Multiple-view Geometry 2

Epipolar geometry, F-matrix estimation, RANSAC

RRC Summer Sessions

Last class

Visual odometry

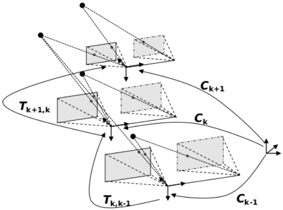


Image sequence

↓

Feature detection

↓

Feature matching (tracking)

↓

Motion estimation		
2D-2D	3D-3D	3D-2D

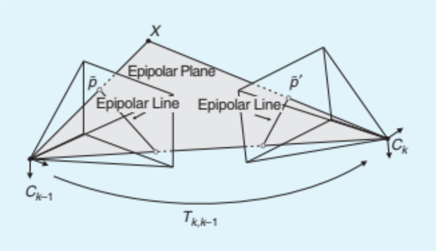
↓

Local optimization

Last class
Fundamental matrix

- The fundamental matrix encodes the relative orientation between two views. It is a 3x3 matrix of rank 2, and with 7 dof.
- $F = K'^{-T} R' S_b R''^{T} K''^{-1}$
- It is derived from the co-planarity constraint $x'^{T} F x'' = 0$.

Epipolar geometry



Courtesy: Scaramuzza

Essential matrix

- The essential matrix is a specialization of the fundamental matrix for calibrated cameras.
- $E = R' S_b R''^T$. We simply drop the calibration matrices.
- The corresponding co-planarity constraint: ${}^k x'^T E \, {}^k x'' = 0$
- It has 5 dof, is homogeneous and singular.
- Both non-zero singular values are indentical.

$$E = U \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} V^T$$

Application

Motion estimation

Courtesy: Kundu et al.

Application

Motion estimation

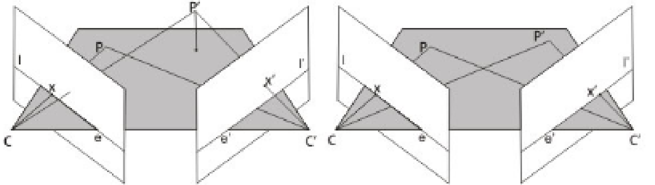
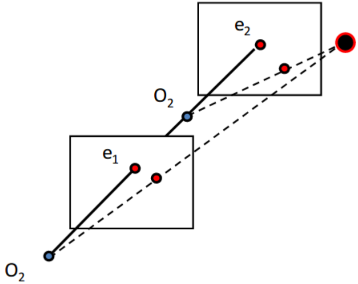


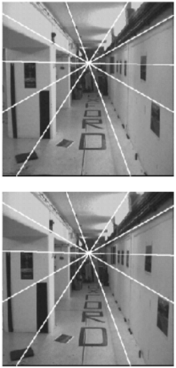
Fig. 2. LEFT: The world point P moves non-degenerately to $P^{'}$ and hence $x^{'}$, the image of $P^{'}$ does not lie on the epipolar line corresponding to x. RIGHT: The point P moves degenerately in the epipolar plane to $P^{'}$. Hence, despite moving, its image point lies on the epipolar line corresponding to the image of P.

Courtesy: Kundu et al.

Application

Motion estimation



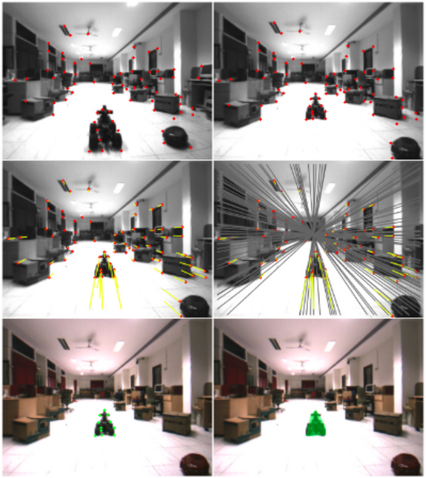


- The epipoles have same position in both images
- Epipole called FOE (focus of expansion)

Courtesy: Fei Fei Li

Application

Motion estimation



Courtesy: Kundu et al.

