2.1.7 Corollary (Uniqueness of Strong Solutions) A solution $u \in C^2(\Omega) \cap C(\bar{\Omega})$ to the Poisson-Dirichlet problem (2.1) is unique.

Proof. Take two solutions u_1, u_2 and set $v = u_1 - u_2$.

$$\Rightarrow \begin{cases} -\Delta v = 0 & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases}$$

$$\Rightarrow \begin{cases} \text{max principle: } & v(x) \le 0 & \forall x \in \bar{\Omega} \\ \text{min principle: } & v(x) \ge 0 & \forall x \in \bar{\Omega} \end{cases}$$

$\Rightarrow$ $v = 0$, i.e. $u_1 = u_2$.

□

The maximum principle can be used to derive further well-posedness results:

2.1.8 Corollary (Continuous Dependence on the Boundary Data) Solutions to (2.1) depend continuously on the boundary data, i.e. if u and $\tilde{u}$ are solutions with boundary values g and $\tilde{g}$, respectively, (but the same right hand side f), then

$$\max_{x \in \bar{\Omega}} |u(x) - \tilde{u}(x)| \le \max_{x \in \partial\Omega} |g(x) - \tilde{g}(x)| \,.$$

(Actually, equality holds.)

Proof. Set $v = u - \tilde{u}$.

$$\Rightarrow \begin{cases} -\Delta v = 0 & \text{in } \Omega \\ v = g - \tilde{g} & \text{on } \partial\Omega \end{cases}$$

$$\Rightarrow \begin{cases} \text{max principle: } & v(x) \le \max_{\partial\Omega} (g - \tilde{g}) \le \max_{\partial\Omega} |g - \tilde{g}| & \forall x \in \bar{\Omega} \\ \text{min principle: } & v(x) \ge \min_{\partial\Omega} (g - \tilde{g}) \ge -\max_{\partial\Omega} |g - \tilde{g}| & \forall x \in \bar{\Omega} \end{cases}$$

$\Rightarrow$ both $+v(x)$ and $-v(x) \le \max_{\partial\Omega} |g - \tilde{g}|$, so $|v(x)| \le \max_{\partial\Omega} |g - \tilde{g}| \quad \forall x \in \bar{\Omega}$

□

Of course, this result can be extended to continuous dependence on the right hand side f, too. It actually holds not only for the Poisson problem, but can be generalised to uniformly elliptic operators without zeroth-order terms. We present it without proof here:

2.1.9 Corollary (Continuous Dependence on the Right Hand Side and Boundary Data) Solutions to (2.1) depend continuously on the data, i.e. if u and $\tilde{u}$ are solutions with data f, g and $\tilde{f}, \tilde{g}$, respectively, then there exists a constant $C > 0$ such that

$$\max_{x \in \bar{\Omega}} |u(x) - \tilde{u}(x)| \le C \sup_{x \in \Omega} \left| f(x) - \tilde{f}(x) \right| + \max_{x \in \partial\Omega} |g(x) - \tilde{g}(x)| \,.$$

(I.e. if $|f - \tilde{f}|$ is small and $|g - \tilde{g}|$ is small, then $|u - \tilde{u}|$ must be small.)