MOODLE 2 : VECTOR SPACE

What is a Vector Space?

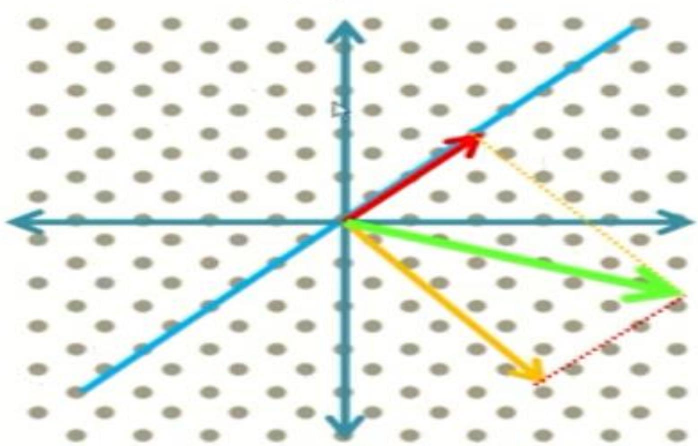
- A vector space is a set of "objects" on which two operations are defined:
  1. Addition
  2. Scalar Multiplication

- *These operations are subject to a set of rules*
- *Before we review these rules, let's look at some examples of vector spaces...*

- The set of real numbers → $\mathbb{R}$

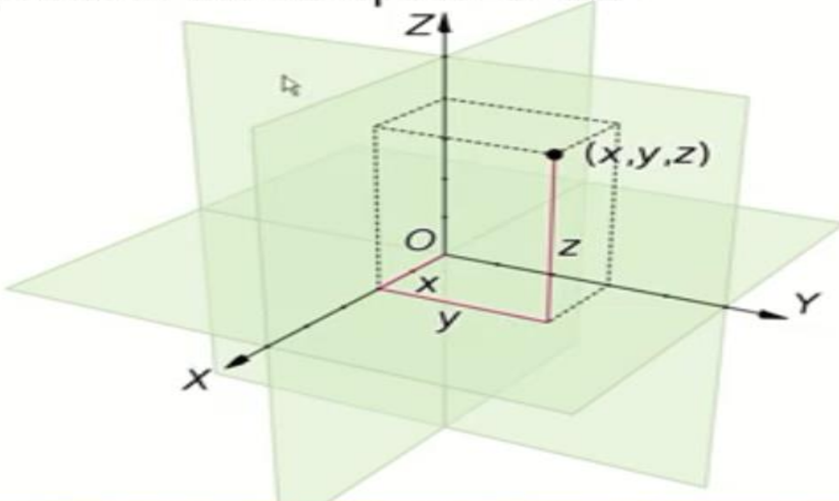
- How many elements are there in this set?
- What happens if you add two real numbers?
- What happens if you multiply a real number by a scalar?

- All the points on the x-y plane → $\mathbb{R}^2$



$$\begin{bmatrix} x \\ y \end{bmatrix}$$

- All the points in the 3D space → $\mathbb{R}^3$



$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

- A vector space V is a set of "**objects**" on which two operations are defined:
  1. **Addition**
     1. $(\mathbf{u}+\mathbf{v}) \in V$ for all $\mathbf{u}, \mathbf{v}$ in V → **Closed under addition**
     2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$ → **Commutative**
     3. $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$ → **Associative**
     4. $\mathbf{0} \in V$ where $\mathbf{0}+\mathbf{u}=\mathbf{u}$ → **Additive Identity**
     5. **For each u, there is** $-\mathbf{u} \in V$ such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$ → **Additive Inverse**
  2. **Scalar Multiplication**
     1. $c\mathbf{u} \in V$ for all $\mathbf{u} \in V, c \in \mathbb{R}$ → **Closed under scalar multiplication**
     2. $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$ → **Distributive**
     3. $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$ → **Distributive**
     4. $c(d\mathbf{u})=(cd)\mathbf{u}$ → **Associative**
     5. $1\mathbf{u}=\mathbf{u}$ → **Multiplicative Identity**

- So is any set of vectors a vector space? NO
- Take for example:

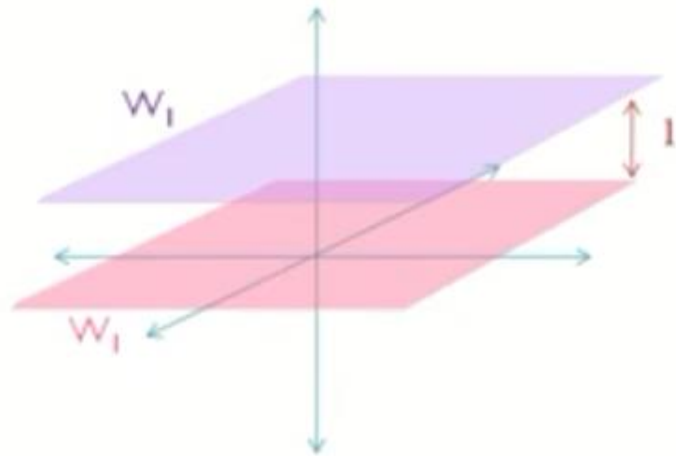
W_1 = Set of all vectors of the form $< x, y, 1 >$

This is NOT a vector space… Why?

