MECH511 Assignment 3 Compressible Flow

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April 1, 2019

1 AUSM

The AUSM scheme can be written as

$$F^{AUSM}_{i+\frac{1}{2}} = M_{i+\frac{1}{2}}\Phi_{i+\frac{1}{2}} + \begin{pmatrix} 0 \\ p_{i+\frac{1}{2}} \\ 0 \end{pmatrix} \tag{1}$$

where $M_{i+\frac{1}{2}}$ is an interface Mach number defined by

$$M_{i+\frac{1}{2}} = \mathscr{M}^{+}_{i} + \mathscr{M}^{-}_{i+1} \tag{2}$$

$$\mathscr{M}^{\pm}_{i}(M) = \begin{cases} \frac{1}{2}(M \pm |M|) & |M| > 1 \\ \pm\frac{1}{4}(M \pm 1)^2 & -1 \leq M \leq 1 \end{cases} \tag{3}$$

$p_{i+\frac{1}{2}}$ is the interface pressure defined by

$$p_{i+\frac{1}{2}} = \mathscr{P}^{+}_{i} p_i + \mathscr{P}^{-}_{i+1} p_{i+1} \tag{4}$$

$$\mathscr{P}^{\pm}_{i}(M) = \begin{cases} \frac{1}{2}(1 \pm sign(M)) & |M| > 1 \\ \frac{1}{4}(M \pm 1)^2(2 \mp M) & -1 \leq M \leq 1 \end{cases} \tag{5}$$

and the convected quantity $\Phi_{i+\frac{1}{2}}$ is defined using upwinding:

$$\Phi_{i+\frac{1}{2}} = \begin{cases} \Phi_i & M_{i+\frac{1}{2}} \geq 0 \\ \Phi_{i+1} & M_{i+\frac{1}{2}} < 0 \end{cases} \tag{6}$$

$$\Phi = \begin{pmatrix} \rho c \\ \rho c u \\ c(E+P) \end{pmatrix} \tag{7}$$

Now for 2 cases: $0 \leq M_i, M_{i+1} \leq 1$ and $1 \leq M_i, M_{i+1}$ we want to recast equation (1) to find the dissipation term $f(U_i, U_{i+1})$ for:

$$F^{AUSM}_{i+\frac{1}{2}} = \frac{F_i + F_{i+1}}{2} - \frac{1}{2}f(U_i, U_{i+1}) \tag{8}$$

First for the supersonic case ($1 \leq M_i, M_{i+1}$) we get:
$M_{i+\frac{1}{2}}$:

$$\mathscr{M}_i^+(M_i) = \frac{1}{2}(M_i + |M_i|) = M_i$$
$$\mathscr{M}_{i+1}^-(M_{i+1}) = \frac{1}{2}(M_{i+1} - |M_{i+1}|) = 0$$

Leaving us with:

$$M_{i+\frac{1}{2}} = M_i$$

$p_{i+\frac{1}{2}}$:

$$\mathscr{P}_i^+(M_i) = \frac{1}{2}(1 + sign(M_i)) = 1$$
$$\mathscr{P}_{i+1}^-(M_{i+1}) = \frac{1}{2}(1 - sign(M_{i+1})) = 0$$

Leaving us with:

$$p_{i+\frac{1}{2}} = p_i$$

$M_{i+\frac{1}{2}}$ is indeed positive so $\Phi_{i+\frac{1}{2}} = \Phi_i$ Putting this all together we get:

$$\begin{aligned}
F_{i+\frac{1}{2}}^{AUSM} &= M_{i+\frac{1}{2}}\Phi_{i+\frac{1}{2}} + \begin{pmatrix} 0 \\ p_{i+\frac{1}{2}} \\ 0 \end{pmatrix} \\
&= M_i\Phi_i + \begin{pmatrix} 0 \\ p_i \\ 0 \end{pmatrix} \\
&= M_i \begin{pmatrix} \rho c \\ \rho c u \\ c(E+P) \end{pmatrix}_i + \begin{pmatrix} 0 \\ p_i \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} \rho u \\ \rho u^2 + p \\ u(E+P) \end{pmatrix}_i = F_i
\end{aligned}$$

