After today's class you should be able to

- contrast the analytical techniques used to study strong vs weak formulations of elliptic boundary value problems
- apply a Poincaré-type inequality and the Lax-Milgram lemma to prove well-posedness of elliptic boundary value problems
- explain under what circumstances solutions to second-order elliptic equations may be more regular than H^1, and when singularities may arise

Well-Posedness of Weak Formulations

Problems (S) and (W) representing the strong or weak form of the Poisson-Dirichlet problem, respectively, are of rather different form. Accordingly, very different analytical toolkits apply to study existence, uniqueness, continuous dependence on data and regularity of solutions to these problems. For strong formulations, one can show existence of solutions by means of Green's functions and maximum principles yield uniqueness and continuous dependence on data. In the context of weak formulations, two famous results from functional analysis immediately give the well-posedness of (W): the Lax-Milgram lemma together with the Poincaré inequality. In our numerical analysis, we will make use of these two results over and over again.

2.1.20 Lemma (Poincaré Inequality)

(a) Let Ω be a bounded domain. There exists a constant $c_P > 0$ such that for all $u \in H_0^1(\Omega)$

$$\|u\|_{L^2(\Omega)} \leq c_P \|\nabla u\|_{L^2(\Omega)}. \tag{2.2}$$

(b) Let Ω be a bounded Lipschitz domain. There exists a constant $c_P > 0$ such that for all $u \in H^1(\Omega)$ with $\int_\Omega u \, \mathrm{d}x = 0$

$$\|u\|_{L^2(\Omega)} \leq c_P \|\nabla u\|_{L^2(\Omega)}. \tag{2.3}$$

Proof. A very useful general form of these results is derived in Lemma 1.36 of H Gajewsky, K Gröger, K Zacharias: *Nichtlineare Operatorgleichungen und Operatordifferentialgleichungen*, Mathematische Lehrbücher und Monographien II, Vol 38, Akademie-Verlag Berlin, 1974. The authors worked in East Germany, so other than the German original there is no English, but a Russian translation! □

2.1.21 Remark

- Version (a) of this inequality, which is already valid if $u = 0$ on a part $\Gamma \subset \partial\Omega$ of the boundary provided that Γ is not a set of measure zero, is useful for problems with Dirichlet boundary values. Version (b), which is already valid if $\int_{\Omega_\circ} u \, \mathrm{d}x = 0$ on a subset $\Omega_\circ \subset \Omega$ provided that $\Omega_\circ$ is not a subset of measure zero, comes in handy for problems with Neumann boundary conditions but no terms of order zero in the PDE.
- It is important to note that the Poincaré inequality only holds for functions under these restrictions (zero boundary values or zero mean), not for all functions in H^1! A simple counterexample is given by

2.1.22 Remark (Minimisation Principle) The weak form (W) of Poisson's equation is equivalent to the convex optimisation problem

$$\min_{u \in H_0^1(\Omega)} \left\{ J(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 \, \mathrm{d}x - \langle f, u \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \right\}$$

(assuming here, for simplicity but without loss of generality[2], homogeneous boundary conditions $g = 0$). The weak formulation (W) is just the optimality condition $\nabla_u J(u) = 0$. Since the minimisation problem is convex, $u \in H_0^1(\Omega)$ satisfies the optimality condition if and only if it minimises J.

More generally, analogous minimisation principles hold for elliptic equations with no first-order terms, (the latter would destroy the symmetry of the left hand side in the weak formulation). We call such functionals J *energy functionals*, since they often correspond to physical energy minimisation principles. We could now use optimisation techniques to prove e.g. existence and uniqueness, but we'll use the following, more general approach that also covers equations with first-order (advection-like) terms.

2.1.23 Theorem (Lax-Milgram Lemma) Let V be a Hilbert space with scalar product $(\cdot,\cdot)_V$ and norm $\|\cdot\|_V = \sqrt{(\cdot,\cdot)_V}$. If $B : V \times V \to \mathbb{R}$ is a

- *continuous*

- and *coercive* aka *(V-)elliptic*

bilinear form and $f \in V^*$, then the problem

$$B(u, v) = \langle f, v\rangle_{V^*,V}, \qquad \forall v \in V$$

admits a unique solution $u \in V$. This solution satisfies the a priori estimate

$$\|u\|_V \le \frac{1}{c_e}\|f\|_{V^*}. \tag{2.4}$$

Proof. (In English!) Evans, pp 317–319. □

2.1.24 Group Work Consider the particular special case where the space V is the d-dimensional Euclidean space $\mathbb{R}^d$, the bilinear form is defined by $B(x, y) = (Ax)^\top y$ and the linear continuous "functional" is $f = b^\top$. A is a given symmetric $d \times d$ matrix and $b \in \mathbb{R}^d$ a given (column) vector.