W_2 = Set of all vectors of the form $< x, y, 0 >$

This IS a vector space… Why?

W_1 = Set of all vectors of the form $< x, y, 1 >$
W_2 = Set of all vectors of the form $< x, y, 0 >$



- There are some well-known vector spaces that satisfy the 10 conditions for addition and scalar multiplication

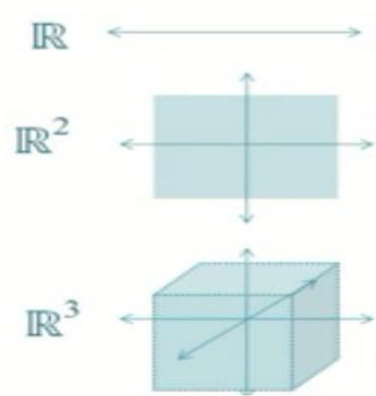
Known Vector Spaces

$\mathbb{R}^n$	$\mathbb{M}_{m\times n}$	P_n
Vectors	***Matrices***	***Polynomials***

- **The Vector Space $\mathbb{R}^n$**
- It is the set of all $n \times 1$ vectors.

$$\mathbf{x}_{n\times 1} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$x_i \in \mathbb{R}$$



The Vector Space $\mathbb{M}_{m \times n}$ "Matrix Space"

- It is the set of all matrices of size $m \times n$
- Also written as $\mathbb{R}^{m \times n}$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \qquad a_{ij} \in \mathbb{R}$$

The Vector Space $\mathrm{P_n}$ "Polynomial Space"

- It is the set of all polynomials of **degree at most n**

$$p(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$$

$$a_i \in \mathbb{R}$$

I. Addition

1. $(\mathbf{u}+\mathbf{v}) \in V$ for all $\mathbf{u}, \mathbf{v}$ in V → **Closed under addition**
2. $\mathbf{u}+\mathbf{v} = \mathbf{v}+\mathbf{u}$ → **Commutative**
3. $(\mathbf{u}+\mathbf{v})+\mathbf{w} = \mathbf{u}+(\mathbf{v}+\mathbf{w})$ → **Associative**
4. $\mathbf{0} \in V$ where $\mathbf{0}+\mathbf{u} = \mathbf{u}$ → **Additive Identity**
5. **For each u, there is** $-\mathbf{u} \in V$ such that $\mathbf{u}+(-\mathbf{u}) = \mathbf{0}$ → **Additive Inverse**

$$\boxed{\begin{aligned} p_1 &= a_0 + a_1 x + \ldots + a_n x^n \\ p_2 &= b_0 + b_1 x + \ldots + b_n x^n \end{aligned}}$$

$$p_1 + p_2 = (a_0+b_0) + (a_1+b_1)x + \ldots + (a_n+b_n)x^n \in P_n$$

$$p_1 + p_2 = p_2 + p_1$$

$$(p_1+p_2)+p_3 = p_1+(p_2+p_3)$$

$p(x) = 0$ is the "zero" of the space

p_1 and $-p_1$ are additive inverses and both belong to the space

2. Scalar Multiplication

1. $c\mathbf{u} \in V$ for all $\mathbf{u} \in V, c \in \mathbb{R}$ → **Closed under scalar multiplication**
2. $c(\mathbf{u}+\mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ → **Distributive**
3. $(c+d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$ → **Distributive**
4. $c(d\mathbf{u}) = (cd)\,\mathbf{u}$ → **Associative**
5. $1\mathbf{u} = \mathbf{u}$ → **Multiplicative Identity**

$$\boxed{\begin{aligned} p_1 &= a_0 + a_1x + \ldots + a_nx^n \\ p_2 &= b_0 + b_1x + \ldots + b_nx^n \end{aligned}}$$

$$cp_1 = ca_0 + ca_1x + \ldots + ca_nx^n \in P_n$$
$$c(p_1 + p_2) = cp_1 + cp_2$$
$$(cd)p_1 = c(dp_1)$$
$$(c+d)p_1 = cp_1 + dp_1$$

2. Scalar Multiplication

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$$cp_1 = ca_0 + ca_1x + \ldots + ca_nx^n \in P_n$$
$$c(p_1 + p_2) = cp_1 + cp_2$$
$$(cd)p_1 = c(dp_1)$$

$$(c+d)p_1 = cp_1 + dp_1 \qquad 1p_1 = p_1$$

- **Example:** Is the following set considered a vector space?
  - The set of polynomials of degree exactly 3

$$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

$$a_0, a_1, a_2 \in \mathbb{R} \qquad a_3 \neq 0$$

➔ **NOT a vector space**

Subspaces

- **Subspace**
  - *A subspace W is a subset of a vector space V (W⊂V) that satisfies the following conditions:*
    - **W is closed under addition**
    - **W is closed under scalar multiplication**

