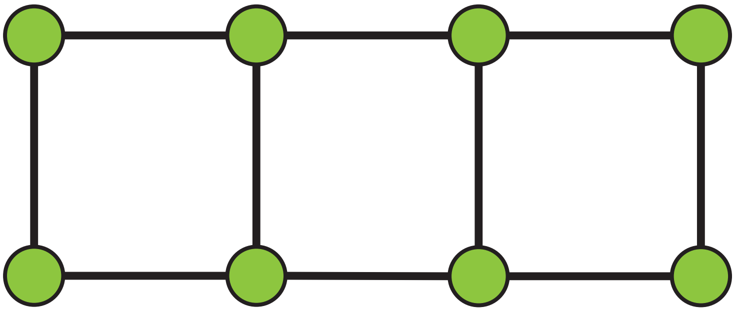
1.

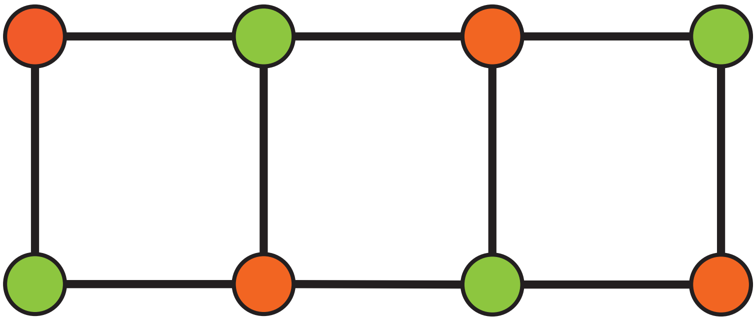
(a)

This problem can be illustrated mathematically as a graph of 8 nodes, with each node having edges connected to its closest neighbours, shown as:



(b)

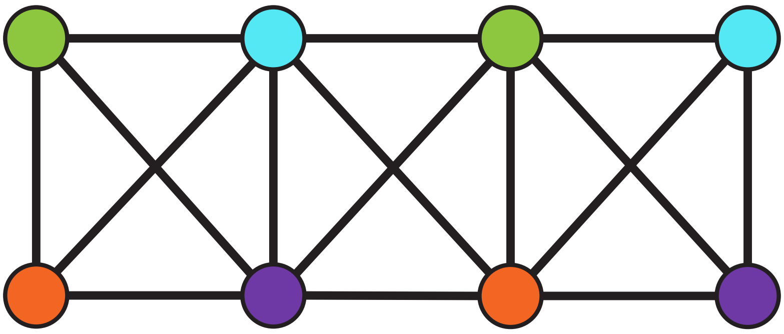
The number of Wi-Fi channels needed equals to the chromatic number of the above graph, which is 2.



(c)

Each vertex will also be connected to its diagonal vertices.

The largest clique size will become 4, and the chromatic number will be 4.



2.

(a)

A_v = $P_{v,red} \lor P_{v,green} \lor P_{v,blue}$

(b)

B_v

$= \neg (P_{v,red} \land P_{v,green}) \land \neg (P_{v,red} \land P_{v,blue}) \land \neg (P_{v,green} \land P_{v,blue})$

$= (\neg P_{v,red} \lor \neg P_{v,green}) \land (\neg P_{v,red} \lor \neg P_{v,blue}) \land (\neg P_{v,green} \lor \neg P_{v,blue})$

(c)

$C_{u,v}$

$= \neg(P_{u,red} \land P_{v,red}) \land \neg(P_{u,green} \land P_{v,green}) \land \neg(P_{u,blue} \land P_{v,blue})$)

$=(\neg P_{u,red} \lor \neg P_{v,red}) \land (\neg P_{u,green} \lor \neg P_{v,green}) \land (\neg P_{u,blue} \lor \neg P_{v,blue})$

(d)

With the three-color problem, each vertex has and only has 1 color ($A_v \land B_v$), and each pair of vertices $\{u,v\} \in E$ must have different colors ($C_{u,v}$).

