MATH521: Numerical Analysis of Partial Differential Equations
Winter 2018/19, Term 2

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After today's class you should be able to

- explain the significance of weak solutions
- derive weak formulations of elliptic PDEs given their strong form
- define Lebesgue and Sobolev spaces and motivate their use for weak formulations

From Strong Solutions to Weak Solutions

A solution to the strong formulation of Poisson's equation with Dirichlet boundary conditions

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \\ u &= g \quad \text{on } \partial\Omega \end{aligned} \tag{S}$$

has to be twice continuously differentiable in the interior of the domain $\Omega \subset \mathbb{R}^d$ (so that the Laplacian can be applied to it) and it must be continuous up to the boundary (so that it actually approaches the boundary values from the interior).

However, asking for two continuous derivatives of the solution u is often too strong a condition. For example, if material parameters (the coefficients A, b or c in a quasi-linear second-order elliptic operator) suddenly change across an interface that cuts through the domain Ω, then this will normally affect the smoothness of the solution u as well: For instance, solutions should be allowed to have 'kinks', meaning discontinuities in the gradient. It would therefore be desirable to have a relaxed formulation, which generalises the notion of solutions to the Poisson-Dirichlet problem (S), so that such functions which are not twice continuously differentiable also qualify as solutions.

In order to find such a generalised *variational* aka *weak formulation* corresponding to a given PDE problem, we apply the *method of weighted residuals*. Let us illustrate this technique by deriving a weak formulation of (S).

- The residual of the PDE, e.g. $r = -\Delta u - f$ should be zero, that is, it should be orthogonal to all possible *weighting functions* aka *test functions* v chosen from a certain space:

$$r \perp v \quad \text{for all test functions } v$$

- Orthogonality means that the scalar product vanishes:

$$(r, v) = 0 \quad \text{for all test functions } v$$

- A suitable scalar product for functions is the integral of their function values multiplied together:

$$\int_\Omega rv \, \mathrm{d}x = 0 \quad \text{for all test functions } v$$

- Now integration by parts can be used to remove derivatives of u contained in the residual, and to transfer (some of) them to the test function v instead:

$$\int_\Omega (-\Delta u - f)v \, \mathrm{d}x = -\int_{\partial\Omega} \frac{\partial u}{\partial n} v \, \mathrm{d}s + \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x - \int_\Omega fv \, \mathrm{d}x = 0 \quad \text{for all test functions } v$$

  If we select a space of test functions that vanish on the boundary—we always use test functions that are zero on those parts of the boundary where Dirichlet conditions are imposed—then the boundary integral is zero.

In the last step we used the integration-by-parts identity (sometimes called Green's first identity)

$$\int_\Omega (\operatorname{div} F)v \, \mathrm{d}x = \int_{\partial\Omega} (F \cdot n)v \, \mathrm{d}s - \int_\Omega F \cdot \nabla v \, \mathrm{d}x$$

for sufficiently regular vector fields $F : \Omega \to \mathbb{R}^d$ and scalar functions $v : \Omega \to \mathbb{R}$, which follows from the divergence theorem.

Overall, we have now shown that every strong solution u to problem (S) also satisfies the equation

$$\int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x = \int_\Omega fv \, \mathrm{d}x \quad \text{for all test functions } v \text{ with } v = 0 \text{ on } \partial\Omega \tag{W*}$$

as well as the boundary condition $u = g$ on $\partial\Omega$. Let us refer to this equation as preliminary weak formulation—preliminary, since we still have to work out in what set to look for solutions u and from what exact space to draw test functions.

It is important to note that in (W*) all second derivatives have disappeared! It is not even necessary to ask for *continuous* derivatives of u and v. Instead, the integral expressions in (W*) are well-defined for all functions u and v and data f with the following properties:

①

②

③

(but we will shortly relax this already weak assumption further to allow for even more general source terms!)

In view of these three requirements, the spaces C^k of continuous(ly differentiable) functions are not well suited for weak formulations. Instead, the natural function spaces to work with are those that impose suitable integrability conditions on the functions they contain. These are the so-called Lebesgue spaces L^p and Sobolev spaces H^k and $W^{k,p}$.

2.1.10 Definition (Lebesgue Space of Square-Integrable Functions) For a function $u : \Omega \to \mathbb{R}$ over a domain $\Omega \subset \mathbb{R}^d$, we define the L^2-norm

$$\|u\|_{L^2(\Omega)} = \left(\int_\Omega |u(x)|^2 \, \mathrm{d}x \right)^{1/2}.$$

The set

$$L^2(\Omega) = \left\{ u : \Omega \to \mathbb{R} \;\middle|\; \|u\|_{L^2(\Omega)} < \infty \right\}$$

is called the *Lebesgue space of order 2*.

