After today's class you should be able to

- define (uniform) ellipticity
- select boundary conditions that may lead to well-posed elliptic problems
- explain the analytical, numerical and physical significance of a maximum principle

2 Second-Order Elliptic Equations

2.1 Characteristic Features

Executive Summary The prototypical representative of elliptic PDEs is the Poisson equation

$$-\Delta u = f,$$

which will serve as our elliptic model equation. You will look at more complicated elliptic equations in your homework assignments and maybe your project. In this section we discuss how to obtain a well-posed problem, and what important properties (e.g. regularity, maximum principle) solutions may possess. Understanding these properties is crucial for us, so that we can later design numerical schemes which lead to equally well-posed discrete problems with discrete solutions that exhibit analogous characteristic features.

Elliptic Operators and Boundary Conditions

We have already defined second-order quasi-linear equations as elliptic if the coefficient matrix A is either positive definite or negative definite. If we just consider the differential *operator*, not the entire *equation*, then it is actually important to distinguish between the positive and the negative case, e.g. for maximum principles. The mathematical convention reads as follows:

2.1.1 Definition (Elliptic Operator)

The quasi-linear operator

$$L = -\sum_{i,j=1}^{d} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i \frac{\partial}{\partial x_i} + c$$

(note the minus sign in front of the second partial derivatives) is said to be

- *elliptic* if all eigenvalues of the coefficient matrix A are positive,
- *uniformly elliptic* if all eigenvalues of A are greater than or equal to a positive constant C.

2.1.2 Example

- $-\Delta u = f$ is an elliptic equation
- $\Delta u = f$ is an elliptic equation
- $-\Delta$ is a uniformly elliptic operator, in particular elliptic
- Δ is not an elliptic operator
- $-e^{-(x^2+y^2)} \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}$ is elliptic, but over unbounded domains not uniformly elliptic
- $-(1-x_2^2)\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2}$ over the unit disk is

From now on, we consider the domain Ω to always be bounded.

2.1.3 Boundary Conditions Let $g, a : \partial\Omega \to \mathbb{R}$. A boundary condition of the form

(a)

is called *Dirichlet boundary condition* or *boundary condition of the first kind*,

(b)

is called *Neumann boundary condition* or *boundary condition of the second kind*,

(c)

is called *Robin boundary condition* or *boundary condition of the third kind*.

These boundary conditions are said to be *homogeneous* if $g = 0$, otherwise *inhomogeneous*.

Well-Posedness of Strong Formulations

The *classical* or *strong formulation* of Poisson's equation with Dirichlet boundary conditions reads: find $u \in C^2(\Omega) \cap C(\bar{\Omega})$ such that

$$-\Delta u = f \qquad \text{in } \Omega \tag{2.1a}$$

$$u = g \qquad \text{on } \partial\Omega. \tag{2.1b}$$

Proving existence of strong solutions of (2.1) on general domains is difficult and not exactly elegant. It also requires (often too) strong assumptions on the regularity of the domain Ω and the data f. We'll skip the discussion and refer to the literature on PDEs—the keyword is *Green's functions*.

The classical uniqueness proof for strong solutions of Poisson's equation equipped with Dirichlet boundary conditions relies on a *maximum principle* for elliptic equations:

2.1.4 Theorem (Elliptic Maximum Principle) Let

$$L = -\sum_{i,j=1}^{d} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i \frac{\partial}{\partial x_i} = -A : \nabla^2 + b \cdot \nabla$$

be elliptic (note that $c \equiv 0$) and $u \in C^2(\Omega) \cap C(\bar{\Omega})$ with a bounded domain Ω. Then

$$Lu \leq 0 \quad \text{in } \Omega \qquad \Rightarrow \qquad \max_{x \in \bar{\Omega}} u(x) \leq \max_{x \in \partial\Omega} u(x),$$

i.e. for source terms that are nowhere positive, the solution u assumes its maximum on the boundary. Vice versa, the minimum principle

$$Lu \geq 0 \quad \text{in } \Omega \qquad \Rightarrow \qquad \min_{x \in \bar{\Omega}} u(x) \geq \min_{x \in \partial\Omega} u(x)$$

guarantees that for such elliptic PDEs with source terms that are nowhere negative, the solution u assumes its minimum on the boundary.

Proof. The basic idea for proving the maximum principle is the following: First consider the stronger assumption $Lu < 0$ and note that if the solution u has a local maximum at a point x in the interior of the domain Ω, then

Now the statement of this theorem can be obtained with some small dirty analytical tricks. For full details, you may consult the book *Partial Differential Equations* by Lawrence C Evans, pp 346–348.