F-matrix estimation

F-matrix estimation

8 point algorithm

Problem Formulation

- **Given:** N corresponding points

  $$(x', y')_n, (x'', y'')_n \quad \text{with} \quad n = 1, \ldots, N$$

- **Wanted:** fundamental matrix F

Courtesy: Stachniss

F-matrix estimation

8 point algorithm

Fundamental Matrix From Corresponding Points

- For each point, we have the coplanarity constraint

$$\mathbf{x}'^{\mathsf{T}}_n \mathsf{F} \mathbf{x}''_n = 0 \qquad n = 1, ..., N$$

Courtesy: Stachniss

F-matrix estimation

8 point algorithm

Fundamental Matrix From Corresponding Points

- For each point, we have the coplanarity constraint

$$\mathbf{x'}_n^{\mathsf{T}} \mathsf{F} \mathbf{x}''_n = 0 \qquad n = 1, ..., N$$

- or

$$[x'_n, y'_n, 1] \left[\underbrace{\boxed{\begin{matrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{matrix}}}_{\text{unknowns!}} \right] \begin{bmatrix} x''_n \\ y''_n \\ 1 \end{bmatrix} = 0$$

Courtesy: Stachniss

F-matrix estimation

8 point algorithm

- **Linear function** in the unknowns F_{ij}

$$[\boxed{x'_n}, y'_n, 1] \begin{bmatrix} \boxed{F_{11}} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{bmatrix} \begin{bmatrix} \boxed{x''_n} \\ y''_n \\ 1 \end{bmatrix} = 0$$

$$\boxed{x''_n F_{11} x'_n} + x''_n F_{21} y'_n + \ldots = 0$$

Courtesy: Stachniss

F-matrix estimation

8 point algorithm

Linear Dependency

- **Linear function** in the unknowns F_{ij}

$$[x'_n \; \boxed{y'_n} \; 1] \begin{bmatrix} F_{11} & F_{12} & F_{13} \\ \boxed{F_{21}} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{bmatrix} \begin{bmatrix} \boxed{x''_n} \\ y''_n \\ 1 \end{bmatrix} = 0$$

$$x''_n F_{11} x'_n + \boxed{x''_n F_{21} y'_n} + \ldots = 0$$

Courtesy: Stachniss

F-matrix estimation

8 point algorithm

Linear Dependency

- **Linear function** in the unknowns F_{ij}

$$[x'_n, y'_n, 1] \begin{bmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{bmatrix} \begin{bmatrix} x''_n \\ y''_n \\ 1 \end{bmatrix} = 0$$

$$\Downarrow$$

$$[x''_n x'_n, x''_n y'_n, x''_n, y''_n x'_n, y''_n y'_n, y''_n, x'_n, y'_n, 1] \cdot$$
$$[F_{11}, F_{21}, F_{31}, F_{12}, F_{22}, F_{32}, F_{13}, F_{23}, F_{33}] = 0$$
$$n = 1, \ldots, N$$

Courtesy: Stachniss

F-matrix estimation

8 point algorithm

Linear Dependency

- Linear function in the unknowns F_{ij}

$$\boldsymbol{a}_n^\mathsf{T} \rightarrow [x''_n x'_n, x''_n y'_n, x''_n, y''_n x'_n, y''_n y'_n, y''_n, x'_n, y'_n, 1] \cdot$$

$$\mathbf{f}^\mathsf{T} \rightarrow [F_{11}, F_{21}, F_{31}, F_{12}, F_{22}, F_{32}, F_{13}, F_{23}, F_{33}] = 0$$

$$n = 1, ..., N$$

$$\Downarrow$$

$$\boldsymbol{a}_n^\mathsf{T} \cdot \mathbf{f}^\mathsf{T} = 0 \qquad n = 1, ..., N$$

Courtesy: Stachniss

F-matrix estimation

8 point algorithm

Using the Kronecker Product

- Linear function in the unknowns F_{ij}

$$\boldsymbol{a}_n^\mathsf{T} \quad [x''_n x'_n, x''_n y'_n, x''_n, y''_n x'_n, y''_n y'_n, y''_n, x'_n, y'_n, 1] \cdot$$

$$\mathbf{f}^\mathsf{T} \quad [F_{11}, F_{21}, F_{31}, F_{12}, F_{22}, F_{32}, F_{13}, F_{23}, F_{33}] = 0$$

$$n = 1, \ldots, N$$

$$(\mathbf{x}''_n \otimes \mathbf{x}'_n)^\mathsf{T} \mathrm{vecF} = \underbrace{\boldsymbol{a}_n^\mathsf{T}}_{(\mathbf{x}''_n \otimes \mathbf{x}'_n)^\mathsf{T}} \underbrace{\mathbf{f}}_{\mathrm{vecF}} = 0 \qquad n = 1, \ldots, N$$