So we get that $F_{i+\frac{1}{2}}^{AUSM} = F_i$ that is:

$$f(U_i, U_{i+1}) = F_{i+1} - F_i$$

$$F_{i+\frac{1}{2}}^{AUSM} = \frac{F_i + F_{i+1}}{2} - \frac{F_{i+1} - F_i}{2} = F_i$$

Now the subsonic case ($0 \leq M_i, M_{i+1} \leq 1$) we get:

$M_{i+\frac{1}{2}}$:

$$\mathscr{M}_i^+(M_i) = \frac{1}{4}(M_i+1)^2 = \frac{1}{4}(M_i^2+2M_i+1)$$
$$\mathscr{M}_{i+1}^+(M_{i+1}) = -\frac{1}{4}(M_{i+1}-1)^2 = -\frac{1}{4}(M_{i+1}^2-2M_{i+1}+1)$$

Leaving us with:

$$M_{i+\frac{1}{2}} = \frac{M_i^2-M_{i+1}^2}{4} + \frac{M_i+M_{i+1}}{2}$$

$p_{i+\frac{1}{2}}$:

$$\mathscr{P}_i^+(M_i) = \frac{1}{4}(M_i+1)^2(2-M_i) = \frac{1}{4}(-M_i^3+3M_i+2)$$
$$\mathscr{P}_{i+1}^+(M_{i+1}) = \frac{1}{4}(M_{i+1}+1)^2(2-M_{i+1}) = \frac{1}{4}(M_{i+1}^3-3M_{i+1}+2)$$

Leaving us with:

$$p_{i+\frac{1}{2}} = \frac{1}{4}\left[(-M_i^3+3M_i+2)p_i + (M_{i+1}^3-3M_{i+1}+2)p_{i+1}\right]$$

$M_{i+\frac{1}{2}}$ is indeed positive so $\Phi_{i+\frac{1}{2}} = \Phi_i$ Putting this all together we get:

$$F_{i+\frac{1}{2}}^{AUSM} = \left(\frac{M_i^2-M_{i+1}^2}{4} + \frac{M_i+M_{i+1}}{2}\right)\Phi_i + \frac{1}{4}(-M_i^3+3M_i+2)\begin{pmatrix}0\\p_i\\0\end{pmatrix} + \frac{1}{4}(M_{i+1}^3-3M_{i+1}+2)\begin{pmatrix}0\\p_{i+1}\\0\end{pmatrix}$$

From this we can gather one F_i, with a bunch of other stuff afterwards:

$$= \frac{F_i}{2} + \left(\frac{M_i^2-M_{i+1}^2}{4} + \frac{M_{i+1}}{2}\right)\Phi_i + \frac{1}{4}(-M_i^3+3M_i)\begin{pmatrix}0\\p_i\\0\end{pmatrix} + \frac{1}{4}(M_{i+1}^3-3M_{i+1}+2)\begin{pmatrix}0\\p_{i+1}\\0\end{pmatrix}$$

It's hard to conclude if there is a F_{i+1} term somewhere in that mess, so for now we will conclude that:

$$f(U_i,U_{i+1}) = F_{i+1} - \left(\frac{M_i^2-M_{i+1}^2}{4} + \frac{M_{i+1}}{2}\right)\Phi_i + \frac{1}{4}(-M_i^3+3M_i)\begin{pmatrix}0\\p_i\\0\end{pmatrix} + \frac{1}{4}(M_{i+1}^3-3M_{i+1}+2)\begin{pmatrix}0\\p_{i+1}\\0\end{pmatrix}$$

Now, looking at both forms of f, no, it is not clear to me whether classifying AUSM as a flux difference splitting scheme or matrix dissipation scheme is possible.

