

Winter 2018/19, Term 2

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Finite Difference Discretisation of the Poisson-Dirichlet Problem on a Rectangular Domain

$$\begin{split} -\Delta u &= f &\quad \text{in } \Omega & u = g &\quad \text{on } \partial \Omega \\ \Leftrightarrow \left(-\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} \right) u &= f &\quad \text{in } \Omega & u = g &\quad \text{on } \partial \Omega \\ \overset{\text{FDM}}{\leadsto} \left(L_1^h + L_2^h \right) u^h &= f^h & \end{split}$$

Discrete Laplacian

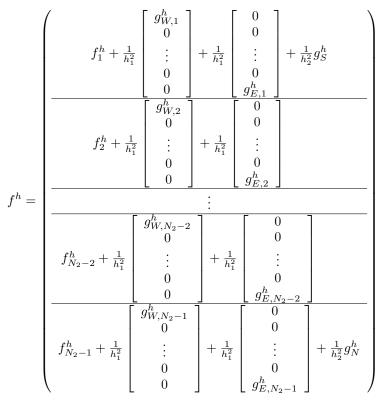
$L_1^h=rac{1}{h_1^2}$	$ \begin{array}{cccccccccccccccccccccccccccccccccccc$	$ \begin{array}{c ccccccccccccccccccccccccccccccccccc$			
					$ \begin{array}{cccccccccccccccccccccccccccccccccccc$
$L_2^h = \frac{1}{h_2^2}$	$ \begin{pmatrix} 2 & & & & -1 \\ & & \ddots & & & \\ & & 2 & & \\ \hline -1 & & & 2 & & \\ & & \ddots & & & \\ & & & -1 & & \\ \hline & & & & & $	··1 -1 ··. 2	·. —1 ——————————————————————————————————		
	rices are composed of	$(N_2-1) \times (N_2-1)$	-1	· 2 ·1	$ \begin{array}{c c} \hline -1 & & \\ & \ddots & \\ \hline 2 & & \\ & \ddots & \\ \hline (N_1-1)\times(N_1-1) \text{ entries.} \end{array} $

They can also be expressed in more compact form by using a Kronecker product

$$L_1^h = \frac{1}{h_1^2} I_{N_2 - 1} \otimes A_{N_1 - 1}$$
$$L_2^h = \frac{1}{h_2^2} A_{N_2 - 1} \otimes I_{N_1 - 1}$$

where I_n is the $n \times n$ identity matrix and A_n is the $n \times n$ second-difference matrix (with entries -1, 2, -1).

Discrete Source Term with Boundary Data



where f_j^h is the vector of length $N_1 - 1$ with the source term f evaluated on the j-th row of interior grid points, g_N^h, g_S^h are vectors of length $N_1 - 1$ with the boundary values on the top or bottom and g_W^h, g_E^h are vectors of length $N_2 - 1$ with the boundary values on the left or right.

The boundary contributions to f^h can also be expressed in more compact form using Kronecker products between vectors g_X^h of boundary values and standard unit vectors $e_i \dots$