**MATH521: Numerical Analysis of Partial Differential Equations**
Winter 2018/19, Term 2

Thursday, 3 January 2019
Timm Treskatis

**After today's class you should be able to**

- recognise the type of (non)linearity of a given PDE
- define well-posedness of a PDE problem and explain what is meant by the regularity of a solution
- identify scalar conservation laws and describe some qualitative properties of their solutions
- distinguish elliptic, parabolic and hyperbolic (scalar, quasilinear, first-order or second-order) equations
- name characteristic features of PDEs or their solutions that may be desirable to preserve after discretisation

# 1 Classification of PDEs

**Executive Summary** Partial differential equations (PDEs) model a vast range of natural phenomena and industrial problems. Even though analytical solutions are not normally available, just the equation itself already reveals crucial information on some characteristic features of the (unknown) solution. In this course we will learn to compute numerical approximations that reflect these properties accurately.

Today we will find out what some of these characteristic features are, and how they can be used to classify PDEs in a meaningful manner:

- Certain (important!) PDEs can be interpreted as conservation laws, others cannot. We will look at some examples of such conservation equations first.
- Another way of classifying PDEs relies on the notions of elliptic, parabolic and hyperbolic equations. We will learn about these in the last section of this first chapter.

## 1.1 Basic Properties

The prototypical equations we study in this course are

**1**. Poisson's equation
$$-\Delta u = f,$$

**2**. the heat equation
$$\frac{\partial u}{\partial t} - \Delta u = f,$$

**3**. and the wave equation
$$\frac{\partial^2 u}{\partial t^2} - \Delta u = f,$$

that are all of second order (they contain derivatives up to and including order two) and that we solve for a scalar, real-valued function $u$.

The Laplacian
$$\Delta = \operatorname{div} \nabla = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}$$
arises in all three of these equations, where $d$ is the number of (physically: spatial) variables, which we collect in a vector $x = (x_1, x_2, \dots, x_d)^\top \in \mathbb{R}^d$. To emphasise the special role of the $(d+1)$-st (physically: time) variable in the heat and wave equations, we used the notation $t$ here instead of yet another $x_i$.

If no special notation is used for $t$, then all of the above equations are of the form

$$\sum_{i,j=1}^{d} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i \frac{\partial u}{\partial x_i} + cu = f \qquad \text{in } \Omega \subseteq \mathbb{R}^d \tag{$\star$}$$

e.g. in 2D

$$a_{11} \frac{\partial^2 u}{\partial x_1^2} + 2a_{12} \frac{\partial^2 u}{\partial x_1 \partial x_2} + a_{22} \frac{\partial^2 u}{\partial x_2^2} + b_1 \frac{\partial u}{\partial x_1} + b_2 \frac{\partial u}{\partial x_2} + cu = f \qquad \text{in } \Omega \subseteq \mathbb{R}^2$$

(where, for this equation to be genuinely of second order, not all of the $a_{ij}$'s are simultaneously equal to zero).

Note that without loss of generality, the matrix $A$ consisting of the coefficients $a_{ij}$ can be assumed to be symmetric with $a_{ij} = a_{ji}$ for all $i, j \in \{1, \ldots, d\}$, because

Keep this assumption of a symmetric matrix $A$ in mind for later on!

We also use the short operator notation for such PDEs

$$Lu = f \qquad \text{in } \Omega.$$

**1.1.1 Definition (Linear and Nonlinear PDEs)** A PDE

$$F(x, u, \nabla u, \nabla^2 u) = 0$$

is said to be

- *linear*, if it can be cast in the form ($\star$) where the coefficients
- *nonlinear*, otherwise
  - *semi-linear*, if it can be cast in the form ($\star$) where the coefficients
  - *quasi-linear*, if it can be cast in the form ($\star$) where the coefficients
  - *fully nonlinear*, if it cannot be cast in the form ($\star$).

In practice, the domain $\Omega$ is often bounded, or unbounded in one direction only (namely, the 'time-direction'). While the PDE describes the behaviour of a solution $u$ in the interior of $\Omega$, additional conditions prescribe certain values on the boundary $\partial\Omega$ or parts thereof. The question what kind of boundary conditions are 'admissible' for a given PDE is not straightforward to answer. The answer will depend on the type and the exact form of the PDE. For now, let us clarify what 'admissible' means in this context:

**1.1.2 Definition (Well-Posed Problem)** A PDE equipped with boundary conditions is said to be *well-posed in the sense of Hadamard*, if

In addition to these three properties, we are also interested in the *regularity* a.k.a. *smoothness* of solutions, i.e. how many derivatives they possess.

