.6419

4-Year Window

- **Weights**: The weights are assigned to two portfolios: ME2 INV2 (46.48%) and BIG HiINV (53.52%), indicating a strong preference for these two portfolios.
- **Return**: The achieved return is 1.2898, which is higher than the target return set for mean-variance optimization.
- **Standard Deviation**: The standard deviation is 5.3642, indicating moderate risk relative to the returns.

8-Year Window

- **Weights**: The weights are heavily allocated to the BIG HiINV portfolio (100%), showing an exclusive preference for this portfolio.
- **Return**: The return is 0.8436, which is lower than the 4-year window but still reasonably high.
- **Standard Deviation**: The standard deviation is 4.7727, showing lower risk compared to the 4-year window, making it more attractive risk-return-wise.

12-Year Window

- **Weights**: The weights are mostly allocated to ME2 INV2 (10.83%) and BIG HiINV (89.17%), showing a strong preference for these portfolios.
- **Return**: The return is 0.7445, which is the lowest among the three windows.
- **Standard Deviation**: The standard deviation is 4.6419, which is the lowest among the three windows, indicating the least risk.

Reasoning for Asset Weights Allocation:

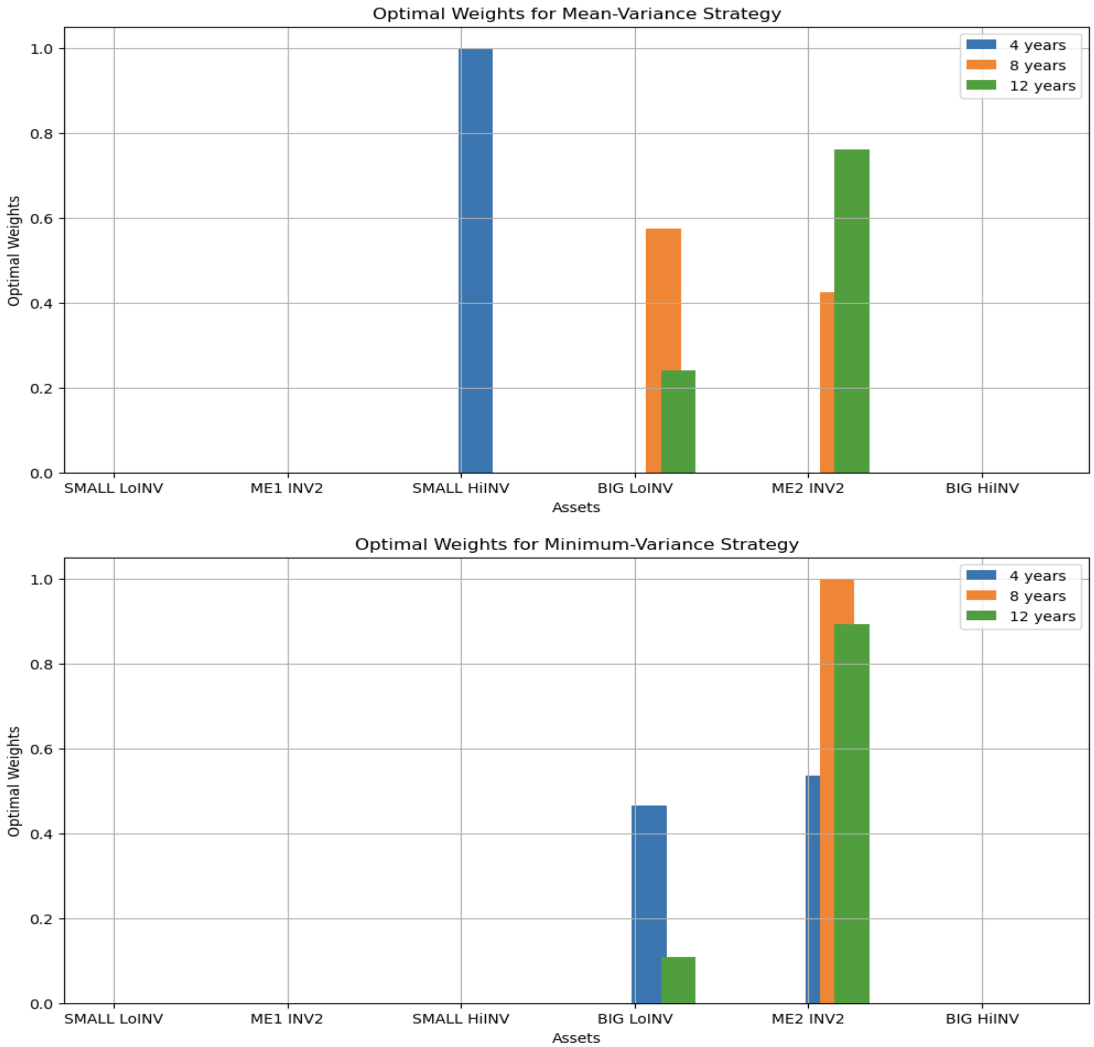


Fig 2. Optimal Weight Allocation Across 6 Portfolios in 2 Strategies

The graphs reveal that certain portfolios, such as SMALL HiINV and ME2 INV2, consistently receive higher weights in both the mean-variance and minimum-variance strategies, indicating their favorable performance in terms of risk-return trade-off and low volatility, respectively. Over shorter windows, portfolios tend to be more concentrated in specific high-performing assets, while longer windows show more diversified allocations.

3. Naive (Equal-Weighted) Portfolio:

This strategy simplifies portfolio construction by equally weighting all assets. Each asset is assigned an equal weight (1/N), regardless of historical data. This approach is advantageous due to its simplicity and ease of implementation. However, it does not consider risk-return optimization and may underperform compared to more sophisticated strategies.

Objective: Equally allocate weights to all assets.

Formulation:

$$w_i = \frac{1}{N} \ \forall i$$

Where N is the number of assets.

Return Analysis:

Table 4. Naive Portfolio Performance Over 3 Windows

