2.1.10 Definition (Lebesgue Space of Square-Integrable Functions) For a function $u : \Omega \to \mathbb{R}$ over a domain $\Omega \subset \mathbb{R}^d$, we define the L^2-norm

$$\|u\|_{L^2(\Omega)} = \left(\int_\Omega |u(x)|^2 \, \mathrm{d}x \right)^{1/2} .$$

vector norm for vector-valued functions $u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} \begin{array}{l} \leftarrow L^2(\Omega) \\ \leftarrow L^2(\Omega) \\ \\ \leftarrow L^2(\Omega) \end{array} \in L^2(\Omega)^n$

The set

$$L^2(\Omega) = \{ \, u : \Omega \to \mathbb{R} \mid \|u\|_{L^2(\Omega)} < \infty \, \}$$

is called the *Lebesgue space of order 2*.

2.1.11 Theorem (L^2 is a Hilbert Space) With the scalar product

$$(u, v)_{L^2(\Omega)} = \int_\Omega uv \, \mathrm{d}x$$

for vector fields $u, v \in L^2(\Omega)^n$:

$(u,v)_{L^2(\Omega)^n} = \int_\Omega u \cdot v \, dx$

the Lebesgue space $L^2(\Omega)$ is a Hilbert space. (Note that $\|u\|_{L^2} = \sqrt{(u,u)_{L^2}}$.)