2 1D Roe Scheme Discretised

Given the compressible Euler equations in 1D, we want to fully discretise using the implicit Euler time advance. The compressible Euler equations are:

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} = 0$$

Where

$$U = \begin{pmatrix} \rho \\ \rho u \\ E \end{pmatrix} \qquad F = \begin{pmatrix} \rho u \\ \rho u^2 + P \\ u(E+P) \end{pmatrix}$$

$$E = \rho c_v T + \rho \frac{u^2}{2} \qquad P = \rho R T$$

And of course, implicit Euler, looks like:

$$\frac{U_i^{n+1} - U_i^n}{\Delta t} + \frac{F_{i+\frac{1}{2}}^{n+1} - F_{i-\frac{1}{2}}^{n+1}}{\Delta x} = 0$$

We must linearise our fluxes:

$$F_{i+\frac{1}{2}}^{n+1} \equiv F(U_i, U_{i+1})^{n+1} \approx F_{i+\frac{1}{2}}^n + \Delta t \left. \frac{\partial F_{i+\frac{1}{2}}}{\partial t} \right|^n + O(\Delta t^2)$$

and by the chain rule we get:

$$\left. \frac{\partial F_{i+\frac{1}{2}}}{\partial t} \right|^n = \left. \frac{\partial F_{i+\frac{1}{2}}}{\partial U_i} \frac{\partial U_i}{\partial t} \right|^n + \left. \frac{\partial F_{i+\frac{1}{2}}}{\partial U_{i+1}} \frac{\partial U_{i+1}}{\partial t} \right|^n$$

where we can use δ-form that is:

$$\frac{\partial U_i}{\partial t} = \frac{U_i^{n+1} - U_i^n}{\Delta t} = \frac{\delta U_i^{n+1}}{\Delta t}$$

Now linearising and rearranging our implicit scheme we get:

$$\left(\frac{I}{\Delta t} + \frac{1}{\Delta x} \left. \frac{\partial F_{i+\frac{1}{2}}}{\partial U_i} \right|^n - \frac{1}{\Delta x} \left. \frac{\partial F_{i-\frac{1}{2}}^n}{\partial U_i} \right|^n \right) \delta U_i^{n+1} + \frac{1}{\Delta x} \left. \frac{\partial F_{i+\frac{1}{2}}^n}{\partial U_{i+1}} \right|^n \delta U_{i+1}^{n+1} - \frac{1}{\Delta x} \left. \frac{\partial F_{i-\frac{1}{2}}^n}{\partial U_{i-1}} \right|^n \delta U_{i-1}^{n+1} = -\frac{F_{i+\frac{1}{2}}^n - F_{i-\frac{1}{2}}^n}{\Delta x}$$

Using Roe's scheme to evaluate the fluxes:

$$F_{i+\frac{1}{2}} = \frac{1}{2}\left(F_{i+\frac{1}{2};i} + F_{i+\frac{1}{2};i+1} \right) - \frac{1}{2} |\tilde{A}| \left(U_{i+\frac{1}{2};i+1} + U_{i+\frac{1}{2};i} \right)$$

Using first-order spatial accuracy on the LHS, for evaluating the interface $i+1/2$ we get simply that $F_{i+\frac{1}{2};i} = F_i$ and $F_{i+\frac{1}{2};i+1} = F_{i+1}$ and likewise for U. This gives:

$$F_{i+\frac{1}{2}} = \frac{1}{2}\left(F_i + F_{i+1}\right) - \frac{1}{2}|\tilde{A}|\left(U_{i+1} + U_i\right)$$

And similar for $F_{i-\frac{1}{2}}$. Now evaluating the Jacobians of the LHS we get:

$$\frac{1}{\Delta x}\frac{\partial F_{i+\frac{1}{2}}}{\partial U_i} = \frac{A_i + |\tilde{A}|_{i+\frac{1}{2}}}{2\Delta x} \qquad \frac{1}{\Delta x}\frac{\partial F_{i+\frac{1}{2}}}{\partial U_{i+1}} = \frac{A_{i+1} - |\tilde{A}|_{i+\frac{1}{2}}}{2\Delta x}$$

$$\frac{1}{\Delta x}\frac{\partial F_{i-\frac{1}{2}}}{\partial U_i} = \frac{A_i - |\tilde{A}|_{i-\frac{1}{2}}}{2\Delta x} \qquad \frac{1}{\Delta x}\frac{\partial F_{i-\frac{1}{2}}}{\partial U_{i-1}} = \frac{A_{i-1} - |\tilde{A}|_{i-\frac{1}{2}}}{2\Delta x}$$

