Fourier Transform

- The Fourier transform is an *analysis* process, decomposing a complex-valued function f(x) into its constituent frequencies and their amplitudes.
- The inverse process is ***synthesis*, ** which recreates f(x) from its transform.

- In physics, engineering and mathematics, the **FT** is an integral transform that takes the input as a function and outputs another function.
- This output that describes the extent to which various frequencies are present in the original function.
- The output of the transform is a complex-valued function of frequency.
- The term ***Fourier transform*** refers to both this complex-valued function and the mathematical operation.

FOURIER TRANDFORM

- Fourier transform is sometimes called the frequency domain representation of the original function.
- The Fourier transform is analogous to decomposing the sound of a musical chord into the intensities of its constituent pitches.

- In general, the Fourier transform is a very useful tool when solving differential equations on domains ranging from $-\infty \ldots + \infty$.
- This is due to the fact that the Fourier transform contains an integral.
- This integral leads to very useful features when put into a differential equation

Fourier Transform Applications

- Circuit Analysis
- Signal Analysis
- Cell phones
- Image Processing
- Signal Processing& LTI system

***Synthesis* and *Analysis* of signals using Fourier *transform*:**

Frequency domain analysis and Fourier Transform

How to Represent Signals?

- *Option 1: Taylor series represents any function using polynomials.*

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}$$

$$(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \ldots.$$

- *Polynomials are not the best - unstable and not very physically meaningful.*
- *Easier to talk about "signals" in terms of its "frequencies" (how fast/often signals change, etc).*

Jean Baptiste Joseph Fourier (1768-1830)

- *Had crazy idea (1807):*

  *Any periodic function can be rewritten as a weighted sum of **Sines** and **Cosines** of different frequencies.*

- *Neither did Lagrange, Laplace, Poisson etc.*
  - *Not translated into English until 1878!*
- *But it was true!*
  - *Called **Fourier Series***
  - *Possibly the greatest tool used in Engineering industry nowadays.*

Fourier Transform

- *We want to understand the frequency ω of our signal. So, let's reparametrize the signal by ω instead of x:*

$$f(x) \longrightarrow \boxed{\text{Fourier Transform}} \longrightarrow F(\omega)$$

- *For every ω from 0 to infinity, $F(\omega)$ holds the amplitude A and phase ϕ of the corresponding sine* $A\sin(\omega x + \phi)$
  - How can *F* hold both? Complex number trick!

$$F(\omega) = R(\omega) + iI(\omega)$$

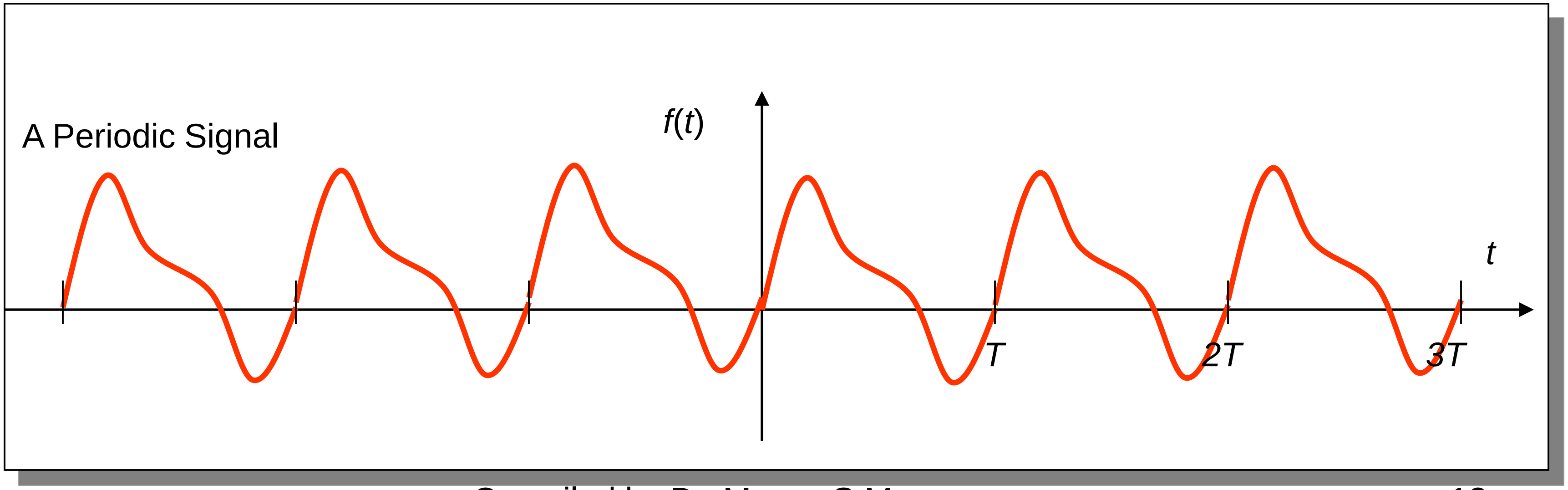
$$A = \pm\sqrt{R(\omega)^2 + I(\omega)^2} \qquad \phi = \tan^{-1}\frac{I(\omega)}{R(\omega)}$$

