Homework Assignment 10

Please submit the following files as indicated below: source code PDF file image file video file

Question 1 | **2 marks** | On the assignment page you can find videos of four animated solutions of the parabolic problem

$$\begin{aligned} \frac{\partial u}{\partial t}(t) - a\Delta u(t) &= f(t) && \text{in } Q =]0,T[\times\Omega \\ u(0) &= u_0 && \text{in } \Omega \\ \frac{\partial u}{\partial n} &= 0 && \text{on } \Sigma =]0,T[\times\partial\Omega \end{aligned} \tag{H}$$

with the data from Assignment 9. However, the initial condition has been replaced with the function

$$u_0(x) = \begin{cases} 50 & \text{if } |x - (1,1)^\top| < 0.5 \\ 20 & \text{elsewhere} \end{cases}$$

Explain your observations, using the proper terminology.

- Crank-Nicolson method, $h \approx 1/50$, $\Delta t = 1/50$:

  The coarse Crank-Nicholson method experiences some oscillations along the discontinuity where the initial condition changes from 50 to 20, however, they are quickly damped out as the solution converges.

- Crank-Nicolson method, $h \approx 1/250$, $\Delta t = 1/50$:

  The fine Crank-Nicholson method experience these same oscillations except that they do not damp out likely due to too large a time-step for the finer grid.

- θ-method with $\theta = 0.51$, $h \approx 1/250$, $\Delta t = 1/50$:

  The θ-method with $\theta = 0.51$, experiences identical oscillations as the fine Crank-Nicholson method at the start, but they do dampen as the solution converges, because the method is now slightly more implicit, the time-step restriction is not as great.

- TR-BDF2 method with $\alpha = 2 - \sqrt{2}$, $h \approx 1/250$, $\Delta t = 1/50$:

  The strong A-stability of the TR-BDF2 method leaves almost no restriction on the size of the time-step, so this method dampens the discontinuity immediately, then converges very smoothly.

Question 2 | **1 mark** | We have seen that the homogeneous wave equation

$$\begin{aligned}
\frac{\partial^2 u}{\partial t^2} - c^2 \Delta u &= 0 && \text{in } Q =]0,T[\times \Omega \\
u(0) &= u_0 && \text{in } \Omega \\
\frac{\partial u}{\partial t}(0) &= v_0 && \text{in } \Omega \\
u &= 0 && \text{on } \Sigma =]0,T[\times \partial\Omega
\end{aligned} \tag{W}$$

with propagation speed $c > 0$ can equivalently be re-written as

$$\begin{aligned}
\frac{\partial u}{\partial t} - v &= 0 && \text{in } Q =]0,T[\times \Omega \\
\frac{\partial v}{\partial t} - c^2 \Delta u &= 0 && \text{in } Q =]0,T[\times \Omega \\
u(0) &= u_0 && \text{in } \Omega \\
v(0) &= v_0 && \text{in } \Omega \\
u &= 0 && \text{on } \Sigma =]0,T[\times \partial\Omega \\
v &= 0 && \text{on } \Sigma =]0,T[\times \partial\Omega.
\end{aligned} \tag{W'}$$

Discretising with the θ-method in time and linear finite elements in space leads to the coupled system for the vectors of nodal values $\vec{u}^h_+$ and $\vec{v}^h_+$

$$M^h \vec{u}^h_+ - \theta \Delta t M^h \vec{v}^h_+ = M^h \vec{u}^h_\circ + (1-\theta)\Delta t M^h \vec{v}^h_\circ \tag{1}$$

$$\theta \Delta t c^2 K^h \vec{u}^h_+ + M^h \vec{v}^h_+ = -(1-\theta)\Delta t c^2 K^h \vec{u}^h_\circ + M^h \vec{v}^h_\circ \tag{2}$$

which has to be solved at every time step. Show that this is equivalent to the two smaller, successively solvable problems

$$\left(M^h + (\theta \Delta t c)^2 K^h\right) \vec{u}^h_+ = M^h \left(\vec{u}^h_\circ + \Delta t \vec{v}^h_\circ\right) - \left(\theta\,(1-\theta)\,(\Delta t c)^2\right) K^h \vec{u}^h_\circ \tag{3}$$

$$M^h \vec{v}^h_+ = M^h \vec{v}^h_\circ - \Delta t c^2 K^h \left(\theta \vec{u}^h_+ + (1-\theta)\,\vec{u}^h_\circ\right) \tag{4}$$