## 1.2 Conservation Equations

Very often, PDE problems model physical, chemical or biological systems which are governed by certain *conservation laws*, such as conservation of

In mathematical terms, conservation of a (mass, energy, momentum, ...) density $u$ implies that if this quantity is transported through the domain with a flux $F$, the density increases inside any arbitrary control volume $V$ if there is a net influx across the boundary of $V$ and it decreases otherwise:

$$\int_V \frac{\partial u}{\partial t}\,\mathrm{d}x = -\int_{\partial V} F \cdot n\,\mathrm{d}s.$$

Provided that the flux $F$ and the control volume $V$ satisfy the assumptions of the divergence theorem, we obtain

$$\int_V \frac{\partial u}{\partial t}\,\mathrm{d}x + \int_V \operatorname{div} F\,\mathrm{d}x = 0.$$

Since this balance has to hold for arbitrary such volumes, we have now derived a rather general form of a conservation equation

$$\frac{\partial u}{\partial t} + \operatorname{div} F = 0,$$

which is called the *continuity equation*. PDEs that can be written in this form (or its steady-state version with no time derivative) are called *conservation equations*.

The flux function $F$ often depends on the solution $u$ or its gradient $\nabla u$. If we assume a linear flux, then we obtain the following two particularly important fluxes:

- advective flux $F = ua$, with an advection (velocity) field $a$
- diffusive flux $F = -D\nabla u$, with a positive diffusivity $D$. More generally, $D$ could also be a symmetric positive definite matrix, a so-called diffusion tensor.

By comparing the physical dimensions of the terms $\frac{\partial u}{\partial t}$ and $\operatorname{div} F$, you can check that the proportionality constants $a$ and $D$ do have the dimension of a velocity or a diffusivity, indeed.

Sometimes, the quantity $u$ is not actually conserved, but sources or sinks such as chemical reactions or external forces give rise to an inhomogeneity $r$ on the right-hand side of the equation.

Below, we list some typical conservation equations:

- Unsteady advection-diffusion-reaction equation

$$\frac{\partial u}{\partial t} + \operatorname{div}(ua) - \operatorname{div}(D\nabla u) = r$$

- Steady advection-diffusion-reaction equation

$$\operatorname{div}(ua) - \operatorname{div}(D\nabla u) = r$$

- Unsteady advection equation

$$\frac{\partial u}{\partial t} + \operatorname{div}(ua) = 0$$

- Steady advection equation

$$\operatorname{div}(ua) = 0$$

Many advection fields in applications are incompressible, which means that $\operatorname{div} a = 0$. Then, using the product rule, the above advective terms can also be written as

**1.2.1 Group Work** What is the mysterious PDE, the solution of which is shown in the video? Can you (qualitatively) guess what the coefficients are, plus any initial or boundary conditions?

## 1.3 Elliptic, Parabolic and Hyperbolic PDEs of Second Order

All first-order PDEs of the form ($\star$), i.e. scalar advection-reaction equations, are generally defined to be hyperbolic. Second-order PDEs of the form ($\star$), i.e. scalar, quasi-linear PDEs of second order, can be elliptic, parabolic, hyperbolic or neither.

Recall that we assume the $(d \times d)$-matrix $A$ of coefficients in the principal part to be symmetric, therefore it has

**1.3.1 Definition (Elliptic, Parabolic and Hyperbolic Equations)** In a point $x \in \Omega$ the second-order PDE ($\star$) is said to be

- *elliptic*, if

- *parabolic*, if

- *hyperbolic*, if

It is important to note that this criterion gives a pointwise classification of a quasi-linear second-order PDE, since the coefficients in $A$ may depend on $x$ and maybe even on the solution $u$ itself. Hence, a single (but somewhat exotic) equation may be elliptic in some parts of the domain $\Omega$, parabolic in other regions and hyperbolic elsewhere. In this course, we normally study equations that do not change type, such as equations with constant coefficients.

**1.3.2 Group Work** Label all of the above PDEs (Poisson's equation, the heat and wave equations and the examples of typical conservation equations) as elliptic, parabolic or hyperbolic!

**1.3.3 Group Work** Where do the labels 'elliptic', 'parabolic' and 'hyperbolic' come from? To find out, consider the second-order PDE in 2D

$$a_{11}\frac{\partial^2 u}{\partial x_1^2} + 2a_{12}\frac{\partial^2 u}{\partial x_1 \partial x_2} + a_{22}\frac{\partial^2 u}{\partial x_2^2} + b_1\frac{\partial u}{\partial x_1} + b_2\frac{\partial u}{\partial x_2} + cu = f \qquad \text{in } \Omega \subseteq \mathbb{R}^2.$$

What conditions on the coefficients of $A$ can you derive from Definition 1.3.1 for each case? How does this relate to the defining equation of the conic sections

$$\begin{aligned} &a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2 + b_1x_1 + b_2x_2 + c = 0 \\ \Leftrightarrow &\begin{pmatrix} x_1 & x_2 \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} b_1 & b_2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + c = 0? \end{aligned}$$