MATH521: Numerical Analysis of Partial Differential Equations
Winter 2018/19, Term 2

Due Date: Thursday, 24 January 2019
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Homework Assignment 3

Please submit the following files as indicated below: source code PDF file image file video file

Question 1 | 3 marks | Let's gain some more practice with weak formulations, Lebesgue and Sobolev spaces and different kinds of boundary conditions.

(a) Let $\Omega \subset \mathbb{R}^d$ be a domain, $f \in C_b(\Omega)$ a bounded and continuous source term and $g \in C(\partial\Omega)$ a continuous function on the boundary.

Show that every solution of the problem

Find a function $u \in C^2(\Omega) \cap C(\bar{\Omega})$ such that

$$\begin{aligned} -\Delta u + u &= f \quad \text{in } \Omega \\ \frac{\partial u}{\partial n} &= g \quad \text{on } \partial\Omega \end{aligned} \tag{S}$$

also solves the problem[1]

Find a function $u \in H^1(\Omega)$ such that for all $v \in H^1(\Omega)$

$$\int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x + \int_\Omega uv \, \mathrm{d}x = \langle f, v \rangle_{(H^1(\Omega))^*, H^1(\Omega)} + \int_{\partial\Omega} gv \, \mathrm{d}s \tag{W}$$

Hint: Unlike Dirichlet conditions, which are imposed on weak solutions by restricting the function space H^1 e.g. to H^1_0, Neumann and Robin boundary conditions are simply plugged into the boundary integral that arises after integration by parts.

$-\Delta u + u - f = 0$

$\frac{\partial u}{\partial n} = g$

$\int_\Omega (-\Delta u + u - f) v \, dx = 0$

$-\int_\Omega \Delta u v \, dx + \int_\Omega uv dx - \int_\Omega fv \, dx = 0$

$\Downarrow$

Green's Identity

[1]NB: This is not the most general weak form. Just like the source term f does not need to be a proper function, g could also be chosen from the dual space of the boundary traces of the test functions. Since the test functions have traces $v|_{\partial\Omega} \in H^{1/2}(\partial\Omega)$, any abstract functional $g \in H^{-1/2}(\partial\Omega) = \left(H^{1/2}(\partial\Omega)\right)^*$ could be used as generalised boundary values.

$$\Rightarrow -\int_{\partial\Omega} \underbrace{\frac{\partial u}{\partial n}}_{\overset{\|}{g}} v\, ds + \int_\Omega \nabla u \cdot \nabla v\, dx - \int_\Omega f v\, dx + \int_\Omega u v\, dx = 0$$

Given $f \in H^{-1}(\Omega)$ and $g \in H^{1/2}(\partial\Omega)$, find $u \in H^1(\Omega)$ such that $v \in H^1(\Omega)$

$$\int_\Omega \nabla u \cdot \nabla v\, dx + \int_\Omega u v\, dx = \int_\Omega f v\, dx + \int_{\partial\Omega} g v\, ds$$

$$\int_\Omega \nabla u \cdot \nabla v\, dx + \int_\Omega u v\, dx = \langle f, v \rangle_{H^{-1}(\Omega), H^1(\Omega)} + \int_{\partial\Omega} g v\, ds$$

(b) Show that for given data $f \in \left(H^1(\Omega)\right)^*$ and $g \in L^2(\partial\Omega)$, Problem (W) has a unique solution.

From Theorem 2.1.23 (Lax-Milgram), a solution exists and is unique if the Bilinear form B on the LHS is continuous and coercive.

$$B(u,v) = \int_\Omega \nabla u \cdot \nabla v\, dx + \int_\Omega u v\, dx$$

Continuity:

$$|B(u,v)| = \left| \int_\Omega \nabla u \cdot \nabla v\, dx + \int_\Omega u v\, dx \right| \overset{\substack{\text{triangle}\\ \text{ineq.}}}{\leq} |(\nabla u, \nabla v)_{L^2}| + |(u,v)_{L^2}|$$

$$= \|\nabla u\|_{L^2} \|\nabla v\|_{L^2} + \|u\|_{L^2} \|v\|_{L^2} \underset{\substack{\uparrow \\ \text{C-S ineq.}}}{\leq} \|u\|_{H^1} \|v\|_{H^1}$$

$C_b = 1$

Coercivity:

$$|B(u,u)| = \left| \int_\Omega \nabla u^2\, dx + \int_\Omega u^2\, dx \right| = \|\nabla u\|_{L^2}^2 + \|u\|_{L^2}^2 \overset{\text{def}^{\text{n}} \text{ of } H^1\text{-norm}}{=} \|u\|_{H^1}^2$$

$C_e = 1$

(c) In part (a) you have shown that all strong solutions are weak solutions. Let's look at an example that shows that not all weak solutions are strong solutions.

