MATH521: Numerical Analysis of Partial Differential Equations
Winter 2018/19, Term 2

Due Date: Thursday, 17 January 2019
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Homework Assignment 2

Please submit the following files as indicated below: source code PDF file image file video file

Question 1 | 3 marks | Let $\Omega \subset \mathbb{R}^d$ be a bounded domain in one, two or three dimensions ($d \in \{1, 2, 3\}$), $a : \Omega \to \mathbb{R}^d$ a continuously differentiable vector field, $D : \Omega \to [D_{\min}, D_{\max}]$ a continuously differentiable and $g : \partial\Omega \to [0, 1]$ a continuous function. $D_{\min} > 0$ and $D_{\max} > 0$ are given constants with $D_{\min} \leq D_{\max}$.

The steady advection-diffusion problem

$$\begin{aligned} \operatorname{div}(ua) - \operatorname{div}(D\nabla u) &= 0 && \text{in } \Omega \\ u &= g && \text{on } \partial\Omega \end{aligned}$$

describes how a certain density u is transported through the domain Ω.

(a) Show that any strong solution $u \in C^2(\Omega) \cap C(\bar{\Omega})$ of this problem satisfies

$$0 \leq u(x) \leq 1 \qquad \text{for all } x \in \bar{\Omega}$$

provided that $\operatorname{div} a = 0$ ("incompressibility condition"). Show all working and clearly indicate what results, e.g. Theorem X.Y.Z or Equation (X.Y) from the notes, you use in each step. Also make sure you verify all assumptions of theorems that you apply.

Given that the source term is nowhere positive or negative, ie. $f=0$, and the steady advection-diffusion equation is indeed elliptic, the elliptic maximum/minimum principle (Thm 2.1.4) applies:

$$\Rightarrow \max_{x \in \Omega} u(x) \leq \max_{x \in \partial\Omega} u(x)$$

$$\Rightarrow \min_{x \in \Omega} u(x) \geq \min_{x \in \partial\Omega} u(x)$$

meaning that every value of u in Ω must be within the range of g, $[0,1]$.

(b) Why do we have to assume $\operatorname{div} a = 0$ to derive these bounds?

Without the incompressibility assumption, thinking in the physical sense, as a substance moves through the domain if it were to compress, the density could increase in an area to a value greater than the boundary max. This way, what goes in must come out, also satisfying conservation of mass.

(c) Derive a weak formulation of the steady advection-diffusion problem with homogeneous Dirichlet boundary condition. (This is not a very interesting problem because its unique solution is $u \equiv 0$, but the purpose of this question is to familiarise you with weak formulations!)

$$\int_\Omega \left(\operatorname{div}(ua) - \operatorname{div}(D\nabla u)\right) v \, dx = 0$$

$$\int_\Omega \operatorname{div}(ua) v \, dx - \int_\Omega \operatorname{div}(D\nabla u)\, v \, dx = 0$$

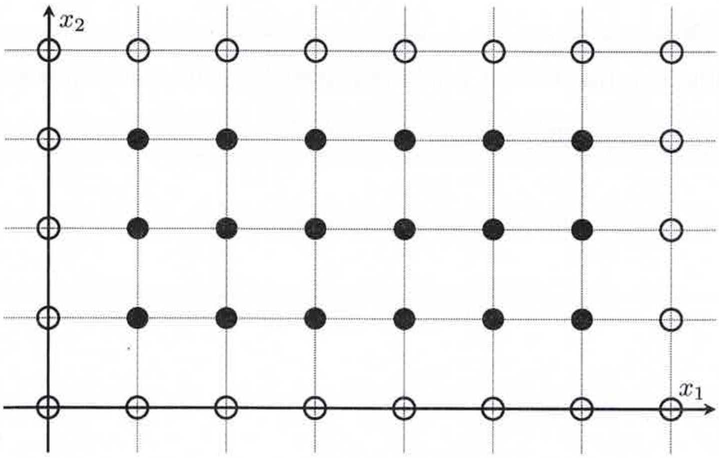
Using Integration by parts:

V goes to zero on the boundary as it is from the same space as U.

$$\underbrace{\int_{\partial\Omega} (ua\cdot n)\, v \, ds}_{0 \rightarrow u=0 \text{ on boundary}} - \int_\Omega ua\cdot\nabla v \, dx - \underbrace{\int_{\partial\Omega} D \frac{\partial u}{\partial n} \cancel{v} \, ds}_{0} + \int_\Omega D\nabla u\cdot\nabla v \, dx = 0$$

$$\boxed{\Rightarrow -\int_\Omega ua\cdot\nabla v \, dx + \int_\Omega D\nabla u\cdot\nabla v \, dx = 0 \qquad u \in H_0^1(\Omega),\ v \in H_0^1(\Omega)}$$

Question 2 | 2 marks | Today we will implement two more auxiliary functions for the finite difference method on rectangular domains. Initially, `X1`, `X2`, a discretisation `F` of the source term and an initial guess `U` for the solution of the PDE are usually given as rectangular arrays. For our computations, we have to re-arrange some of these arrays to column vectors. For plotting and post-processing, we have to re-arrange some column vectors back to rectangular arrays.



If the PDE is equipped with Dirichlet boundary conditions, then the solution values on the boundary (empty circles) are already known. Only the interior solution values (filled circles) are the actual unknowns.

(a) Write a function `msh2vec` that applies so-called *lexicographical ordering* to re-arrange a rectangular array to a column vector. This function takes the two input variables

`U`: an array of size (`msh.N(2)` $-$ 1) $\times$ (`msh.N(1)` $-$ 1) corresponding to function values on the *interior* nodes of the grid only, not the boundary nodes

`msh`: the output of `meshRectangle`

and returns

`u`: an array of size (`msh.N(1)` $-$ 1)(`msh.N(2)` $-$ 1) $\times$ 1

The first component of `u` should contain the value of `U` that corresponds to the bottom left interior grid point (filled circle) of the rectangular domain, the second component should correspond to the next grid point to the right etc. Moving from left to right row by row, the last component of `u` will be equal to the value of `U` that corresponds to the interior grid point (filled circle) in the top right corner of the domain.

(b) Implement a function `U = vec2msh(u,msh)` that undoes the action of `msh2vec`. Write a script `hw2.m` that tests whether for the sample function from `hw1.m` the arrays

```
U = u(msh.X1(2:end-1,2:end-1),msh.X2(2:end-1,2:end-1))
```

and

```
V = vec2msh(msh2vec(U,msh),msh)
```

are indeed the same.

Hint: In GNU Octave / MATLAB, the command `reshape` may be helpful.

Please submit all files that are needed to run your code, even if you have submitted them previously.

Your Course Project It's time to start thinking about a small research project. Please consult the course outline for details, and if you have an idea or need one, please come and talk to me!

Your Learning Progress | What is the one most important thing that you have learnt from this assignment?

The most important thing that I (think I) learned is the use of Lebesgue and Sobolev spaces and how they relate to each other and other spaces.

What is the most substantial new insight that you have gained from this course this week? Any *aha moment*?

Why the weak formulation is so useful.