2.1.11 Theorem ($\mathbf{L^2}$ is a Hilbert Space) With the scalar product

$$(u, v)_{L^2(\Omega)} = \int_\Omega uv \, \mathrm{d}x$$

the Lebesgue space $L^2(\Omega)$ is a Hilbert space. (Note that $\|u\|_{L^2} = \sqrt{(u,u)_{L^2}}$.)

2.1.12 Group Work Check whether or not the following functions u are in $L^2(\Omega)$ with the given domains Ω:

(a) the Heaviside step function

$$u(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

over $\Omega =]-1, 1[$

(b) the Heaviside step function over $\Omega = \mathbb{R}$

(c) an arbitrary function $u \in C(\Omega)$ over a bounded domain Ω

(d) an arbitrary function $u \in C(\bar{\Omega})$ over a bounded domain Ω

(e) $u(x) = \frac{1}{|x|^p}$ over the interval $\Omega =]-1, 1[$ (where $p > 0$)

(f) $u(x) = \frac{1}{|x|^p}$ over the unit disk $\Omega = B(0,1) \subset \mathbb{R}^2$

(g) $u(x) = \frac{1}{|x|^p}$ over the unit ball $\Omega = B(0,1) \subset \mathbb{R}^3$

2.1.13 Remark (Properties of L^2-functions)

An important observation from (e), (f) and (g) is that some behaviour of L^2-functions is dimension-dependent:

If you have just encountered L^2 for the first time ever, you may find the following property of these functions particularly odd:

- In Lebesgue spaces, we usually cannot evaluate function values at points. The expression $u(x)$ is not meaningful. Consider for instance the two functions $u, v \in L^2(]-1,1[)$:

$$u(x) = x^2 \qquad \text{and} \qquad v(x) = \begin{cases} x^2 & \text{for } x \neq 0.3 \\ 0.8 & \text{for } x = 0.3 \end{cases}.$$

  These two functions are not distinguishable in $L^2(]-1,1[)$ since

Since the set consisting only of the point $x = 0.3$ "weighs nothing" (if you're familiar with measure theory: this set has measure zero), function values on this set and other sets of measure zero could be modified arbitrarily without actually changing the function. In this example, too, u and v are exactly the same L^2-function!

While this nonuniqueness of point values may seem puzzling initially, it is actually most convenient if you want to compute derivatives of functions that are not differentiable in the classical sense, e.g. $f(x) = |x|$ at $x = 0$. Since the set $\{0\}$ containing only the origin has measure zero, it simply does not matter what value you assign to $f'(0)$. Whichever number you choose, f' as an L^2-function is unaffected by this choice.

For a formal definition of *weak derivatives*, which uses integration by parts again, please refer to the literature on functional analysis.

2.1.14 Definition ($\mathbf{L^2}$-based Sobolev Spaces) For a domain $\Omega \subset \mathbb{R}^d$ and $k \in \mathbb{N}_0$ we define the *Sobolev space* $H^k(\Omega)$ as the set of all functions $u \in L^2(\Omega)$, of which all weak partial derivatives up to and including order k are in $L^2(\Omega)$ as well.

2.1.15 Theorem ($\mathbf{H^k}$ is a Hilbert Space) With the scalar product

$$(u, v)_{H^k(\Omega)} = \int_\Omega uv \, \mathrm{d}x + \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x + \int_\Omega \nabla^2 u : \nabla^2 v \, \mathrm{d}x + \ldots \text{ (up to order } k)$$

the Sobolev space $H^k(\Omega)$ is a Hilbert space.

2.1.16 Remark (Further Lebesgue and Sobolev Spaces) By replacing all 2's in Definition 2.1.10 with a general exponent $p \in [1, \infty[$, one obtains other L^p-spaces. $L^\infty(\Omega)$ is the space of all essentially bounded functions over Ω. *Essentially* bounded means that out of all those indistinguishable equivalent functions with possibly different function values on sets of measure zero, there is at least one function which is bounded in the usual sense.

Similarly, the Sobolev spaces H^k can be generalised to sets of L^p functions, which have derivatives up to order k in L^p. These are denoted $W^{k,p}(\Omega)$, and the L^2-based spaces above are the special cases $H^k(\Omega) = W^{k,2}(\Omega)$.

2.1.17 Group Work Looking at the properties ①-③ which are required for the well-posedness of (W*), can you identify the "right" function spaces for the solution u, the test functions v and the right hand side f in the preliminary weak form (W*)?

①

②

$$\left| \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x \right| =$$

③

$$\left| \int_\Omega f v \, \mathrm{d}x \right| =$$

Therefore, we need

$$u \in \qquad\qquad v \in \qquad\qquad f \in$$

(Recall that, additionally, $u = g$ and $v = 0$ on $\partial\Omega$).

