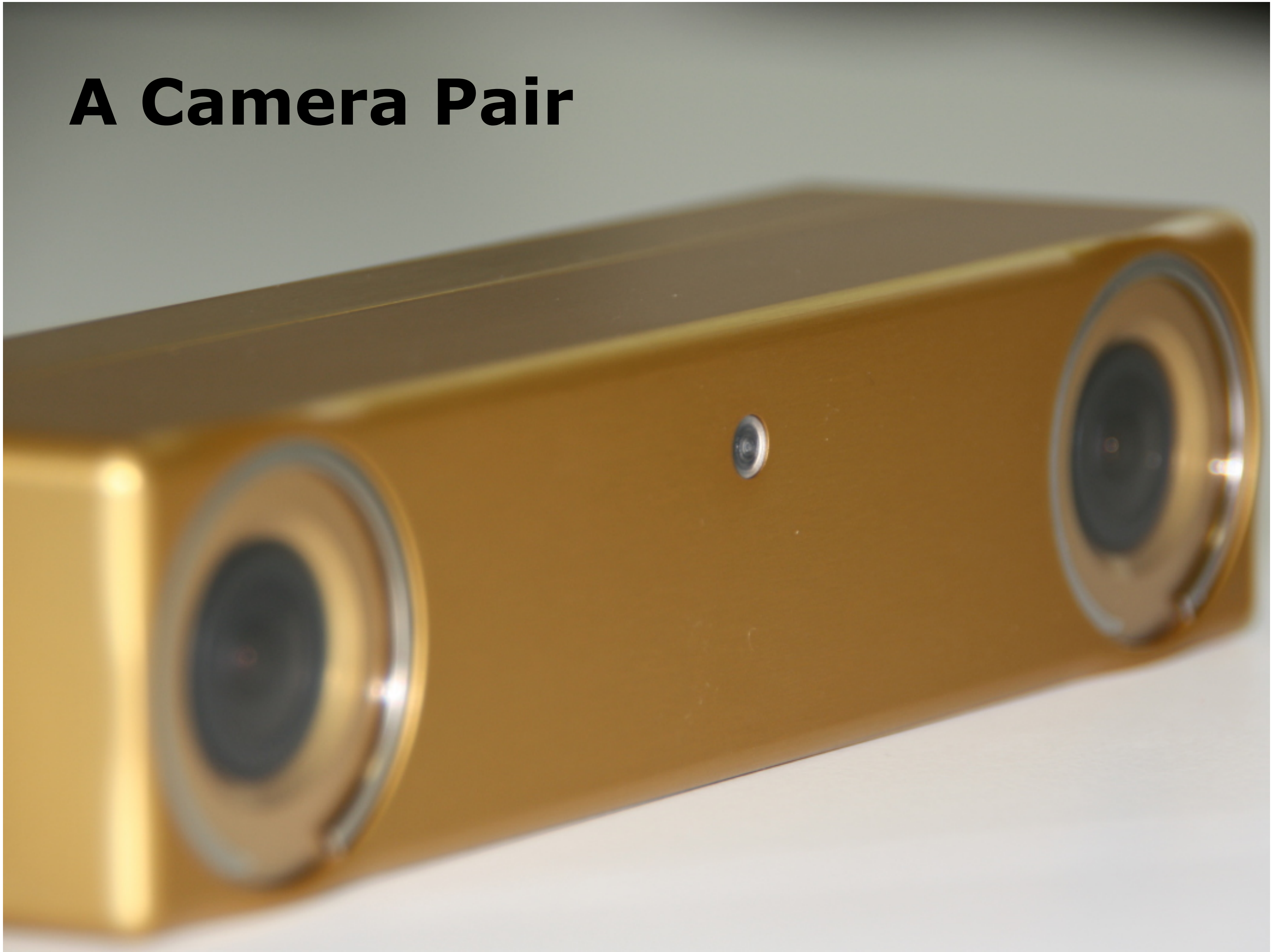
Photogrammetry II

Relative Orientation and the Fundamental Matrix

Cyrill Stachniss

The slides have been created by Cyrill Stachniss.

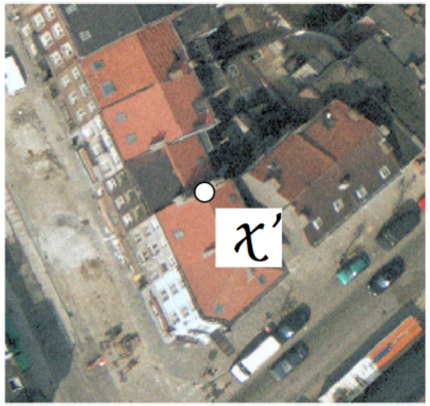
A Camera Pair

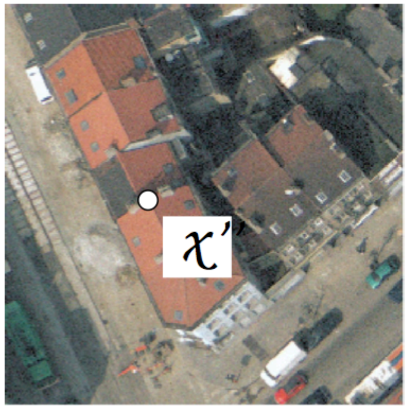


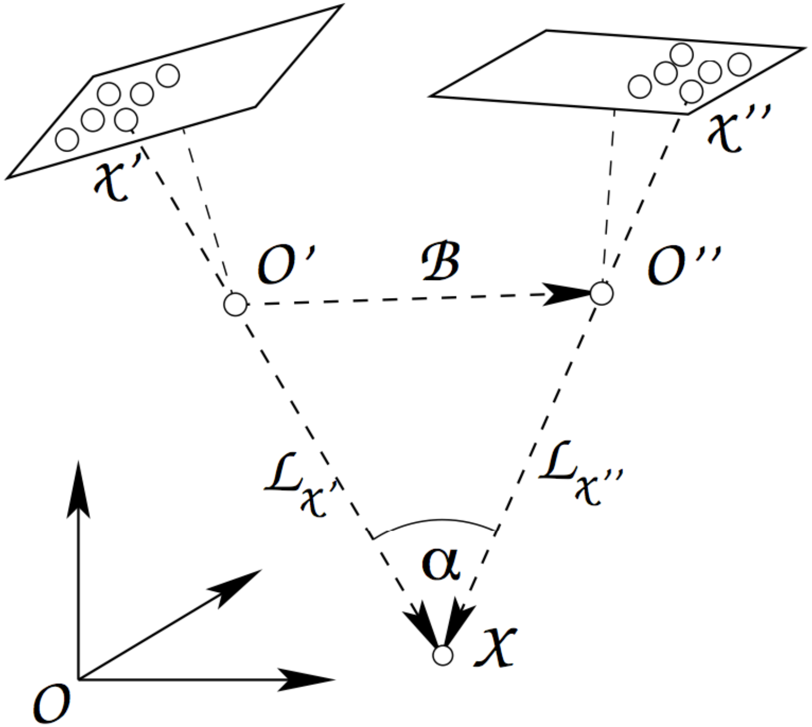
Camera Pair

- In the Photogrammetry I course, we computed the camera orientation for **single camera**
- We are now considering situation in which we have two images, potentially taken from **two cameras**

We Observe Bundles of Rays







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Orientation Parameters for the Camera Pair and Relative Orientation

Orientation

- The orientation of the camera pair can be described using independent orientations for each camera

How many parameters are needed?