$\varphi_G = \bigwedge_{v \in V}(A_v \land B_v) \land \bigwedge_{\{u,v\} \in E} C_{u,v}$

3.

(a)

(R) $x \lor x = x, x \in A$

$\sqsubseteq$ is reflective.

(AS) For $x,y \in A$, if $x \lor y = y$ and $y \lor x = x$:

$\because x \lor y = y$:

$y \lor x = y$

$\because y \lor x = x$:

$y \lor x = y = x$

$\therefore x = y$

$\sqsubseteq$ is asymmetry.

(T)

For x,y,z $\in$ A, if $x \vee y = y$ and $y \vee z = z$:

$y = x \vee y$

$\because y \vee z = z$

$x \vee y \vee z = z$

$\because y \vee z = z$

$x \vee z = z$

$\sqsubseteq$ is transisive.

Therefore, $\sqsubseteq$ is a partial order.

(b)

$\sqsubseteq$ corresponds to $x \subseteq y$ for $x \subseteq X$, $y \subseteq X$.

(c)

$$
\begin{aligned}
&(x \vee y) \leftrightarrow y \\
&= (\neg(x \vee y) \vee y) \wedge ((x \vee y) \vee \neg y) \\
&= ((\neg x \wedge \neg y) \vee y) \wedge (x \vee (y \vee \neg y)) \\
&= (\neg x \wedge \neg y) \vee y \\
&= (\neg x \vee y) \wedge (\neg y \vee y) \\
&= \neg x \vee y \\
&= x \rightarrow y
\end{aligned}
$$

4.

(a)

[B] n=0:

$add(n, 0) = n = 0$

$add(0, n) = add(0, 0) = 0$

$\therefore$ for n=0, $add(n, 0) = add(0, n) = 0$

[I] $\forall$ n $\in \mathbb{N}$, hypothesize P(n) $\rightarrow$ P(n + 1)

add(0, n + 1)

= add(0, n) + 1

= add(n, 0) + 1 (IH)

= n + 1

= add(n + 1, 0)

Conclusion: $\forall$ n $\in \mathbb{N}$, add(n, 0) = add(0, n)

(b)

Thereom 1:

Let R(n) be proposition:

R(n): for n $\in \mathbb{N}$, hypothesize add(m+1, n) = add(m, n) + 1

[B] n = 0

add(m+1, 0) = m + 1

= add(m, 0) + 1

[I] $\forall$ n $\geqslant$ 0 , R(n) $\rightarrow$ R(n + 1)

add(m + 1, n + 1)

= add(m + 1, n) + 1

= (add(m, n) + 1) + 1 (IH)

= add(m, n+1) + 1

Conclusion: for n $\geqslant$ 0 , add(m+1, n) = add(m, n) + 1

Thereom 2:

Let S(x) be proposition:

x $\in \mathbb{Z}$, x+m $\geqslant$ 0, add(m + x, n) = add(m, n)+ x

[B]

x =0:

add(m, n) = add(m, n)

x =1:

$\text{add}(m + 1, n) = \text{add}(m, n) + 1$

$x = -1$ ($m>0$):

$\text{add}(m, n)$

$= \text{add}(m-1 +1, n)$

$=\text{add}(m-1, n) +1$

$\therefore \text{add}(m-1, n) = \text{add}(m, n) -1$

[I]

$\forall x \geqslant 1$, hypothesize $S(x) \rightarrow S(x + 1)$

$\text{add}(m + x + 1, n)$

$= \text{add}(m + x, n) + 1$ (Thereom 1)

$= (\text{add}(m, n) + x) + 1$ (IH)

$= \text{add}(m, n) + (x + 1)$

$\forall x \leq -1, x + m \geqslant 1$, hypothesize $S(x) \rightarrow S(x - 1)$

$\text{add}(m + x, n)$

$= \text{add}(m + x - 1+1, n)$

$= \text{add}(m + x - 1, n) +1$ $(x + m \geqslant 1)$(Thereom 1)