      ➔ This implies that the "zero" of the space V must also be in W

- *Basically, if we take a subset of a vector space, all the conditions naturally apply except for the two closure conditions*

1. **Addition**
   1. $(\mathbf{u}+\mathbf{v}) \in V$ for all $\mathbf{u}, \mathbf{v}$ in V → **Closed under addition**
   2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$ → **Commutative**
   3. $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$ → **Associative**
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2. **Scalar Multiplication**
   1. $c\mathbf{u} \in V$ for all $\mathbf{u} \in V$, $c \in \mathbb{R}$ → **Closed under scalar multiplication**
   2. $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$ → **Distributive**
   3. $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$ → **Distributive**
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   5. $1\mathbf{u}=\mathbf{u}$ → **Multiplicative Identity**

- *So, given a subset of a vector space, we just need to check the 2 conditions to verify that it is a subspace.*
- ***A subspace is also a vector space***
- **Example: Verify whether the following subsets of $\mathbb{R}^3$ are subspaces:**
  - $V_1 = \{ <x, y, x+y>; x, y \in \mathbb{R} \}$

$\mathbf{u}_1 \in \mathbf{V}_1 \qquad \mathbf{u}_2 \in \mathbf{V}_1$

$$\begin{bmatrix} x_1 \\ y_1 \\ x_1+y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \\ x_2+y_2 \end{bmatrix} = \begin{bmatrix} x_1+x_2 \\ y_1+y_2 \\ x_1+y_1+x_2+y_2 \end{bmatrix}$$

$$= \begin{bmatrix} x_1+x_2 \\ y_1+y_2 \\ (x_1+x_2)+(y_1+y_2) \end{bmatrix}$$

$(\mathbf{u}_1 + \mathbf{u}_2) \in \mathrm{V}$

Closed under addition

$\mathbf{u}_1 \in \mathbf{V}_1$

$$c\begin{bmatrix} x_1 \\ y_1 \\ x_1 + y_1 \end{bmatrix} = \begin{bmatrix} cx_1 \\ cy_1 \\ c(x_1 + y_1) \end{bmatrix} = \begin{bmatrix} cx_1 \\ cy_1 \\ cx_1 + cy_1 \end{bmatrix}$$

$c\mathbf{u}_1 \in \mathbf{V}_1$

Closed under scalar multiplication

- If we put $x = 0$ and $y = 0$, the third term becomes $x+y=0$ as well, so the zero vector is included in V_1 because it follows the general form $<x, y, x+y>$.

So V_1 is a subspace of $\mathbb{R}^3$ which also means it is a vector space ✓

- **Example: Verify whether the following subsets of $\mathbb{R}^3$ are subspaces:**
  - $\mathrm{V}_2 = \{ <x, y, 0>; x > y \}$

SOLUTION

$\mathbf{u}_1 \in V_1 \quad \mathbf{u}_2 \in V_1$

$$\begin{bmatrix} x_1 \\ y_1 \\ 0 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ 0 \end{bmatrix}$$

If $x_1 > y_1$ and $x_2 > y_2$
then $(x_1 + x_2) > (y_1 + y_2)$
➔ Closed under addition

$$c\begin{bmatrix} x_1 \\ y_1 \\ 0 \end{bmatrix} = \begin{bmatrix} cx_1 \\ cy_1 \\ 0 \end{bmatrix}$$

We know that $x_1 > y_1$
But that doesn't mean that
$cx_1 > cy_1$ (if c is negative or 0)
➔ NOT closed under scalar multiplication

- V_3 = The points on the plane $x + y - z = 0$

SOLUTION

This is a homogeneous equation → a system of 1 equation and 3 unknowns
➔Infinite number of solutions ➔ "The points on the plane" are the solutions
➔ Free variables y & z

$$x + y - z = 0$$
$$x = -y + z$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -y + z \\ y \\ z \end{bmatrix} = y\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$V_3 = \left\{ \mathbf{v} = y\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} ; \; y, z \in \mathbb{R} \right\}$$

Now let's check the conditions ➔

$$V_3 = \left\{ \mathbf{v} = y \underbrace{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}}_{\mathbf{w}_1} + z \underbrace{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}_{\mathbf{w}_2} ;\ y, z \in \mathbb{R} \right\}$$

$\mathbf{u}_1 = a_1\mathbf{w}_1 + a_2\mathbf{w}_2 \qquad \mathbf{u}_2 = b_1\mathbf{w}_1 + b_2\mathbf{w}_2$

$\Rightarrow \mathbf{u}_1 + \mathbf{u}_2 = a_1\mathbf{w}_1 + a_2\mathbf{w}_2 + b_1\mathbf{w}_1 + b_2\mathbf{w}_2 = (a_1+b_1)\,\mathbf{w}_1 + (a_2+b_2)\,\mathbf{w}_2$