- Calibrated cameras: **?** parameters
- Uncalibrated cameras: **?** parameters

Orientation

- The orientation of the camera pair can be described using independent orientations for each camera

How many parameters are needed?

- Calibrated cameras: **12** parameters
- Uncalibrated cameras: **22** parameters

Orientation with Control Points

- The orientation of the camera pair can be described using independent orientations for each camera

- Calibrated cameras: 12 parameters
- Can be computed via two separate spatial resection/P3P steps
- Requires 3(4) known control points

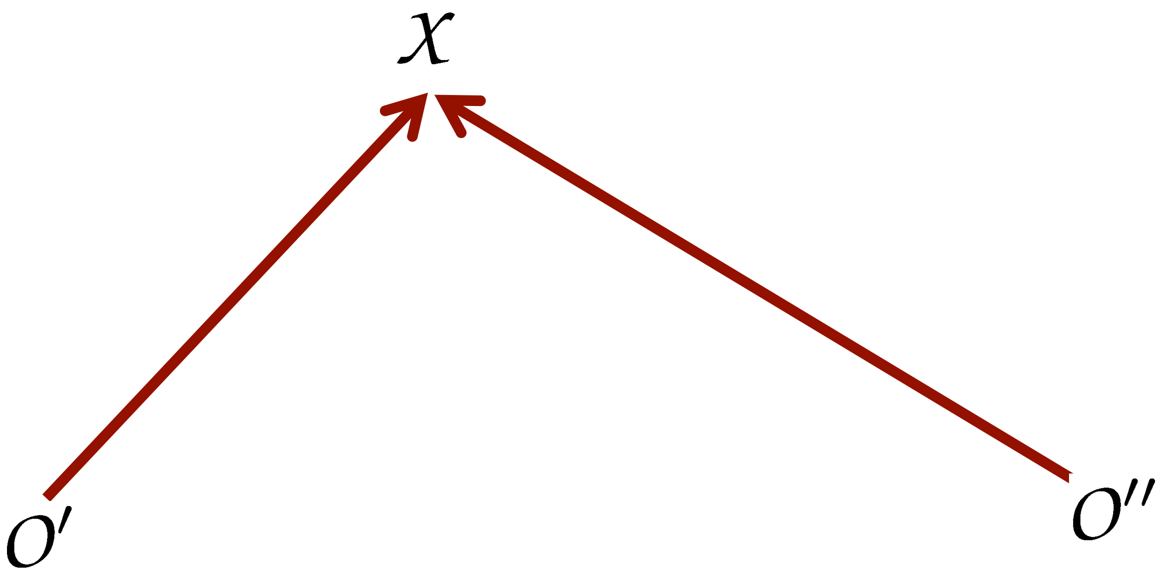
Orientation with Control Points

- The orientation of the camera pair can be described using independent orientations for each camera

- Uncalibrated cameras: 22 parameters
- Can be computed via two separate DLT steps
- Requires 6 known control points

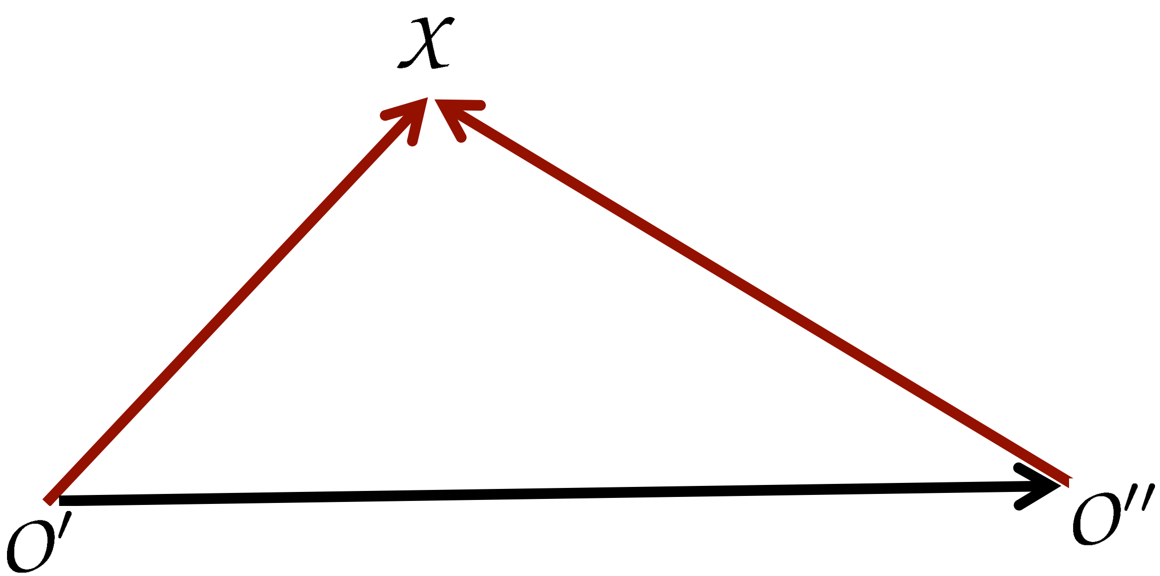
Which Parameters Can We Compute **Without Additional** Information About the Scene?

We start with a perfect orientation and the intersection of two corresponding rays (DE: homologe Strahlen)



Coplanarity Constraint
(DE: Koplanaritätsbedingung)

- Consider perfect orientation and the intersection of two corresponding rays
- Both rays lie within one plane in 3D



For Calibrated Cameras

- We need 2x6=12 parameters for two calibrated cameras for the orientation
- Mapping of the calibrated camera is angle-preserving
- Angle-preserving model of the object
- Angle-preserving mapping is a 7 DoF similarity transformation
- Without additional information, **we cannot obtain all 12 parameters**

Which Parameters Can We Obtain?