Window	Strategy	Mean Returns	Covariance Matrix	Weights	Return	Std Dev
4	Naive	{'SMALL LoINV': 1.2358, 'ME1 INV2': 1.3919, 'SMALL HiINV': 1.049, 'BIG LoINV': 1.2029, 'ME2 INV2': 1.3652, 'BIG HiINV': 1.4617}	[[34.578, 34.3542, 39.8516, 30.2217, 30.2036, 33.8814], [34.3542, 34.9559, 40.4661, 30.3048, 30.5539, 34.8063], [39.8516, 40.4661, 48.6404, 33.7868, 34.6713, 40.8991], [30.2217, 30.3048, 33.7868, 29.2092, 28.3966, 30.9234], [30.2036, 30.5539, 34.6713, 28.3966, 29.1022, 32.0665], [33.8814, 34.8063, 40.8991, 30.9234, 32.0665, 39.1433]]	[0.1667, 0.1667, 0.1667, 0.1667, 0.1667, 0.1667]	1.2844	5.8367
8	Naive	{'SMALL LoINV': 0.5962, 'ME1 INV2': 0.7941, 'SMALL HiINV': 0.6053, 'BIG LoINV': 0.6316, 'ME2 INV2': 0.8436, 'BIG HiINV': 0.8243}	[[30.5564, 29.4258, 32.5373, 26.4023, 24.8782, 27.8695], [29.4258, 29.1023, 32.0537, 25.7048, 24.5456, 27.6249], [32.5373, 32.0537, 36.8757, 27.5273, 26.7988, 31.0533], [26.4023, 25.7048, 27.5273, 25.2087, 23.091, 25.1341], [24.8782, 24.5456, 26.7988, 23.091, 22.7789, 24.96], [27.8695, 27.6249, 31.0533, 25.1341, 24.96, 29.7923]]	[0.1667, 0.1667, 0.1667, 0.1667, 0.1667, 0.1667]	0.7159	5.2534
12	Naive	{'SMALL LoINV': 0.6786, 'ME1 INV2': 0.8266, 'SMALL HiINV': 0.6508, 'BIG LoINV': 0.5894, 'ME2 INV2': 0.7633, 'BIG HiINV': 0.7261}	[[27.0464, 25.4563, 27.524, 23.2775, 22.4269, 24.2247], [25.4563, 24.6392, 26.6289, 22.1838, 21.5974, 23.5573], [27.524, 26.6289, 30.0396, 23.3439, 23.0911, 25.9282], [23.2775, 22.1838, 23.3439, 22.8372, 21.39, 22.4846], [22.4269, 21.5974, 23.0911, 21.39, 21.5659, 22.7586], [24.2247, 23.5573, 25.9282, 22.4846, 22.7586, 25.9883]]	[0.1667, 0.1667, 0.1667, 0.1667, 0.1667, 0.1667]	0.7058	4.8986

Weights Assigned: Each portfolio is assigned an equal weight of approximately 0.1667, reflecting the naive strategy's principle of equal distribution among assets.

4-Year Window

- **Returns**: The portfolio achieves a return of 1.2844, which is high compared to other strategies.
- **Standard Deviation**: The standard deviation of 5.8367 indicates moderate volatility, slightly higher than the minimum-variance strategy but lower than the mean-variance strategy for the same window.

8-Year Window

- **Returns**: The return drops to 0.7159, showing reduced performance over a longer window compared to the 4-year window.
- **Standard Deviation**: The standard deviation decreases to 5.2534, indicating reduced volatility compared to the 4-year window.

