Finite Difference Discretisation of the Poisson-Dirichlet Problem on a Rectangular Domain

$$
\begin{aligned}
-\Delta u &= f \quad \text{in } \Omega & u &= g \quad \text{on } \partial\Omega \\
\Leftrightarrow \left(-\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2}\right) u &= f \quad \text{in } \Omega & u &= g \quad \text{on } \partial\Omega \\
\overset{\text{FDM}}{\rightsquigarrow} \left(L_1^h + L_2^h\right) u^h &= f^h
\end{aligned}
$$

Discrete Laplacian

$$
L_1^h = \frac{1}{h_1^2}\left(\begin{array}{ccccc|ccccc|c|ccccc}
2 & -1 & & & & & & & & & & & & & & \\
-1 & 2 & -1 & & & & & & & & & & & & & \\
 & & \ddots & & & & & & & & & & & & & \\
 & & -1 & 2 & -1 & & & & & & & & & & & \\
 & & & -1 & 2 & & & & & & & & & & & \\
\hline
 & & & & & 2 & -1 & & & & & & & & & \\
 & & & & & -1 & 2 & -1 & & & & & & & & \\
 & & & & & & & \ddots & & & & & & & & \\
 & & & & & & & -1 & 2 & -1 & & & & & & \\
 & & & & & & & & -1 & 2 & & & & & & \\
\hline
 & & & & & & & & & & \ddots & & & & & \\
\hline
 & & & & & & & & & & & 2 & -1 & & & \\
 & & & & & & & & & & & -1 & 2 & -1 & & \\
 & & & & & & & & & & & & & \ddots & & \\
 & & & & & & & & & & & & & -1 & 2 & -1 \\
 & & & & & & & & & & & & & & -1 & 2
\end{array}\right)
$$

$$
L_2^h = \frac{1}{h_2^2}\left(\begin{array}{ccc|ccc|ccc|c|ccc|ccc}
2 & & & -1 & & & & & & & & & & & & \\
 & \ddots & & & \ddots & & & & & & & & & & & \\
 & & 2 & & & -1 & & & & & & & & & & \\
\hline
-1 & & & 2 & & & -1 & & & & & & & & & \\
 & \ddots & & & \ddots & & & \ddots & & & & & & & & \\
 & & -1 & & & 2 & & & -1 & & & & & & & \\
\hline
 & & & & & & & & & & & & & & & \\
\hline
 & & & & & & & & & & 2 & & & -1 & & \\
 & & & & & & & & & & & \ddots & & & \ddots & \\
 & & & & & & & & & & & & 2 & & & -1 \\
\hline
 & & & & & & & & & & -1 & & & 2 & & \\
 & & & & & & & & & & & \ddots & & & \ddots & \\
 & & & & & & & & & & & & -1 & & & 2
\end{array}\right)
$$

Note that these matrices are composed of $(N_2 - 1) \times (N_2 - 1)$ blocks, and each block has $(N_1 - 1) \times (N_1 - 1)$ entries.

They can also be expressed in more compact form by using a Kronecker product

$$L_1^h = \frac{1}{h_1^2} I_{N_2-1} \otimes A_{N_1-1}$$
$$L_2^h = \frac{1}{h_2^2} A_{N_2-1} \otimes I_{N_1-1}$$

where I_n is the $n \times n$ identity matrix and A_n is the $n \times n$ second-difference matrix (with entries -1, 2, -1).

Discrete Source Term with Boundary Data

$$f^h = \left(\begin{array}{c}
f_1^h + \frac{1}{h_1^2}\begin{bmatrix} g_{W,1}^h \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} + \frac{1}{h_1^2}\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ g_{E,1}^h \end{bmatrix} + \frac{1}{h_2^2} g_S^h \\ \hline
f_2^h + \frac{1}{h_1^2}\begin{bmatrix} g_{W,2}^h \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} + \frac{1}{h_1^2}\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ g_{E,2}^h \end{bmatrix} \\ \hline
\vdots \\ \hline
f_{N_2-2}^h + \frac{1}{h_1^2}\begin{bmatrix} g_{W,N_2-2}^h \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} + \frac{1}{h_1^2}\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ g_{E,N_2-2}^h \end{bmatrix} \\ \hline
f_{N_2-1}^h + \frac{1}{h_1^2}\begin{bmatrix} g_{W,N_2-1}^h \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} + \frac{1}{h_1^2}\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ g_{E,N_2-1}^h \end{bmatrix} + \frac{1}{h_2^2} g_N^h
\end{array} \right)$$

where f_j^h is the vector of length $N_1 - 1$ with the source term f evaluated on the j-th row of interior grid points, g_N^h, g_S^h are vectors of length $N_1 - 1$ with the boundary values on the top or bottom and g_W^h, g_E^h are vectors of length $N_2 - 1$ with the boundary values on the left or right.

The boundary contributions to f^h can also be expressed in more compact form using Kronecker products between vectors g_X^h of boundary values and standard unit vectors e_i ...