$$F(\omega) \longrightarrow \boxed{\text{Inverse Fourier Transform}} \longrightarrow f(x)$$

Continuous-Time Fourier Transform

Review of Fourier Series

- Deal with continuous-time periodic signals.
- Discrete frequency spectra.



Two Forms for Fourier Series

Sinusoidal Form

$$\boxed{f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{2\pi nt}{T} + \sum_{n=1}^{\infty} b_n \sin\frac{2\pi nt}{T}}$$

$$a_0 = \frac{2}{T}\int_{-T/2}^{T/2} f(t)dt$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos n\omega_0 t dt$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin n\omega_0 t dt$$

Complex Form:

$$\boxed{f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}}$$

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-jn\omega_0 t} dt$$

Continuous-Time Fourier Transform

Fourier Integral

$$f_T(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t} \qquad c_n = \frac{1}{T}\int_{-T/2}^{T/2} f_T(t) e^{-jn\omega_0 t}\,dt$$

$$= \sum_{n=-\infty}^{\infty} \left[\frac{1}{T}\int_{-T/2}^{T/2} f_T(\tau) e^{-jn\omega_0 \tau}\,d\tau\right] e^{jn\omega_0 t}$$

$$\omega_0 = \frac{2\pi}{T} \Rightarrow \frac{1}{T} = \frac{\omega_0}{2\pi}$$

$$= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} \left[\int_{-T/2}^{T/2} f_T(\tau) e^{-jn\omega_0 \tau}\,d\tau\right] \omega_0 e^{jn\omega_0 t}$$

Let $\Delta\omega = \omega_0 = \dfrac{2\pi}{T}$

$$= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} \left[\int_{-T/2}^{T/2} f_T(\tau) e^{-jn\omega_0 \tau}\,d\tau\right] e^{jn\omega_0 t}\Delta\omega$$

$$T \to \infty \Rightarrow d\omega = \Delta\omega \approx 0$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f_T(\tau) e^{-j\omega \tau}\,d\tau\right] e^{j\omega t}\,d\omega$$

Fourier Integral

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{\left[\int_{-\infty}^{\infty} f(\tau)e^{-j\omega\tau}d\tau\right]}_{F(j\omega)} e^{j\omega t}d\omega$$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)e^{j\omega t}d\omega$$ Synthesis

$$F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$ Analysis

Fourier Series vs. Fourier Integral

Fourier Series:

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

Period Function

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f_T(t) e^{-jn\omega_0 t}\, dt$$

Discrete Spectra

Fourier Integral:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega) e^{j\omega t}\, d\omega$$

Non-Period Function

$$F(j\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t}\, dt$$

Continuous Spectra

Relationship between exponentials and sinusoids

- Euler's formula:

