MATH521: Numerical Analysis of Partial Differential Equations



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After today's class you should be able to

- recognise the type of (non)linearity of a given PDE
- define well-posedness of a PDE problem and explain what is meant by the regularity of a solution
- identify scalar conservation laws and describe some qualitative properties of their solutions
- distinguish elliptic, parabolic and hyperbolic (scalar, quasilinear, first-order or second-order) equations
- name characteristic features of PDEs or their solutions that may be desirable to preserve after discretisation

1 Classification of PDEs

Executive Summary Partial differential equations (PDEs) model a vast range of natural phenomena and industrial problems. Even though analytical solutions are not normally available, just the equation itself already reveals crucial information on some characteristic features of the (unknown) solution. In this course we will learn to compute numerical approximations that reflect these properties accurately.

Today we will find out what some of these characteristic features are, and how they can be used to classify PDEs in a meaningful manner:

- Certain (important!) PDEs can be interpreted as conservation laws, others cannot. We will look at some examples of such conservation equations first.
- Another way of classifying PDEs relies on the notions of elliptic, parabolic and hyperbolic equations. We will learn about these in the last section of this first chapter.

1.1 Basic Properties

The prototypical equations we study in this course are

1. Poisson's equation

$$-\Delta u = f,$$

2. the heat equation

$$\frac{\partial u}{\partial t} - \Delta u = f,$$

3. and the wave equation

$$\frac{\partial^2 u}{\partial t^2} - \Delta u = f,$$

that are all of second order (they contain derivatives up to and including order two) and that we solve for a scalar, real-valued function u.

The Laplacian

$$\Delta = \operatorname{div} \nabla = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}$$

arises in all three of these equations, where d is the number of (physically: spatial) variables, which we collect in a vector $x = (x_1, x_2, \dots, x_d)^{\top} \in \mathbb{R}^d$. To emphasise the special role of the (d+1)-st (physically: time) variable in the heat and wave equations, we used the notation t here instead of yet another x_i .

If no special notation is used for t, then all of the above equations are of the form

$$\sum_{i,j=1}^{d} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i \frac{\partial u}{\partial x_i} + cu = f \quad \text{in } \Omega \subseteq \mathbb{R}^d$$
 (*)

e.g. in 2D

$$a_{11}\frac{\partial^2 u}{\partial x_1^2} + 2a_{12}\frac{\partial^2 u}{\partial x_1\partial x_2} + a_{22}\frac{\partial^2 u}{\partial x_2^2} + b_1\frac{\partial u}{\partial x_1} + b_2\frac{\partial u}{\partial x_2} + cu = f \qquad \text{in } \Omega \subseteq \mathbb{R}^2$$

(where, for this equation to be genuinely of second order, not all of the a_{ij} 's are simultaneously equal to zero).

Note that without loss of generality, the matrix A consisting of the coefficients a_{ij} can be assumed to be symmetric with $a_{ij} = a_{ji}$ for all $i, j \in \{1, ..., d\}$, because

Keep this assumption of a symmetric matrix A in mind for later on!

We also use the short operator notation for such PDEs

$$Lu = f$$
 in Ω .

