2.1.1 Definition (Elliptic Operator)

The quasi-linear operator

$$L = -\sum_{i,j=1}^{d} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i \frac{\partial}{\partial x_i} + c$$

(note the minus sign in front of the second partial derivatives) is said to be

- *elliptic* if all eigenvalues of the coefficient matrix A are positive,
- *uniformly elliptic* if all eigenvalues of A are greater than or equal to a positive constant C.

2.1.2 Example

- $-\Delta u = f$ is an elliptic equation
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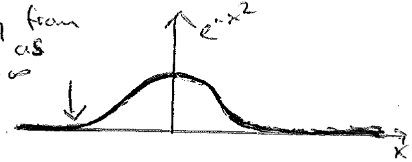
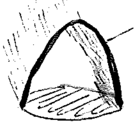
- $-\Delta$ is a uniformly elliptic operator, in particular elliptic
- Δ is not an elliptic operator
- $-e^{-(x^2+y^2)} \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}$ is elliptic, but over unbounded domains not uniformly elliptic
- $-(1-x_2^2)\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2}$ over the unit disk is elliptic, but not uniformly.

not bounded away from zero as $|x| \to \infty$



From now on, we consider the domain Ω to always be bounded.



$1-x_2^2 > 0$ in Ω, but not $\geq C > 0$

$-x_2^2 \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2}$ is only elliptic in points (x_1, x_2) with $x_2 \neq 0$