Follow the steps outlined below to find out what the assumptions of the Lax-Milgram lemma mean in this case, and what the statement of the theorem is!

(a) *The problem.* What are the scalar product $(\cdot,\cdot)_V$ and the norm $\|\cdot\|_V = \sqrt{(\cdot,\cdot)_V}$ in this case? What is the dual space V^*? Write out the variational problem

$$B(x, y) = \langle f, y\rangle_{V^*,V}, \qquad \forall y \in V$$

in terms of the given data. This is the "weak formulation" of what "strong formulation"? Convince yourself and your teammates that both are equivalent here!

[2] Note that $u = g + u_0 \in g + H_0^1\Omega$ is a solution to the problem with inhomogeneous boundary conditions if and only if $u_0 \in H_0^1(\Omega)$ is a solution to the problem with homogeneous boundary conditions but modified right hand side $f + \Delta g$. This is why, for simplicity, homogeneous Dirichlet conditions are usually assumed for theoretical considerations and the analysis of problems. In the numerical practice, however, it would not normally be easy to transform a problem with inhomogeneous boundary values to a problem with homogeneous boundary values, which is why we keep the original form.

(b) *The assumptions.* What do continuity and coercivity mean in terms of the matrix A? What are the constants c_b and c_e?

(c) *The statement.* If these assumptions are met, what does the Lax-Milgram lemma tell you about the above problem?

2.1.25 Corollary (Well-Posedness of (W)) Given $f \in H^{-1}(\Omega)$ and $g \in H^1(\Omega)$, there is a unique solution $u \in g + H_0^1(\Omega)$ such that for all test functions $v \in H_0^1(\Omega)$

$$\int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x = \langle f, v\rangle_{H^{-1}(\Omega), H_0^1(\Omega)}. \tag{W}$$

There exists a constant $C > 0$ depending only on the domain Ω such that for all perturbations $\tilde{f} \in H^{-1}(\Omega)$ and $\tilde{g} \in H^1(\Omega)$ of the data, the corresponding perturbed solution $\tilde{u} \in \tilde{g} + H_0^1(\Omega)$ is close to the unperturbed solution u in the sense

$$\|u - \tilde{u}\|_{H^1} \leq C\left(\|f - \tilde{f}\|_{H^{-1}} + \|g - \tilde{g}\|_{H^1}\right).$$

Proof. Let us first transform the problem to an equivalent one with homogeneous boundary conditions: $u \in g + H_0^1(\Omega)$ solves (W) if and only if $w = u - g \in H_0^1(\Omega)$ solves

To apply the Lax-Milgram lemma, we have to verify that the bilinear form B on the left hand side

$$B(w,v) = \int_{\Omega} \nabla w \cdot \nabla v \, \mathrm{d}x$$

meets the assumptions of the theorem:

- continuity

- coercivity

Now the Lax-Milgram lemma implies existence and uniqueness of the solution w (hence $u = w + g$ is a unique solution of (W)) along with the a priori estimate

Therefore, perturbations $f - \tilde{f}$ and $g - \tilde{g}$ lead to an error

and, noting that $u = g + w$, $\tilde{u} = \tilde{g} + \tilde{w}$,

$\square$

Regularity of Solutions

In the most general setting, weak solutions to an elliptic, linear, second-order PDE lie in the space $H^1(\Omega)$. In 2D and higher dimensions, such solutions need not be continuous and higher derivatives may not exist. They may not even be bounded. Certain geometries and boundary conditions are particularly notorious for generating unbounded peaks in derivatives of the solution.

In many cases, the data of a PDE and its domain are smoother than in the most general setting. For example, we often have a proper function $f \in L^2(\Omega)$ instead of just $f \in H^{-1}(\Omega)$ on the right hand side of the PDE. We then hope that this higher regularity of f will propagate through to the solution u so that u will have one extra derivative as well. Provided that the domain is not too irregular, this is indeed the case:

2.1.26 Theorem (H^2 and Higher Regularity of Solutions to Elliptic Boundary Value Problems)

(a) Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with C^2-boundary. Let L be a uniformly elliptic operator given in divergence form

$$Lu = -\operatorname{div}(A\nabla u) + b \cdot \nabla u + cu \tag{2.5}$$

with $a_{ij} \in C^1(\bar{\Omega})$, $b_i, c \in L^\infty(\Omega)$ $(i, j \in \{1, \dots, d\})$ and let $f \in L^2(\Omega)$.