$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$

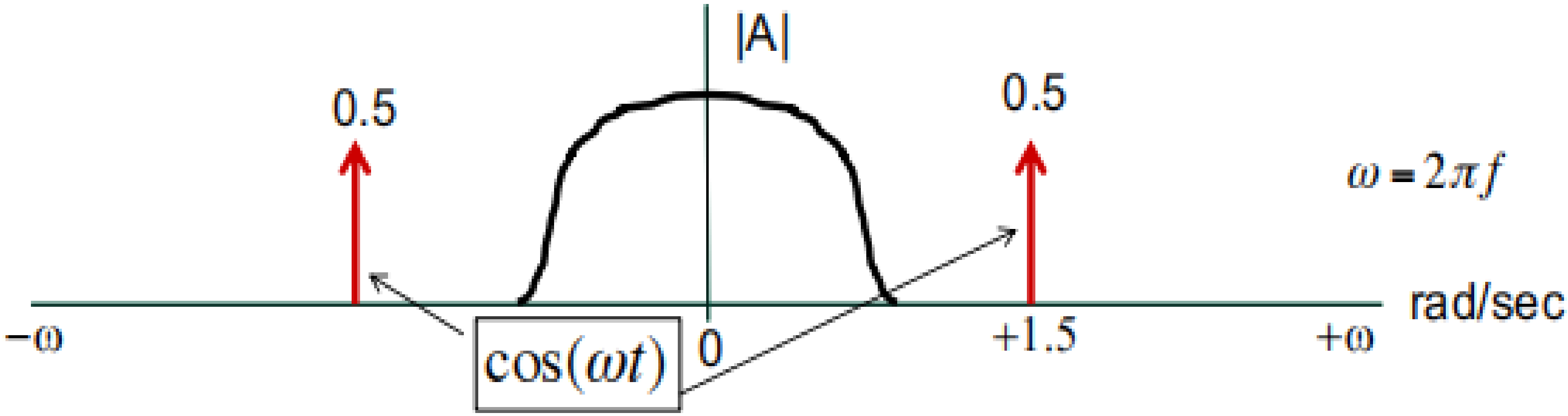
$$e^{-j\omega t} = \cos(-\omega t) + j\sin(-\omega t)$$
$$= \cos(\omega t) - j\sin(\omega t)$$

$$\cos(\omega t) = \frac{e^{j\omega t} + e^{-j\omega t}}{2}$$

$$\sin(\omega t) = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}$$

- Therefore, in signal analysis, we usual regard "frequency" to be ω in the exponential vector:

- The frequency spectrum is therefore a plot of the amplitude (and phase) projected onto exponential components $e^{j\omega t}$ for different ω.



Continuous-Time Fourier Transform

Fourier Transform Pair

Inverse Fourier Transform:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)e^{j\omega t}d\omega$$

Synthesis

Fourier Transform:

$$F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$

Analysis

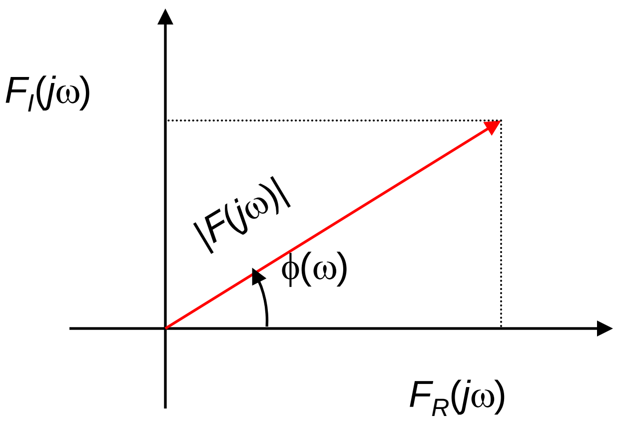
Continuous Spectra

$$F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$

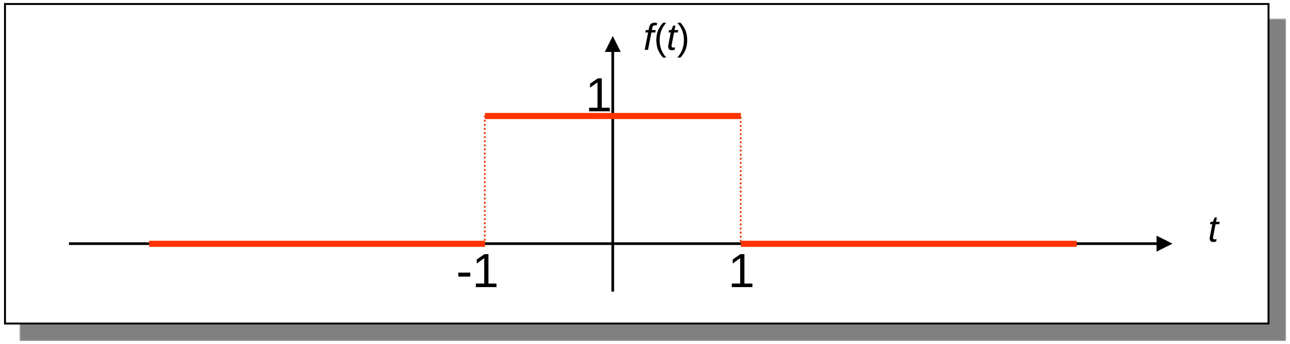
$$F(j\omega) = F_R(j\omega) + jF_I(j\omega)$$

$$= \underbrace{|F(j\omega)|}_{\text{Magnitude}}\, e^{\overbrace{j\phi(\omega)}^{}}$$