12-Year Window

- **Returns**: The return further declines to 0.7058, reflecting a continuing trend of lower returns with increasing window length.
- **Standard Deviation**: The standard deviation reduces to 4.8986, suggesting that the portfolio's volatility decreases over longer periods.

Comparison of All Strategies:

The Efficient Frontier provides a visual comparison of the performance of different portfolio strategies. From the graph, we can conclude that standard deviation range is quite higher than the mean scale. This indicates that historical dataset exhibits a high level of volatility relative to the returns achieved. A few of the pointers which might be behind this can be historical instability due to some major event, low risk-adjusted returns, etc.

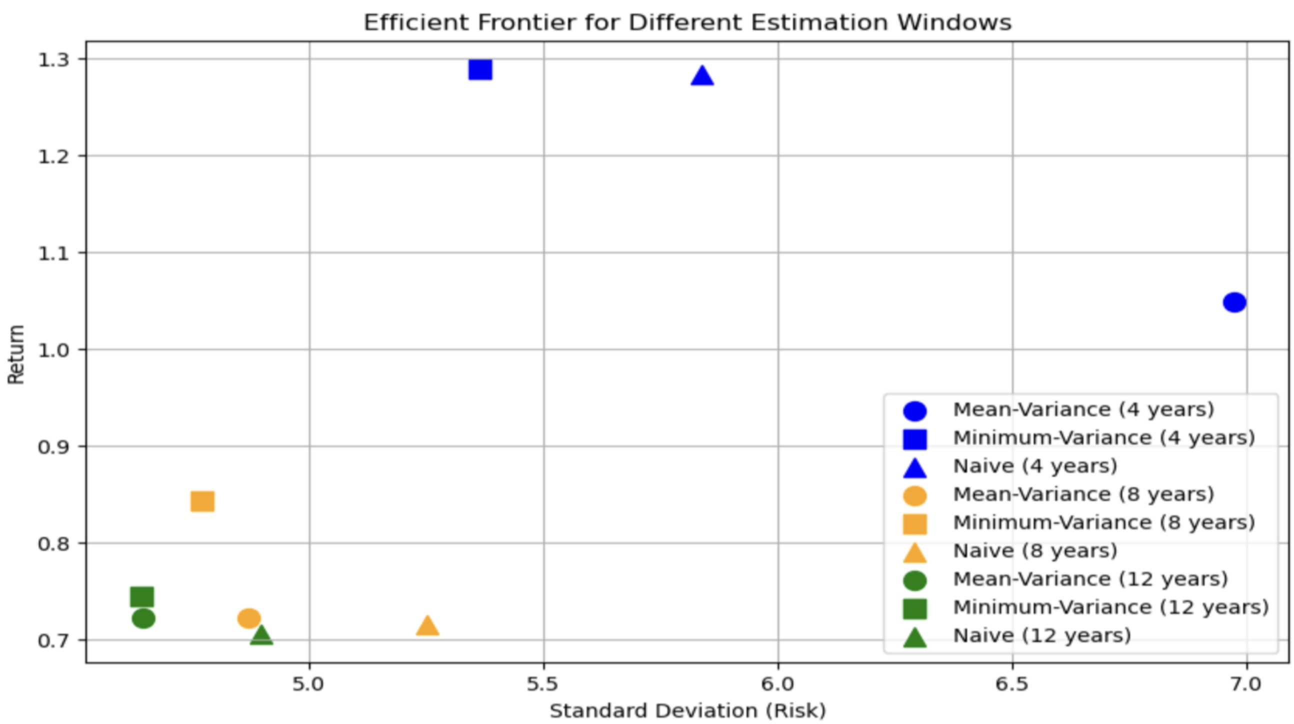


Fig- 1.A (Mean-Deviation Graph)

Table 5. Nine Policies Return and Risk Comparison

Strategy	Window (Years)	Return	Std Dev (Risk)	Performance Summary
Mean-Variance	4	1.049	6.9743	High return but with the highest risk.
	8	0.7217	4.8718	Moderate return and risk.
	12	0.7217	4.6448	Similar return with slightly decreased risk.
Minimum-Variance	4	1.2898	5.3642	Highest return with significantly lower risk.
	8	0.8436	4.7727	Balanced return and risk, better than Mean-Variance.
	12	0.7445	4.6419	Lower risk, stable performance.
Naive	4	1.2844	5.8367	High return with moderate risk.
	8	0.7159	5.2534	Moderate return with decreased risk.
	12	0.7058	4.8986	Consistent return with further decreased risk.

Interpretation:

Mean-Variance Strategy:

- Provides competitive returns but incurs the highest risk, especially over shorter estimation windows.
- Shows decreasing returns and risk with longer estimation windows.