A weak solution $u \in H_0^1(\Omega)$ of

$$\begin{aligned} Lu &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{2.6}$$

is even in $H^2(\Omega)$ and there exists a constant $C > 0$, depending only on the coefficients A, b, c and the domain Ω, such that the a priori estimate

$$\|u\|_{H^2(\Omega)} \leq C\|f\|_{L^2(\Omega)}$$

holds for unique solutions u of (2.6).

(b) Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with C^{2+k}-boundary $(k \in \mathbb{N}_0)$. Let L be a uniformly elliptic operator given in divergence form (2.5) with $a_{ij}, b_i, c \in C^{1+k}(\bar{\Omega})$ $(i, j \in \{1, \dots, d\})$ and let $f \in H^k(\Omega)$.

A weak solution $u \in H_0^1(\Omega)$ of (2.6) is even in $H^{2+k}(\Omega)$ and there exists a constant $C > 0$, depending only on the coefficients A, b, c and the domain Ω, such that the a priori estimate

$$\|u\|_{H^{2+k}(\Omega)} \leq C\|f\|_{H^k(\Omega)}$$

holds for unique solutions u of (2.6).

Proof. See pp 336–345 in the book of Evans. □

In practice, domains are rarely this smooth. Very often, we solve PDE problems over domains with corners, such as rectangles or other polygons. Mathematically speaking, the boundaries of these domains are merely Lipschitz, but certainly not C^2. At least for Poisson's equation we can still obtain an analogous result on such rougher domains:

2.1.27 Theorem (H^2-Regularity of Solutions to the Poisson-Dirichlet Problem) Let $\Omega \subset \mathbb{R}^2$ be a bounded domain, the boundary of which is either a convex polygon or a C^2-curve. Let $u \in H_0^1(\Omega)$ be the unique solution to the Poisson-Dirichlet problem

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

with $f \in L^2(\Omega)$.

Then u is even in $H^2(\Omega)$ and there exists a constant $C > 0$, depending only on Ω, such that the a priori estimate

$$\|u\|_{H^2(\Omega)} \leq C\|f\|_{L^2(\Omega)}$$

holds.

Proof. For smooth domains, this statement is a special case of the previous theorem. Convex polygonal domains are rather complicated! The analysis of this case can be found in the book of Pierre Grisvard, *Singularities in Boundary Value Problems*, Research Notes in Applied Mathematics, Masson 1992, Paris. □

Domains that both have corners and that are also non-convex are a classical source of trouble. Around *re-entrant corners* of the domain, i.e. corners where the interior angle is larger than π, solutions tend to develop singularities that prevent them from being H^2. Therefore, the assumption of a convex domain in Theorem 2.1.27 is absolutely crucial, even with otherwise arbitrarily smooth data, as the following example shall demonstrate:

2.1.28 Example (Corner Singularities) On the circular sector, described in polar coordinates by

$$\Omega = \left\{ (r,\vartheta) \in \mathbb{R}^2 \,\middle|\, 0 < r < 1 \text{ and } 0 < \vartheta < \omega \right\}$$

we consider the problem

$$-\Delta u = 0 \qquad \text{in } \Omega$$

with boundary conditions

$$u(r,\vartheta) = 0 \qquad \text{for } \vartheta \in \{ 0, \omega \} \qquad\qquad u(r,\vartheta) = \sin\frac{\pi\vartheta}{\omega} \qquad \text{for } r = 1$$

where $\omega \in]0, 2\pi]$.

The unique strong and weak solution is given by

$$u(r,\vartheta) = r^{\pi/\omega} \sin\frac{\vartheta\pi}{\omega}.$$

For all angles $\omega \in]0, 2\pi]$, $u \in C^2(\Omega) \cap C(\bar{\Omega})$ and $u \in H^1(\Omega)$, but as soon as $\omega > \pi$, a corner singularity arises in the gradient

$$\left(\frac{\partial u}{\partial r}, \frac{1}{r}\frac{\partial u}{\partial \vartheta}\right) = \frac{\pi}{\omega} r^{\frac{\pi}{\omega}-1} \left(\sin\frac{\pi\vartheta}{\omega}, \cos\frac{\pi\vartheta}{\omega}\right)$$

at the origin. Then $u \notin C^1(\bar{\Omega})$ and $u \notin H^2(\Omega)$.

The extreme case $\omega = 2\pi$ describes a circular disk with a crack. Such problems are of great importance in mechanical and civil engineering.