Now that we have made the weak formulation rigorous as far as the interior of the domain is concerned, let us address the problem of imposing Dirichlet boundary conditions. There are a few more mathematical technicalities involved when it comes to the equally rigorous definition of boundary values. Since the boundary $\partial\Omega$ is of one dimension lower than the domain Ω, it is a set of measure zero. Therefore, the function values on the boundary could be altered arbitrarily without actually changing the function. It appears like this is catastrophic news for us, as we are trying to impose boundary values on u and v that do have an impact on the solution! Luckily, there is a sound way of assigning boundary values (aka traces) to H^1-functions. The existence of one weak derivative is crucial here—L^2-functions don't generally have meaningful boundary values! If you are interested in the mathematical details, look up the trace theorem e.g. in the book of Evans on pages 273–277.

2.1.18 Definition Let Ω be a bounded domain with C^1 or Lipschitz boundary $\partial\Omega$.

(a) The space of all functions $u \in H^1(\Omega)$ with $u|_{\partial\Omega} = 0$ is denoted by $H_0^1(\Omega)$.

(b) The space of all functions $g \in H^k(\partial\Omega)$ (with $k \in \mathbb{N}_0$), which can be obtained as traces $g = f|_{\partial\Omega}$ of functions $f \in H^{k+1}(\partial\Omega)$ is denoted by $H^{k+1/2}(\partial\Omega)$.

First consider the problem with homogeneous Dirichlet boundary conditions $g = 0$. Part (a) of Definition 2.1.18 provides us with the space for the solution and the test functions, both of which are supposed to vanish on $\partial\Omega$. Building upon all our preliminary work, the final weak form of the problem

$$\begin{aligned} -\Delta u &= f && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega \end{aligned}$$

reads:

Given $f \in L^2(\Omega)$, find $u \in H_0^1(\Omega)$ such that for all test functions $v \in H_0^1(\Omega)$

$$\int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x = \int_\Omega f v \, \mathrm{d}x.$$

Part (b) of Definition 2.1.18 allows us to extend given boundary values $g \in H^{1/2}(\partial\Omega)$ (non-uniquely) to a function $g \in H^1(\Omega)$ over the entire domain. This way, a solution $u \in H^1(\Omega)$ with boundary values g could be decomposed into

$$u = g + u_0$$

Thus, a weak formulation of Problem (S) reads:

Given $f \in L^2(\Omega)$ and $g \in H^{1/2}(\partial\Omega)$, find $u \in g + H^1_0(\Omega)$ such that for all test functions $v \in H^1_0(\Omega)$

$$\int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x = \int_\Omega f v \, \mathrm{d}x.$$

So far, we have assumed that the source term f of the PDE is an L^2-function. While this covers a wide range of scenarios, we may sometimes want to assume even less regularity than that. To figure out how the weak formulations can be generalised further to allow for such data, let us analyse the formal structure of the right hand side of the weak form. Thinking of the right hand side as an operator,

- if f is an L^2-function:

  This mapping is
- if f

  This mapping[1] is

2.1.19 Definition The dual space of $H^1_0(\Omega)$ is denoted by $H^{-1}(\Omega)$:

$$H^{-1}(\Omega) = \left(H^1_0(\Omega)\right)^* = \left\{\, f : H^1_0(\Omega) \to \mathbb{R} \;\middle|\; f \text{ is linear and continuous} \,\right\}.$$

For the action of $f \in H^{-1}(\Omega)$ on $v \in H^1_0(\Omega)$ we use the symmetric notation $\langle f, v\rangle_{H^{-1}(\Omega), H^1_0(\Omega)}$ reminiscent of a scalar product instead of $f(v)$.

We have obtained our most general weak formulation of Problem (S):

Given $f \in H^{-1}(\Omega)$ and $g \in H^{1/2}(\partial\Omega)$, find $u \in g + H^1_0(\Omega)$ such that for all test functions $v \in H^1_0(\Omega)$

$$\int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x = \langle f, v\rangle_{H^{-1}(\Omega), H^1_0(\Omega)}. \tag{W}$$

[1] Note that for this f to be well-defined, we must be able to evaluate the test functions at a point. As we have seen, this is not normally possible, but another theorem from functional analysis comes to our rescue: *Sobolev embeddings* (cf Evans on pp 277 onwards, or, for a compact overview: Robert A Adams and John JF Fournier, *Sobolev Spaces*, Elsevier 2003 on pp 85–86). These embeddings have many important applications. In particular they guarantee that H^k-functions are bounded and continuous (almost everywhere) provided that k is big enough. If the domain Ω is 1D then all H^1-functions are continuous, but the higher the dimension the more regularity (larger k) is needed for guaranteed continuity. Thus, the latter f is well-defined if Ω is a 1D domain, but not otherwise.