2.2.19 Lemma (M-Criterion) Every weakly chained diagonally dominant L-matrix is monotone (and hence an M-matrix).

Proof. This criterion for monotonicity was presented and proved by JH Bramble and BE Hubbard in the article *On a finite difference analogue of an elliptic boundary value problem which is neither diagonally dominant nor of non-negative type*, Journal of Mathematics and Physics 43(1-4), 1964, pp 117–132.

Let $A \in \mathbb{R}^{n\times n}$ be a weakly chained diagonally dominant L-matrix.

It will be convenient to introduce the notation

$$SDD = \left\{ i \in \{ 1,\ldots,n \} \,\middle|\, \sum_j a_{ij} > 0 \right\}$$

for the index set of strongly diagonally dominant rows, and for all other rows in which A is weakly but not strictly diagonally dominant we write

$$WDD = \{ 1,\ldots,n \} \setminus SDD = \left\{ i \in \{ 1,\ldots,n \} \,\middle|\, \sum_j a_{ij} = 0 \right\}.$$

Furthermore, for the set of nonzero elements in row i (i.e. the stencil) we use the notation

$$\mathcal{S}_i = \{ j \in \{ 1,\ldots,n \} \mid a_{ij} \neq 0 \}$$

and $\mathcal{S}_i^\circ = \mathcal{S}_i \setminus \{ i \}$ for the stencil with its centre removed.

The L-matrix property allows us to simplify the (in)equalities of diagonal dominance:

- for all rows $i \in SDD$:
$$a_{ii} > \sum_{j\neq i} |a_{ij}| = \sum_{j\in\mathcal{S}_i^\circ} |a_{ij}| \qquad \Leftrightarrow \qquad \sum_{j\in\mathcal{S}_i^\circ} \frac{|a_{ij}|}{a_{ii}} < 1$$

- for all rows $i \in WDD$:
$$a_{ii} = \sum_{j\neq i} |a_{ij}| = \sum_{j\in\mathcal{S}_i^\circ} |a_{ij}| \qquad \Leftrightarrow \qquad \sum_{j\in\mathcal{S}_i^\circ} \frac{|a_{ij}|}{a_{ii}} = 1$$

To show that A is monotone, let $x \in \mathbb{R}^n$ be such that $Ax \geq 0$. That is, for all rows $i \in \{ 1,\ldots,n \}$

Now select a row i where $x_i = \min_{j=1,\ldots,n} x_j = m$. Assume that $m < 0$.

- if $i \in SDD$:

- if $i \in WDD$:

Therefore, $x \geq 0$. $\square$

2.2.20 Theorem (Monotonicity of the Discrete Laplacian) The finite-difference approximation L^h of the Laplacian with Dirichlet boundary conditions is monotone:

$$L^h u^h \geq 0 \qquad \Rightarrow \qquad u^h \geq 0.$$

Proof. Please convince yourself that the matrix L^h is

- strongly diagonally dominant in all rows corresponding to grid points adjacent to the boundary,
- weakly, but not strongly diagonally dominant in all rows corresponding to the other grid points 'further away' from the boundary,
- irreducibly diagonally dominant[6], and in particular weakly chained diagonally dominant[7]
- an L-matrix since it has only nonnegative off-diagonal entries and only positive diagonal entries.

The M-criterion now implies that L^h is an M-matrix and in particular monotone, as asserted. $\square$

[6] "One can walk from any interior grid point to any other interior grid point, taking only steps that are covered by the stencil."
[7] "One can walk from any interior grid point to a point adjacent to the boundary where the matrix is strictly diagonally dominant."

2.2.21 Lemma (Algebraic Maximum Principle) Let $A \in \mathbb{R}^{n\times n}$ be a weakly chained diagonally dominant L-matrix. The matrix A satisfies the algebraic maximum principle

$$(Ax)_i \le 0 \quad \forall i \in WDD \qquad \Rightarrow \qquad \max_{i=1,\dots,n} x_i \le \max_{i\in SDD} x_i$$

i.e. provided that the right hand side is nonpositive in rows where A is not strictly diagonally dominant, the solution vector x assumes its maximum in a row where A is strictly diagonally dominant. Vice versa, the algebraic minimum principle

$$(Ax)_i \ge 0 \quad \forall i \in WDD \qquad \Rightarrow \qquad \min_{i=1,\dots,n} x_i \ge \min_{i\in SDD} x_i$$

guarantees that for such linear systems with nonnegative right hand side in rows of only weak diagonal dominance, the solution vector x assumes its minimum in a row where A is strictly diagonally dominant.

Proof. Let us prove the minimum principle here. Then the maximum principle follows by applying the minimum principle with the vector $-x$ instead of x.

The proof follows the same steps as the proof of the M-criterion. Let $x \in \mathbb{R}^n$ be given such that $(Ax)_i \ge 0$ for all rows $i \in WDD$. Now select one such row $i \in WDD$ where $x_i = \min_{j\in WDD} x_j = m$. Assume that $m < \min_{j\in SDD} x_j$.