(it holds in general: $\mathbf{x}^\mathsf{T} \mathsf{F} \mathbf{y} = (\mathbf{y} \otimes \mathbf{x})^\mathsf{T} \mathrm{vecF}$)

Courtesy: Stachniss

F-matrix estimation

8 point algorithm

Solving the Linear System

- Singular value decomposition solves

$$A\mathbf{f} = \mathbf{0}$$

- and thus provides a solution for

$$\mathbf{f} = [F_{11}, F_{21}, F_{31}, F_{12}, F_{22}, F_{32}, F_{13}, F_{23}, F_{33}]^\mathsf{T}$$

- SVD: $\mathbf{f}$ can be characterized as a right-singular vector corresponding to a singular value of A that is zero

Courtesy: Stachniss

F-matrix estimation

8 point algorithm

How Many Points Are Needed?

- The vector $\mathbf{f}$ has 9 dimensions

$$A = \begin{bmatrix} \boldsymbol{a}_1^\top \\ \cdots \\ \boldsymbol{a}_n^\top \\ \cdots \\ \boldsymbol{a}_N^\top \end{bmatrix} \Rightarrow A\mathbf{f} = \mathbf{0}$$

- Fundamental matrix is **homogenous**
- Matrix A has a rank of at most **8**
- **We need 8 corresponding points**

Courtesy: Stachniss

F-matrix estimation

8 point algorithm

More Than 8 Points...

- In reality: noisy measurements
- With more than 8 points, the matrix A will become regular (but should not!)
- Use the singular vector $\hat{\mathbf{f}}$ of A that corresponds to the **smallest** singular value is the solution $\hat{\mathbf{f}} \rightarrow \hat{\mathsf{F}}$

Courtesy: Stachniss

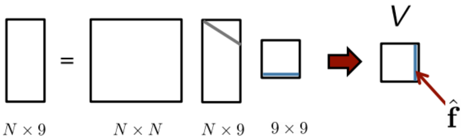
F-matrix estimation

8 point algorithm

Singular Vector

- Use the singular vector $\hat{\mathbf{f}}$ of A that corresponds to the **smallest** singular value is the solution $\hat{\mathbf{f}} \to \hat{\mathsf{F}}$

$$A = UDV^{\mathsf{T}}$$



Courtesy: Stachniss

F-matrix estimation

8 point algorithm

Enforcing Rank 2

- We want to **enforce** a matrix F with
  $\mathrm{rank}(\mathsf{F}) = 2$
- F should **approximate** our computed matrix $\hat{\mathsf{F}}$ as close a possible
- Use a second SVD (this time of $\hat{\mathsf{F}}$)

  $\mathsf{F} = UD^a V^\mathsf{T} = U\mathrm{diag}(D_{11}, D_{22}, 0)V^\mathsf{T}$

  with $\mathrm{svd}(\hat{\mathsf{F}}) = UDV^\mathsf{T}$
  and $D_{11} \geq D_{22} \geq D_{33}$

Courtesy: Stachniss

F-matrix estimation

8 point algorithm

8-Point Algorithm

```
function F = F_from_point_pairs(xs, xss)
% xs, xss: Nx3 homologous point coordinates, N > 7
% F:       3x3 fundamental matrix

% coefficient matrix
for n = 1 : size(xs, 1)
    A(n, :) = kron(xss(n, :), xs(n, :));
end

% singular value decomposition
[U, D, V] = svd(A);

% approximate F, possibly regular
Fa = reshape(V(:, 9), 3, 3)';

% svd decomposition of F
[Ua, Da, Va] = svd(Fa);

% algebraically best F, singular
F = Ua * diag([Da(1, 1), Da(2, 2), 0]) * Va';
```

