Restricted Power Domination on Trees via Mingling Power Domination

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March 21, 2025

Abstract

The special form of the restricted power domination problem, where vertices in the forbidden zone are adjacent to no more than one other forbidden vertex, is examined and solved on tree graphs. A linear-time algorithm is provided to find power-dominating sets given certain forbidden zones over the vertices of the graph, and its correctness is proven.

1 Introduction

The problem of restricted power domination arises when calculating the placement of phasor management units, or PMUs, on a power grid. Initially, the PMU placement problem was proposed by power companies, as building and maintaining PMU devices can be very costly. Any given placement of PMUs should observe the entire network. That is, to be able to calculate, with physical laws, the state of any electrical node in the network using the limited data that the PMUs provide. By modeling the power grid as a graph G, with the vertices $V(G)$ representing electrical nodes, and the edges $E(G)$ representing the transmission lines, the PMU placement can be rewritten in terms of a graph-theoretical problem. For any vertex in G, the corresponding electrical node may or may not have a PMU, and in the case that it does have a PMU, we say that the vertex is "observed". Other vertices and edges can also be observed, according to a set of rules, outlined by Haynes et al., which are as follows [2]:

1. Any vertex that is incident to an observed edge is observed.
2. Any edge joining two observed vertices is observed.
3. If a vertex is incident to a total of $k > 1$ edges and if $k - 1$ of these edges are observed, then all k edges are observed.

However, these rules can be a bit difficult to work with. Because of this, there has been much research done to simplify the rules outlined by Haynes et al. [1, 3, 4]. This research was then summarized by Pai et al. in terms of a graph coloring game [6].

The game starts by choosing an initial set of vertices, $S \subseteq V(G)$, to color blue, where each vertex in S corresponds to a placement of a PMU. The blue-colored vertices will represent any electrical nodes observed by the PMU. At this point, all other vertices not in S are colored white. The game then proceeds by repeatedly applying a set of rules until either the entire graph is colored blue, or until applying the rules do not cause any more vertices to be colored. The rules, known as observation rules, are defined below.

Definition 1.1. *Observation rules*

1. OR1: Any vertex u adjacent to a vertex $v \in S$ is colored blue at the very beginning.
2. OR2: If a blue vertex u with degree $d > 1$ has only one white neighbor v, then v also gets colored blue.

The goal of the game is then to find a set S so that applying the observation rules will color all of a graph G blue, which in terms of the electrical network corresponds to all nodes being observed. Such a set S is known as a power dominating set of G, or a PD set of G for short. The power-dominating number of G, which will be denoted as $\gamma_P(G)$, is the cardinality of the smallest possible PD set of G. Note that there might be multiple smallest possible PD sets of G, and we'll refer to such sets as minimum power-dominating sets of G, or MPD sets of G for short.

Finding MPD sets of graphs can be a hard problem. For example, Haynes et al. showed it to be an NP-complete problem for bipartite and chordal graphs [2]. However, linear-time solutions have been found for certain classes of graphs, such Lyu's label-based algorithm for tree graphs [5]. The algorithm shown in this paper is based off of Lyu's algorithm, which has been modified to work with situations where the placement of PMUs are restricted in a certain way. We first lay down some preliminary information, and then go on to describe the algorithm itself. After that, the correctness of the algorithm is proven, and then the paper ends with a course reflection.

2 Preliminaries

It is important to first define the restricted power domination problem, as well as a variant of the power domination problem, known as the mingling power domination problem.

Definition 2.1. *Forbidden zones, restricted power domination, and over restriction*

A forbidden zone F of a graph G is a subset of $V(G)$ where no PMUs can be placed. A set $S \subseteq V(G)$ is a restricted power dominating set, or RPD-F set, of G with respect to a forbidden zone F if S is a PD set such that $S \cap F = \varnothing$. The restricted power dominating number of a graph G with respect to a forbidden zone F is the cardinality of the smallest RPD-F set of G, which we'll denote as $\gamma_P(G, F)$. Such an RPD-F set is called a minimum restricted power dominating set with respect to F, or a MRPD-F set for short. Just like with MPD sets, it is possible for a graph to have multiple MRPD-F sets.