Like in the proof of Lemma 2.2.19 we derive the estimate

$$m = x_i \ge m \sum_{j\in\mathcal{S}_i^\circ} \frac{|a_{ij}|}{a_{ii}} + \sum_{j\in\mathcal{S}_i^\circ} \frac{|a_{ij}|}{a_{ii}}(x_j - m). \tag{2.17}$$

Since $i \in WDD$, $\sum_{j\in\mathcal{S}_i^\circ} \frac{|a_{ij}|}{a_{ii}} = 1$ and thus (2.17) can only hold if all $x_j = m$ ($j \in \mathcal{S}_i^\circ$).

Let $i = i_0 \to i_1 \to \dots \to i_s \in SDD$ be a chain from row i to a row i_s with strict diagonal dominance, where all $a_{i_{l-1},i_l} \ne 0$ ($l = 1,\dots,s$).

Since $i_1 \in \mathcal{S}_i^\circ$, $x_{i_1} = m$, too. If $i_1 \in SDD$, (2.17) leads to a contradiction, else we find that all $x_j = m$ in the stencil around i_1 ($j \in \mathcal{S}_{i_1}^\circ$), including $x_{i_2} = m$. Continuing with this argument, we eventually arrive at row $i_s \in SDD$, and a final contradiction becomes inevitable. Therefore, the assumption of $m < \min_{j\in SDD} x_j$ cannot hold, so x cannot have a strict global minimum in a row $i \in WDD$. $\square$

Don't you agree that these are really neat analogies? The weak chain property captures the propagation of information in at least one direction (like for first-order hyperbolic equations), irreducibility global propagation of information (like for elliptic equations). Strict diagonal dominance arises in our finite-difference discretisations due to imposed boundary conditions or a positive zeroth-order coefficient. Weakly but not strictly diagonally dominant rows correspond to points in the interior of the domain where there is no zeroth-order reaction term.

2.2.22 Theorem (Discrete Maximum Principle for the Laplacian) Let $-\Delta^h$ be the finite-difference discretisation of the Laplacian. Then

$$\begin{aligned}
-\Delta^h u^h &\le 0 \quad \text{in } \Omega^h \qquad &\Rightarrow \qquad &\max_{x\in\bar\Omega^h} u^h(x) \le \max_{x\in\partial\Omega^h} u^h(x)\\
-\Delta^h u^h &\ge 0 \quad \text{in } \Omega^h \qquad &\Rightarrow \qquad &\min_{x\in\bar\Omega^h} u^h(x) \ge \min_{x\in\partial\Omega^h} u^h(x).
\end{aligned}$$

Proof. We present the explicit proof for the problem in 1D. The extension to higher dimensions is straightforward.

To obtain a linear system that also includes the values of the numerical solution u^h on the boundary $\partial\Omega^h$, we do not eliminate the boundary values this time. The 'even bigger matrix' reads

This matrix

-
-
-
-

and now the maximum principle for the discrete Laplacian follows from the general algebraic maximum principle. □

As alluded to before, most of our results in this section are equally applicable to certain discretisations e.g. of hyperbolic equations and not specific to discretised elliptic operators. It is, however, possible to use exploit additional structure of discretised elliptic problems to refine these monotonicity results and discrete maximum principles further. You may investigate this area in the analytical version of Homework Assignment 4!

On to the bad news: despite this beautiful theory of monotone discretisations, very many discretisations are not monotone. We usually look for schemes that are at least second-order consistent and convergent, because the accuracy of solutions from first-order schemes is often insufficient. However, there is a conflict between higher-order consistency and monotonicity of the discretisation. This applies not only to finite-difference methods, but to all possible discretisation schemes:

2.2.23 Theorem (Godunov's Order Barriers)

(a) A linear and monotone discretisation of a first-order derivative is at most first-order consistent.

(b) A linear and monotone discretisation of a second-order derivative is at most second-order consistent.

Proof. A very clear presentation of these order barriers for finite-difference methods can be found in the book W Hundsdorfer and JG Verwer: *Numerical Solution of Time-Dependent Advection-Diffusion-Reaction Equations*, Springer-Verlag Berlin Heidelberg, 2003 with the actual result on pp 118–121. □

Since both high accuracy and monotonicity are usually important, we will have to resort to *nonlinear* schemes. This means that the coefficients in the 'big, no longer linear system' will have to depend on the solution. Nonlinear schemes are commonly used for accurate solutions of PDE problems with a dominant advection term (i.e. derivatives of first order), in order to attain higher than first-order consistency while still preserving monotonicity. Prominent examples of nonlinear schemes include flux or slope limiters, streamline diffusion methods, ENO (essentially non-oscillatory) and WENO (weighted ENO) schemes.

2.2.24 Remark (Advantages and Disadvantages of Finite-Difference Methods)

⊕ Finite-difference methods are easy to set up and implement on equidistant, rectangular grids.

⊕ In these cases, the discrete matrices possess a lot of structure, which can be exploited for efficient solution algorithms of the resulting linear systems.

⊕ Some finite-difference methods accurately reflect characteristic features of an elliptic PDE, e.g. well-posedness, maximum principle, symmetry, positive definiteness, global propagation of information.

⊖ The equations become very complicated on non-equidistant grids or more complex domains. Then much of the structure of the discrete matrices is lost as well.

⊖ Finite-difference methods only approximate some point values of the solution, they do not return a function that is defined over the entire domain.

⊖ C^4 regularity up to the boundary is required to guarantee 2^{nd} order convergence for the Poisson problem. This is extremely unrealistic in practice.