$\Rightarrow c\mathbf{u}_1 = ca_1\mathbf{w}_1 + ca_2\mathbf{w}_2$

Subspace of $\mathbb{R}^3$

- **Example:** Prove that the solution set of any homogeneous system $\mathbf{Ax} = \mathbf{0}$ for $\mathbf{A}_{m\times n}$ is a subspace of $\mathbb{R}^n$:

SOLUTION

- If $\mathbf{x}_1$ and $\mathbf{x}_2$ are solutions of the homogeneous system $\mathbf{Ax}=\mathbf{0}$
  - $\mathbf{Ax}_1 = \mathbf{0}$ and $\mathbf{Ax}_2 = \mathbf{0}$
- Is $(\mathbf{x}_1+\mathbf{x}_2)$ also a solution?
  - $\mathbf{A}(\mathbf{x}_1+\mathbf{x}_2) = \mathbf{Ax}_1 + \mathbf{Ax}_2 = \mathbf{0} + \mathbf{0} = \mathbf{0} \rightarrow$ closed under addition
- Is $c\mathbf{x}_1$ also a solution?
  - $\mathbf{A}(c\mathbf{x}_1) = c(\mathbf{Ax}_1) = c(\mathbf{0}) = \mathbf{0} \rightarrow$ closed under scalar multiplication

- If the system has a trivial solution $\rightarrow$ the zero vector is the only vector space with only ONE vector.

The solution set of any homogeneous system of linear equations Ax = 0 forms a subspace

Review Questions

- Which of these statements is true?
  - The vector space $\mathbb{R}^2$ is a subspace of $\mathbb{R}^3$
  - A straight line passing through the origin in $\mathbb{R}^3$ is a vector space
  - If the non-homogeneous system Ax=b has an infinite number of solutions, then its solution set forms a subspace

LINEAR COMBINATIONS AND LINEAR INDEPENDENCE

Linear Combinations

- **Linear Combination**

  Let's say we have four vectors $\mathbf{x}$, $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$, where

$$\boxed{\mathbf{x} = 2\mathbf{u} + 3\mathbf{v} - \mathbf{w}}$$

- We say that $\mathbf{x}$ is a linear combination of $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$
- Similarly, any of the four vectors in this example is a linear combination of the other three (e.g. $\mathbf{u}$ is a linear combination of $\mathbf{x}$, $\mathbf{v}$ and $\mathbf{w}$).

- ***Linear combination*** ➔ A combination made using ***addition and/or scalar multiplication***
- If the vectors $\mathbf{w}, \mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n} \in$ a vector space V, and

$$\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n$$

  where $c_1, c_2, \ldots\ c_n \in \mathbb{R}$, then we say that $\mathbf{w}$ is a *linear combination* of $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}$

- We can also say that $\mathbf{v_1}$ is a linear combination of $\mathbf{w}, \mathbf{v_2}, \ldots, \mathbf{v_n}$

Remember the vector equation?

$$a_{11}x + a_{12}y + a_{13}z = b_1$$
$$a_{21}x + a_{22}y + a_{23}z = b_2$$
$$a_{31}x + a_{32}y + a_{33}z = b_3$$

$$x\begin{bmatrix}a_{11}\\a_{21}\\a_{31}\end{bmatrix} + y\begin{bmatrix}a_{12}\\a_{22}\\a_{32}\end{bmatrix} + z\begin{bmatrix}a_{13}\\a_{23}\\a_{33}\end{bmatrix} = \begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$$

Vector Equation

$$x\mathbf{a}_1 + y\mathbf{a}_2 + z\mathbf{a}_3 = \mathbf{b}$$

- ***We say that if the system has a solution then* $\mathbf{b}$ *is a linear combination of the columns of* $\mathbf{A}$**

- **Example 1**: Determine whether $\mathbf{x} = <8, 7, -2>$ is a linear combination of the vectors in $\mathbf{S} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$

SOLUTION

$$\mathbf{v}_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix}-1\\2\\1\end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix}3\\4\\-1\end{bmatrix}$$

$$\boxed{\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3}$$

If there are values for c_1, c_2 and c_3 that satisfy this equation, then $\mathbf{x}$ can be written as a linear combination of the vectors in S

If not, then $\mathbf{x}$ is not a linear combination of the vectors in S

$$\begin{bmatrix} 8 \\ 7 \\ -2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 3 \\ 4 \\ -1 \end{bmatrix} \Rightarrow \begin{aligned} c_1 - c_2 + 3c_2 &= 8 \\ c_1 + 2c_2 + 4c_3 &= 7 \\ c_1 + c_2 - c_3 &= -2 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 3 & 8 \\ 1 & 2 & 4 & 7 \\ 1 & 1 & -1 & -2 \end{array}\right] \Leftarrow \begin{aligned} c_1 - c_2 + 3c_2 &= 8 \\ c_1 + 2c_2 + 4c_3 &= 7 \\ c_1 + c_2 - c_3 &= -2 \end{aligned}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 3 & 8 \\ 0 & 1 & -2 & -5 \\ 0 & 3 & 1 & -1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 3 & 8 \\ 0 & 1 & -2 & -5 \\ 0 & 0 & 7 & 14 \end{array}\right]$$

