MATH521: Numerical Analysis of Partial Differential Equations
Winter 2018/19, Term 2

Due Date: Friday, 5 April 2019
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Homework Assignment 12

Please submit the following files as indicated below: source code PDF file image file video file

Question 1 | 2 marks | If we discretise a linear convection[1] equation, such as the heat equation, the advection-diffusion equation or the first-order hyperbolic advection equation, with the θ-scheme in time and finite differences, finite elements or finite volumes in space, then we typically obtain a fully discrete scheme of the form

$$\left(M^h + \Delta t\theta C^h\right)\vec{u}^h_+ = \left(M^h - \Delta t(1-\theta)C^h\right)\vec{u}^h_\circ + \vec{r}^h$$

with a strictly diagonally dominant mass matrix $M^h \geq 0$, a weakly chained diagonally dominant L-matrix C^h and a source term $\vec{r}^h \geq 0$.

This iteration is called *positivity preserving* if

$$\vec{u}^h_\circ \geq 0 \quad \Longrightarrow \quad \vec{u}^h_+ \geq 0.$$

Assuming that the sparsity pattern of the mass matrix M^h is contained in the sparsity pattern of the discretised convection matrix C^h, i.e. $m^h_{ij} \neq 0 \Rightarrow c^h_{ij} \neq 0$, show that a sufficient condition for positivity preservation of the scheme is the time-step restriction

$$\max_{i,j:c^h_{ij}<0} \frac{m^h_{ij}}{\theta|c^h_{ij}|} < \Delta t < \min_i \frac{m^h_{ii}}{(1-\theta)c^h_{ii}}.$$

[1]For mathematicians, convection = advection + diffusion.

Question 2 | **3 marks** | The Fisher equation

$$
\begin{aligned}
\frac{\partial u}{\partial t} - \Delta u &= u(1-u) && \text{in } Q =]0,T[\times\Omega \\
u(0) &= u_0 && \text{in } \Omega \\
\frac{\partial u}{\partial n} &= 0 && \text{on } \Sigma =]0,T[\times\partial\Omega
\end{aligned}
$$

is a semilinear reaction-diffusion equation that is used in mathematical biology to model the spread and growth of a population u that lives in the domain Ω. The term on the right hand side models logistic growth / decay for a system with a carrying capacity scaled to 1. It follows from Sobolev embeddings[2] that the nonlinear source term $u(1-u) \in L^2(\Omega)$ provided that $u \in H^1(\Omega)$. Therefore, we may continue to work with the same function spaces that we already used for linear elliptic or parabolic PDEs.

After a discretisation with an implicit θ-scheme in time ($\theta > 0$), we are confronted with the following semilinear elliptic problem:

Find the solution $u_+ \in V$ at the next time level such that for all $v \in V$

$$
\int\limits_\Omega u_+ v \, \mathrm{d}x + \Delta t \theta \left(\int\limits_\Omega \nabla u_+ \cdot \nabla v \, \mathrm{d}x - \int\limits_\Omega u_+(1-u_+) v \, \mathrm{d}x \right) = \langle f_\circ, v \rangle_{V^*,V} \tag{1}
$$

with the known right hand side

$$
\langle f_\circ, v \rangle_{V^*,V} = \int\limits_\Omega u_\circ v \, \mathrm{d}x + \Delta t (1-\theta) \left(- \int\limits_\Omega \nabla u_\circ \cdot \nabla v \, \mathrm{d}x + \int\limits_\Omega u_\circ (1-u_\circ) v \, \mathrm{d}x \right).
$$

If this is only a semi-discretisation, then $V = H^1(\Omega)$. If a conforming finite-element discretisation is applied in space, then $V = V^h \subset H^1(\Omega)$.

[2] On a sufficiently regular 2D or 3D domain, all functions $u \in H^1(\Omega)$ are also in $L^p(\Omega)$ for $p \leq 6$. In particular $u \in L^2(\Omega)$ and $u \in L^4(\Omega)$. The latter implies $u^2 \in L^2(\Omega)$, and hence $u(1-u) = u - u^2 \in L^2(\Omega)$

Since (1) is not a linear equation, we cannot simply solve a big linear system $Au_+ = b$ for the next time step u_+. Instead, we now have to solve a big nonlinear system $F(u_+) = 0$ e.g. with Newton's method to compute u_+.

Newton's method for the solution of the nonlinear equation $F(u_+) = 0$

1. Start with an initial guess u_+^0 (normally $u_+^0 = u_\circ$) and $n = 0$.
2. Given u_+^n, solve the linear problem
$$DF(u_+^n)\delta = -F(u_+^n). \tag{2}$$
3. Update $u_+^{n+1} = u_+^n + \delta$.
4. If a stopping criterion is met, e.g. if
$$\|u_+^{n+1} - u_+^n\|_V \leq \text{abstol} + \text{reltol}\ \|u_+^{n+1}\|_V$$
then stop, else set $n \leftarrow n + 1$ and go to step 2.

If the initial guess is sufficiently close to the true solution, i.e. if $\|u_+^0 - u_+\|_V$ is sufficiently small, and if the (Fréchet) derivative DF is invertible at the solution u_+ and Lipschitz continuous in a neighbourhood around it, then the Newton iterates u_+^n converge to the exact solution u_+ at a locally quadratic rate.

We apply Newton's method to solve (1) for the next time step of Fisher's equation. Derive the weak form of the linear elliptic equation (2) that you have to solve for the increment $\delta \in V$ in each Newton iteration.

Your Learning Progress | **0 marks, but -1 mark if unanswered** | What is the one most important thing that you have learnt from this assignment?

What is the most substantial new insight that you have gained from this course this week? Any *aha moment*?