2.1.12 Group Work Check whether or not the following functions u are in $L^2(\Omega)$ with the given domains Ω:

(a) the Heaviside step function

$$u(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

over $\Omega =]-1, 1[$

$\|u\|_{L^2}^2 = \int_{-1}^{1} u(x)^2 \, dx = \int_0^1 1^2 \, dx = 1 < \infty$

yes!

(b) the Heaviside step function over $\Omega = \mathbb{R}$

$\|u\|_{L^2}^2 = \int_{-\infty}^{\infty} u(x)^2 \, dx = \int_0^\infty 1^2 \, dx = \infty$

12 no!

(c) an arbitrary function $u \in C(\Omega)$ over a bounded domain Ω no!

Take e.g. $u(x) = \frac{1}{x}$ on $\Omega =]0,1[$.

$\|u\|_{L^2}^2 = \int_0^1 \frac{1}{x^2} \, dx = \infty$

(d) an arbitrary function $u \in C(\bar{\Omega})$ over a bounded domain Ω yes!

u assumes its max & min over the compact set $\bar{\Omega} \Rightarrow u$ is bounded on Ω

$\|u\|_{L^2}^2 = \int_\Omega u^2 \, dx \leq \left(\sup_\Omega |u|^2\right) |\Omega| < \infty$

(e) $u(x) = \frac{1}{|x|^p}$ over the interval $\Omega =]-1, 1[$ (where $p > 0$)

$\|u\|_{L^2}^2 = \int_{-1}^{1} \frac{1}{x^{2p}} \, dx < \infty \iff 2p < 1 \iff p < \frac{1}{2}$

(f) $u(x) = \frac{1}{|x|^p}$ over the unit disk $\Omega = B(0,1) \subset \mathbb{R}^2$

$\|u\|_{L^2}^2 = \int_{B(0,1)} \frac{1}{|x|^{2p}} \, dx = \int_0^{2\pi} \int_0^1 \underbrace{\frac{1}{r^{2p}} r}_{= \frac{1}{r^{2p-1}}} \, dr \, d\varphi < \infty \iff 2p-1 < 1 \iff p < 1$

(g) $u(x) = \frac{1}{|x|^p}$ over the unit ball $\Omega = B(0,1) \subset \mathbb{R}^3$

$\|u\|_{L^2}^2 = \int_{B(0,1)} \frac{1}{|x|^{2p}} \, dx = \int_0^{\pi} \int_0^{2\pi} \int_0^1 \underbrace{\frac{1}{r^{2p}} r^2}_{= \frac{1}{r^{2p-2}}} \, dr \, d\varphi \, d\vartheta < \infty \iff 2p-2 < 1 \iff p < \frac{3}{2}$