2.1.5 Group Work Imagine you are cooking some soup on a stovetop. At steady state, the temperature distribution T inside the metal volume of the pot Ω can be found by solving the equation

$$-\Delta T = 0 \qquad \text{in } \Omega.$$

On the surface of the pot, the temperature is fixed as follows:

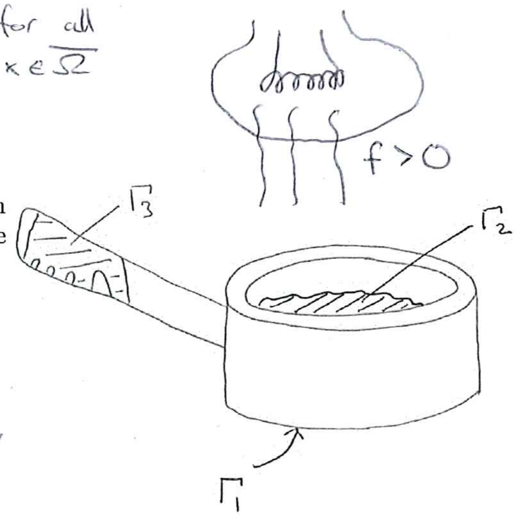
- $T = 200$ °C on the bottom surface Γ_1 that is in contact with the element
- $T = 93$ °C on the inner surface Γ_2 that is in contact with the hot soup
- $T = 32$ °C on the surface of the handle Γ_3 that is in contact with your hand
- $T = 22$ °C everywhere else on the pot surface $\partial\Omega \setminus (\Gamma_1 \cup \Gamma_2 \cup \Gamma_3)$ that is in contact with the surrounding air.

(a) What is the best lower bound and the best upper bound you can derive from the elliptic maximum principle for the temperature distribution T in the metal volume of the pot?

Both the max & the min principle apply.

Max principle $\Rightarrow T(x) \leq \max_{\partial\Omega} T = 200°C$ } for all $x \in \overline{\Omega}$

Min principle $\Rightarrow T(x) \geq \min_{\partial\Omega} T = 22°C$



(b) What changes in your answer to part (a) if you switch on an infrared lamp above the stove, as a consequence of which the PDE model now reads $-\Delta T = f$ with $f > 0$?

Only the min principle applies.

$T(x) \geq 22°C$ for all $x \in \overline{\Omega}$, but

temperatures $> 200°C$ could arise now

due to the positive source term

$\rightarrow$ extra heating

2.1.6 Group Work (Outlook: Discrete Maximum Principle) Assume that we want to solve the PDE problem

$$\begin{aligned} Lu &= f \qquad &&\text{in } \Omega \\ u &= 0 &&\text{on } \partial\Omega \end{aligned}$$

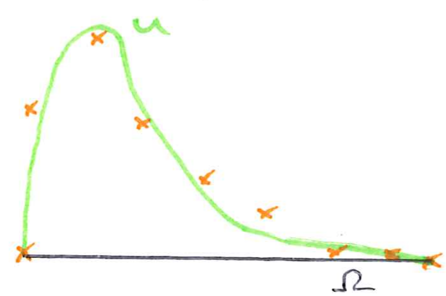
numerically, with an elliptic operator L that satisfies the assumptions of the maximum principle. This could e.g. be a temperature problem like above or steady advection-diffusion-reaction problem. After discretisation e.g. with finite differences or finite elements, we obtain a linear system

$$Ax = b. \qquad \text{Hint: } x = A^{-1}b$$

In analogy to the continuous maximum / minimum principle, whenever the the right hand side vector b does not go below zero, nor should the numerical solution vector x. Can you derive a condition on the matrix A that guarantees just that, i.e. $x \geq 0$ whenever $b \geq 0$? (NB: This $\geq$ notation for vectors should be understood componentwise. $x \geq 0$ means that all entries of the vector x should be non-negative.)

If $A^{-1} \geq 0$ (componentwise), then all entries of x satisfy

$$x_i = \sum_j \underbrace{(A^{-1})_{ij}}_{\geq 0} \underbrace{b_j}_{\geq 0} \geq 0$$



We'll soon learn about more practical criteria that are easier to check than the signs in A^{-1}!