The system has a solution
→ x is a linear combination of the vectors in S

The solution is unique
→ There is only ONE possible combination

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 3 & 8 \\ 0 & 1 & -2 & -5 \\ 0 & 0 & 7 & 14 \end{array}\right] \Rightarrow \begin{aligned} c_1 - c_2 + 3c_2 = 8 &\rightarrow c_1 = 1 \\ c_2 - 2c_3 = -5 &\rightarrow c_2 = -1 \\ 7c_3 = 14 &\rightarrow c_3 = 2 \end{aligned}$$

$$\boxed{\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3}$$

$$\Downarrow$$

$$\boxed{\mathbf{x} = \mathbf{v}_1 - \mathbf{v}_2 + 2\mathbf{v}_3}$$

$$c_1 = 1$$
$$c_2 = -1$$
$$c_3 = 2$$

- **Example 2:** Given the vectors

$$\mathbf{x} = <4, 3, 4> \text{ and } \mathbf{y} = <1, 2, 3>$$

Which of them is a linear combination of the vectors in the following set?

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}$$

SOLUTION

$$\left[\begin{array}{ccc|c|c} 1 & 1 & 1 & 4 & 1 \\ 0 & 1 & 2 & 3 & 2 \\ 1 & 1 & 1 & 4 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c|c} 1 & 1 & 1 & 4 & 1 \\ 0 & 1 & 2 & 3 & 2 \\ 0 & 0 & 0 & 0 & 2 \end{array}\right]$$

$$\begin{array}{c} \begin{array}{cccc} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{x} \end{array} \\ \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \qquad \begin{array}{c} \begin{array}{cccc} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{y} \end{array} \\ \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 2 \end{array}\right] \end{array}$$

$\mathbf{x}$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ $\qquad$ $\mathbf{y}$ is <u>NOT</u> a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \boxed{2\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 4 \end{bmatrix}}$$

$c_3 \rightarrow$ free variable

$c_2 + 2c_3 = 3 \rightarrow c_2 = 3 - 2c_3$

$c_1 + c_2 + c_3 = 4 \rightarrow c_1 = 4 - c_2 - c_3 = 1 + c_3$

For example: $c_3 = 1$, then $c_2 = 1$ and $c_1 = 2$

- **Linear Independence:**
  - If we have a set of vectors:

$$S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$$

$$\boxed{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n = \mathbf{0}}$$

Trivial "zero" solution
$c_1 = c_2 = \ldots = c_n = 0$
➔ **Independent Set**

Infinite number of solutions $(c_1, c_2, \ldots\ c_n)$
➔ **Dependent Set**

- For example, the following three vectors are independent; it is impossible to form any linear combinations using these three vectors:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$$

Example: Is the set $\mathbf{S} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ linearly independent?

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 3 \\ 4 \\ -1 \end{bmatrix}$$

SOLUTION

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 3 & 0 \\ 1 & 2 & 4 & 0 \\ 1 & 1 & -1 & 0 \end{array}\right] \sim \cdots \sim \left[\begin{array}{ccc|c} 1 & -1 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 3 & 0 \\ 1 & 2 & 4 & 0 \\ 1 & 1 & -1 & 0 \end{array}\right] \sim \cdots \sim \left[\begin{array}{ccc|c} 1 & -1 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$c_1 = 0 \quad c_2 = 0 \quad c_3 = 0$$

- **Example:** Is this set linearly independent?

$$\{<1,0,0>,<0,1,1>\}$$

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad c_1 = c_2 = 0$$

linearly <u>independent set</u>

NOTE!

If we have only TWO vectors just check if they are scalar multiples, if they are not, then they are independent.

Important note no.1:

- If you want to check whether a set of only TWO vectors is an independent set, then:
  - If $\mathbf{v}_1 \neq c\mathbf{v}_2$ → Independent
  - If $\mathbf{v}_1 = c\mathbf{v}_2$ → Dependent

- **Example:** Is this set linearly independent?

$$\{<1,2,2>,<2,4,5>\}$$

$$\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \neq c \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}$$

→ **independent**

- **Example:** Is this set linearly independent?

$$\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = 0.5 \begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix}$$

➔ **Dependent**

- **Example:** Which of the following sets are independent?

$$S = \{< 1, 0, 1 >, < -1, 0, 1 >, < -1, 0, -1 >\}$$

$$S = \{< 1, 0, 1 >, < -1, 0, 1 >, < -1, 0, -1 >\}$$

Scalar multiples

➔ **Dependent**

- ***Important note no.2:***
  - If the number of vectors is greater than their dimension, then they form a dependent set.

