Homework Assignment 7

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5 marks | Let $D > 0$, $a \in \mathbb{R}^2$, $r \geq 0$ and $f \in L^2(\Omega)$, where $\Omega \subset \mathbb{R}^2$ is a convex, polygonal domain.

We use conforming linear finite elements with exact integration to solve the steady diffusion-advection-reaction problem

$$
\begin{aligned}
-D\Delta u + \operatorname{div}(au) + ru &= f && \text{in } \Omega \\
u &= 0 && \text{on } \partial\Omega.
\end{aligned}
$$

(Recall that the assumption of homogeneous boundary conditions is no loss of generality, since any inhomogeneous boundary conditions could be subtracted from u to obtain the same PDE with homogeneous boundary conditions but a new source term.)

Follow the methodology from pp 67–68 in our notes to show that the numerical solution u^h converges to u at a linear rate in the H^1-norm and at a quadratic rate in the L^2-norm, provided that $u \in H^2(\Omega)$:

$$\|u^h - u\|_{H^1(\Omega)} \leq ch\|\nabla^2 u\|_{L^2(\Omega)} \tag{1}$$

$$\|u^h - u\|_{L^2(\Omega)} \leq ch^2\|\nabla^2 u\|_{L^2(\Omega)}. \tag{2}$$

Hints:

1. To show that the nonsymmetric bilinear form of this elliptic operator is coercive in the H^1-norm, prove and then use that
$$\int_\Omega (a \cdot \nabla u)v \, \mathrm{d}x = -\int_\Omega (a \cdot \nabla v)u \, \mathrm{d}x \quad \text{for all } u, v \in H_0^1(\Omega). \tag{3}$$

2. You may assume that
$$\|u\|_{H^2(\Omega)} \leq c\|f\|_{L^2(\Omega)}. \tag{4}$$

First things first, let's find the weak formulation:

$$-D\Delta u + \operatorname{div}(au) + ru = f$$

Expanding the divergence term:

$$-D\Delta u + a \cdot \nabla u + \underbrace{\cancel{(\operatorname{div} a)}}_{0}u + ru = f$$

Multiplying the residual by a test function and integrating over the domain:

$$\int_\Omega (-D\Delta u + a \cdot \nabla u + ru - f)v \, \mathrm{d}x = 0$$

Applying Green's first identity:

$$\underbrace{\cancel{\int_{\partial\Omega} D\frac{\partial u}{\partial n}v \, \mathrm{d}s}}_{0} + \int_\Omega (D\nabla u \cdot \nabla v + (a \cdot \nabla u)v + ruv - fv) \, \mathrm{d}x = 0$$

Now we have our weak formulation, given $f \in L^2(\Omega)$, find $u \in H_0^1(\Omega)$ such that for all $v \in H_0^1(\Omega)$:

$$\int_\Omega (D\nabla u \cdot \nabla v + (a \cdot \nabla u)v + ruv) \, \mathrm{d}x = \langle f, v\rangle_{H^{-1}, H_0^1(\Omega)}$$

Now to prove equation (3) using Green's first identity again:

$$\int_\omega \mathrm{div}(au)v \, \mathrm{d}x = \cancelto{0}{\int_{\partial\Omega} (au \cdot n)v ds} - \int_\Omega au \cdot \nabla v \, \mathrm{d}x$$

And againg, expanding the divergence:

$$\mathrm{div}(au) = a \cdot \nabla u + \cancelto{0}{(\mathrm{div}\, a)u}$$

Leaving us with:

$$\int_\Omega (a \cdot \nabla u)v \, \mathrm{d}x = -\int_\Omega (a \cdot \nabla v)u \, \mathrm{d}x \quad \text{for all } u, v \in H_0^1(\Omega)$$

Now putting our problem in bilinear form we get one of two forms:

$$B(u,v) = \int_\Omega D\nabla u \cdot \nabla v \, \mathrm{d}x + \int_\Omega (a \cdot \nabla u)v \, \mathrm{d}x + \int_\Omega ruv \, \mathrm{d}x$$

or:

$$B(u,v) = \int_\Omega D\nabla u \cdot \nabla v \, \mathrm{d}x - \int_\Omega (a \cdot \nabla v)u \, \mathrm{d}x + \int_\Omega ruv \, \mathrm{d}x$$

We must now prove coercivity and continuity in order to use Céa's Lemma for convergence.

Continuity:

$$\begin{aligned} |B(u,v)| &= |D\int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x + \int_\Omega (a \cdot \nabla u)v \, \mathrm{d}x + r\int_\Omega uv \, \mathrm{d}x \\ &= |D\langle \nabla u, \nabla v\rangle_{L^2} + r\langle u, v\rangle_{L^2} + \langle a \cdot \nabla u, v\rangle_{L^2}| \end{aligned}$$

Using Cauchy-Schwartz inequality:

$$\leq D\|\nabla u\|_{L^2}\|\nabla v\|_{L^2} + r\|u\|_{L^2}\|v\|_{L^2} + \|a \cdot \nabla u\|_{L^2}\|v\|_{L^2}$$

Note that:

$$\|a \cdot \nabla u\| \leq max(|a|)\|\nabla u\|_{L^2}$$

Now using the definition of the H^1-norm we get:

$$|B(u,v)| \leq \underbrace{max(D, max(r, |a|))}_{C_b > 0}\|u\|_{H^1}\|v\|_{H^1}$$

We can conclude that because D is strictly positive, C_b is also strictly positive, thus proving continuity.