Minimum-Variance Strategy:

- Consistently offers the highest returns with significantly lower risk compared to Mean-Variance.
- Exhibits the most favorable risk-return balance, especially over shorter periods (4-year window).

Naive Strategy:

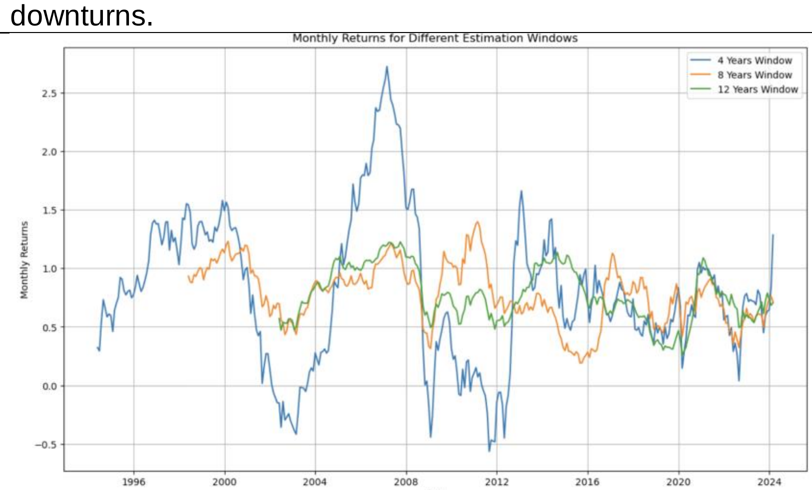
- Provides high returns with moderate risk, particularly in the 4-year window.
- Maintains a good balance of returns and risk, with stability over longer periods.

Conclusion:

This analysis revealed that minimum-variance portfolios outperform mean-variance portfolios in terms of lower standard deviation and higher returns, especially for shorter windows. This suggests minimizing risk is more beneficial in volatile markets. The naive portfolio showed consistent but lower performance compared to optimized strategies. Optimal weights analysis indicated significant concentration in assets like SMALL HiINV and ME2 INV2, highlighting their stability and favorable performance. (Check Appendix A.3 and A.4 for more graphical comparison)

In summary, minimum-variance optimization is advantageous in turbulent periods, while mean-variance optimization is beneficial for specific return targets. The naive strategy, although simpler, serves as a useful benchmark. This study underscores the importance of quantitative methods in financial decision-making and the need to adapt strategies to changing market dynamics.

Appendix: A

Index	Topic	Comment
1	Target Return Calculation	Historical data for many equity markets, including European markets, often show average annual returns in the range of 8-12%. Setting the average monthly target aligns well with the historical performance, making it a realistic goal.
2	Major Events	Dot-Com Bubble (2000-2002): A significant market downturn caused by the collapse of internet-related stocks, affecting global markets. Global Financial Crisis (2008): A severe worldwide economic crisis precipitated by the collapse of Lehman Brothers, leading to widespread financial instability. European Debt Crisis (2010-2012): A period of financial turmoil in the Eurozone, marked by high sovereign debt levels in several European countries. Brexit Referendum (2016): The UK's vote to leave the European Union, causing significant market volatility and uncertainty. COVID-19 Pandemic (2020-present): A global health crisis leading to unprecedented market disruptions and economic downturns.
3	Monthly Return for Different Estimation Windows	

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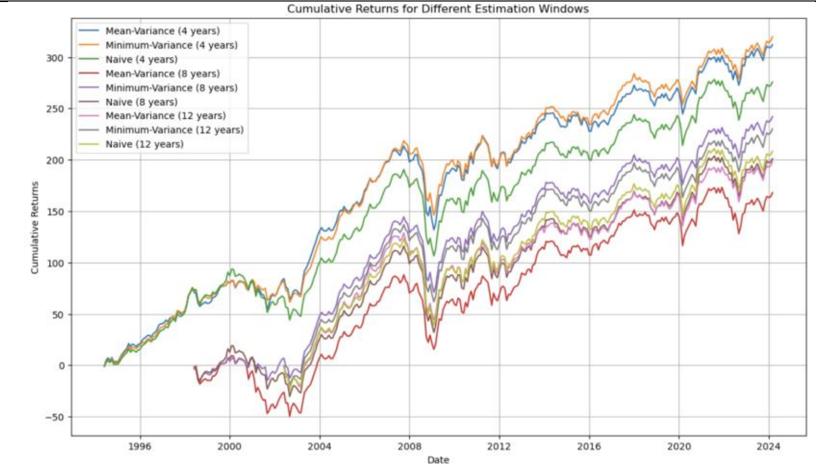
<td>
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