Example: 4 vectors in $\mathbb{R}^3$

$$S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \\ -2 \end{bmatrix} \right\}$$

S ➔ ***Dependent set***

- **Example:** Find the value of k that makes the following set linearly independent:

$$S = \left\{ \begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} -2 & 0 \\ -1 & k \end{bmatrix} \right\}$$

$$c_1 \begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix} + c_2 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + c_3 \begin{bmatrix} -2 & 0 \\ -1 & k \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{aligned} (1,1): &\quad c_1 + c_2 - 2c_3 = 0 \\ (1,2): &\quad -c_1 + c_2 = 0 \\ (2,1): &\quad c_2 - c_3 = 0 \\ (2,2): &\quad 3c_1 + c_2 + kc_3 = 0 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 3 & 1 & k & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -2 & k+6 & 0 \end{array}\right] \quad \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & k+4 & 0 \end{array}\right]$$

The system has a unique solution (only the zero solution) if:

$$k + 4 \neq 0$$

Otherwise the system has an infinite number of solutions

For S to be an independent set:

$$k \neq -4$$

If we take $k=-4$ then the set becomes dependent

- **Example:** Which of the following sets are independent?

$$S = \{t, t^2+1, 2t-t^2-1\}$$

$$c_1(t)+c_2(t^2+1)+c_3(2t-t^2-1)=0$$

Coefficients of t^2: $c_2 - c_3 = 0$

Coefficients of t: $c_1 + 2c_3 = 0$

Coefficients of t^0: $c_2 - c_3 = 0$

$$\left[\begin{array}{ccc|c} 0 & 1 & -1 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 0 & 1 & -1 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

→ Dependent

$$2t-t^2-1 = 2(t)-(t^2+1)$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

c_3 → Free variable $\quad c_3 = 1$

$c_2 = c_3 \quad c_2 = 1$

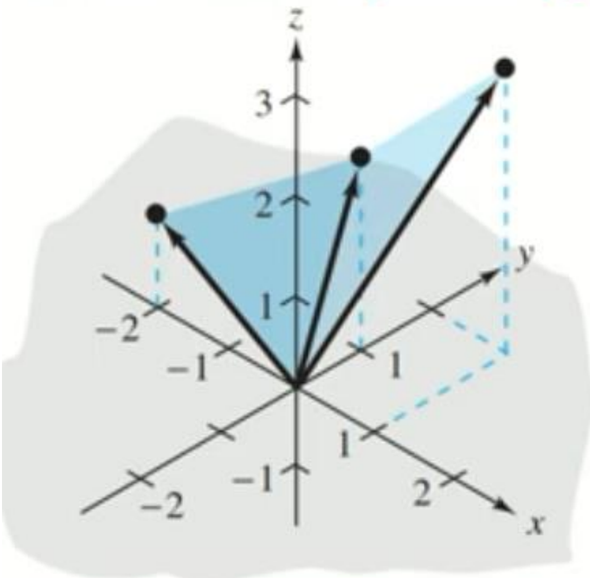
$c_1 = -2c_3 \quad c_1 = -2$

$$\boxed{-2(t)+(t^2+1)+(2t-t^2-1)=0}$$

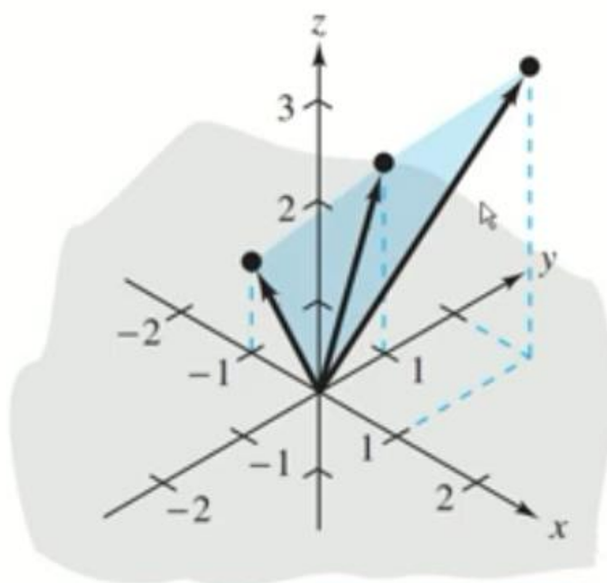
SPANNING SET AND BASIS OF A VECTOR SPACE

Spanning Sets and Bases

▸ What is a spanning set?