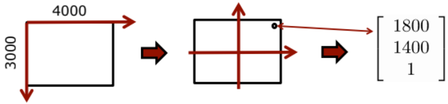
Courtesy: Stachniss

F-matrix estimation

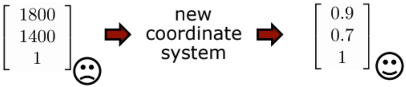
8 point algorithm

Well-Conditioned Problem

- Example image 12MPixel camera



- Ill-conditioned, numerically instable



Courtesy: Stachniss

F-matrix estimation

8 point algorithm

Conditioning/Normalization to Obtain a Well-Conditioned Problem

- Normalization of the point coordinates substantially **improves** the **stability**
- **Transform** the points so that the center of mass of all points is at (0,0)
- **Scale** the image so that the x and y coordinated are within [-1,1]

☺

Courtesy: Stachniss

F-matrix estimation

8 point algorithm

Conditioning/Normalization

- Define $\mathsf{T} : \mathsf{T}\mathbf{x} = \hat{\mathbf{x}}$ so that coordinates are zero-centered and in [-1,1]
- Determine fundamental matrix $\hat{\mathsf{F}}$ from the transformed coordinates

$$\begin{aligned} \mathbf{x}'^{\mathsf{T}}\mathsf{F}\mathbf{x}'' &= (\mathsf{T}^{-1}\hat{\mathbf{x}}')^{\mathsf{T}}\mathsf{F}(\mathsf{T}^{-1}\hat{\mathbf{x}}'') \\ &= \hat{\mathbf{x}}'^{\mathsf{T}}\mathsf{T}^{-\mathsf{T}}\mathsf{F}\mathsf{T}^{-1}\hat{\mathbf{x}}'' \\ &= \hat{\mathbf{x}}'^{\mathsf{T}}\hat{\mathsf{F}}\hat{\mathbf{x}}'' \end{aligned}$$

- Obtain essential matrix F through

$$\Rightarrow \quad \begin{aligned} \hat{\mathsf{F}} &= \mathsf{T}^{-\mathsf{T}}\mathsf{F}\mathsf{T}^{-1} \\ \mathsf{F} &= \mathsf{T}^{\mathsf{T}}\hat{\mathsf{F}}\mathsf{T} \end{aligned}$$

Courtesy: Stachniss

Obtaining R,t

Solution by Hartley & Zisserman

- We know that

$$E = U \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} V^T$$

rotation matrices

- Define $Z = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad W = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

- So that $ZW = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Courtesy: Stachniss

Obtaining R,t

Solution by Hartley & Zisserman

$$
\begin{aligned}
\mathsf{E} &= U \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} V^{\mathsf{T}} \\
&= U \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{Z} \underbrace{\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{W} V^{\mathsf{T}}
\end{aligned}
$$

Courtesy: Stachniss

Obtaining R,t

Solution by Hartley & Zisserman

$$
\begin{aligned}
\mathsf{E} &= U \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} V^{\mathsf{T}} \\
&= U \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{Z} \underbrace{\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{W} V^{\mathsf{T}} \\
&= UZ \underbrace{U^{\mathsf{T}} U}_{I} W V^{\mathsf{T}}
\end{aligned}
$$

Courtesy: Stachniss

Obtaining R,t

Solution by Hartley & Zisserman

$$
\begin{aligned}
\mathrm{E} &= U \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} V^{\mathsf{T}} \\
&= U \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{Z} \underbrace{\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{W} V^{\mathsf{T}} \\
&= U Z \underbrace{U^{\mathsf{T}} U}_{I} W V^{\mathsf{T}} \\
&= \underbrace{U Z U^{\mathsf{T}}}_{S_B} \underbrace{U W V^{\mathsf{T}}}_{R^{\mathsf{T}}}
\end{aligned}
$$

Courtesy: Stachniss

Obtaining R,t

Four Possibilities to Define Z, W

$$
\begin{aligned}
\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} & = ZW & = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
& = Z^{\mathsf{T}} W^{\mathsf{T}} & = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{\mathsf{T}} \\
& = -Z^{\mathsf{T}} W & = -\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
& = -Z W^{\mathsf{T}} & = -\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{\mathsf{T}}
\end{aligned}
$$

