MATH521: Numerical Analysis of Partial Differential Equations
Winter 2018/19, Term 2

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After today's class you should be able to

- assemble the "big linear system" corresponding to a finite-element method
- compute L^2, H^1 and energy norms of functions from finite-element spaces

Linear Finite Elements

Before we dive further into more abstract properties and the error analysis of Galerkin methods, let us consider an important representative and its practical implementation.When we speak of *linear finite elements*, we consider a space V^h that is composed of certain globally continuous and piecewise linear functions.

Let us re-visit the Poisson-Dirichlet problem once again. Assuming that the given right hand side $f \in L^2(\Omega)$, we look for a solution $u \in V = H^1_0(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x = \int_\Omega f v \,\mathrm{d}x, \qquad \forall v \in H^1_0(\Omega). \tag{2.18}$$

Following the three-step recipe for generic Galerkin methods, we first have to find a suitable subspace $V^h \subset H^1_0(\Omega)$. For the time being we assume that Ω is a polygon, because then we may decompose it exactly into triangles as follows:

2.3.3 Definition (Regular Triangulation) A finite set $\mathcal{T}^h$ of closed triangles is said to be a *regular triangulation*[8] of the domain Ω, if

$$\bar{\Omega} = \bigcup_{T \in \mathcal{T}^h} T$$

and if for any two triangles $T, T' \in \mathcal{T}^h$ that are not identical, their intersection $T \cap T'$ is either empty, one common corner point of T and T' or one common edge of T and T'.

The mesh-size parameter of this triangulation is defined as $h = \max\limits_{T \in \mathcal{T}^h} \operatorname{diam} T$.

[8] Of course, this construction is not limited to two dimensions. The more general concept of such sets of line segments (1D), triangles (2D), tetrahedra (3D) and higher-dimensional simplices is called *simplicial complex*.

We can now build a finite-dimensional space V^h from continuous functions on $\bar{\Omega}$ that are linear when restricted to a $T \in \mathcal{T}^h$: One possible (and very convenient) basis of this space is given by the functions $\phi_i^h \in V^h$ defined by

This space is a subspace of $V = H_0^1(\Omega)$, indeed:

2.3.4 Lemma Let $\mathcal{T}^h$ be a regular triangulation of the bounded domain Ω. If $v^h \in C(\bar{\Omega})$ and if for all $T \in \mathcal{T}^h$ the restriction $v^h|_T \in C^1(T)$, then $v^h \in H^1(\Omega)$.

Proof.

- $v^h \in L^2(\Omega)$:

- $\nabla v^h \in L^2(\Omega)$: The function $g : \Omega \to \mathbb{R}^d$ defined piecewise

$$g|_T = \nabla v^h|_T \qquad \text{for } T \in \mathcal{T}^h$$

is, of course, the weak gradient of v^h (this is easily checked using the definition of the weak gradient).

□

The same reasoning applies to show that every function $v^h \in C^{k-1}(\bar{\Omega})$ with traces $v^h|_T \in C^k(T)$ for all $T \in \mathcal{T}^h$ is in $H^k(\Omega)$.

Moving on to the second step in the construction of a Galerkin approximation, we may now decompose any function $v^h \in V^h$ (including the discrete solution u^h) in this basis as

$$v^h = \sum_{j=1}^{N} v^h(x^j)\phi_j^h.$$

Obviously, v^h is fully determined by its function values $v^h(x^j)$ on the interior vertices x^j of the triangulation.

Therefore, we also collect these nodal values in a column vector

$$\vec{v}^h = \begin{pmatrix} v^h(x^1) \\ \vdots \\ v^h(x^N) \end{pmatrix}$$

for an alternative representation of v^h.

Thirdly and lastly, the Galerkin equations for the model problem discretised with linear finite elements read

or, in matrix form,

with

2.3.5 Group Work (Linear Finite Elements in 1D) In one dimension, the model problem reads: find $u \in H_0^1(]0,1[)$ such that for all $v \in H_0^1(]0,1[)$:

$$\int_0^1 u'v' \,\mathrm{d}x = \int_0^1 fv \,\mathrm{d}x.$$

We discretise this problem on the equidistant grid

$$0, h, 2h, 3h, \ldots, (N-1)h, 1$$

with N subintervals and grid spacing $h = 1/N$.

The Galerkin equations read

$$\sum_{j=1}^{N-1} \left(\int_0^1 \left(\phi_j^h\right)' \left(\phi_i^h\right)' \,\mathrm{d}x \right) u_j^h = \int_0^1 f\phi_i^h \,\mathrm{d}x \qquad \forall i \in \{\, 1, \ldots, N-1 \,\}.$$

(a) Calculate the entries of the stiffness matrix K^h!

$$k_{ij}^h = \int_0^1 \left(\phi_j^h\right)' \left(\phi_i^h\right)' \, \mathrm{d}x = \begin{cases} & \text{if } i = j \\ & \text{if } i = j+1 \\ & \text{if } i = j-1 \\ & \text{if } |i-j| > 1 \end{cases}$$

(b) Now compute the entries of the load vector $\vec{f}^h$:

$$f_i^h = \int_0^1 f \phi_i^h \, \mathrm{d}x =$$

(c) The "big linear system" reads

(d) Given the coefficient vector $\vec{v}^h$ of a function $v^h \in V^h$, how can you easily compute the L^2-norm of v^h from $\vec{v}^h$?

$$\|v^h\|_{L^2}^2 = \int_0^1 \left(v^h\right)^2 \, \mathrm{d}x =$$