Homework Assignment 9

Please submit the following files as indicated below: source code PDF file image file video file

Question 1 | **2 marks** | We consider the initial boundary value problem for the heat equation

$$\begin{aligned} \frac{\partial u}{\partial t}(t) - a\Delta u(t) &= f(t) \quad && \text{in } Q =]0,T[\times\Omega \\ u(0) &= u_0 && \text{in } \Omega \\ \frac{\partial u}{\partial n} &= 0 && \text{on } \Sigma =]0,T[\times\partial\Omega \end{aligned} \tag{H}$$

where u is a temperature field, u_0 an initial temperature distribution, the diffusion-like parameter $a > 0$ the heat conductivity of the material, f a source term e.g. due to thermal radiation and $T > 0$ a final time. The homogeneous Neumann boundary conditions mean that the domain Ω is perfectly insulated so that no thermal energy is radiated into the environment.

The θ-method is a class of Runge-Kutta schemes for integrating ODEs of the form

$$\dot{U} = F(t, U)$$

by using the iteration

$$U_+ = U_\circ + \Delta t\left(\theta F(t_+, U_+) + (1-\theta)F(t_\circ, U_\circ)\right).$$

The parameter $\theta \in [0,1]$ can be interpreted as the 'degree of implicitness', since $\theta = 0$ gives the forward Euler method, $\theta = \frac{1}{2}$ the Crank-Nicolson method (aka implicit trapezium rule in the ODE context) and $\theta = 1$ the backward Euler method.

For the discretisation in space, we apply linear finite elements.

Show that if the spatial triangulation $\mathcal{T}^h$ remains fixed, then both the method of lines and Rothe's method lead to the same discrete problems

$$\left(M^h + \theta\Delta t a K^h\right)\vec{u}^h_+ = \left(M^h - (1-\theta)\Delta t a K^h\right)\vec{u}^h_\circ + \Delta t\left(\theta \vec{f}^h_+ + (1-\theta)\vec{f}^h_\circ\right).$$

You don't have to include any details about the components of the discrete vectors and matrices. We all know what they are!

<u>Method of Lines</u>

Discretise space then time:

$$u_t - a\Delta u = f$$

$$\int_\Omega (u_t - a\Delta u - f)v \,\mathrm{d}x = 0$$

$$\int_\Omega u_t v \,\mathrm{d}x + a\int_\Omega \nabla u\cdot\nabla v \,\mathrm{d}x - \cancel{\int_{\partial\Omega}\frac{\partial u}{\partial n} v \,\mathrm{d}s}^{\,0} = \int_\Omega f v \,\mathrm{d}x$$

Weak formulation:

$$\int_\Omega u_t v \,\mathrm{d}x + a\int_\Omega \nabla u\cdot\nabla v \,\mathrm{d}x = \int_\Omega f v \,\mathrm{d}x$$

Using:

$$v^h = \sum_{j=1}^{N} v^h(x^j)\phi_j^h$$

$$\sum_{j=1}^{N} \int_\Omega \phi_j^h \cdot u_{t,j}^h \, \mathrm{d}x + a \sum_{j=1}^{N} \left(\int_\Omega \nabla \phi_j^h \cdot \nabla \phi_i^h \, \mathrm{d}x \right) u_j^h = \sum_{j=1}^{N} \int_\Omega f_j^h \cdot \phi_i^h \, \mathrm{d}x$$

$$=> M^h \vec{u}_t^h + aK^h \vec{u}_j^h = \vec{f}_j^h$$

Discretise in time:

$$M^h \frac{\left(\vec{u}_+^h - \vec{u}_o^h\right)}{\Delta t} = \theta \left(-aK^h \vec{u}_+^h + \vec{f}_+^h\right) + (1-\theta)\left(-aK^h \vec{u}_o^h + \vec{f}_o^h\right)$$

$$M^h \vec{u}_+^h = M^h \vec{u}_o^h + \Delta t \theta \left(-aK^h \vec{u}_+^h + \vec{f}_+^h\right) + \Delta t(1-\theta)\left(-aK^h \vec{u}_o^h + \vec{f}_o^h\right)$$

Leaving us with:

$$\left(M^h + a\Delta t \theta K^h\right) \vec{u}_+^h = \left(M^h - a\Delta t(1-\theta) K^h\right) \vec{u}_o^h + \Delta t \left(\theta \vec{f}_+^h + (1-\theta)\vec{f}_o^h\right)$$

<u>Rothe's Method</u>

Discretise in time then space:

$$u_t - a\Delta u = f$$

$$\frac{u_+ - u_o}{\Delta t} = \theta \left(a\Delta u_+ + f_+\right) + (1-\theta)\left(a\Delta u_o + f_o\right)$$

$$u_t = u_o + \Delta t \theta \left(a\Delta u_+ + f_+\right) + \Delta t(1-\theta)\left(a\Delta u_o + f_o\right)$$

$$u_t - \Delta t \theta a \Delta u_+ = u_o + \Delta t(1-\theta) a \Delta u_o + \Delta t \theta f_+ + \Delta t(1-\theta) f_o$$

$$\int_\Omega u_+ v \, \mathrm{d}x - \Delta t \theta a \int_\Omega \Delta u_+ v \, \mathrm{d}x = \int_\Omega u_o v \, \mathrm{d}x + \Delta t(1-\theta) a \int_\Omega \Delta u_o v \, \mathrm{d}x + \Delta t \theta \int_\Omega f_+ v \, \mathrm{d}x + \Delta t(1-\theta) \int_\Omega f_o v \, \mathrm{d}x$$