$\therefore \text{add}(m + x - 1, n)$

$= \text{add}(m + x, n) -1$ $(x + m \geqslant 1)$

$= \text{add}(m, n) + (x - 1)$ (IH)

Conclusion: For $x\in\mathbb{Z}$, $x+m \geqslant 0$, $\text{add}(m + x, n) = \text{add}(m, n)+ x$

Thereom 3:

Let $T(x)$ be proposition:

$x\in\mathbb{Z}$, $x+n \geqslant 0$, $\text{add}(m, n + x) = \text{add}(m, n)+ x$

[B]

$x =0$:

$\text{add}(m, n) = \text{add}(m, n)$

$x =1$:

$add(m, n + 1) = add(m, n) + 1$

$x = -1$ $(n>0)$:

$add(m, n)$

$= add(m, n - 1 + 1)$

$= add(m, n - 1) + 1$

$\therefore$ $add(m, n - 1) = add(m, n) - 1$

[I]

$\forall x \geqslant 1$, hypothesize $T(x) \to T(x + 1)$

$add(m, n + x + 1)$

$= add(m, n + x) + 1$

$= (add(m, n) + x) + 1$ (IH)

$=$ $add(m, n) + (x + 1)$

$\forall x \leq -1, x + n \geqslant 1$, hypothesize $T(x) \to T(x - 1)$

$add(m , n + x)$

$= add(m, n + x - 1 + 1)$

$= add(m, n + x - 1) + 1$ $(x + n \geqslant 1)$(Thereom 1)

$\therefore add(m, n + x - 1)$

$= add(m, n + x) - 1$ $(x + n \geqslant 1)$

$= add(m, n) + (x-1)$ (IH)

Conclusion: For $x \in \mathbb{Z}$, $x+n \geqslant 0$, $add(m, n + x) = add(m, n) + x$

Proof of question:

Let $U(n) = P(n) \wedge Q(n)$

$U(n)$: If $a + b = n$ and $a, b \geqslant 0$ then $add(a, b) = add(b, a)$, $n \in \mathbb{N}$

[B] $n = 0$:

$add(a, 0) = add(0, a)$

[I] $\forall n \geqslant 0$, hypothesize $U(n) \to U(n+1)$

$\because n = a + b$

Let $x, y \in \mathbb{Z}$, $x + y = 1$, $x \geq -a$, $y \geq -b$, $a1 = a + x$, $b1 = b + y$.

$\therefore a1 + b1 = n + 1$, $a1 \in \mathbb{N}$, $b1 \in \mathbb{N}$

add(a1, b1)

= add(a + x, b + y)

= add(a, b + y) + x (Thereom 2)

= add(a, b) + x + y (Thereom 3)

= add(a, b) + x + y

add(b1, a1)

= add(b + y, a + x)

= add(b, a + x) + y (Thereom 2)

= add(b, a) + y + x (Thereom 3)

= add(a, b) + x + y (IH)

= add(a1, b1)

$\because a + b = n$ and $a1 + b1 = n + 1$,

$U(n) \rightarrow U(n+1)$

Conclusion: $\forall n \in \mathbb{N}, U(n)$

$(P(n) \wedge Q(n)) \rightarrow Q(n)$

$= \neg P(n) \vee \neg Q(n) \vee Q(n)$

$= T$

$\therefore U(n) \vDash Q(n)$

$\therefore Q(n)$ holds for all n

5.