Phase



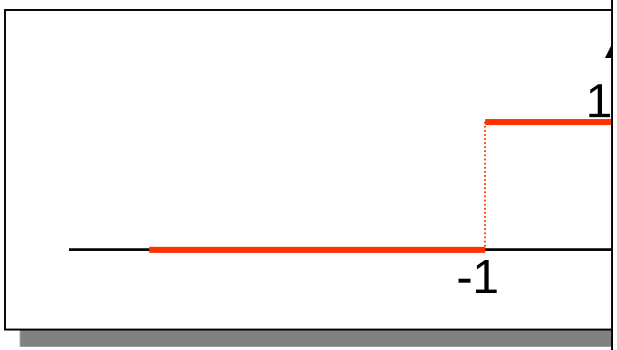
Example

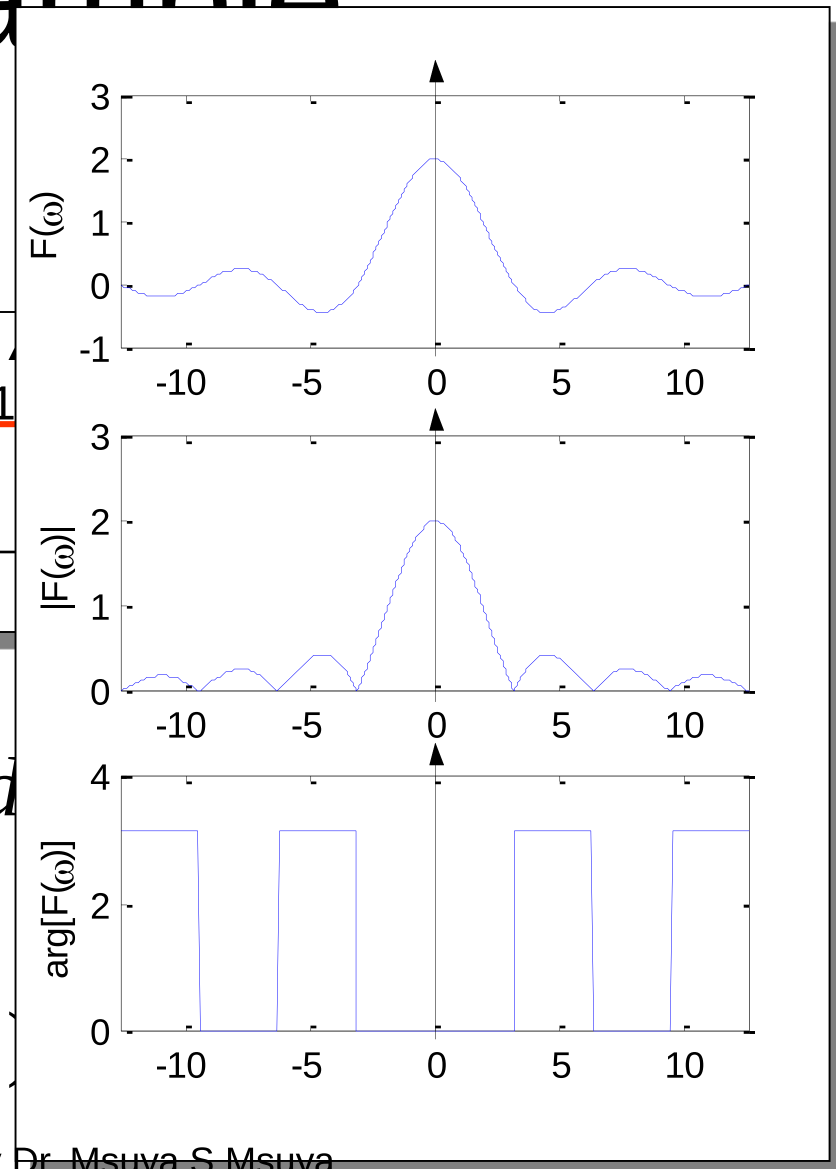


$$F(j\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt = \int_{-1}^{1} e^{-j\omega t} dt = \frac{1}{-j\omega} e^{-j\omega t} \Big|_{-1}^{1}$$

$$= \frac{j}{\omega} (e^{-j\omega} - e^{j\omega}) = \frac{2 \sin \omega}{\omega}$$