Courtesy: Stachniss

Obtaining R,t

Yields Four Solutions

$$E = \underbrace{UZU^\top}_{S_B} \underbrace{UWV^\top}_{R^\top}$$

2 solutions for S_B 2 solutions for R

$$S^1_{\hat{B}} = UZU^\top \quad S^2_{\hat{B}} = UZ^\top U^\top \quad R^\top_1 = UWV^\top \quad R^\top_2 = UW^\top V^\top$$

4 solutions

$$E^1 = UZU^\top \; UWV^\top$$
$$E^2 = UZ^\top U^\top \; UWV^\top$$
$$E^3 = UZU^\top \; UW^\top V^\top$$
$$E^4 = UZ^\top U^\top \; UW^\top V^\top$$

Courtesy: Stachniss

Obtaining R,t

Solution by Hartley & Zisserman

- Compute the SVD of E

$$UDV^\top = \mathrm{svd}(\mathsf{E})$$

- Compute the four solutions

$$\mathsf{S}^1_{\hat{B}} = UZU^\top \quad \mathsf{S}^2_{\hat{B}} = UZ^\top U^\top \quad R_1^\top = UWV^\top \quad R_2^\top = UW^\top V^\top$$

- Test for which solutions all points are in front of both cameras
- Return the only physically plausible (unique) configuration

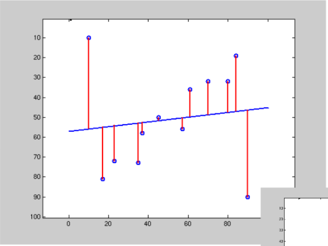
Courtesy: Stachniss

Robust estimation

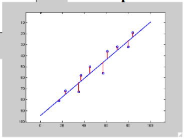
Robust estimation

Least squares estimation is sensitive to outliers, so that a few outliers can greatly skew the result.

Least squares regression with outliers



compare



Solution: Estimation methods that are robust to outliers.

Courtesy: Collins

Robust estimation

Random Sample Consensus (RANSAC)

Objective

Robust fit of a model to a data set S which contains outliers.

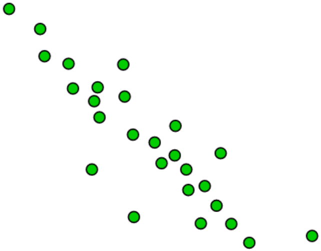
Algorithm

(i) Randomly select a sample of s data points from S and instantiate the model from this subset.
(ii) Determine the set of data points S_i which are within a distance threshold t of the model. The set S_i is the consensus set of the sample and defines the inliers of S.
(iii) If the size of S_i (the number of inliers) is greater than some threshold T, re-estimate the model using all the points in S_i and terminate.
(iv) If the size of S_i is less than T, select a new subset and repeat the above.
(v) After N trials the largest consensus set S_i is selected, and the model is re-estimated using all the points in the subset S_i.

