Programming Assignment 5: Adjoint Methods

Mech 511 — Spring, 2019

Due date: April 22

This assignment will explore some, though not nearly all, of the interesting uses of adjoint methods. The primal problem will be a Poisson problem in a square $[0,1]\times[0,1]$, with homogeneous Dirichlet boundary conditions on three sides ($x=0$, $y=0$, and $x=1$); we'll call the solution to the primal problem u and the solution to the adjoint problem v. Other details will vary for different cases. You can use any Poisson solver you have lying around (from 510, for example, or from the multigrid assignment) or write one from scratch.

1. **Warm-up problem**. (Easy) For this case, the boundary condition at $y=1$ is $u=0$ and the source term for the primal problem is $f=-\frac{\pi^3}{4}\sin\pi x\sin\pi y$. For the dual problem, we also have homogeneous Dirichlet boundary conditions, with source term $g=\frac{\pi^5}{2}x(1-x)y(1-y)$. The output functional in this case is

$$J=(u,g)=(v,f)$$

Solve both the primal and dual problems, and demonstrate that you get the same value for the (discrete) output functional evaluated both ways, and that the discrete output functional converges to its exact value at the order of accuracy of your numerical scheme.

2. **Error estimation**. (Medium) For the previous case, use a high-order (i.e., more than second order) discretization of the Laplace operator to compute the correction / error estimate term

$$\delta J=(Lu_h-f_h)$$

In this case, your high-order operator will fill the role of Lu_h without any need for high-order interpolation of the solution.[1] Use your error estimate to get an improved value of the functional J and show that this corrected functional converges with a higher order of accuracy than the uncorrected functional.

3. **Functionals with boundary integrals**. (Easy) For this case, use $u=\sin\pi x$ as a boundary condition at $y=1$; retain the same source terms as in Question 1. Your primal output functional is now

$$J=(u,g)_\Omega+(u_n,h)_{\partial\Omega}$$

[1] Note that my experiments suggest that this level of approximation, while successful here, isn't sufficient for error estimation for functionals with boundary integrals.

where $h = x(1-x)$ on $y = 0$ and zero elsewhere; keep in mind that u_n is defined as the directional derivative in the *outward* normal direction, so $-u_y$ in this case.

It is easy to show that the dual problem in this case has a boundary condition of $v = x(1-x)$ on $y = 0$, and $v = 0$ elsewhere, and that the dual functional is

$$J = (v, f)_\Omega + (v_n, e)_{\partial\Omega}$$

where $e = \sin \pi x$ on $y = 1$ and zero elsewhere.

Compute this functional using both the primal and dual problems, and show that this formulation is adjoint consistent and converges with the correct order of accuracy.

4. **Sensitivity**. (Medium to hard) Suppose now that we modify the primal problem from Question 1 so that

$$f = (1+c_1)\left(-\frac{\pi^3}{4}\sin \pi x \sin \pi y\right) + c_2 g$$

In the context of an optimization problem, we would need to be able to compute $\frac{\partial J}{\partial \vec{c}}$. Writing this partial derivative out using the chain rule:

$$\begin{aligned} J &= (u, g)_\Omega = J(u(\vec{c})) \\ \frac{\partial J}{\partial \vec{c}} &= \frac{\partial J}{\partial u}\frac{\partial u}{\partial \vec{c}} \end{aligned} \tag{1}$$

This would require an additional primal solve to estimate the derivative of the solution with respect to each component $\vec{c}$. But we can take advantage of the zero-residual condition:

$$\begin{aligned} L(u) &= f \\ \frac{\partial L}{\partial u}\frac{\partial u}{\partial \vec{c}} &= \frac{\partial f}{\partial \vec{c}} \\ \frac{\partial u}{\partial \vec{c}} &= \left(\frac{\partial L}{\partial u}\right)^{-1}\left(\frac{\partial f}{\partial \vec{c}}\right) \end{aligned}$$

This doesn't reduce the amount of work yet, but substituting this into Eqn. 1 and taking the transpose does:

$$\left(\frac{\partial J}{\partial \vec{c}}\right)^T = \left(\frac{\partial f}{\partial \vec{c}}\right)^T \left(\frac{\partial L}{\partial u}\right)^{-T} \left(\frac{\partial J}{\partial u}\right)^T$$

where the last two terms represent the solution to the adjoint problem with $\frac{\partial J}{\partial u}$ as a source term.

Use the adjoint formulation to find the sensitivity of this objective function for $c_1 = 0$, $c_2 = 0.5$; use finite differences to verify your sensitivity result.