Example



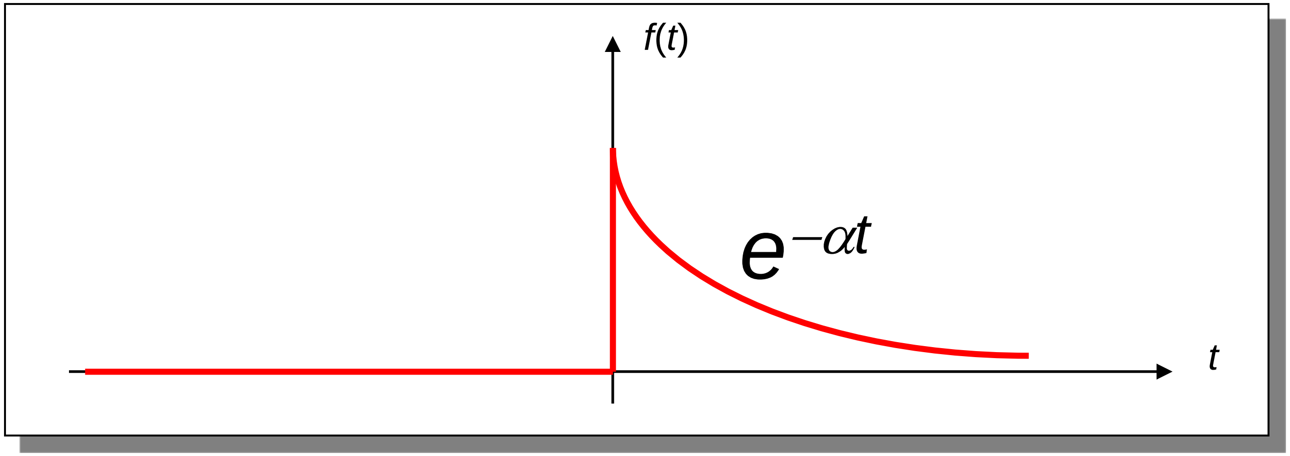


$$F(j\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} d$$

$$t \Big|_{-1}^{1}$$

$$= \frac{j}{\omega}\left(e^{-j\omega} - e^{j\omega}\right.$$

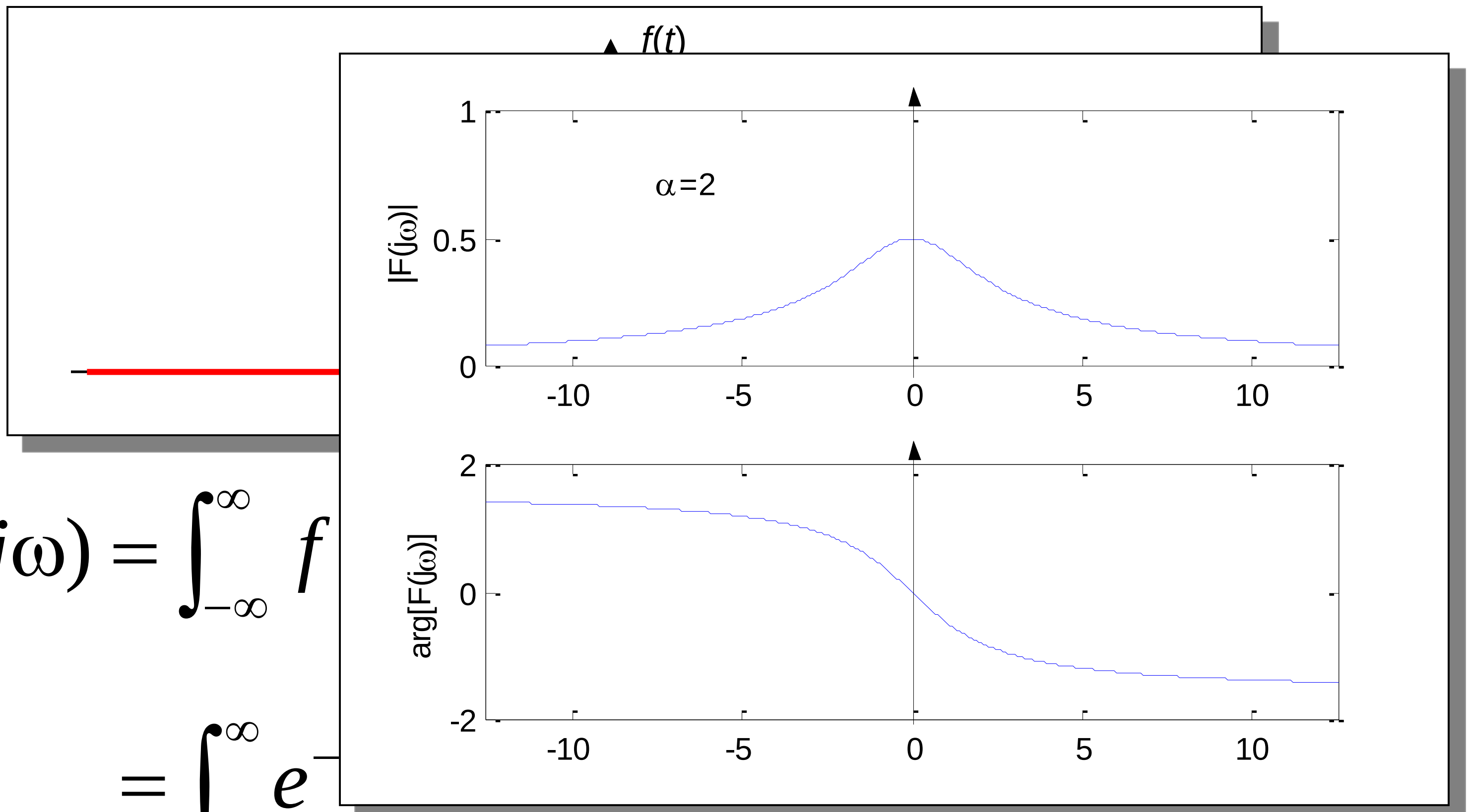
Example



$$F(j\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt \quad = \int_{0}^{\infty} e^{-\alpha t} e^{-j\omega t} dt$$

$$= \int_{0}^{\infty} e^{-(\alpha + j\omega)t} dt \quad = \frac{1}{\alpha + j\omega}$$

Example



$$F(j\omega)=\int_{-\infty}^{\infty} f$$

$$=\int_{0}^{\infty} e^{-}$$

$$\alpha + j\omega$$

Continuous-Time Fourier Transform

Notation

$$\mathcal{F}[f(t)] = F(j\omega)$$

$$\mathcal{F}^{-1}[F(j\omega)] = f(t)$$

$$f(t) \xleftrightarrow{\mathcal{F}} F(j\omega)$$

Frequency Shifting (Modulation)

$$\boxed{f(t)e^{j\omega_0} \xleftrightarrow{\mathcal{F}} F\left[j(\omega-\omega_0)\right]}$$

Pf)

$$\mathcal{F}[f(t)e^{j\omega_0 t}] = \int_{-\infty}^{\infty} f(t)e^{j\omega_0 t}e^{-j\omega t}dt$$

$$= \int_{-\infty}^{\infty} f(t)e^{-j(\omega-\omega_0)t}dt$$

$$= F\left[j(\omega-\omega_0)\right]$$