It is possible for no RPD-F set to exist for some forbidden zone F. If this is the case, we say that F is overly restrictive of G. An easy way to tell if F is overly restrictive of G is to first choose a starting set of vertices $S = V(G) \setminus F$. Then, by applying the observation rules, one can determine whether S is an RPD-F set. If S is not an RPD-F set of G, then F is overly restrictive, as S is maximal with respect to the forbidden zone. In this case, an RPD-F set does not exist, so $\gamma_P(G, F)$ does not exist either.

Definition 2.2. *Mingling power domination*

Mingling power domination is a variation of power domination proposed by Lyu that makes use of labels to track the state of vertices in the graph [5]. It allows one to remove vertices from a graph, then update the new graph with new labels that provide useful information about MPD sets of the original graph. The definition provided here is based on Lyu's definition.

Mingling power domination starts with a graph G that has labels $(L_P(u), L_Q(u))$ for each vertex $u \in V(G)$, where $L_P(u) \in \{1, 2\}$ and $L_Q(u) \in \mathbb{Z}^+$. The liberate set, $S_L = \{u : L_P(u) = 2\}$, corresponds to the set of blue-colored vertices in standard power domination, and likewise, the barrier set, $S_B = \{u : L_P(u) = 1\}$, corresponds to the set of white-colored vertices. Lastly, the necessary set is defined as $S_N = \{u : L_Q(u) \geq 3\}$. Just like with standard power domination, there are two observation rules, with the second one being applied until either $V(G) = S_L$ or until no labels change between before and after applying the rule. Likewise, a set S is a mingling power dominating set (which we'll abbreviate as an mPD set) if $S_N \subseteq S$ and S causes all vertices to become part of the liberate set. Furthermore, a set S is a restricted mingling power dominating set with respect to a forbidden zone F (which we'll denote as an mRPD-F set) if $S \cap F = \varnothing$ for some mPD set S. Once again, just like with normal restricted power domination, there is a notion of a minimum restricted mingling power dominating set with respect to a forbidden zone, F, which is an mRPD-F set, S, such that S contains the fewest possible number of vertices. Such a set will be written as an mMRPD-F set. The mingling restricted power domination number of a graph G given a forbidden zone F is the cardinality of mMRPD-F set, S, which we will denote as $\gamma_P^m(G, F)$.

Below are the modified observation rules for mingling power domination:

1. OR1': For any $v \in N_G[S]$, we have $L_P(v) = 2$
2. OR2': If $L_P(v) = 2$, $L_Q(v) = 1$, and $N_G(v) \cup S_B = \{u\}$, then $L_P(u) = 2$

Note that these are actually similar to the rules of standard power domination, except that it is rewritten in terms of labels, and there is the added condition in OR2' that $L_Q(v) = 1$ in order for a vertex u to enlighten a vertex v. Furthermore, the problem of mingling power domination is the exact same as the problem of regular power domination if one sets the labels of all vertices in a graph G to $(1, 1)$. Thus, if one can find an mMRPD-F set of such a graph G, that set is also an MRPD-F set, and so $\gamma_P^m(G, F) = \gamma_P(G, F)$.

Definition 2.3. *Notation for leafs*

Let T be a tree, and $uv \in E(T)$ such that $\deg v = 1$. Then we call v a leaf of T, and uv a branch of T. We'll denote such an edge uv by writing $u \to v$.

It's important to note that for any $u \to v$ branch in T, $T - v$ is also a tree. This is true because removing v from T doesn't cause the tree to become disconnected, as $\deg v = 1$, so removing v will not result in more components. Furthermore, $T - v$ will not contain any cycles, as it is a subgraph of a graph with no cycles, which satisfies the second condition for being a tree.