where A and $\tilde{A}$ are as given in the notes. Subbing these back into our discretisation we get:

$$\left(\frac{I}{\Delta t} + \frac{|\tilde{A}|_{i+\frac{1}{2}} + |\tilde{A}|_{i-\frac{1}{2}}}{2\Delta x}\right)\delta U_i^{n+1} + \frac{A_{i+1} - |\tilde{A}|_{i+\frac{1}{2}}}{2\Delta x}\delta U_{i+1}^{n+1} - \frac{A_{i-1} - |\tilde{A}|_{i-\frac{1}{2}}}{2\Delta x}\delta U_{i-1}^{n+1} = -\frac{F^n_{i+\frac{1}{2}} - F^n_{i-\frac{1}{2}}}{\Delta x}$$

Now using a second-order spatial discretisation for Roe's scheme on the RHS:

$$F_{i+\frac{1}{2}} = \frac{1}{2}\left(F_{i+\frac{1}{2};i} + F_{i+\frac{1}{2};i+1}\right) - \frac{1}{2}|\tilde{A}|\left(U_{i+\frac{1}{2};i+1} + U_{i+\frac{1}{2};i}\right)$$

we get:

$$F_{i+\frac{1}{2};i} = \frac{3F_i - F_{i-1}}{2} \qquad F_{i+\frac{1}{2};i+1} = \frac{-3F_{i+1} + F_{i+2}}{2}$$

and likewise for U. Subbing these into Roe's scheme we get:

$$F_{i+\frac{1}{2}} = \frac{1}{4}\left(-F_{i-1} + 3F_i - 3F_{i+1} + F_{i+2}\right) - \frac{1}{4}|\tilde{A}|\left(-U_{i-1} + 3U_i - 3U_{i+1} + U_{i+2}\right)$$

$$F_{i-\frac{1}{2}} = \frac{1}{4}\left(-F_{i-2} + 3F_{i-1} - 3F_i + F_{i+1}\right) - \frac{1}{4}|\tilde{A}|\left(-U_{i-2} + 3U_{i-1} - 3U_i + U_{i+1}\right)$$

And finally a discretised RHS of:

$$\begin{aligned}-\frac{F^n_{i+\frac{1}{2}} - F^n_{i-\frac{1}{2}}}{\Delta x} &= -\frac{1}{4}\left(F_{i-2} - 4F_{i-1} + 6F_i - 4F_{i+1} + F_{i+2}\right) \\ &\quad - \frac{1}{4}|\tilde{A}|_{i+\frac{1}{2}}\left(-U_{i-1} + 3U_i - 3U_{i+1} + U_{i+2}\right) + \frac{1}{4}|\tilde{A}|_{i-\frac{1}{2}}\left(-U_{i-2} + 3U_{i-1} - 3U_i + U_{i+1}\right)\end{aligned}$$

Combining the LHS and RHS we find the full discretisation:

$$\begin{aligned}&\left(\frac{I}{\Delta t} + \frac{|\tilde{A}|_{i+\frac{1}{2}} + |\tilde{A}|_{i-\frac{1}{2}}}{2\Delta x}\right)\delta U_i^{n+1} + \frac{A_{i+1} - |\tilde{A}|_{i+\frac{1}{2}}}{2\Delta x}\delta U_{i+1}^{n+1} - \frac{A_{i-1} - |\tilde{A}|_{i-\frac{1}{2}}}{2\Delta x}\delta U_{i-1}^{n+1} \\ &\quad = -\frac{1}{4}\left(F_{i-2} - 4F_{i-1} + 6F_i - 4F_{i+1} + F_{i+2}\right) - \frac{1}{4}|\tilde{A}|_{i+\frac{1}{2}}\left(-U_{i-1} + 3U_i - 3U_{i+1} + U_{i+2}\right) \\ &\qquad + \frac{1}{4}|\tilde{A}|_{i-\frac{1}{2}}\left(-U_{i-2} + 3U_{i-1} - 3U_i + U_{i+1}\right)\end{aligned}$$