Cameras Measure Directions

- We cannot obtain the (global) **translation** and **rotation** (if the cameras maintain their relative transformation) as well as the **scale**

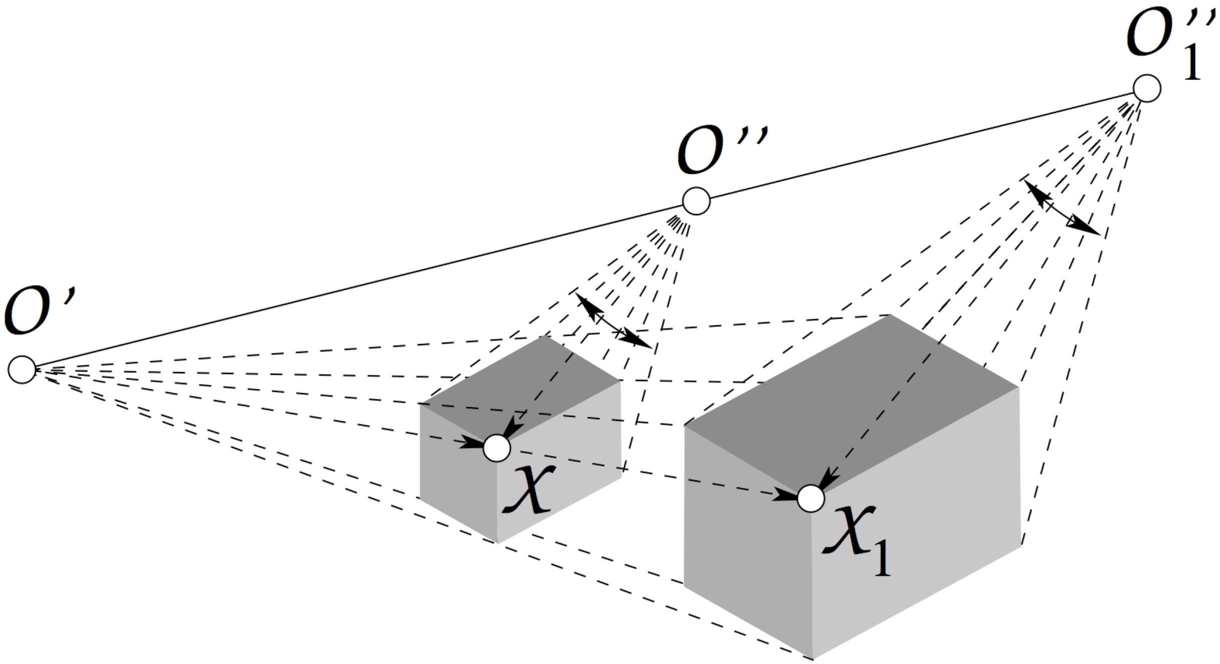


Image courtesy: Förstner & Wrobel

What We Can Compute

- The **rotation** R of the second camera w.r.t. the first one (3 parameters)
- The **direction** B of the line connecting the to centers of projection
- We do **not know** their **distance**

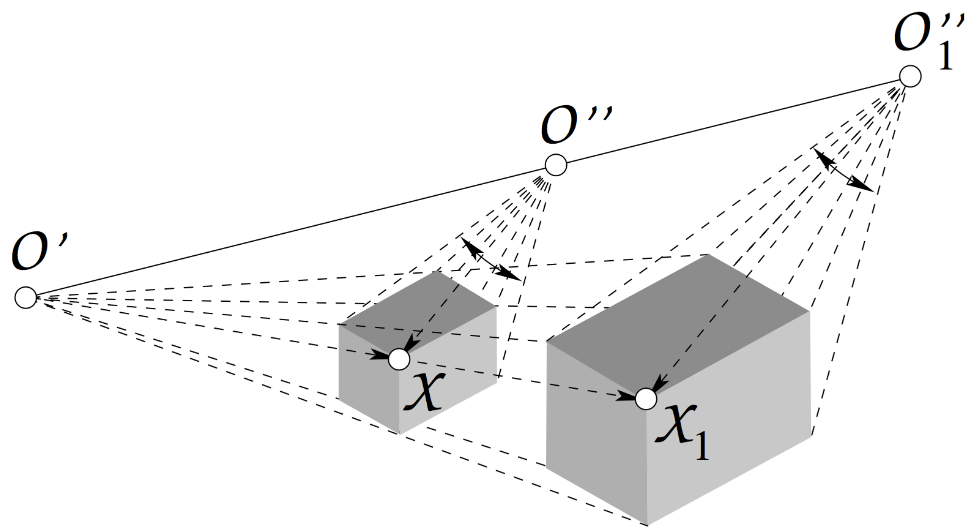


Image courtesy: Förstner & Wrobel

For Calibrated Cameras

- We need 2x6=12 parameters for two calibrated cameras for the orientation
- With a calibrated camera, we obtain an angle-preserving model of the object
- Without additional information, we can **only obtain** 12-7 = **5 parameters** (7=translation, rotation, scale)