Courtesy: Collins

Robust estimation

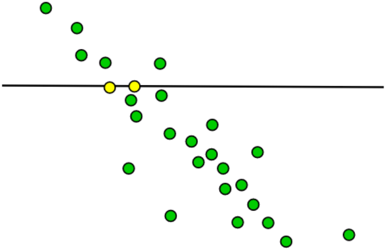
RANSAC



Courtesy: Collins

Robust estimation

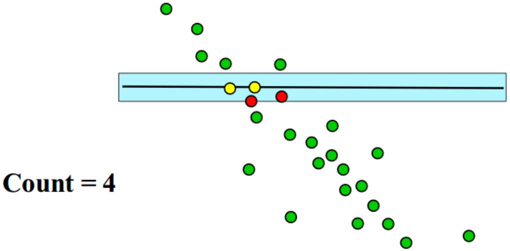
RANSAC



Courtesy: Collins

Robust estimation

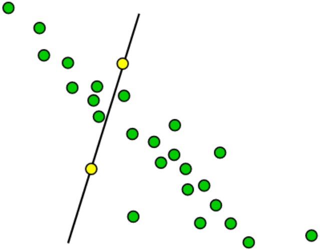
RANSAC



Courtesy: Collins

Robust estimation

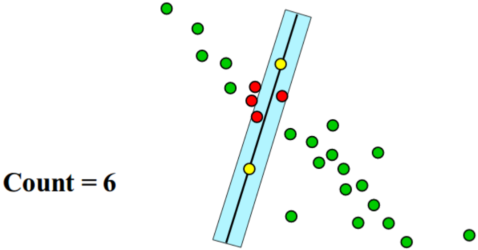
RANSAC



Courtesy: Collins

Robust estimation

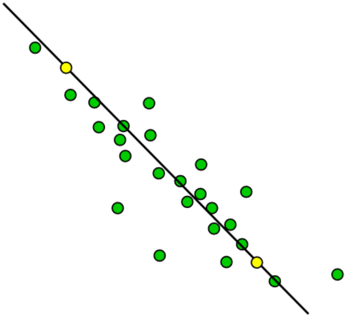
RANSAC



Courtesy: Collins

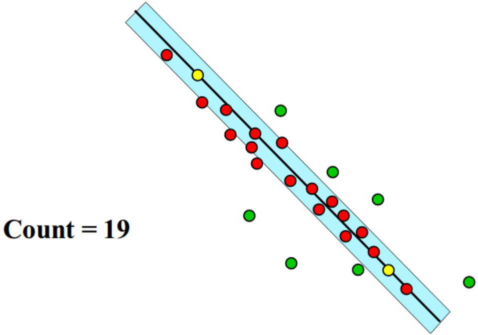
Robust estimation

RANSAC



Courtesy: Collins

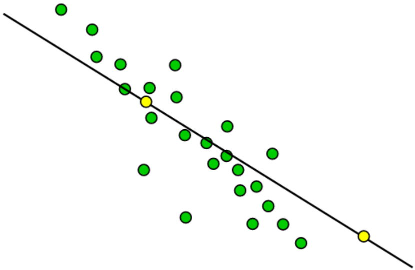
Robust estimation
RANSAC



Courtesy: Collins

Robust estimation

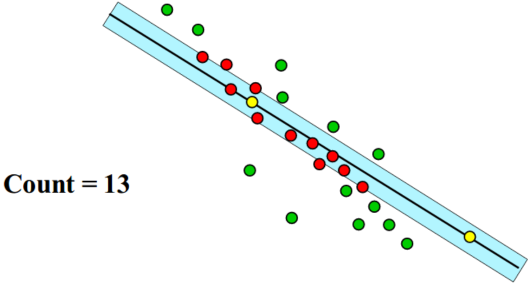
RANSAC



Courtesy: Collins

Robust estimation

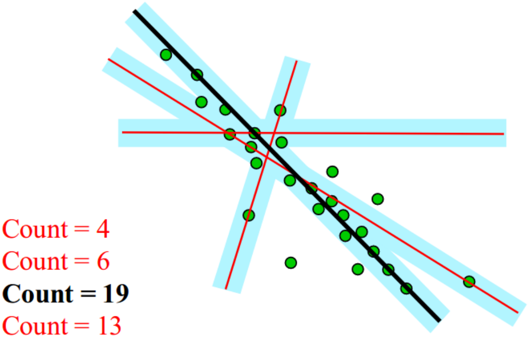
RANSAC



Courtesy: Collins

Robust estimation

RANSAC



Courtesy: Collins

Robust estimation

RANSAC

How many tries?

Choose N so that, with probability p, at least one random sample is free from outliers. e.g. p=0.99

$$(1-(1-e)^s)^N = 1-p$$

$$N = \frac{\log(1-p)}{log(1-(1-e)^s)}$$

	proportion of outliers e						
s	5%	10%	20%	25%	30%	40%	50%
2	2	3	5	6	7	11	17
3	3	4	7	9	11	19	35
4	3	5	9	13	17	34	72
5	4	6	12	17	26	57	146
6	4	7	16	24	37	97	293
7	4	8	20	33	54	163	588
8	5	9	26	44	78	272	1177

Courtesy: Collins

Robust estimation

RANSAC

How large is an acceptable consensus set?

- We have seen that we don't have to exhaustively sample subsets of points, we just need to randomly sample N subsets.
- However, typically, we don't even have to sample N sets!
- Early termination: terminate when inlier ratio reaches expected ratio of inliers

$$T = (1 - e) * (\textit{total number of data points})$$

Courtesy: Collins

How to estimate F-matrix using RANSAC?