For simplicity, let $\Omega =]-1, 1[$. Then the weak formulation becomes

Find a function $u \in H^1(]-1,1[)$ such that for all $v \in H^1(]-1,1[)$

$$\int_{-1}^{1} u'v' \, dx + \int_{-1}^{1} uv \, dx = \langle f, v \rangle_{(H^1)^*, H^1} + g(1)v(1) - g(-1)v(-1) \tag{W}$$

The function $u \in H^1(]-1,1[)$ with

$$u(x) = \frac{1}{2}(1 - |x|)$$

is not twice continuously differentiable, so it cannot solve Problem (S). Plug this function into the left hand side of the weak form to find a source term f and boundary values g for which u solves Problem (W), though.

$u(x) = \frac{1}{2}(1 - |x|)$

$u'(x) = -\frac{1}{2} \operatorname{sgn}(x)$

$$\int_{-1}^{1} -\tfrac{1}{2} \operatorname{sgn}(x) \, v' dx + \int_{-1}^{1} \tfrac{1}{2}(1 - |x|) \, v \, dx = \langle f, v \rangle_{(H^1)^*, H^1} + g(1)v(1) - g(-1)v(-1)$$

use IBP for first term and split

$$\int_{-1}^{0} -\tfrac{1}{2} \operatorname{sgn}(x) v' dx + \int_{0}^{1} -\tfrac{1}{2} \operatorname{sgn}(x) v' dx = \frac{-1}{2} \operatorname{sgn}(x) v \Big|_{-1}^{0} + \frac{-1}{2} \operatorname{sgn}(x) v \Big|_{0}^{1} - \int_{-1}^{0} \delta(x) v dx - \int_{0}^{1} \delta(x) v dx$$

$$= \tfrac{1}{2} v(-1) - \tfrac{1}{2} v(1) - v(0) \cancel{\operatorname{sgn}(0)}^{\,0} - v(0) \cancel{\operatorname{sgn}(0)}^{\,0} \Rightarrow \tfrac{1}{2} v(-1) - \tfrac{1}{2} v(1)$$

From here if we say $g(1) = -\frac{1}{2}$ & $g(-1) = \frac{1}{2}$

then we are left with:

$$\int_{-1}^{1} \tfrac{1}{2}(1 - |x|) v \, dx = \langle f, v \rangle_{(H^1)^*, H^1}$$

$$\Rightarrow \int_{\Omega} uv \, dx = \int_{\Omega} fv \, dx$$

$\therefore u = f$

$$\boxed{\begin{aligned} &\Rightarrow f = \tfrac{1}{2}(1 - |x|) \\ &g(1) = -\tfrac{1}{2} \\ &g(-1) = \tfrac{1}{2} \end{aligned}}$$

Question 2 | 2 marks | We will now solve the Poisson-Dirichlet problem

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega \\ u &= g \quad \text{on } \partial\Omega \end{aligned}$$

on a rectangular domain with the classical finite-difference method.

(a) Implement a function `discretisePoisson` with inputs

`f` and `g`: function handles for f and g

`msh`: the output of `meshRectangle`

and outputs

`A`: a sparse (`msh.N(1)` − 1)(`msh.N(2)` − 1) × (`msh.N(1)` − 1)(`msh.N(2)` − 1) array

`b`: a (`msh.N(1)` − 1)(`msh.N(2)` − 1) × 1 array

that assembles the "big linear system" for the Poisson-Dirichlet problem, as derived in class.

(b) Write a script `hw3.m` to solve the above boundary value problem on the rectangle $\Omega =]0,1[\times]2,3[$ with $f(x_1, x_2) = 40\pi^2 \cos(2\pi x_1)\cos(6\pi x_2)$ and $g(x_1, x_2) = \cos(2\pi x_1)\cos(6\pi x_2)$. Use the same meshing parameters as in `hw1.m`. You may solve the linear system e.g. with the built-in \ command.

Provide a surface plot of the numerical solution u^h including its boundary values and a plot of the sparsity pattern of `A`. Check if both graphs agree with your expectations (zoom into the sparsity plot to see the detail).

Hint: In GNU Octave / MATLAB, the commands `speye`, `spdiags`, `kron` and `spy` may be helpful. Very similar commands are available in NumPy or SciPy. Avoid commands like `eye` or `diag` for large matrices, since they also store all zeros. Else your program would take minutes to run (instead of a fraction of a second) and use up all memory.

Your Project Proposal Please upload your proposal as a one-page PDF document. We do not need large line spacing or wide margins, but please do not make the font size smaller than 10pt. Before submission, please check carefully that you have covered all requirements as defined in the rubric for this assessment component.

Your Learning Progress | 0 marks, but -1 mark if unanswered | What is the one most important thing that you have learnt from this assignment?

This assignment taught me to always pay attention to the conditions at which a certain theory / lemme applies

What is the most substantial new insight that you have gained from this course this week? Any *aha moment*?

The notes tend to have all of the info needed to complete the assignment (for the most part).