The minimum principle follows from the maximum principle by noting that $Lu \geq 0 \Leftrightarrow L(-u) \leq 0$. $\square$

2.1.5 Group Work Imagine you are cooking some soup on a stovetop. At steady state, the temperature distribution T inside the metal volume of the pot Ω can be found by solving the equation

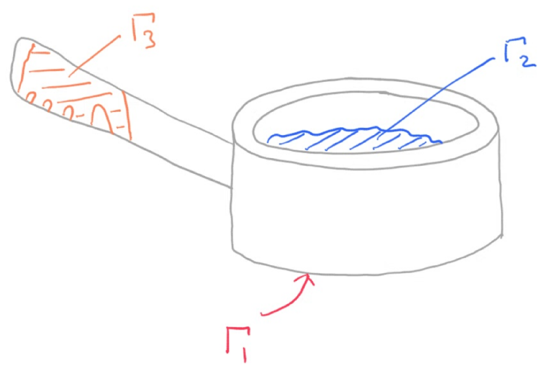
$$-\Delta T = 0 \qquad \text{in } \Omega.$$

On the surface of the pot, the temperature is fixed as follows:

- $T = 200$ °C on the bottom surface Γ_1 that is in contact with the element
- $T = 93$ °C on the inner surface Γ_2 that is in contact with the hot soup
- $T = 32$ °C on the surface of the handle Γ_3 that is in contact with your hand
- $T = 22$ °C everywhere else on the pot surface $\partial\Omega \setminus (\Gamma_1 \cup \Gamma_2 \cup \Gamma_3)$ that is in contact with the surrounding air.

(a) What is the best lower bound and the best upper bound you can derive from the elliptic maximum principle for the temperature distribution T in the metal volume of the pot?

(b) What changes in your answer to part (a) if you switch on an infrared lamp above the stove, as a consequence of which the PDE model now reads $-\Delta T = f$ with $f > 0$?



2.1.6 Group Work (Outlook: Discrete Maximum Principle) Assume that we want to solve the PDE problem

$$\begin{aligned} Lu &= f && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega \end{aligned}$$

numerically, with an elliptic operator L that satisfies the assumptions of the maximum principle. This could e.g. be a temperature problem like above or steady advection-diffusion-reaction problem. After discretisation e.g. with finite differences or finite elements, we obtain a linear system

$$Ax = b.$$

In analogy to the continuous maximum / minimum principle, whenever the the right hand side vector b does not go below zero, nor should the numerical solution vector x. Can you derive a condition on the matrix A that guarantees just that, i.e. $x \geq 0$ whenever $b \geq 0$? (NB: This $\geq$ notation for vectors should be understood componentwise. $x \geq 0$ means that all entries of the vector x should be non-negative.)

2.1.7 Corollary (Uniqueness of Strong Solutions) A solution $u \in C^2(\Omega) \cap C(\bar{\Omega})$ to the Poisson-Dirichlet problem (2.1) is unique.

Proof.

□

The maximum principle can be used to derive further well-posedness results:

2.1.8 Corollary (Continuous Dependence on the Boundary Data) Solutions to (2.1) depend continuously on the boundary data, i.e. if u and $\tilde{u}$ are solutions with boundary values g and $\tilde{g}$, respectively, (but the same right hand side f), then

$$\max_{x \in \bar{\Omega}} |u(x) - \tilde{u}(x)| \leq \max_{x \in \partial\Omega} |g(x) - \tilde{g}(x)| .$$

(Actually, equality holds.)

Proof.

□

Of course, this result can be extended to continuous dependence on the right hand side f, too. It actually holds not only for the Poisson problem, but can be generalised to uniformly elliptic operators without zeroth-order terms. We present it without proof here:

2.1.9 Corollary (Continuous Dependence on the Right Hand Side and Boundary Data) Solutions to (2.1) depend continuously on the data, i.e. if u and $\tilde{u}$ are solutions with data f, g and $\tilde{f}, \tilde{g}$, respectively, then there exists a constant $C > 0$ such that

$$\max_{x \in \bar{\Omega}} |u(x) - \tilde{u}(x)| \leq C \sup_{x \in \Omega} \left| f(x) - \tilde{f}(x) \right| + \max_{x \in \partial\Omega} |g(x) - \tilde{g}(x)| .$$