Coercivity:

Using both of our bilinear forms above:

$$\begin{aligned} B(u,u) &= \int_\Omega D\nabla u^2 \, \mathrm{d}x + \int_\Omega au \cdot \nabla u \, \mathrm{d}x + \int_\Omega ru^2 \, \mathrm{d}x \\ B(u,u) &= \int_\Omega D\nabla u^2 \, \mathrm{d}x - \int_\Omega au \cdot \nabla u \, \mathrm{d}x + \int_\Omega ru^2 \, \mathrm{d}x \end{aligned}$$

Because these two equations are equal we can conclude that $\int_\Omega au \cdot \nabla u \, \mathrm{d}x = 0$, leaving us with:

$$\begin{aligned} |B(u,u)| &= D\int_\Omega |\nabla u|^2 \, \mathrm{d}x + r\int_\Omega |u|^2 \, \mathrm{d}x \\ &= D\|\nabla u\|_{L^2}^2 + r\|u\|_{L^2}^2 \end{aligned}$$

Now because r can equal zero we must separate and use Poicaré's inequality with the first term to prove coercivity:

$$D\|\nabla u\|_{L^2}^2 = \frac{D}{2}\|\nabla u\|_{L^2}^2 + \frac{D}{2}\|\nabla u\|_{L^2}^2$$

Using Poincaré's inequality on one of these terms:

$$\frac{D}{2}\|\nabla u\|_{L^2}^2 \geq \frac{DC_p^2}{2}\|u\|_{L^2}^2$$

Putting this back into our bilinear form:

$$B(u,u) = \frac{D}{2}\|\nabla u\|_{L^2}^2 + \left(\frac{DC_p^2}{2} + r\right)\|u\|_{L^2}^2 \geq C_e\|u\|_{H^1}^2$$

where:

$$C_e = min\left(\frac{D}{2}, \frac{DC_p^2}{2} + r\right)$$

And because $D > 0$ and $C_p > 0$, we can conclude that it is indeed coercive.

Now we can apply Céa's lemma to find the convergence in the H^1 and L^2 norms. Starting with the H^1-norm, following the procedure in 2.3.28:

$$\|u^h - u\|_{H^1} = \frac{c_b}{c_e}\inf_{v^h \in H_0^1}\left(\|v^h - u\|_{H^1}\right)$$

Choose $v^h = I^h u$

$$= \frac{c_b}{c_e}\|I^h u - u\|_{H^1}$$

By the definition of the H^1-norm:

$$= \frac{c_b}{c_e}\left(\|\nabla(I^h u - u)\|_{L^2} + \|(I^h u - u)\|_{L^2}\right)$$

And from proof 2.3.26:

$$= \frac{c_b}{c_e}\left(ch\|\nabla^2 u\|_{L^2} + ch^2\|\nabla^2 u\|_{L^2}\right)$$

Now as $h \to 0$ the second term goes to $\mathcal{O}(h^2)$ and can be eaten up by the constant out front leaving us with:

$$\|u^h - u\|_{H^1(\Omega)} = ch\|\nabla^2 u\|_{L^2(\Omega)}$$

Moving on to the convergence in the L^2-norm, following the theorem in 2.3.31:

$$\|u^h - u\|_{L^2} = \int_\Omega \frac{e^h}{\|e^h\|_{L^2}} e^h \, \mathrm{d}x = B(e^h, z)$$

Galerkin Orthogonality gives:

$$= B(e^h, z - I^h z)$$

Then Cauchy-Schwartz gives:

$$= \|e^h\|_B\|z - I^h z\|_B$$

Now from 2.3.30 and the definition of the B-norm we get:

$$= ch\|\nabla^2 z\|_{L^2}\|\nabla(z - I^h z)\|_{L^2}$$

Then from theorem 2.3.26, we find:

$$= ch\|\nabla^2 z\|_{L^2} ch\|\nabla^2 u\|_{L^2}$$

Then finally from theorem 2.1.27, combining the constants:

$$\|u^h - u\|_{L^2(\Omega)} = ch^2\|\nabla^2 u\|_{L^2(\Omega)}$$

Your Learning Progress | **0 marks, but -1 mark if unanswered** | What is the one most important thing that you have learnt from this assignment?

Always make sure that *all* conditions are met for a certain theory/ lemma before pretending to use it :).

What is the most substantial new insight that you have gained from this course this week? Any *aha moment*?

The finite element theory for parabolic PDEs is way over my head at the moment, I think I'll stick to figuring out elliptic equations for now.