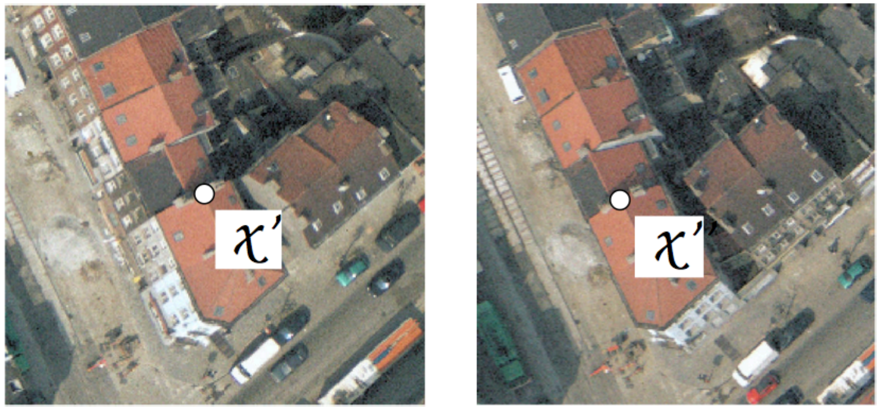
first camera

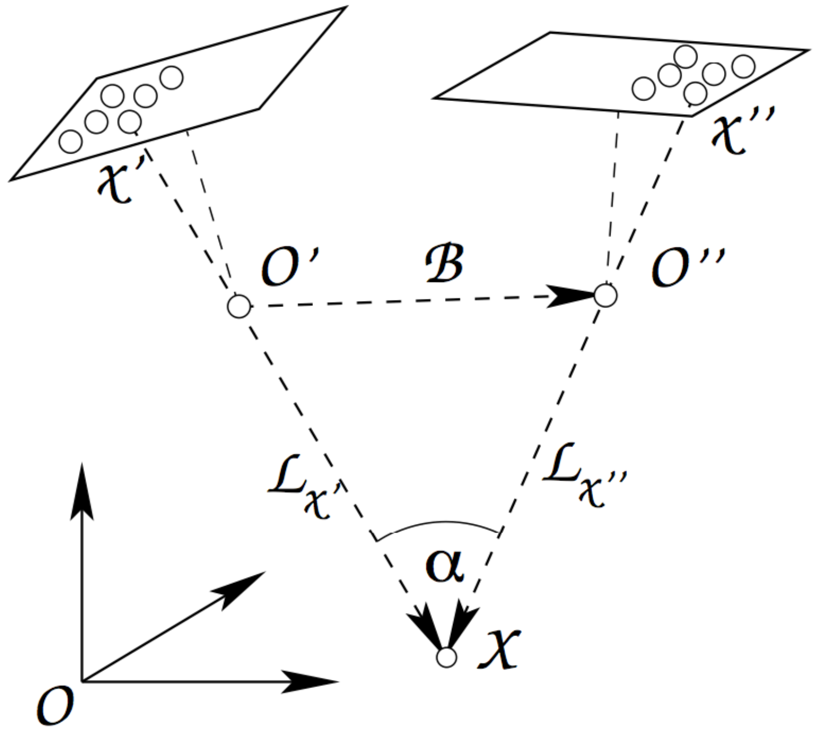
distance between cameras

Photogrammetric Model

- Given two cameras images, we can reconstruct the object only **up to a similarity transform**
- Called a **photogrammetric model**
- The **orientation of the photogrammetric model** is called the **absolute orientation**
- For obtaining the absolute orientation, we need at least **3 points** in 3D (for 7 parameters)

What do we need for computing a **3D model** of the scene?





For Uncalibrated Cameras

- **Straight-line preserving** but **not angle preserving**
- Object can be reconstructed up to a straight-line preserving mapping
- Projective transform (15 parameters)
- Thus, for uncalibrated cameras, we can only **obtain 22-15=7** parameters given two images
- We need at **least 5 points** in 3D (15 coordinates) for the absolute o.

Relative and Absolute Orient.

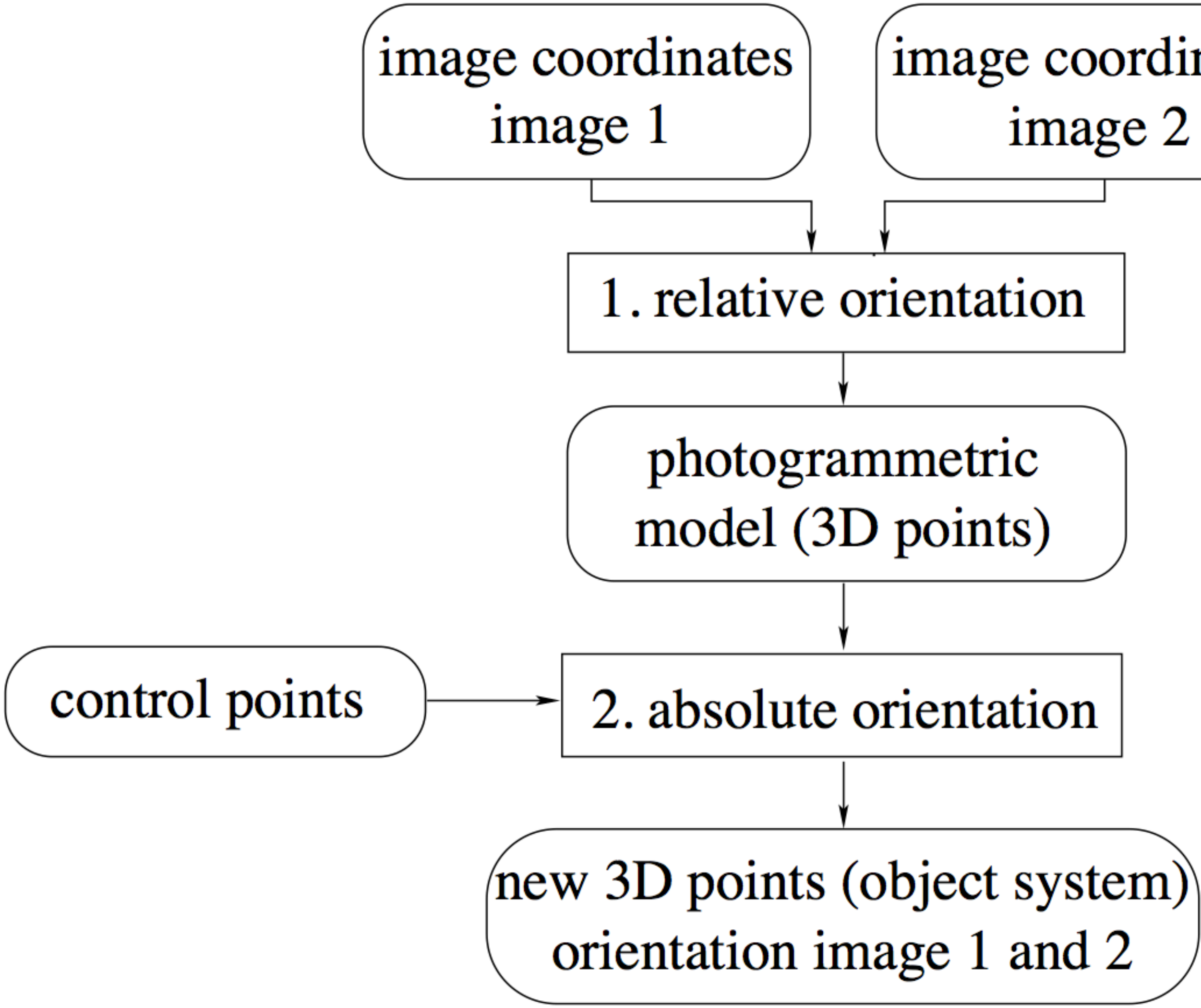


Image courtesy: Förstner & Wrobel

Summary

Cameras	#params /img	#params /img pair	#params for RO	#params for AO	min #P
calibrated	**6**	**12**	**5**	**7**	**3**
not calibrated	**11**	**22**	**7**	**15**	**5**

