Therefore, we also collect these nodal values in a column vector

$$\vec{v}^h = \begin{pmatrix} v^h(x^1) \\ \vdots \\ v^h(x^N) \end{pmatrix}$$

for an alternative representation of v^h.

Thirdly and lastly, the Galerkin equations for the model problem discretised with linear finite elements read

$$\sum_{j=1}^{N} \left(\int_\Omega \nabla\phi_j^h \cdot \nabla\phi_i^h \, dx \right) \underbrace{u_j^h}_{u^h(x^j)} = \int_\Omega f \phi_i^h \, dx \qquad \forall i \in \{1, \dots, N\}$$

or, in matrix form,

$$\underbrace{K^h}_{\text{stiffness matrix}} \vec{u}^h = \underbrace{\vec{f}^h}_{\text{load vector}}$$

with

$$k_{ij}^h = \int_\Omega \nabla\phi_j^h \cdot \nabla\phi_i^h \, dx$$

$$f_i^h = \int_\Omega f \phi_i^h \, dx \qquad (i, j \in \{1, \dots, N\})$$

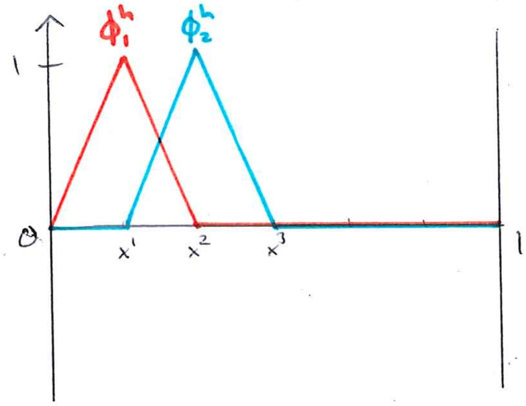
2.3.5 Group Work (Linear Finite Elements in 1D) In one dimension, the model problem reads: find $u \in H_0^1(]0,1[)$ such that for all $v \in H_0^1(]0,1[)$:

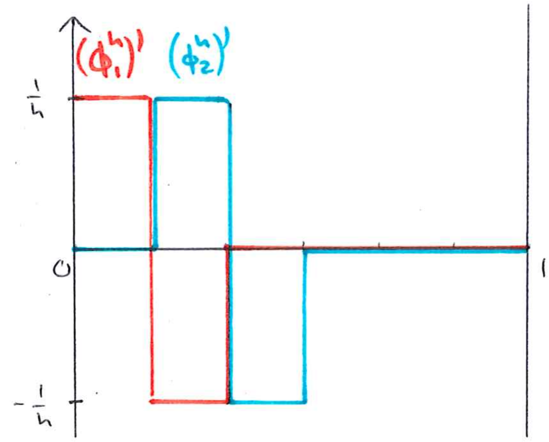
$$\int_0^1 u'v' \, \mathrm{d}x = \int_0^1 fv \, \mathrm{d}x.$$

We discretise this problem on the equidistant grid

$$0, h, 2h, 3h, \dots, (N-1)h, 1$$

with N subintervals and grid spacing $h = 1/N$.





The Galerkin equations read

$$\sum_{j=1}^{N-1} \left(\int_0^1 (\phi_j^h)' \, (\phi_i^h)' \, \mathrm{d}x \right) u_j^h = \int_0^1 f\phi_i^h \, \mathrm{d}x \qquad \forall i \in \{1, \dots, N-1\}.$$

(a) Calculate the entries of the stiffness matrix K^h!

$$k_{ij}^h = \int_0^1 (\phi_j^h)' \, (\phi_i^h)' \;\mathrm{d}x = \begin{cases} \displaystyle\int_{x^{i-1}}^{x^{i+1}} \frac{1}{h^2}\,dx = \frac{2}{h} & \text{if } i = j \\ \displaystyle\int_{x^{i-1}}^{x^{i}} -\frac{1}{h}\frac{1}{h}\,dx = -\frac{1}{h} & \text{if } i = j+1 \\ \displaystyle\int_{x^{i}}^{x^{i+1}} -\frac{1}{h}\frac{1}{h}\,dx = -\frac{1}{h} & \text{if } i = j-1 \\ 0 & \text{if } |i-j| > 1 \end{cases}$$

(b) Now compute the entries of the load vector $\vec{f^h}$:

$$f_i^h = \int_0^1 f\phi_i^h \;\mathrm{d}x = \int_{x^{i-1}}^{x^i} f\,\phi_i^h\,dx + \int_{x^i}^{x^{i+1}} f\,\phi_i^h\,dx$$

$$\approx \begin{cases} \underbrace{(x^i - x^{i-1})}_{h}\, f\!\left(\frac{x^{i-1}+x^i}{2}\right) \underbrace{\phi_i^h\!\left(\frac{x^{i-1}+x^i}{2}\right)}_{1/2} + \underbrace{(x^{i+1}-x^i)}_{h}\, f\!\left(\frac{x^i+x^{i+1}}{2}\right) \underbrace{\phi_i^h\!\left(\frac{x^i+x^{i+1}}{2}\right)}_{1/2} & \text{[midpoint rule]} \\ \underbrace{\frac{x^i - x^{i-1}}{2}}_{\frac{h}{2}} \left(f(x^{i-1}) \underbrace{\phi_i^h(x^{i-1})}_{0} + f(x^i) \underbrace{\phi_i^h(x^i)}_{1} \right) + \underbrace{\frac{x^{i+1}-x^i}{2}}_{\frac{h}{2}} \left(f(x^i) \underbrace{\phi_i^h(x^i)}_{1} + f(x^{i+1}) \underbrace{\phi_i^h(x^{i+1})}_{0} \right) & \text{[trapezium rule]} \end{cases}$$

$$= \begin{cases} \frac{h}{2}\left(f\!\left(\frac{x^{i-1}+x^i}{2}\right) + f\!\left(\frac{x^i+x^{i+1}}{2}\right)\right) & \text{[midpoint rule]} \\ h\, f(x^i) & \text{[trapezium rule]} \end{cases}$$

(c) The "big linear system" reads (using the trapezium rule)

$$\frac{1}{h}\begin{pmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & & \ddots & \\ & & -1 & 2 \end{pmatrix}\begin{pmatrix} u_1^h \\ u_2^h \\ \vdots \\ u_{N-1}^h \end{pmatrix} = h\begin{pmatrix} f(x^1) \\ f(x^2) \\ \vdots \\ f(x^{N-1}) \end{pmatrix}$$

where $x^i = ih \quad (i=1,\ldots,N-1)$.

NB: If the trapezium rule is used to integrate the source term, then the discrete problem is equivalent to a finite-difference approximation. Other quadrature formulae give a different right hand side in the discrete linear system.

(d) Given the coefficient vector $\vec{v}^h$ of a function $v^h \in V^h$, how can you easily compute the L^2-norm of v^h from $\vec{v}^h$?

$$\|v^h\|_{L^2}^2 = \int_0^1 \left(v^h\right)^2 \, dx =$$