Rearranging equation 2 above for $M^h \vec{v}^h_+$ we get:

$$M^h \vec{v}^h_+ = -\theta \Delta t c^2 K^h \vec{u}^h_+ - (1-\theta)\Delta t c^2 K^h \vec{u}^h_\circ + M^h \vec{v}^h_\circ$$

Then subbing this into equation 1:

$$M^h \vec{u}^h_+ - \theta \Delta t \left(-\theta \Delta t c^2 K^h \vec{u}^h_+ - (1-\theta)\Delta t c^2 K^h \vec{u}^h_\circ + M^h \vec{v}^h_\circ\right) = M^h \vec{u}^h_\circ + (1-\theta)\Delta t M^h \vec{v}^h_\circ$$

Rearranging this equation:

$$M^h \vec{u}^h_+ + (\theta \Delta t c)^2 K^h \vec{u}^h_+ = -\theta \Delta t (1-\theta)\Delta t c^2 K^h \vec{u}^h_\circ + \theta \Delta t M^h \vec{v}^h_\circ + M^h \vec{u}^h_\circ + (1-\theta)\Delta t M^h \vec{v}^h_\circ$$

Rearranging and Combining terms once more and we get equation 3:

$$\left(M^h + (\theta \Delta t c)^2 K^h\right) \vec{u}^h_+ = M^h \left(\vec{u}^h_\circ + \Delta t \vec{v}^h_\circ\right) - \left(\theta\,(1-\theta)\,(\Delta t c)^2\right) K^h \vec{u}^h_\circ$$

To obtain equation 4, simply rearrange equation 2, by moving the $\vec{u}^h_+$ term to the RHS and rearrange:

$$M^h \vec{v}^h_+ = -\theta \Delta t c^2 K^h \vec{u}^h_+ - (1-\theta)\Delta t c^2 K^h \vec{u}^h_\circ + M^h \vec{v}^h_\circ$$

$$M^h \vec{v}^h_+ = M^h \vec{v}^h_\circ - \Delta t c^2 K^h \left(\theta \vec{u}^h_+ + (1-\theta)\,\vec{u}^h_\circ\right)$$

Question 3 | **2 marks**

(a) Download and complete the FEniCS script `hw10.py` to solve Problem (W) with the data provided.

(b) Solve the wave equation

- with the Crank-Nicolson method
- with the backward Euler method
- with the forward Euler method

and look at the solutions in ParaView.

Hint: Use the 'Warp by Scalar' filter, re-scale the colour map to the range $[-1, 1]$ and tick the box 'enable opacity mapping for surfaces' in the colour map editor.

Explain your observations. As always, take care to use the terminology correctly.

Using the Forward Euler method the wave starts to propagate outwards very coarsely until frame 11, when the domain disappears, leaving us with some faint oscillations in the centre until frame 61, when it disappears and an error message pops up at frame 462. Likely due to a time-step that is too large for the domain.

Using the Crank-Nicholson method we get a nice wave that propagates outward then inwards with no decay over time and more and more ripples being produced each time it is rebounded off the boundary.

Using the Backward Euler method we get visually the same output as the Crank-Nicholson until the first rebound, then the wave starts to diffuse and completely flattens by the last time step, due to the artificial dissipation from the strong A-stability of the BE method.

Your Learning Progress | **0 marks, but -1 mark if unanswered** | What is the one most important thing that you have learnt from this assignment?

The strong A-stability of the Backward Euler method can cause some real problems with artificial dissipation, as the wave equation shouldn't decay at all. This obviously could have serious implications to other problems as well.

What is the most substantial new insight that you have gained from this course this week? Any *aha moment*?

If you are pushing the time limit with your dry-runs of the presentation, don't expect it to be any faster for real.