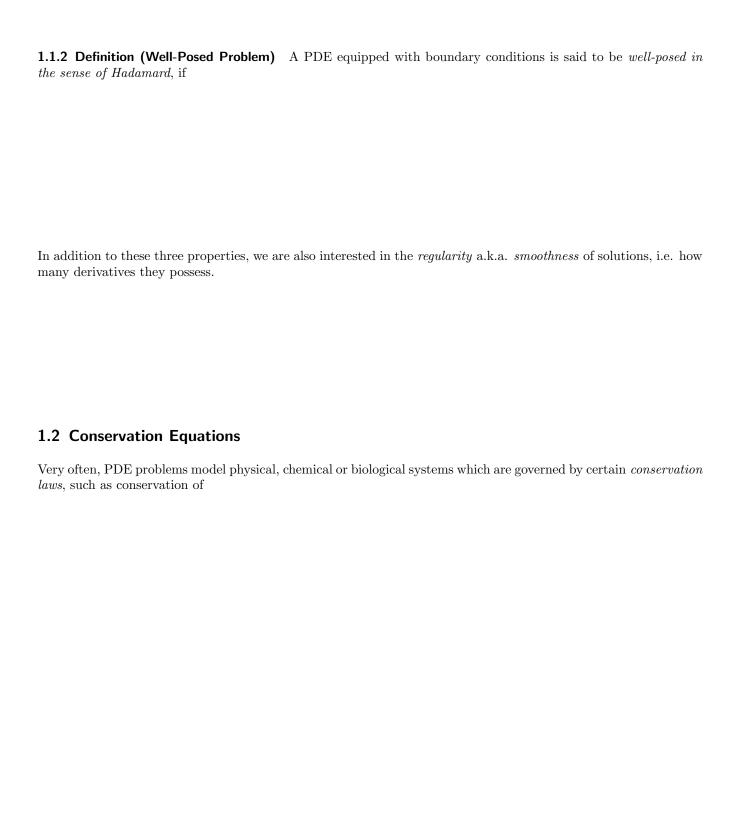
1.1.1 Definition (Linear and Nonlinear PDEs) A PDE

$$F(x, u, \nabla u, \nabla^2 u) = 0$$

is said to be

- linear, if it can be cast in the form (\star) where the coefficients
- nonlinear, otherwise
 - semi-linear, if it can be cast in the form (\star) where the coefficients
 - quasi-linear, if it can be cast in the form (\star) where the coefficients
 - fully nonlinear, if it cannot be cast in the form (\star) .

In practice, the domain Ω is often bounded, or unbounded in one direction only (namely, the 'time-direction'). While the PDE describes the behaviour of a solution u in the interior of Ω , additional conditions prescribe certain values on the boundary $\partial\Omega$ or parts thereof. The question what kind of boundary conditions are 'admissible' for a given PDE is not straightforward to answer. The answer will depend on the type and the exact form of the PDE. For now, let us clarify what 'admissible' means in this context:



In mathematical terms, conservation of a (mass, energy, momentum, ...) density u implies that if this quantity is transported through the domain with a flux F, the density increases inside any arbitrary control volume V if there is a net influx across the boundary of V and it decreases otherwise:

$$\int\limits_{V} \frac{\partial u}{\partial t} \, \mathrm{d}x = -\int\limits_{\partial V} F \cdot n \, \mathrm{d}s.$$

Provided that the flux F and the control volume V satisfy the assumptions of the divergence theorem, we obtain

$$\int_{V} \frac{\partial u}{\partial t} \, \mathrm{d}x + \int_{V} \operatorname{div} F \, \mathrm{d}x = 0.$$

Since this balance has to hold for arbitrary such volumes, we have now derived a rather general form of a conservation equation

$$\frac{\partial u}{\partial t} + \operatorname{div} F = 0,$$

which is called the *continuity equation*. PDEs that can be written in this form (or its steady-state version with no time derivative) are called *conservation equations*.

The flux function F often depends on the solution u or its gradient ∇u . If we assume a linear flux, then we obtain the following two particularly important fluxes:

- advective flux F = ua, with an advection (velocity) field a
- diffusive flux $F = -D\nabla u$, with a positive diffusivity D. More generally, D could also be a symmetric positive definite matrix, a so-called diffusion tensor.

By comparing the physical dimensions of the terms $\frac{\partial u}{\partial t}$ and div F, you can check that the proportionality constants a and D do have the dimension of a velocity or a diffusivity, indeed.

Sometimes, the quantity u is not actually conserved, but sources or sinks such as chemical reactions or external forces give rise to an inhomogeneity r on the right-hand side of the equation.