RO = relative orientation

AO = absolute orientation

min #P = min. number of control points

2
Coplanarity Constraint for Straight-Line Preserving (Uncalibrated) Cameras

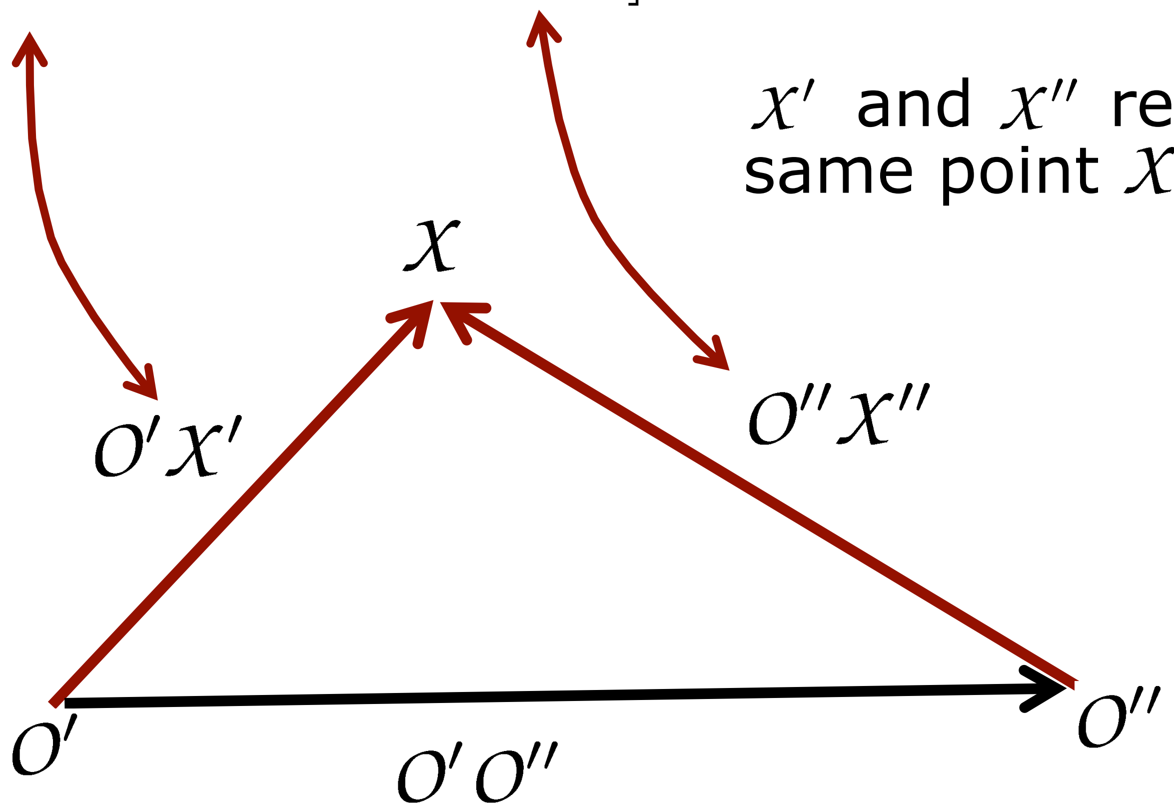
(DE: Koplanaritätsbedingung für geradentreu abbildende Kameras)

Coplanarity Constraint for Uncalibrated Cameras

Coplanarity can be expressed by

$$[O'X' \quad O'O'' \quad O''X''] = 0$$

x' and x'' refer to the same point X in space



Scalar Triple Product (DE: Spatprodukt)

- The operator $[\cdot,\cdot,\cdot]$ is the triple product
- Dot product of one of the vectors with the cross product of the other two

$$[A, B, C] = (A \times B) \cdot C$$

- It is the volume of the parallelepiped of three vectors

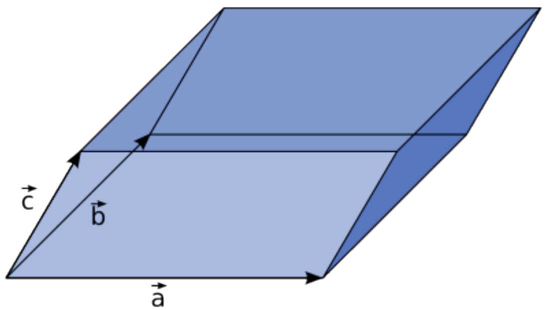


Image courtesy: Wikipedia (Niabot)

Scalar Triple Product Properties

$$[A, B, C] = (A \times B) \cdot C = A \cdot (B \times C)$$

$$[A, B, C] = (A \times B) \cdot C = -(B \times A) \cdot C = -[B, A, C]$$

$$[A, B, C] = \det \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix}$$

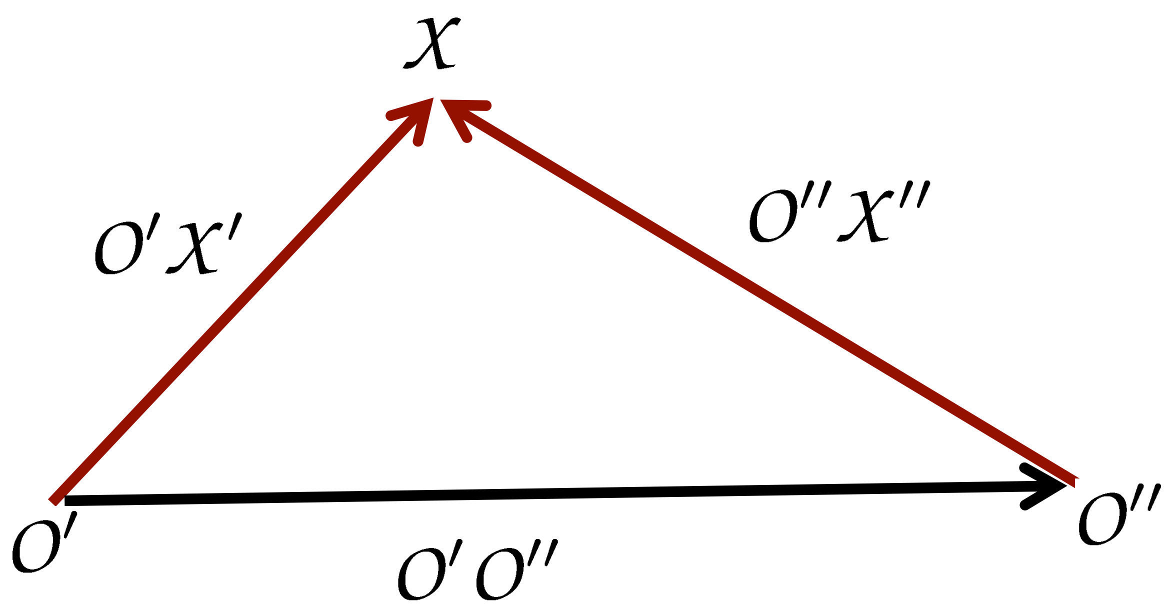
$$[A, A, B] = 0$$

$[A, B, C] = 0$ means that the three vectors lie in one plane

Coplanarity Constraint for Uncalibrated Cameras

Coplanarity can be expressed by

$$[O'X' \quad O'O'' \quad O''X''] = 0$$



Coplanarity Constraint for Uncalibrated Cameras

- The directions of the vectors $O'X'$ and $O''X''$ can be derived from the image coordinates $\mathbf{x}', \mathbf{x}''$

$$\mathbf{x}' = \mathsf{P}'\mathbf{X} \qquad \mathbf{x}'' = \mathsf{P}''\mathbf{X}$$