(a)

rec_a:

$T(0) = O(1)$

$T(1) = O(1)$

$T(n)$

$= T(n-1) + T(n-2) + O(1)$

$= 2T(n-2) + T(n-3) + O(1)$

$= 3T(n-3) + 2T(n-4) + O(1)$

$= 5T(n-4) + 3T(n-5) + O(1)$

$= fib(x)(n-x+1) + fib(x-1)T(n-x) + O(1)$

$= ...$

$= \sum_{x=0}^{n} fib(x) + n\, O(1)$

fib is the fibonacci sequence, $fib(x) = fib(x-1) + fib(x-2)$

Hypothesize T(n) is polynomial, $T(n) = n^a$: a is a constant, $a > 0$, $a \in \mathbb{R}$

$T(n+1)$

$= T(n) + T(n-1) + O(1)$

$= O(n^a + n^{a-1})$

$\because T(n+1) = O(n^{a+1})$

$n^{a+1} = n^a + n^{a-1}$

$n^2 - n - 1 = 0$

This equation cannot apply for any arbitary $n > 1$

$\therefore$ T(n) is not polynomial.

Hypothesize T(n) is exponential, $T(n) = a^n$: a is a constant, $a > 0$, $a \in \mathbb{R}$

$T(n+1)$

$= T(n) + T(n-1) + O(1)$

$= O(a^n + a^{n-1})$

$= O((a+1)a^{n-1})$

$\because T(n+1) = O(a^{n+1})$

$a^{n+1} = (a+1)a^{n-1}$

$a^2 = a+1$

$a^2 - a - 1 = 0$

$a = (\sqrt{5}+1)/2$

$\therefore$ Let $F = \frac{\sqrt{5}+1}{2}$,

$T(n) = O(F^n)$

$\approx O(1.6^n)$

Proof:

Thereom 1:

F^2

$= (\sqrt{5}+1)^2/4$

$= (5+2\sqrt{5}+1)/4$

$= (4+2\sqrt{5}+2)/4$

$= 1+ (\sqrt{5}+1)/2$

$= F + 1$

Let proposition P(m):

$n, m \in \mathbb{N}, \forall\ m > 1$:

$\forall\ n \leqslant m, T(n) = O(F^n)$

[B]

$T(0) = 1$

$T(1) = 1$

$T(2) = 5$

[I]

$\forall\ m > 1$, hypothesize $P(m) \rightarrow P(m+1)$

$T(n+1)$

$= T(n) + T(n\text{-}1) + O(1)$

$= O(F^n) + O(F^{n-1}) + O(1)$ (IH)

$= (F+1)O(F^{n-1}) + O(1)$

$= F^2\ O(F^{n-1}) + O(1)$ (Thereom 1)

$= O(F^{n+1})$

Conclusion: $\forall\ m > 1$, P(m)

iter_a:

$T(n) = T(n-1) + O(n)$

$T(n) = O(n)$

(b)

n	a_n	$a_n + 2^n$
0	0	1
1	1	3
2	5	9
3	19	27
4	65	81
5	211	243
6	665	729
7	2059	2187
8	6305	6561

....

Guess:

$a_n + 2^n = 3^n$, $n = 0, 1, 2, 3, ...$

$a_n = 3^n - 2^n$

Proof:

[B]

$a_0 = 0 = 3^0 - 2^0$

$a_1 = 1 = 3^1 - 2^1$

[I]

hypothesize $a_n \rightarrow a_{n+1}$ for $n \in \mathbb{N}$

a_{n+1}

$= 5 a_n - 6 a_{n-1}$

$= 5 * 3^n – 5 * 2^n – 6* (3^{n-1} - 2^{n-1})$

$= 15 * 3^{n-1} – 6 * 3^{n-1} - 10 * 2^{n-1} + 6* 2^{n-1}$

$= 9 * 3^{n-1} – 4 * 2^{n-1}$

$= 3^{n+1} – 2^{n+1}$

Conclusion: $a_n = 3^n - 2^n$

(c)

Use the binary power algorithm to directly compute: $a_n = 3^n - 2^n$.

```
p = 1
q = a
i = n
while i > 0 do{
      if i is odd then
            {p = p * q}
      q = q * q
      i = floor(i/2)
}
end while
return p
```

Complexity:

$T(n) = T(n / 2) + O(1)$

$T(n) \in O(logn)$.