MATH521: Numerical Analysis of Partial Differential Equations
Winter 2018/19, Term 2

Due Date: Thursday, 28 February 2019
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Homework Assignment 7

Please submit the following files as indicated below: source code PDF file image file video file

5 marks | Let $D > 0$, $a \in \mathbb{R}^2$, $r \geq 0$ and $f \in L^2(\Omega)$, where $\Omega \subset \mathbb{R}^2$ is a convex, polygonal domain.

We use conforming linear finite elements with exact integration to solve the steady diffusion-advection-reaction problem

$$\begin{aligned} -D\Delta u + \operatorname{div}(au) + ru &= f && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega. \end{aligned}$$

(Recall that the assumption of homogeneous boundary conditions is no loss of generality, since any inhomogeneous boundary conditions could be subtracted from u to obtain the same PDE with homogeneous boundary conditions but a new source term.)

Follow the methodology from pp 67–68 in our notes to show that the numerical solution u^h converges to u at a linear rate in the H^1-norm and at a quadratic rate in the L^2-norm, provided that $u \in H^2(\Omega)$:

$$\|u^h - u\|_{H^1(\Omega)} \leq ch\|\nabla^2 u\|_{L^2(\Omega)} \tag{1}$$

$$\|u^h - u\|_{L^2(\Omega)} \leq ch^2\|\nabla^2 u\|_{L^2(\Omega)}. \tag{2}$$

Hints:

1. To show that the nonsymmetric bilinear form of this elliptic operator is coercive in the H^1-norm, prove and then use that
$$\int_\Omega (a \cdot \nabla u)v \,\mathrm{d}x = -\int_\Omega (a \cdot \nabla v)u \,\mathrm{d}x \quad \text{for all } u, v \in H_0^1(\Omega). \tag{3}$$

2. You may assume that
$$\|u\|_{H^2(\Omega)} \leq c\|f\|_{L^2(\Omega)}. \tag{4}$$

Your Learning Progress | **0 marks, but -1 mark if unanswered** | What is the one most important thing that you have learnt from this assignment?

What is the most substantial new insight that you have gained from this course this week? Any *aha moment*?