MATH521: Numerical Analysis of Partial Differential Equations
Winter 2018/19, Term 2

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After today's class you should be able to

- explain the concepts of Ritz projections and Galerkin approximations
- interpret Galerkin orthogonality and apply it in manipulations of weak formulations
- define conformity of an approximation
- derive the Galerkin equations and the discrete linear system for a given linear elliptic PDE

2.3 Finite Elements for Poisson's Equation

Executive Summary The finite-element method follows an approach which is completely different from the finite-difference method. Instead of the strong formulation of a PDE, finite-element discretisations start from the *weak formulation* of a PDE. To obtain a finite-dimensional problem, we simply restrict the spaces of solutions and test functions to finite-dimensional subspaces. We no longer approximate any derivatives.

From the practical viewpoint, finite-element methods are characterised by great versatility. Their error analysis yields convergence results under far weaker assumptions than what is required for convergence estimates of finite-difference methods. Finite elements are backed by a very rich and not only in my opinion elegant theory, which allows for far-reaching and mathematically rigorous predictions to an extent that is probably unparalleled in numerical analysis.

Ritz Projections and Galerkin Approximations

How to obtain a (finite-dimensional) linear system from the weak form of a linear elliptic equation? Let us look at the two closely related approaches that were developed at the beginning of the 20^{th} century. Let us compare them in the abstract setting of some infinite-dimensional Hilbert space V (e.g. $H_0^1(\Omega)$), where $B : V \times V \to \mathbb{R}$ is a continuous and coercive bilinear form and $f \in V^*$:

The Swiss way[a]

Variational method
(applies to symmetric B only, but also to optimisation problems that are not equivalent to PDEs)

$$\min_{u \in V} \frac{1}{2} B(u,u) - \langle f, u \rangle_{V^*,V}$$

[a] W Ritz: *Über eine neue Methode zur Lösung gewisser Variationsprobleme der mathematischen Physik*, Journal für die reine und angewandte Mathematik 135, 1909, pp 1–61 `https://doi.org/10.1515/crll.1909.135.1`

The Russian way[b]

Method of weighted residuals
(applies to PDEs with symmetric or nonsymmetric B)

Find $u \in V$ such that

$$B(u,v) = \langle f, v \rangle_{V^*,V}, \qquad \forall v \in V$$

[b] BG Galerkin: *On electrical circuits for the approximate solution of the Laplace equation (in Russian)*, Vestnik Inzh 19, 1915, pp 897–908

Recall that if B is symmetric, then the two formulations are equivalent, since the equation on the right is just the first-order optimality condition (Euler-Lagrange equation) of the convex optimisation problem on the left. In this case of a symmetric bilinear form, continuity and coercivity of B imply that B is a valid inner product with which V still has its Hilbert space structure. The norm

$$\|u\|_B = \sqrt{B(u,u)}$$

that is then naturally defined by the PDE is called the *energy norm* of the problem.

The fundamental idea (proposed by Walter Ritz and then also applied by Boris Galerkin) that reduces each infinite-dimensional problem to a finite-dimensional problem is the following:

The Swiss way	*The Russian way*
$$\min_{u^h \in V^h} \frac{1}{2} B(u^h, u^h) - \langle f, u^h \rangle_{V^*,V}$$	Find $u^h \in V^h$ such that $$B(u^h, v^h) = \langle f, v^h \rangle_{V^*,V}, \qquad \forall v^h \in V^h$$

2.3.1 Group Work (Consistency of Galerkin Approximations) Determine the consistency error of the Galerkin approximation!

2.3.2 Definition (Conformity) A Galerkin approximation with $V^h \subset V$ that uses the bilinear form B and source term f of the original problem is called *conforming*.

What are possible finite-dimensional subspaces V^h of function spaces V such as $L^2(\Omega)$ or $H^1(\Omega)$?

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To obtain a linear system of equations from the problem

$$B(u, v) = \langle f, v \rangle_{V^*,V}, \qquad \forall v \in V$$

we proceed in three steps:

1st Step Choose an N-dimensional subspace $V^h \subset V$ and a basis $\left(\phi_i^h\right)_{i=1}^N$ of V^h

2nd Step Write

$$u^h =$$

3rd Step Sub this u^h into the weak formulation and use the basis functions as test functions to obtain the *Galerkin equations*