MATH521: Numerical Analysis of Partial Differential Equations
Winter 2018/19, Term 2

Due Date: Thursday, 14 February 2019
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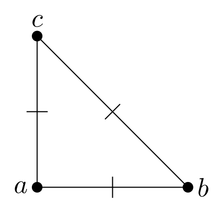
Homework Assignment 6

Please submit the following files as indicated below: source code PDF file image file video file

Question 1 | 2 marks | Given three points $a, b, c \in \mathbb{R}^2$ that are not collinear (not all on one line) and that are sorted in anticlockwise order, we define

$$T = \Delta(a, b, c) \qquad \text{(the triangle with these vertices)}$$
$$P = P_2(T)$$
$$L = \left\{ p \mapsto p(a), \quad p \mapsto p(b), \quad p \mapsto p(c), \quad p \mapsto \frac{\partial p}{\partial n}\left(\frac{a+b}{2}\right), \quad p \mapsto \frac{\partial p}{\partial n}\left(\frac{b+c}{2}\right), \quad p \mapsto \frac{\partial p}{\partial n}\left(\frac{c+a}{2}\right) \right\} \subset P^*$$



(a) Show that given any data for

$$p(a), \quad p(b), \quad p(c), \quad \frac{\partial p}{\partial n}\left(\frac{a+b}{2}\right), \quad \frac{\partial p}{\partial n}\left(\frac{b+c}{2}\right) \quad \text{and} \quad \frac{\partial p}{\partial n}\left(\frac{c+a}{2}\right)$$

there exists a unique interpolant $p \in P$.

1st Hint: Where does a parabola with $p(a) = 0$ and $p(b) = 0$ have its vertex?

2nd Hint: There is a video on checking unisolvence in the Media Gallery.

To apply the unisolvence criterion, we note that the dimension of P and the number of degrees of freedom in L are both six. We now set all degrees of freedom to zero and show that this already implies $p \equiv 0$ on T.

1. On each triangle edge, p is a parabola with identical function values (zero) at the two endpoints. Hence, its vertex is located at the edge midpoint. Consequently, the tangential derivative of p vanishes in the three edge midpoints.
2. Since the normal derivatives vanish in the edge midpoints, too, the entire gradient is zero in these three points. But the gradient of a quadratic polynomial is a linear polynomial, which is fully determined by its values in the three edge midpoints (that are not collinear). Hence, $\nabla p \equiv 0$.
3. Now it follows that p is constant, and due to the prescribed values in the corner points, this constant must be zero.

(b) Now let Ω^h be a domain with a regular triangulation $\mathcal{T}^h$ such that

$$\bar{\Omega}^h = \bigcup_{T \in \mathcal{T}^h} T.$$

Show that the space

$$V^h = \left\{ v^h : \bar{\Omega}^h \to \mathbb{R} \;\middle|\; v^h\big|_T \in P_2(T), v^h \text{ is continuous in all vertices}, \frac{\partial v^h}{\partial n} \text{ is continuous in all edge midpoints} \right\}$$

is not H^1-conforming by giving me a specific example of a function $v^h \in V^h$ that has a jump across a triangle edge.

Consider a quadrilateral domain Ω^h composed of only two triangles T_+ and T_- with the common edge $[0,1] \times \{0\}$, where T_+ is located above the x_1-axis ($x_2 \geq 0$ on T_+) and T_- is located below the x_1-axis ($x_2 \leq 0$ on T_-). We define

$$v^h = \begin{cases} x_1(1-x_1) & \text{for } x_2 \geq 0 \\ 0 & \text{for } x_2 < 0. \end{cases}$$

Clearly, $v^h\big|_{T_\pm} \in P_2(T_\pm)$ and v^h jumps discontinuously when crossing the x_1-axis at $0 < x_1 < 1$. Also,

$$\begin{aligned} v^h\big|_{T_+}(0,0) &= v^h\big|_{T_-}(0,0) \\ v^h\big|_{T_+}(1,0) &= v^h\big|_{T_-}(1,0) \\ \frac{\partial v^h\big|_{T_+}}{\partial x_2}(1/2,0) &= \frac{\partial v^h\big|_{T_-}}{\partial x_2}(1/2,0) \end{aligned}$$

and thus $v^h \in V^h$ despite its discontinuity.