Fourier Transform for Real Functions

If $f(t)$ is a real function, and $F(j\omega) = F_R(j\omega) + jF_I(j\omega)$

$\Rightarrow$ $F(-j\omega) = F^*(j\omega)$

$$F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$

$$F^*(j\omega) = \int_{-\infty}^{\infty} f(t)e^{j\omega t}dt = F(-j\omega)$$

Fourier Transform for Real Functions

If $f(t)$ is a real function, and $F(j\omega) = F_R(j\omega) + jF_I(j\omega)$

- $F(-j\omega) = F^*(j\omega)$

- $F_R(j\omega)$ is even, and $F_I(j\omega)$ is odd.

$$\underbrace{F_R(-j\omega) = F_R(j\omega)} \quad \underbrace{F_I(-j\omega) = -F_I(j\omega)}$$

- *Magnitude spectrum* $|F(j\omega)|$ is even, and *phase spectrum* $\phi(\omega)$ is odd.

Example:

$$\mathcal{F}[f(t)] = F(j\omega) \qquad \mathcal{F}[f(t)\cos\omega_0 t] = ?$$

Sol)

$$f(t)\cos\omega_0 t = \frac{1}{2} f(t)(e^{j\omega_0 t} + e^{-j\omega_0 t})$$

$$\mathcal{F}[f(t)\cos\omega_0 t] = \frac{1}{2}\mathcal{F}[f(t)e^{j\omega_0 t}] + \frac{1}{2}\mathcal{F}[f(t)e^{-j\omega_0 t}]$$

$$= \frac{1}{2}F[j(\omega - \omega_0)] + \frac{1}{2}F[j(\omega + \omega_0)]$$

Continuous-Time Fourier Transform