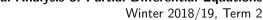
MATH521: Numerical Analysis of Partial Differential Equations



Thursday, 24 January 2019 Timm Treskatis



After today's class you should be able to

- explain the concepts of Ritz projections and Galerkin approximations
- interpret Galerkin orthogonality and apply it in manipulations of weak formulations
- define conformity of an approximation
- derive the Galerkin equations and the discrete linear system for a given linear elliptic PDE

2.3 Finite Elements for Poisson's Equation

Executive Summary The finite-element method follows an approach which is completely different from the finite-difference method. Instead of the strong formulation of a PDE, finite-element discretisations start from the *weak* formulation of a PDE. To obtain a finite-dimensional problem, we simply restrict the spaces of solutions and test functions to finite-dimensional subspaces. We no longer approximate any derivatives.

From the practical viewpoint, finite-element methods are characterised by great versatility. Their error analysis yields convergence results under far weaker assumptions than what is required for convergence estimates of finite-difference methods. Finite elements are backed by a very rich and not only in my opinion elegant theory, which allows for far-reaching and mathematically rigorous predictions to an extent that is probably unparalleled in numerical analysis.

Ritz Projections and Galerkin Approximations

How to obtain a (finite-dimensional) linear system from the weak form of a linear elliptic equation? Let us look at the two closely related approaches that were developed at the beginning of the 20^{th} century. Let us compare them in the abstract setting of some infinite-dimensional Hilbert space V (e.g. $H_0^1(\Omega)$), where $B: V \times V \to \mathbb{R}$ is a continuous and coercive bilinear form and $f \in V^*$:

The Swiss way^a

The Russian way^b

$$\min_{u \in V} \frac{1}{2} B(u,u) - \langle f, u \rangle_{V^*,V}$$

Find
$$u \in V$$
 such that
$$B(u,v) = \langle f,v \rangle_{V^*,V}, \qquad \forall v \in V$$

Recall that if B is symmetric, then the two formulations are equivalent, since the equation on the right is just the first-order optimality condition (Euler-Lagrange equation) of the convex optimisation problem on the left. In this case of a symmetric bilinear form, continuity and coercivity of B imply that B is a valid inner product with which V still has its Hilbert space structure. The norm

$$||u||_B = \sqrt{B(u, u)}$$

that is then naturally defined by the PDE is called the *energy norm* of the problem.



^bBG Galerkin: On electrical circuits for the approximate solution of the Laplace equation (in Russian), Vestnik Inzh 19, 1915, pp 897–908

The fundamental idea (proposed by Walter Ritz and then also applied by Boris Galerkin) that reduces each infinite-dimensional problem to a finite-dimensional problem is the following:

The Swiss way

$$\min_{u^h \in V^h} \frac{1}{2} B(u^h, u^h) - \langle f, u^h \rangle_{V^*, V}$$

Find
$$u^h \in V^h$$
 such that
$$B(u^h, v^h) = \langle f, v^h \rangle_{V^*, V}, \qquad \forall v^h \in V^h$$

2.3.1 Group Work (Consistency of Galerkin Approximations)	Determine the consistency error	of the Galerkin
approximation!		

2.3.2 Definition (Conformity) A Galerkin approximation with $V^h \subset V$ that uses the bilinear form B and source term f of the original problem is called *conforming*.

What are possible finite-dimensional subspaces V^h of function spaces V such as $L^2(\Omega)$ or $H^1(\Omega)$?

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To obtain a linear system of equations from the problem

$$B(u, v) = \langle f, v \rangle_{V^*, V}, \quad \forall v \in V$$

we proceed in three steps:

 $\mathbf{1^{st}}$ Step Choose an N-dimensional subspace $V^h \subset V$ and a basis $\left(\phi_i^h\right)_{i=1}^N$ of V^h

2nd Step Write

$$u^h =$$

 ${f 3^{rd}}$ Step Sub this u^h into the weak formulation and use the basis functions as test functions to obtain the Galerkin equations