Cover ALL of $\mathbb{R}^3$



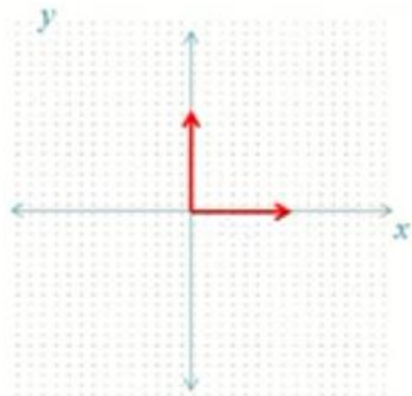
Cover a plane in $\mathbb{R}^3$

If the set S is a spanning set of the vector space V then any vector in V can be written as a ***linear combination*** of the vectors in S

Example:

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

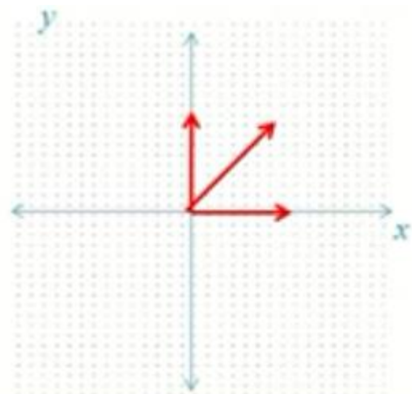
$$\begin{bmatrix} x \\ y \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$



$\mathbb{R}^2$

Example:

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$



$\mathbb{R}^2$

- $V = \text{span}\{v_1, v_2, \ldots v_n\}$
- $S = \{v_1, v_2, \ldots v_n\}$ is a spanning set for V
- V is spanned by $S = \{v_1, v_2, \ldots v_n\}$

- **What is a basis?**

Basis = Spanning + Independent

- A basis is the ***smallest possible spanning set*** (with the least number of vectors to span the space).
- The ***number of vectors*** in a basis of a vector space V is the **dimension** of the vector space ➔ dim V

- Any basis is also a spanning set
- A spanning set is a basis if it is also an independent set

- **Example:**
  - Is this set a basis or a spanning set of $\mathbb{R}^3$?

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Spanning Set ✓   **Basis** ✓

- Is this set a basis or a spanning set of $\mathbb{R}^3$?

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Spanning Set ✓   **Basis** ✗

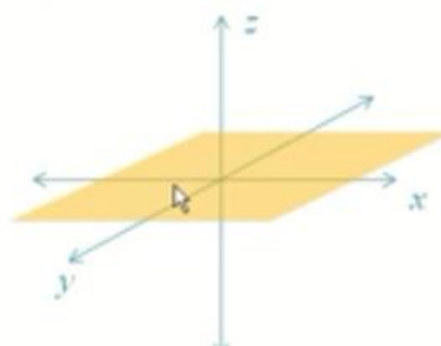
- Is this set a basis or a spanning set of $\mathbb{R}^3$?

$$S = \left\{ \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \right\}$$

Spanning Set ✓ **Basis** ✓

- Is this set a basis or a spanning set of $\mathbb{R}^3$?

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$



Spanning Set ✗ **Basis** ✗

- Is this set a basis or a spanning set of $\mathbb{R}^3$?

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Spanning Set	**Basis**
✗	✗

- ***Theorem:***

If S is an independent set, and the number of vectors in S is equal to dimV, then S is a basis for V

* Given that the vectors in $S \in V$

- Is this set a basis or a spanning set of $\mathbb{R}^3$?

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \right\}$$

- 3 vectors in $\mathbb{R}^3$
- independent

Spanning Set	**Basis**
✓	✓

- *For any vector space:*
  - There is ***more than one spanning set***
  - The number of vectors in a spanning set can be $\geq \dim V$
  - There is ***more than one basis***
  - The ***number of vectors in a basis*** is the SAME for all bases of V, and is equal to dim V
- **Dimension of a Vector Space:**

dim V = the number of vectors in a basis of V

- ***Dimensions of known vector spaces:***
  - $\dim \mathbb{R}^n = n$
  - $\dim \mathbb{M}_{n\times m} = nm$
  - $\dim \mathbb{P}_n = n+1$

- **Dimension of a Vector Space:**

$$\dim \mathbb{M}_{n\times m} = nm$$

Basis of $\mathbb{M}_{2\times 2}$ (dim =4)

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

- **Dimension of a Vector Space:**

$$\dim \mathbb{P}_n = n + 1$$

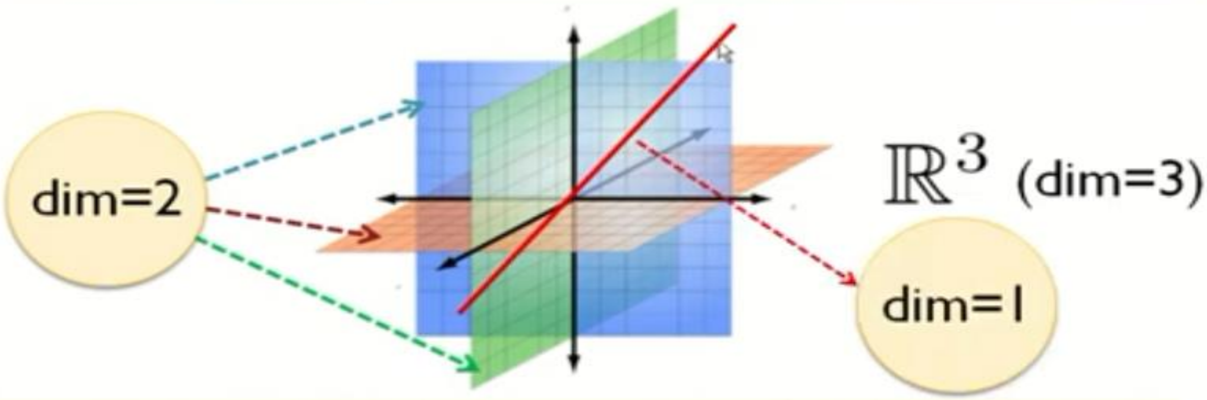
- $\mathbb{P}_2$ ➔ Polynomials of degree ≤ 2