$$\int_\Omega u_+ v \, \mathrm{d}x - \Delta t \theta a \left[\cancelto{0}{\int_{\partial\Omega} \frac{\partial u_+}{\partial n} v \, \mathrm{d}s} - \int_\Omega \nabla u_+ \cdot \nabla v \, \mathrm{d}x \right] = \int_\Omega u_o v \, \mathrm{d}x + \Delta t(1-\theta) a \left[\cancelto{0}{\int_{\partial\Omega} \frac{\partial u_o}{\partial n} v \, \mathrm{d}s} - \int_\Omega \nabla u_o \cdot \nabla v \, \mathrm{d}x \right]$$
$$+ \Delta t \theta \int_\Omega f_+ v \, \mathrm{d}x + \Delta t(1-\theta) \int_\Omega f_o v \, \mathrm{d}x$$

$$\int_\Omega u_+ v \, \mathrm{d}x + \Delta t \theta a \int_\Omega \nabla u_+ \cdot \nabla v \, \mathrm{d}x = \int_\Omega u_o v \, \mathrm{d}x + \Delta t(1-\theta) a - \int_\Omega \nabla u_o \cdot \nabla v \, \mathrm{d}x + \Delta t \theta \int_\Omega f_+ v \, \mathrm{d}x + \Delta t(1-\theta) \int_\Omega f_o v \, \mathrm{d}x$$

Using:

$$v^h = \sum_{j=1}^{N} v^h(x^j)\phi_j^h$$

$$\sum_{j=1}^{N} \int_\Omega \phi_j^h \cdot u_{+j}^h \, \mathrm{d}x + \Delta t \theta a \sum_{j=1}^{N} \left(\int_\Omega \nabla \phi_j^h \cdot \nabla \phi_i^h \, \mathrm{d}x \right) u_{+j}^h = \sum_{j=1}^{N} \int_\Omega \phi_j^h \cdot u_{oj}^h \, \mathrm{d}x - \Delta t(1-\theta) a \sum_{j=1}^{N} \left(\int_\Omega \nabla \phi_j^h \cdot \nabla \phi_i^h \, \mathrm{d}x \right) u_{oj}^h$$
$$+ \Delta t \theta \sum_{j=1}^{N} \int_\Omega \phi_j^h \cdot f_{+j}^h \, \mathrm{d}x + \Delta t(1-\theta) \sum_{j=1}^{N} \int_\Omega \phi_j^h \cdot f_{oj}^h \, \mathrm{d}x$$

$$M^h \vec{u}_+^h + \Delta t \theta a K^h \vec{u}_+^h = M^h \vec{u}_o^h - \Delta t(1-\theta) a K^h \vec{u}_o^h + \Delta t \theta \vec{f}_+^h + \Delta t(1-\theta)\vec{f}_o^h$$

Leaving us with as above:

$$\left(M^h + a\Delta t \theta K^h\right)\vec{u}_+^h = \left(M^h - a\Delta t(1-\theta)K^h\right)\vec{u}_o^h + \Delta t\left(\theta \vec{f}_+^h + (1-\theta)\vec{f}_o^h\right)$$

So if the spatial triangulation is fixed, the entries for M^h and K^h will be the same, resulting in the same full discretisation.

Question 2 | 3 marks

(a) The FEniCS script `hw9.py` implements the backward Euler method for Problem (H). Starting from room temperature ($u_0 \equiv 20$), the bottom left corner of a metal piece Ω with conductivity parameter $a = 0.1$ is held over a flame for one second, then the flame is extinguished. This is modelled by

$$f(t,x) = \begin{cases} 200e^{-5x_1^2 - 2x_2^2} & t \leq 1 \\ 0 & t > 1 \end{cases}$$

Complete the missing commands to compute the evolution of the temperature field over the first five seconds using a time step size of $\Delta t = 10^{-2}$.

Save your results as a video, using a frame rate such that the video time is equal to the physical time. You don't have to submit any other files for this part of Question 2.

Hint: Open the PVD-file in ParaView, click the *Apply* button and make sure that in the *View* menu the *Color Map Editor* is highlighted. Then select a reasonable colour map

colourmap.png

and re-scale the colour values

rescale.png

to the range $[20, 160]$. Use the same range for the following questions, too.

(b) Generalise this script to implement the θ-method from Question 1. Check whether setting $\theta = 1$ still gives you the same results. Using the same parameters as in Question 2(a), solve the problem with the Crank-Nicolson method and the forward Euler method. What do you observe?

Implementing the θ-method as above, when setting $\theta = 1$ it does indeed return the same results as the previous backward Euler method. Changing $\theta = 0.5$ to use the Crank-Nicholson method, the results remain qualitatively the same. The domain heats in the corner then dissipates, although there may be some discrepancies, they are almost impossible to discern at full speed. Finally, setting $\theta = 0$ to leave us with the forward Euler method, this simply didn't work. The method is unstable leaving us with some very weird oscillations all over, then when the source term disappeared the solution blew up.

(c) Solve the problem with the forward Euler method again up to time $T = 0.1$, once with $\Delta t = 1.25 \times 10^{-4}$ and once with $\Delta t = 10^{-4}$. Explain your observations, using the relevant terminology.

Solving the problem again with forward Euler, setting $T = 0.1$ and $\Delta t = 1.25 \times 10^{-4}$, this time-step is much closer to the maximum time-step and the code looked to converge for a time, but eventually became unstable and blew up. Setting $\Delta t = 10^{-4}$ the forward Euler method remained stable up to the final time, producing a very nice dissipation of heat for a tenth of a second.

Your Learning Progress | **0 marks, but -1 mark if unanswered** | What is the one most important thing that you have learnt from this assignment?

Fenics is rather finicky when it comes to applying the constant tag to different terms

What is the most substantial new insight that you have gained from this course this week? Any *aha moment*?

The forward Euler method is almost useless (comparatively) in most numerical simulations, and you would be better served by almost any other method.