3 Solution and Proof of Correctness

The algorithm for finding restricted power dominating sets on trees is similar to that of finding normal power dominating sets on trees. However, there are a few differences. If a tree T has any leafs that are part of the forbidden zone, they must be removed first before any non-forbidden vertices can be considered. This means that, unlike Lyu's approach, one cannot rely on a tree ordering. There are a few additional rules for the updating of labels upon removal of vertices to account for additional restrictions as well.

Firstly, removal of restricted leafs works the same as removal of normal vertices in Lyu's algorithm. The main difference comes to the removal leafs that are not in the forbidden zone, but are adjacent to a vertex that is. The exact algorithm is outlined below:

1. Let T be a tree with a forbidden zone F such that each vertex in F is adjacent to at most one other vertex in F.
2. For each $v \in V(T)$, let $L_Q(v) = 1$ and $L_P(v) = 1$, thus setting the label of each vertex in T to $(1, 1)$.
3. If there exists a $u \to v$ branch such that $v \in F$, follow the steps outlined in part a), otherwise, follow the steps in part b). If there is no $u \to v$ branch in T, then T must only have one vertex, so skip to step 5. For each case, let $T' = T - v$, and let u' be the vertex in T' corresponding to u in T.
   (a) There are a few potential cases for when $v \in F$:
      i. $L_Q(v) = L_P(v) = 1$. In this case, let $L_Q(u') = L_Q(u) + 1$.
      ii. $L_Q(v) = 2$ and $L_P(v) = 2$. In this case, do nothing to u'.
      iii. $L_Q(v) = 1$ and $L_P(v) = 2$. In this case, let $L_P(u') = 2$.
   (b) The cases for when $v \notin F$ are:
      i. If $u \notin F$, and $L_Q(v) = L_P(v) = 1$, then let $L_Q(u') = L_Q(u) + 1$.
      ii. If $L_Q(v) = 2$, and $L_P(v) = 2$, then do nothing to u'.
      iii. If $L_Q(v) = 1$ and $L_P(v) = 2$, then let $L_P(u') = 2$.
      iv. If $L_Q(v) \geq 2$, then color v, and let $L_P(u') = 2$.
      v. If $u \in F$, and $L_Q(u) = 1$ and $L_P(v) = 1$, then let $L_Q(u') = 2$.
      vi. If $u \in F$, and $L_Q(u) = 2$, then color v, and then let $L_P(u') = 2$.
4. Let T' become the new T, and go back to step 3
5. If T only has one vertex, then we have two cases:

(a) If $L_P(u) = 2$, then do nothing.

(b) If $L_P(u) = 1$, then color u.

6. All the vertices that ended up getting colored form an MRPD-F set.

Many of these cases are either from Lyu's algorithm, or equivalent to the cases presented in Lyu's paper, and thus their correctness has already been proven. As for the order of vertex removal, the reason behind removing any leafs in the forbidden zone before any other leafs is to prevent the generation of over-restricted subgraphs of T, specifically with case (b) vi. To complete the proof of correctness for the algorithm, a proof of case (b) vi is given in the following theorem.

Theorem 3.1. *Let T be a tree with a forbidden zone $F \subseteq V(T)$ such that every vertex in the forbidden zone is adjacent to at most one other forbidden vertex. Let T also have labels $(L_P(x), L_Q(x))$ for all $x \in V(T)$. Suppose that T contains a branch $u \to v$ such that $v \notin F$, $u \in F$, and $L_Q(u) = 2$. Then if S is an mMRPD-F set of T, it is true that $v \in S$. Furthermore, for $T' = T - v$ where $L(u') = 2$, it is also true that $\gamma_P^m(T', F) = \gamma_P^m(T, F) - 1$.*

Proof. Suppose that S is an mMRPD-F set of T, and, for the sake of contradiction, assume that $v \notin S$. We have that $u \notin S$, as otherwise $S \cap F \neq \varnothing$, as u is in the forbidden zone. As v is a leaf of T, the only vertex that v is adjacent to is u. Because $u \notin S$, and u is the only vertex that contains v in its open neighborhood, v can not be observed by $OR1'$, and thus v must be observed by u via $OR2'$. However, in order for u to observe v by $OR2'$, we must have that $L_Q(u) = 1$, but this is not the case, as $L_Q(u) = 2$. Thus, v cannot be observed through either observation rule, which is a contradiction, as S is an mMRPD-F set of T. This implies that the assumption that $v \notin S$ was wrong, and so, v must be in S.

