Week 5 HW

① Prove that e is irrational

Assume e is rational

$$e = \frac{a}{b}, \quad \text{where } b \neq 0$$

Proof by contradiction (since it's a negative claim, not rational.)

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}$$ → an indefinite series definition.

$$e = \sum_{n=0}^{b} \frac{1}{n!} + \sum_{n=b+1}^{\infty} \frac{1}{n!}$$

$$S = \sum_{n=0}^{b} \frac{1}{n!}$$

$$R = \sum_{n=b+1}^{\infty} \frac{1}{n!}$$

so $e = S + R$

Multiply both sides by b.

$$b! \cdot e = b! \cdot S + b! \cdot R$$

Let's call :

$A = b! \cdot e$ (this is an int because $e = \frac{a}{b}$, so $b! \cdot e = a(b-1)!$)

$B = b! \cdot S$ - also an int (sum of integers)

so $A - B = b! \cdot R$

Call $x = b! \cdot R$ ⇒ Then $x = A - B$ (x is an integer)

$$X = \sum_{n=b+1}^{\infty} \frac{b!}{n!}$$

$$\frac{b!}{(b+1)!} = \frac{1}{b+1}, \quad \frac{b!}{(b+2)!} = \frac{1}{(b+1)(b+2)},$$

$$\frac{b!}{(b+3)!} = \frac{1}{(b+1)(b+2)(b+3)} \text{ etc.}$$

So:

$$X = \frac{1}{b+1} + \frac{1}{(b+1)(b+2)} + \frac{1}{(b+1)(b+2)(b+3)} + \ldots$$

$$\frac{1}{b+1} + \frac{1}{(b+1)^2} + \frac{1}{(b+1)^3} + \ldots = \frac{\frac{1}{b+1}}{1 - \frac{1}{b+1}} = \frac{1}{b}$$

So, $0 < X < \frac{1}{b}$

But earlier we showed that X is an integer so it's a contradiction.

<u>Problem 2</u>: Use a direct proof to show that every odd integer is the difference of 2 squares.

$p(n) \rightarrow q(n)$

n - is odd (assume).

$n = m^2 - k^2 = (m-k)(m+k)$

$n = 2k+1$

$n = (m-k)(m+k)$

$m = \frac{n+1}{2} \qquad k = \frac{n-1}{2}$

$a^2 - b^2 = n$

③ Use a proof by contradiction to prove that the sum of an irrational number & a rational number is irrational.

$\cancel{\sqrt{2}+n=k}$

$\overset{\text{irr. num}}{s} = \overset{\text{rat.}}{r} + \overset{\text{irr.}}{i}$

assume s is rational.

$i = s - r$

irrat. nums.

if $s = \frac{a}{b}$ and $r = \frac{c}{d}$, $b, d \neq 0$

$s + (-r) = \frac{ad - bc}{bd}$ so $s+(-r)$ is a rat. num.

but $s + (-r) = \underbrace{r + i}_{s} - r = i$ irrat. num

this contradicts our hypothesis that r is rational.

④ Prove or disprove that the ~~sum~~ product of two irrational numbers is irrational.

$\overset{\text{irr}}{s} \times \overset{\text{irr}}{i} = \overset{\text{irr}}{r}$

$\sqrt{2} \times \sqrt{2} = 2$ → the product of 2 irr. nums is a rational number.

This counter example refutes the proposition.

Problem 5: Use a proof by contraposition to show that if $\underbrace{x+y \geq 2}_{p}$, where x and y are real numbers, then $\underbrace{x \geq 1}_{q}$ or $\underbrace{y \geq 1}_{r}$.

$p \to (q \vee r)$
$\neg(q \vee r) \to \neg p$
$\neg q \wedge \neg r = \neg p$

Real nums = rat. nums + irrat. nums.

If it's not true that $x \geq 1$ or $y \geq 1$
then it's not true $x + y \geq 2$

Assume it's not true $x \geq 1$ or $y \geq 1$
then by Morgan's law
$x < 1$ and $y < 1$
$x + y < 2$ this is the negation of $x+y \geq 2$ so proof is complete.

Problem 6: Show that if n is an integer and $\underbrace{n^3+5 \text{ is odd}}_{p}$, then $\underbrace{n \text{ is even}}_{q}$ using

a) proof by contraposition $\neg q \to \neg p$
b) proof by contradiction.

a) Proof by contrapositive:
Assume, if $n \to$ ~~even~~ odd
$n^3+5 \to$ is ~~odd~~ even

$n = 2k + 1.$
$n^3+5 = (2k+1)^3 + 5 = 8k^3 + 8k^2 + 1 + \cancel{5} =$
$= 2(4k^3 + 6k^2 + 3k + 3).$
Thus $n^3+5 \to$ is two times some int, so it is even.

p $\quad\quad\quad\quad\quad\quad\quad\quad$ $\neg q$

b) Suppose than n^3+5 is odd & that n is odd

Since n is odd n^3 is odd

$$\underset{odd}{(n^3+5)} - \underset{odd}{n^3} = \underline{\underline{5}}$$

→ but should be even the supposition was wrong, proof by contradiction is complete.

7 The barber is the one who shaves all those men who do not shave themselves. The Q is, does the barber shave himself?

U - all men in the community

Let: $S: U \to \{T, F\}$ be the

All men such that $\forall x \in U: B(x) \Leftrightarrow x$ is shaved by barber

The initial premises can be written in formal logic as.

$$\underbrace{\forall x \in U: (\neg S(x))}_{\text{all men that don't shave themselves}} \Leftrightarrow \underbrace{B(x)}_{\text{are shaved by barber.}}$$

$B(b) \Leftrightarrow S(b)$

→ Barber is shaved by himself, so he is ~~the~~ of those who shave themselves.

$$\overset{p}{S(b)} \Leftrightarrow \overset{q}{B(b)} \Leftrightarrow \overset{\neg p}{(\neg S(b))}$$

$$\vdash ((p \Leftrightarrow q) \wedge (q \Leftrightarrow r)) \Rightarrow (p \Leftrightarrow r)$$

If p implies q and q implies r then p implies r.

We have: $S(b) \Leftrightarrow (\neg S(b))$ - This is a contradiction, so the initial premises are contradictory & cannot ~~hold~~ hold.

Problem 8: Show that if x and y are integers and both xy and $x+y$ are even, then both x and y are even.

Assume x is odd
$x = 2m + 1$

We need to show xy is odd
or $x+y$ is odd.

Consider 2 cases: 1) y - even
then $y = 2n$
$x+y = (2m+1) + 2n = 2(m+n)+1$ is odd

2) y - odd
$y = 2n+1$
$xy = (2m+1)(2n+1) = 4mn + 2m + 2n + 1 =$
$= 2(2mn + m + n) + 1$ is odd

This proves the contrapositive,
if x and y are not both even, then either $x+y$ or xy is odd.

This means the original st-t is also true.