Homework Assignment 4: Model Answers (Analytical Flavour)

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Question 1 | **1 mark** | Let $u \in C^3(\bar{\Omega})$ with a bounded domain Ω. For simplicity and with no real loss of generality we assume that $\Omega \subset \mathbb{R}$ (because even in higher dimensions the partial derivatives are just ordinary 1D directional derivatives).

In Lemma 2.2.6 we showed that the one-sided difference quotients

$$\partial^{+h}u(x) = \frac{u(x+h) - u(x)}{h} \qquad \text{and} \qquad \partial^{-h}u(x) = \frac{u(x) - u(x-h)}{h}$$

are a first-order consistent approximation of $u'(x)$. Use the same Taylor-series technique to show that these difference quotients actually approximate $u'(x) - Du''(x)$ "better", namely with second-order consistency, than they approximate $u'(x)$. Here $D \in \mathbb{R}$ is a certain number which may depend on h.

Let $x \in \Omega$ and $h > 0$ such that $[x-h, x+h] \subset \bar{\Omega}$. Taylor expansion yields

$$u(x \pm h) = u(x) \pm hu'(x) + \frac{h^2}{2}u''(x) \pm \frac{h^3}{6}u'''(\xi_\pm)$$

where $\xi_+ \in]x, x+h[$ and $\xi_- \in]x-h, x[$.

We now obtain

$$\left|\partial^{\pm h}u(x) - (u'(x) - Du''(x))\right| = \left|u'(x) \pm \frac{h}{2}u''(x) + \frac{h^2}{6}u'''(\xi_\pm) - (u'(x) - Du''(x))\right| \leq \frac{h^2}{6} \max_{[x-h,x+h]} |u'''|$$

with $D = \mp\frac{h}{2}$.

Important conclusion: Approximating an advection term with a downwind difference quotient introduces artificial anti-diffusion (uh oh - that sounds like trouble!!!), an approximation with an upwind difference quotient introduces artificial diffusion (unphysical smoothing / smearing, which is also undesirable, see Q2(d) of the applied version of this assignment).

Question 2 | 4 marks | Let $\Omega \subset \mathbb{R}^d$ be a bounded domain and $L : C^2(\Omega) \cap C(\bar{\Omega}) \to C(\Omega)$ a linear second-order elliptic operator. If—using the notation from the notes—its zeroth-order coefficient $c \geq 0$, then the operator is nonnegativity-preserving

$$Lu \geq 0 \text{ in } \Omega \quad \wedge \quad u \geq 0 \text{ on } \partial\Omega \qquad \Rightarrow \qquad u \geq 0 \text{ in } \Omega \tag{1}$$

(in class we only discussed the case $c = 0$).

For a matrix $A \in \mathbb{R}^{n \times n}$, we have shown that the analogous algebraic property, namely monotonicity

$$Ax \geq 0 \qquad \Rightarrow \qquad x \geq 0, \tag{2}$$

is equivalent to A being nonsingular and inverse-nonnegative. We then proved the (sufficient) monotonicity criterion that every weakly chained diagonally dominant L-matrix is monotone. Hence, a discretisation scheme that turns operators L with the property (1) into weakly chained diagonally dominant L-matrices preserves very important structure of the problem.

In this assignment, I would like you to refine these results to highlight another characteristic feature of elliptic operators[1]. In fact, elliptic operators of the above form have strictly positive Green's functions in the interior of Ω, so in addition to (1) they also have the property

$$Lu \geq 0 \text{ in } \Omega \quad \wedge \quad \exists x \in \Omega : (Lu)(x) > 0 \quad \wedge \quad u \geq 0 \text{ on } \partial\Omega \qquad \Rightarrow \qquad u > 0 \text{ in } \Omega.$$

(a) Formulate the corresponding stronger monotonicity property of matrices and show that it is equivalent to nonsingularity and inverse-positivity.

A matrix $A \in \mathbb{R}^{n \times n}$ satisfies (2) and additionally

$$Ax \geq 0 \quad \wedge \quad \exists i \in \{1, \dots, n\} : (Ax)_i > 0 \qquad \Rightarrow \qquad x > 0 \tag{3}$$

if and only if it is nonsingular and inverse-positive.

Proof. If the matrix satisfies (2) and (3) then it is nonsingular (Lemma 2.2.16) and from $A\left(A^{-1}\right)_i = e_i$ we conclude $\left(A^{-1}\right)_i > 0$ for all columns $i \in \{1, \dots, n\}$.

Conversely, if A is nonsingular and inverse-positive, then (2) follows from Lemma 2.2.16. If $x \in \mathbb{R}^n$ is a vector such that $Ax \geq 0$ with $(Ax)_i > 0$ for one $i \in \{1, \dots, n\}$, then $A^{-1} > 0$ yields, for all $j \in \{1, \dots, n\}$

$$x_j = (A^{-1}Ax)_j = \sum_{k=1}^{n} (A^{-1})_{jk}(Ax)_k \geq (A^{-1})_{ji}(Ax)_i > 0,$$

which shows (3). $\square$

[1] Even a hyperbolic operator like $L = a \cdot \nabla$ preserves nonnegativity and also satisfies a similar maximum principle. What we look at in this question is a feature of elliptic operators only, but not of hyperbolic operators.

(b) Can you also find and prove a sufficient criterion in the style of the M-criterion from Lemma 2.2.19 that implies this stronger form of monotonicity?

Every irreducibly diagonally dominant L-matrix satisfies the monotonicity properties (2) and (3).

Proof. Since every irreducibly diagonally dominant matrix is also weakly chained diagonally dominant, we have (2) due to Lemma 2.2.19.

Let $A \in \mathbb{R}^{n\times n}$ be an irreducibly diagonally dominant L-matrix. Let $x \in \mathbb{R}^n$ be such that $Ax \geq 0$ and $(Ax)_i > 0$ for some $i \in \{1, \ldots, n\}$, i.e. due to the L-matrix property and $x \geq 0$

$$\begin{aligned} & \sum_{j=1}^{n} a_{ij} x_j > 0 \\ \Leftrightarrow \quad & a_{ii} x_i > \sum_{j\neq i} |a_{ij}| x_j = \sum_{j\in \mathcal{S}_i^\circ} |a_{ij}| x_j \\ \Leftrightarrow \quad & x_i > \sum_{j\in \mathcal{S}_i^\circ} \frac{|a_{ij}|}{a_{ii}} x_j \geq 0 \end{aligned} \tag{4}$$

Assume that there exists $i_0 \in \{1, \ldots, n\}$ for which $x_{i_0} = 0$. Then the same re-arrangement leads to

$$0 = x_{i_0} \geq \sum_{j\in \mathcal{S}_{i_0}^\circ} \frac{|a_{i_0 j}|}{a_{i_0 i_0}} x_j \tag{5}$$

which is only possible if all $x_j = 0$, $j \in \mathcal{S}_{i_0}^\circ$.

Let $i_0 \to i_1 \to \cdots \to i_s = i$ be a chain of indices such that $a_{i_{l-1}, i_l} \neq 0$, $l = 1, \ldots, s$. Since $i_1 \in \mathcal{S}_{i_0}^\circ$, we also have $x_{i_1} = 0$. Applying (5) to row i_1 instead of row i_0 gives $x_{i_2} = 0$ and we continue with this argument until we find that $x_{i_s} = x_i = 0$, which is a contradiction to (4). Hence $x > 0$. $\square$

Your Learning Progress | **0 marks, but -1 mark if unanswered** | What is the one most important thing that you have learnt from this assignment?

What is the most substantial new insight that you have gained from this course this week? Any *aha moment*?