For any mMRPD-F set of T', say S', we have that $S' \cup \{v\}$ is a mMRPD-F set of T, as S' power dominates all of T except v, and $L_P(u')$ isn't dependent on v, as it is defined to be 2 in T'. Thus we have that $= |S' \cup \{v\}| = |S'| + 1 = \gamma_P^m(T', F) + 1 = |S| = \gamma_P^m(T, F)$, as both are mMRPD-$F$ sets, and so they both have the same cardinality. Equivalently, $\gamma_P^m(T', F) = \gamma_P^m(T, F) - 1$

□

4 Conclusion

In this paper, a solution to a special case of the restricted power domination on trees has been provided. Without any forbidden zones, this algorithm works the same way as Lyu's algorithm, and uses some minor modifications to work with certain restrictions where forbidden vertices are adjacent to at most one other vertex. Future research could include finding a solution to the case of any forbidden zone over a tree, as well as finding which cases of the graph results in over-restriction.

While studying this problem, I've come across multiple algorithms for solving it, and I've learned a lot about what power domination is and how it works. There have been many times when I've thought I came up with a valid solution, only to find some sort of counterexample. In fact, up until just about a week ago, I thought my proof of a solution for a general case of forbidden zones was correct, until after I showed a friend the handouts (which included the algorithm) that I did for my oral presentation. He figured out a counterexample, which caused my algorithm to break when multiple forbidden vertices were connected together. This is why I then decided to rewrite this paper so that it only included certain cases of forbidden zones, as it removes some of the possible situations that breaks my algorithm. This version of the algorithm contains a lot fewer cases as a result, which also made it a bit easier to work with and prove. Either way, the general problem on trees is one that I still plan on working on even after this paper, as I feel like I was getting somewhere with my previous approach.

This experience has taught me some of the process of solving difficult problems, as well as the importance of learning from counterexamples. And honestly, that is probably one of the most important things I've learned from this class. Overall, there were a lot more topics covered in this class than I expected, and I enjoyed going through a lot of different things to see just how much there is to graph theory. Everything from tournaments, to Hamiltonian paths, and edge coloring problems, has been fascinating to learn about. Having to model things, then go through proofs involving those models has challenged me to think in different ways, and I feel that I'm more comfortable working with graphs and other models as a result.

References

[1] J. Guo, R. Niedermeier, and D. Raible. Improved algorithms and complexity results for power domination in graphs. *Algorithmica*, 52:177–202, 2005.

[2] T. Haynes, S. Hedetniemi, S. Hedetniemi, and M. Henning. Domination in graphs applied to electric power networks. *SIAM J. Discrete Math.*, 15:519–529, 07 2002.

[3] J. Kneis, D. Mölle, S. Richter, and P. Rossmanith. Parameterized power domination complexity. *Information Processing Letters*, 98(4):145–149, 2006.

[4] C.-S. Liao and D.-T. Lee. Power domination problem in graphs. In L. Wang, editor, *Computing and Combinatorics*, pages 818–828, Berlin, Heidelberg, 2005. Springer Berlin Heidelberg.

[5] Y. Lyu. Labeling algorithm for power domination problem of trees. *Applied Mathematics and Computation*, 457:128163, 2023.

[6] K.-J. Pai, J.-M. Chang, and Y.-L. Wang. Restricted power domination and fault-tolerant power domination on grids. *Discrete Applied Mathematics*, 158(10):1079–1089, 2010.