Below, we list some typical conservation equations:

• Unsteady advection-diffusion-reaction equation

$$\frac{\partial u}{\partial t} + \operatorname{div}(ua) - \operatorname{div}(D\nabla u) = r$$

• Steady advection-diffusion-reaction equation

$$\operatorname{div}(ua) - \operatorname{div}(D\nabla u) = r$$

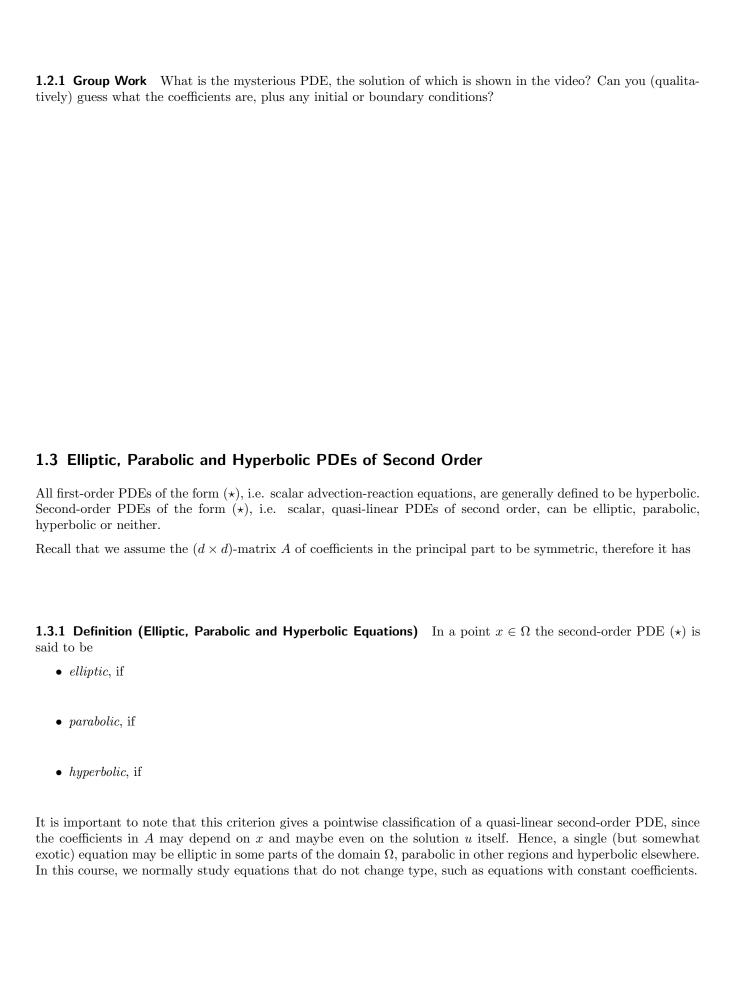
• Unsteady advection equation

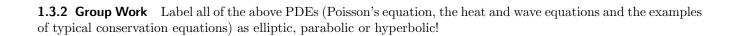
$$\frac{\partial u}{\partial t} + \operatorname{div}(ua) = 0$$

Steady advection equation

$$\operatorname{div}(ua) = 0$$

Many advection fields in applications are incompressible, which means that $\operatorname{div} a = 0$. Then, using the product rule, the above advective terms can also be written as





1.3.3 Group Work Where do the labels 'elliptic', 'parabolic' and 'hyperbolic' come from? To find out, consider the second-order PDE in 2D

$$a_{11}\frac{\partial^2 u}{\partial x_1^2} + 2a_{12}\frac{\partial^2 u}{\partial x_1\partial x_2} + a_{22}\frac{\partial^2 u}{\partial x_2^2} + b_1\frac{\partial u}{\partial x_1} + b_2\frac{\partial u}{\partial x_2} + cu = f \qquad \text{in } \Omega \subseteq \mathbb{R}^2.$$

What conditions on the coefficients of A can you derive from Definition 1.3.1 for each case? How does this relate to the defining equation of the conic sections

$$a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2 + b_1x_1 + b_2x_2 + c = 0$$

$$\Leftrightarrow \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} b_1 & b_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + c = 0?$$