$$a_2x^2 + a_1x + a_0$$

$$\text{Basis} = \{1, x, x^2\}$$

Theorem:

If W is a subspace of V, then $\dim W \le \dim V$



Example 1: Find a suitable basis for the following subspace of $\mathbb{R}^4$

$$W = \left\{ \begin{bmatrix} x \\ y \\ x+y \\ x-y \end{bmatrix}, x, y \in \mathbb{R} \right\}$$

$$\begin{bmatrix} x \\ y \\ x+y \\ x-y \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix} \right\}$$

- Spanning set ✓
- Independent ✓
- Basis ✓

$$\dim W = 2$$

- **Example 2: Find a suitable basis for the following subspace of $\mathbb{R}^4$**

$$W = \left\{ \begin{bmatrix} a-b+c \\ a-b-c \\ a-b \\ c \end{bmatrix}, a, b, c \in \mathbb{R} \right\}$$

$$\begin{bmatrix} a-b+c \\ a-b-c \\ a-b \\ c \end{bmatrix} = a \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} -1 \\ -1 \\ -1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

$$S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \cancel{\begin{bmatrix} -1 \\ -1 \\ -1 \\ 0 \end{bmatrix}}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

- Spanning set ✓
- Independent ?

$$\dim W = 2$$

- **Example 3: Find a basis and the dimension of the solution space of $AX = 0$ where**

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ -3 & 0 & 6 & -1 \\ 3 & 4 & -2 & 1 \\ 2 & 0 & -4 & -2 \end{bmatrix} \sim \ldots \sim \begin{array}{c} \begin{array}{cccc} x & y & z & w \end{array} \\ \begin{bmatrix} \boxed{1} & 3 & 1 & 3 \\ 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$\begin{array}{c} \begin{array}{cccc} x & y & z & w \end{array} \\ \left[\begin{array}{cccc|c} \boxed{1} & 3 & 1 & 3 & 0 \\ 0 & \boxed{1} & 1 & 0 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

$$x = -3(-z) - z - 3(0) = 2z$$
$$y = -z$$
$$w = 0$$

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 2z \\ -z \\ z \\ 0 \end{bmatrix} = z \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

- The solutions of this homogeneous system are multiples of the vector <2, -1, 1, 0>
- The spanning set of the solution space is <2, -1, 1, 0>
- It is also the basis
- dim = 1

- **Example 4: Find a basis and the dimension of the solution space of**

$$x - y + z - w = 0$$

$$\left[\boxed{1} \quad -1 \quad 1 \quad -1 \mid 0\right] \qquad x = y - z + w$$

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} y - z + w \\ y \\ z \\ w \end{bmatrix} = y \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + z \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + w \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\boxed{\dim = 3}$$

INDEPENDENT ➔ Basis

Spanning Set of the Solution Space

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} y - z + w \\ y \\ z \\ w \end{bmatrix} = y \begin{bmatrix} 1 \\ \boxed{1} \\ 0 \\ 0 \end{bmatrix} + z \begin{bmatrix} -1 \\ 0 \\ \boxed{1} \\ 0 \end{bmatrix} + w \begin{bmatrix} 1 \\ 0 \\ 0 \\ \boxed{1} \end{bmatrix}$$

- **Example 5: Find** $\dim W$ **if** $W = \operatorname{span} S$ **and**

$$S = \{1 + t,\ 2 - t^2, t^2 - 2t - 4\}$$

- S is the spanning set of W, so if it is independent, it is also a basis
- If not, then the independent polynomials in S form a basis set
- So: CHECK INDEPENDENCE

$$k_1(1 + t) + k_2(2 - t^2) + k_3(t^2 - 2t - 4) = 0$$

Equations:

$$\begin{aligned} t^2&: \quad -k_2 + k_3 = 0 \\ t&: \ k_1 \quad - 2k_3 = 0 \\ t^0&: \ k_1 + 2k_2 - 4k_3 = 0 \end{aligned} \qquad \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -2 \\ 1 & 2 & -4 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -2 \\ 1 & 2 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & -1 & 1 \\ 1 & 2 & -4 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 0 & -2 \\ 0 & \boxed{-1} & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\dim W = 2$$