- with the projection matrices

$$\mathsf{P}' = \mathsf{K}'R'[I_3 | -\boldsymbol{X}_{O'}] \qquad \mathsf{P}'' = \mathsf{K}''R''[I_3 | -\boldsymbol{X}_{O''}]$$

Reminder: $[I_3 | -\boldsymbol{X}_{O''}] = \begin{bmatrix} 1 & 0 & 0 & -X_{O''} \\ 0 & 1 & 0 & -Y_{O''} \\ 0 & 0 & 1 & -Z_{O''} \end{bmatrix}$

Directions to a Point

- The normalized directions of the vectors $O''X''$ and $O'X'$ are

$$\boxed{{}^{n}\mathbf{x}' = (R')^{-1}(\mathsf{K}')^{-1}\mathbf{x}'} \longleftarrow \text{image coord.}$$

- as the normalized projection

$$ {}^{n}\mathbf{x}' = [I_3 | -\boldsymbol{X}_{O'}]\mathbf{X} \longleftarrow \text{world coord.}$$

- provides the direction to from the center of projection to the point in 3D
- Analogous:

$$ {}^{n}\mathbf{x}'' = (R'')^{-1}(\mathsf{K}'')^{-1}\mathbf{x}''$$

Base Vector

- The base vector $O'O''$ directly results from the coordinates of the projection centers

$$\boxed{\mathrm{b} = \boldsymbol{B} = \boldsymbol{X}_{O''} - \boldsymbol{X}_{O'}}$$

Coplanarity Constraint

- Using the previous relations, the coplanarity constraint

$$[O'X' \quad O'O'' \quad O''X''] \;=\; 0$$

- can be rewritten as

$$\begin{aligned} [\,{}^n\mathbf{x}' \quad \mathbf{b} \quad {}^n\mathbf{x}''] &= 0 \\ {}^n\mathbf{x}' \cdot (\mathbf{b} \times {}^n\mathbf{x}'') &= 0 \\ {}^n\mathbf{x}'^{\mathsf{T}} S_b \, {}^n\mathbf{x}'' &= 0 \end{aligned}$$

skew-symmetric matrix

Derivation

- Why is this correct?

$$
\begin{aligned}
{}^{n}\mathbf{x}' \cdot (\mathbf{b} \times {}^{n}\mathbf{x}'') &= 0 \\
{}^{n}\mathbf{x}'^{\mathsf{T}} S_b \, {}^{n}\mathbf{x}'' &= 0
\end{aligned}
$$

Derivation

- Why is this correct?

$$\begin{aligned} {}^{n}\mathbf{x}' \cdot (\mathbf{b} \times {}^{n}\mathbf{x}'') &= 0 \\ {}^{n}\mathbf{x}'^{\mathsf{T}} S_b \, {}^{n}\mathbf{x}'' &= 0 \end{aligned}$$

- Results from the cross product as

$$\underbrace{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}}_{\mathbf{b}} \times \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\mathbf{x}} = \begin{bmatrix} -b_3 x_2 + b_2 x_3 \\ b_3 x_1 - b_1 x_3 \\ -b_2 x_1 + b_1 x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{bmatrix}}_{S_b} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\mathbf{x}}$$

- with S_b being a skew-symmetric matrix

Coplanarity Constraint

- By combining ${}^n\mathbf{x}' = (R')^{-1}(\mathsf{K}')^{-1}\mathbf{x}'$ and ${}^n\mathbf{x}'^{\mathsf{T}} S_b \, {}^n\mathbf{x}'' \quad = \quad 0$
- we obtain

$$\mathbf{x}'^{\mathsf{T}}(\mathsf{K}')^{-\mathsf{T}}(R')^{-\mathsf{T}}\mathsf{S}_b(R'')^{-1}(\mathsf{K}'')^{-1}\mathbf{x}'' = 0$$

Coplanarity Constraint

- By combining ${}^{n}\mathbf{x}' = (R')^{-1}(\mathsf{K}')^{-1}\mathbf{x}'$ and ${}^{n}\mathbf{x}'^{\mathsf{T}} S_b \, {}^{n}\mathbf{x}'' \quad = \quad 0$
- we obtain

$$\mathbf{x}'^{\mathsf{T}} \underbrace{(\mathsf{K}')^{-\mathsf{T}}(R')^{-\mathsf{T}}\mathsf{S}_b(R'')^{-1}(\mathsf{K}'')^{-1}}_{\mathsf{F}} \mathbf{x}'' = 0$$

$$\begin{aligned} \mathsf{F} &= (\mathsf{K}')^{-\mathsf{T}}(R')^{-\mathsf{T}}\mathsf{S}_b(R'')^{-1}(\mathsf{K}'')^{-1} \\ &= (\mathsf{K}')^{-\mathsf{T}}R'\mathsf{S}_b R''^{\mathsf{T}}(\mathsf{K}'')^{-1} \end{aligned}$$

Fundamental Matrix (DE: Fundamentalmatrix)

- The matrix F is the **fundamental matrix** (for uncalibrated cameras):

$$\boxed{\mathsf{F} \quad = \quad (\mathsf{K}')^{-\mathsf{T}} R' \mathsf{S}_b R''^{\mathsf{T}} (\mathsf{K}'')^{-1}}$$

- It allow for expressing the **coplanarity constraint** by

$$\boxed{\mathbf{x}'^{\mathsf{T}} \mathsf{F} \mathbf{x}'' \quad = \quad 0}$$

Fundamental Matrix (DE: Fundamentalmatrix)

- The **fundamental matrix** is the matrix that fulfills the equation

$$\mathbf{x}'^{\mathsf{T}} \mathsf{F} \mathbf{x}'' = 0$$

  for corresponding points

- The fundamental matrix contains the all the available **information about the relative orientation** of **two images** from uncalibrated cameras

Alternative Definition

- In the context of many images, we will call F_{ij} that fundamental matrix which yields the constraint ${\mathbf{x}'}_i^\top \mathsf{F}_{ij} \mathbf{x}''_j = 0$
- Thus in our case, we have $\mathsf{F} = \mathsf{F}_{12}$
- Our definition of F is not the same as in the book of Hartley and Zisserman (CV)
- The definition in Hartley and Zisserman is based on ${\mathbf{x}''_i}^\top \mathsf{F}_{ij} \mathbf{x}'_j = 0$, i.e. $\mathsf{F} = \mathsf{F}_{21} = \mathsf{F}_{12}^\top$
- The transposition needs to be taken into account when comparing algebraic expressions