NB: This is the so-called Morley element and of great importance for nonconforming approximations of 4^{th}-order elliptic equations like the plate equation or the Cahn-Hilliard equation. H^2-conforming approximations are quite demanding:

- If P should be a space $P_k(T)$ of polynomials, then this must be at least quintic ($k \geq 5$)! The classical quintic element for this purpose is the Argyris element.
- Alternatively, the triangle T could be subdivided into three smaller triangles along the angle bisectors. The functions in P are then defined as cubic C^1-splines: three cubics, each defined on one of the subtriangles, stitched together in a continuously differentiable fashion. This is the so-called Hsieh-Clough-Tocher element.

The Morley element is clearly a lot simpler than those H^2-conforming approximations. It works surprisingly well for 4^{th}-order equations (with modified Galerkin equations), even though it's not even H^1-conforming!

Question 2 | **3 marks** | We will now complete our finite-element solver for the linear elasticity problem

$$\begin{aligned} -c\Delta u + au &= f \quad \text{in } \Omega \\ u &= g \quad \text{on } \partial\Omega. \end{aligned} \tag{1}$$

(a) Remove lines 1-10 from `discretiseLinearElasticity.m` and uncomment the sections of code that are currently commented out. Complete the missing commands, including the subfunction `assembleStiffness`. Also inspect the `assembleLoad` subfunction. Remember that you may use the code from last week's model answers if you are unsure whether your own code works correctly.

(b) Write a script `hw6.m` which

- solves the linear elasticity problem on Ω^h, which you may choose from `kiwi.mat`, `maple.mat`, `pi.mat`, `ubc.mat`. You may also select your own data for $f(x_1, x_2)$, $g(x_1, x_2)$, a and c.

  Hint: You have to set `GammaD = @(x1,x2) true(size(x1))`. For debugging, you might want to use `video.mat` and check the sparsity patterns of the various matrices.

- calculates the L^2, H^1 and energy norms

$$\|u^h\|_{L^2} = \sqrt{\int_{\Omega^h} |u^h|^2 \,\mathrm{d}x}$$

$$\|u^h\|_{H^1} = \sqrt{\|u^h\|_{L^2}^2 + \|\nabla u^h\|_{L^2}^2} = \sqrt{\int_{\Omega^h} |u^h|^2 \,\mathrm{d}x + \int_{\Omega^h} |\nabla u^h|^2 \,\mathrm{d}x}$$

$$\|u^h\|_B = \sqrt{B(u^h, u^h)} = \sqrt{c\int_{\Omega^h} |\nabla u^h|^2 \,\mathrm{d}x + a\int_{\Omega^h} |u^h|^2 \,\mathrm{d}x}$$

  of the solution, where B is the bilinear form corresponding to the elliptic operator

- creates undistorted plots of the mesh, the force f and the solution u^h (including the boundary points). Post your plots of f and u^h in the discussion forum!

(c) What problem do you solve numerically when you set `GammaD = @(x1,x2) false(size(x1))`? Analyse the code to infer its weak formulation:

The hat functions centred on boundary nodes are no longer removed. Therefore, the test functions and the solution now come from $H^1(\Omega)$ instead of $H_0^1(\Omega)$. The weak form reads: find $u \in H^1(\Omega)$ such that for all $v \in H^1(\Omega)$

$$c\int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x + a\int_\Omega uv \,\mathrm{d}x = \int_\Omega fv \,\mathrm{d}x.$$

(Go back to Question 1 in Assignment 3 to see that this is the weak form of the linear elasticity problem with homogeneous Neumann boundary conditions.)

So that you don't get bored during the break... Install FEniCS and ParaView on your computer and bring it with you to our first class after the break on Tuesday 26 February. Please make sure everything is set up and running before that date. Both FEniCS and ParaView are free and open source software.

FEniCS on Ubuntu Linux or Windows 10 Follow the instructions here: `https://fenicsproject.org/download/`.

FEniCS on other Linux distributions, older versions of Windows or macOS It will be easiest to use FEniCS on Docker. Follow the installation instructions here: `https://fenics-containers.readthedocs.io/en/latest/quickstart.html`.

ParaView on Linux ParaView should already be included in the official repositories of your distribution.

ParaView on other operating systems Download it here: `https://www.paraview.org/download/`.

If you need help with troubleshooting, there is a discussion thread on Canvas.

Your Learning Progress | **0 marks, but -1 mark if unanswered** | What is the one most important thing that you have learnt from this assignment?

What is the most substantial new insight that you have gained from this course this week? Any *aha moment*?