Fundamental Matrix From the Camera Projection Matrices

- If the projection matrices are given, we can derive the fundamental matrix
- Let the projection matrices be partitioned into a left 3×3 matrix and a 3-vector as $\mathsf{P}' = [\mathsf{A}'|\mathbf{a}']$. Then, we have

$$\mathsf{F} \quad = \quad (\mathsf{K}')^{-\mathsf{T}} R' \mathsf{S}_b R''^{\mathsf{T}} (\mathsf{K}'')^{-1} \quad = \quad \mathsf{A}'^{-\top} \mathsf{S}_{b'_{12}} \mathsf{A}''^{-1}$$

- with

$$\mathbf{b}'_{12} = \mathsf{A}''^{-1}\mathbf{a}'' - \mathsf{A}'^{-1}\mathbf{a}' \quad \text{and} \quad S_b = \begin{bmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{bmatrix}$$

Fundamental Matrix From the Camera Projection Matrices

- Fundamental matrix of the form

$$\mathsf{F} = \mathsf{A}'^{-\top} \mathsf{S}_{b'_{12}} \mathsf{A}''^{-1}$$

- is a result of the projection centers

$$\boldsymbol{X}_{O'} = -\mathsf{A}'^{-1}\mathbf{a}' \qquad \boldsymbol{X}_{O''} = -\mathsf{A}''^{-1}\mathbf{a}''$$

- and $\mathsf{A}' = \mathsf{K}'R', \quad \boldsymbol{X}_{O''} = -\mathsf{A}''^{-1}\mathbf{a}''$
- This yields $\mathbf{b}'_{12} = \mathsf{A}''^{-1}\mathbf{a}'' - \mathsf{A}'^{-1}\mathbf{a}'$

See: Förstner, Wrobel: Photogrammetric Computer Vision, Chapter 12.2 ("The Geometry of the Image Pair")

4
Testing Point Correspondences

Correspondence Test for Two Points in the Image Plane

- We can exploit the coplanarity constraint to test for the correspondence of two points
- For correspondence, the residual

$$\begin{aligned} w &= \mathbf{x}'^{\mathsf{T}} \mathsf{F} \mathbf{x}'' = \mathrm{vec}(\mathbf{x}'^{\mathsf{T}} \mathsf{F} \mathbf{x}'') \\ &= (\mathbf{x}'' \otimes \mathbf{x}')^{\mathsf{T}} \mathrm{vec} \mathsf{F} = (\mathbf{x}'' \otimes \mathbf{x}')^{\mathsf{T}} \mathbf{f} \end{aligned}$$

should be zero (the operator $\otimes$ is the Kronecker product, see next slide)

Kronecker Product

- The Kronecker product is a special product of matrices and defined as

$$A \otimes B = \begin{bmatrix} A_{11}B & \dots & A_{1n}B \\ \dots & \dots & \dots \\ A_{m1}B & \dots & A_{mn}B \end{bmatrix}$$

- Example

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \otimes \begin{pmatrix} 7 & 8 \\ 9 & 0 \end{pmatrix} = \begin{pmatrix} 1\begin{pmatrix} 7 & 8 \\ 9 & 0 \end{pmatrix} & 2\begin{pmatrix} 7 & 8 \\ 9 & 0 \end{pmatrix} \\ 3\begin{pmatrix} 7 & 8 \\ 9 & 0 \end{pmatrix} & 4\begin{pmatrix} 7 & 8 \\ 9 & 0 \end{pmatrix} \\ 5\begin{pmatrix} 7 & 8 \\ 9 & 0 \end{pmatrix} & 6\begin{pmatrix} 7 & 8 \\ 9 & 0 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 7 & 8 & 14 & 16 \\ 9 & 0 & 18 & 0 \\ 21 & 24 & 28 & 32 \\ 27 & 0 & 36 & 0 \\ 35 & 40 & 42 & 48 \\ 45 & 0 & 54 & 0 \end{pmatrix}$$

Correspondence Test

- In reality, $w = (\mathbf{x}'' \otimes \mathbf{x}')^\mathsf{T}\mathbf{f}$ is seldom =0
- w has the variance

$$\sigma_w^2 = \left(\frac{\partial w}{\partial \mathbf{x}'}\right) \Sigma_{x'x'} \left(\frac{\partial w}{\partial \mathbf{x}'}\right)^\mathsf{T} + \left(\frac{\partial w}{\partial \mathbf{x}''}\right) \Sigma_{x''x''} \left(\frac{\partial w}{\partial \mathbf{x}''}\right)^\mathsf{T} + \left(\frac{\partial w}{\partial \mathbf{f}}\right) \Sigma_{ff} \left(\frac{\partial w}{\partial \mathbf{f}}\right)^\mathsf{T}$$

- Direct result from the variance propagation (DE: Varianzfortpflanzung)
- Assumes known errors on $\mathbf{x}'$ and $\mathbf{x}''$ and in the elements of F

Correspondence Test

- In reality, $w = (\mathbf{x}'' \otimes \mathbf{x}')^{\mathsf{T}}\mathbf{f}$ is seldom =0
- w has the variance

$$\sigma_w^2 = \left(\frac{\partial w}{\partial \mathbf{x}'}\right) \Sigma_{x'x'} \left(\frac{\partial w}{\partial \mathbf{x}'}\right)^{\mathsf{T}} + \left(\frac{\partial w}{\partial \mathbf{x}''}\right) \Sigma_{x''x''} \left(\frac{\partial w}{\partial \mathbf{x}''}\right)^{\mathsf{T}}$$

$$+ \left(\frac{\partial w}{\partial \mathbf{f}}\right) \Sigma_{ff} \left(\frac{\partial w}{\partial \mathbf{f}}\right)^{\mathsf{T}}$$

- where

$$\left(\frac{\partial w}{\partial \mathbf{x}'}\right) = \mathbf{x}''^{\mathsf{T}} \mathsf{F}^{\mathsf{T}} \qquad \left(\frac{\partial w}{\partial \mathbf{x}''}\right) = \mathbf{x}'^{\mathsf{T}} \mathsf{F} \qquad \left(\frac{\partial w}{\partial \mathbf{f}}\right) = (\mathbf{x}'' \otimes \mathbf{x}')^{\mathsf{T}}$$

See: Förstner, Wrobel: Photogrammetric Computer Vision, Chapter 12.2.3 (“The Geometry of the Image Pair”)

Correspondence Test

- Given the variance, we can formulate a significance test with

$$z = \frac{w_i}{\sigma_{w_i}} \sim N(0,1)$$

- where

$$z = \frac{(\mathbf{x}'' \otimes \mathbf{x}')^{\mathsf{T}} \mathbf{f}}{\sqrt{\mathbf{x}''^{\mathsf{T}} \mathsf{F}^{\mathsf{T}} \Sigma_{x'x'} \mathsf{F} \mathbf{x}'' + \mathbf{x}'^{\mathsf{T}} \mathsf{F} \Sigma_{x''x''} \mathsf{F}^{\mathsf{T}} \mathbf{x}' + (\mathbf{x}'' \otimes \mathbf{x}')^{\mathsf{T}} \Sigma_{ff} (\mathbf{x}'' \otimes \mathbf{x}')}}$$

- Note: test value is point-dependent!

Correspondence Test

- Given the variance, we can formulate a significance test with

$$z = \frac{w_i}{\sigma_{w_i}} \sim N(0,1)$$

- The test allows us to discard a hypothesis if $|z| > k_\alpha$ where k_α defines the threshold for the confidence level, e.g., $k_\alpha = 1.96$ for $\alpha = 5\%$

Next Week:
Computing the
Fundamental Matrix
from Corresponding Points

Fundamental Matrix from Corresponding Points

- The coplanarity constraint is bilinear in the homogenous image coordinates $\mathbf{x}'$ and $\mathbf{x}''$ and linear in the elements of the fundamental matrix F
- This is the basis for a simple determination of the fundamental matrix from corresponding points

Degrees of Freedom

- The fundamental F matrix has seven degrees of freedom. This is because F is homogeneous and singular, as the skew symmetric matrix S_b is singular with rank two.
- Any matrix of the form

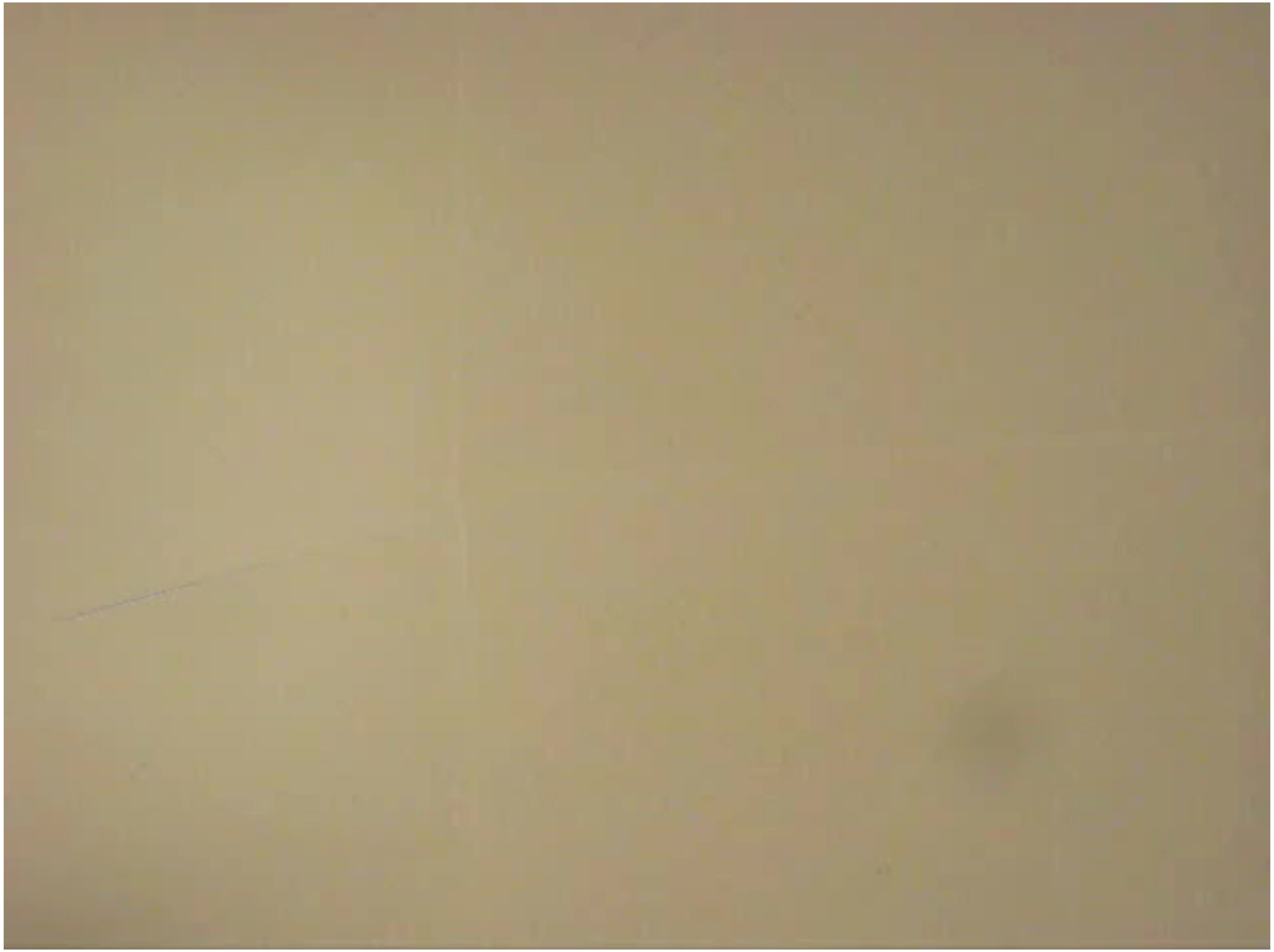
$$\mathsf{F} = U\mathrm{Diag}(s_1, s_2, 0)V^{\mathsf{T}} \quad \text{with } s_i > 0$$

- with orthogonal matrices U and V is a fundamental matrix

Corresponding Points

- We need **7 corresponding points** to compute the fundamental matrix
- We will study a **direct method that needs 8 points** (next week)

The Fundamental Matrix Song



Video courtesy: Daniel Wedge
http://danielwedge.com/fmatrix/

Summary

- Geometry of image pairs
- Relative orientation
- Absolute orientation
- Corresponding points
- Fundamental matrix
- Correspondence test

Literature

- Förstner, Skript Photogrammetrie II, Chapter 1
- Förstner, Wrobel: Photogrammetric Computer Vision, Ch. 12.2.1 – 12.2.2

Slide Information

- The slides have been created by Cyrill Stachniss as part of the photogrammetry and robotics courses.
- I tried to acknowledge all people from whom I used images or videos. In case I made a mistake or missed someone, please let me know.
- The photogrammetry material heavily relies on the very well written lecture notes by Wolfgang Förstner and the Photogrammetric Computer Vision book by Förstner & Wrobel.
- Parts of the robotics material stems from the great Probabilistic Robotics book by Thrun, Burgard and Fox.
- If you are a university lecturer, feel free to use the course material. If you adapt the course material, please make sure that you keep the acknowledgements to others and please acknowledge me as well. To satisfy my own curiosity, please send me email notice if you use my slides.

Cyrill Stachniss, cyrill.stachniss@igg.uni-bonn.de, 2014