SOLVING A SYSTEM OF LINEAR EQUATIONS

- What is a system of linear equations?

$$\begin{aligned} x + y + z &= 3 \\ x - y + 2z &= 2 \\ 3x - 2y + z &= 0 \end{aligned}$$

- Solution: $(x, y, z) = (1, 1, 1)$

$$\begin{aligned} &a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ &a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\ &\vdots \\ &a_{m1}x_1 + a_{n2}x_2 + \ldots + a_{mn}x_n = b_m \end{aligned}$$

$$\begin{array}{ccc} \mathbf{A} & \mathbf{x} & = \quad \mathbf{b} \end{array}$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$\begin{array}{ccc} \mathbf{A} & \mathbf{x} & = & \mathbf{b} \\ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} & \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} & = & \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \end{array}$$

Coefficient Matrix

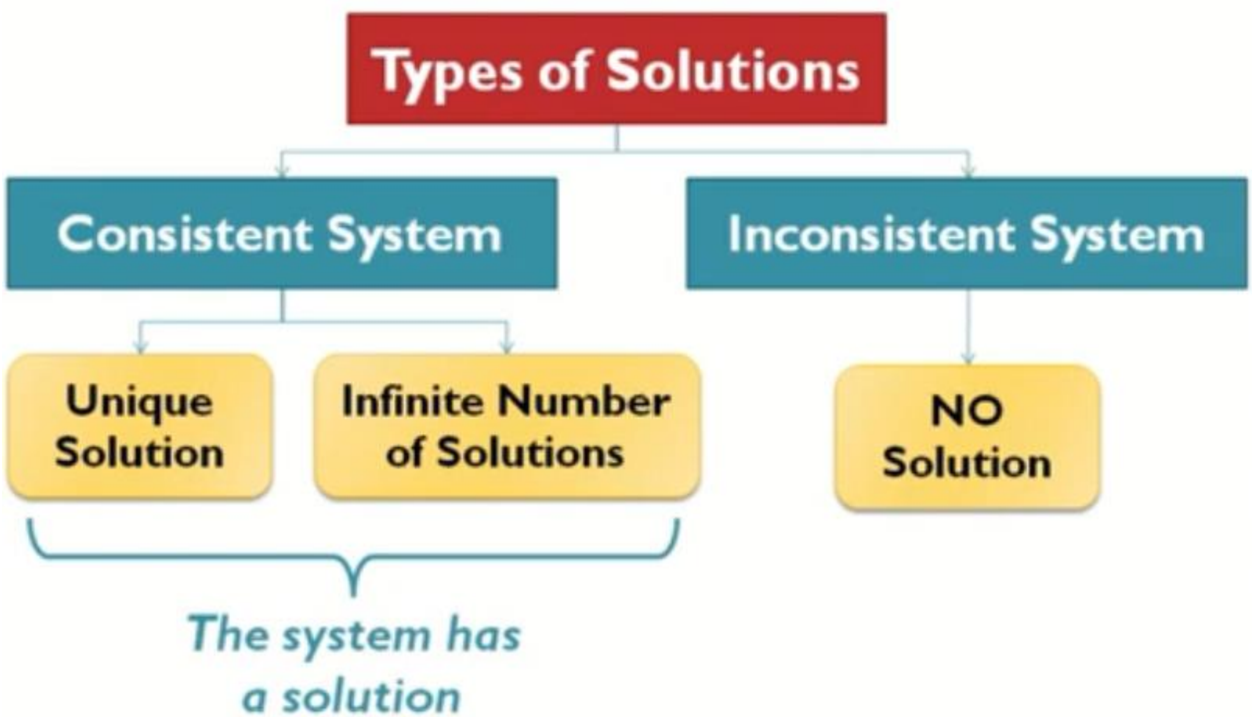
$m \times n$

Vector of Unknowns

$n \times 1$

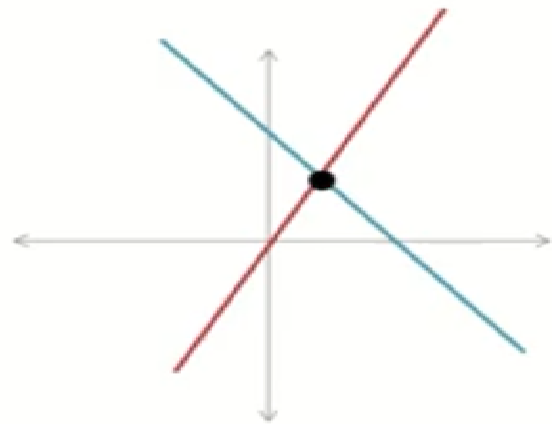
Vector of Free Terms

$m \times 1$



- **Unique Solution:**

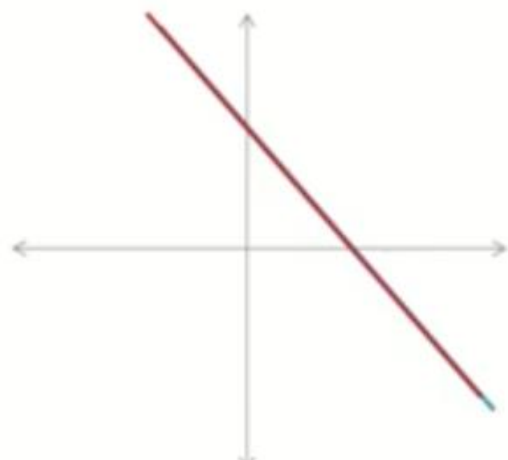
$$x + y = 2$$
$$x - y = 0$$



$x = 1$ and $y = 1$

- **Infinite Number of Solutions:**

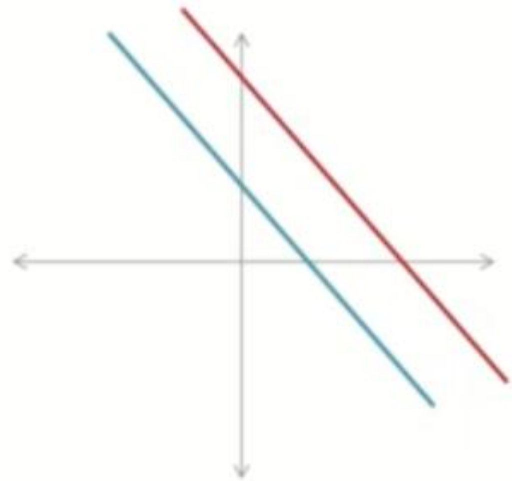
$$x + y = 2$$
$$2x + 2y = 4$$



$(1, 1)$, $(2, 0)$, $(-1, 3)$, $(0.5, 1.5)$, …

NO Solution (Inconsistent System):

$$x + y = 2$$
$$x + y = 1$$



Impossible!

The Augmented Matrix

$$\mathbf{Ax} = \mathbf{b}$$

$$\boxed{\text{aug}\mathbf{A} = [\mathbf{A}|\mathbf{b}]}$$

Elementary Row Operations:

1. Interchange two rows
2. Multiply a row by a scalar
3. Add two rows (multiply by a scalar and add to another)

Gauss Elimination

- In the Gauss elimination method, we use **elementary row operations** to reduce the system to a simpler form

- STEPS:
  1. Forward elimination
  2. Backward substitution

- Simple Example:

Forward Elimination

$$x + y = 2$$
$$x - y = 0$$

$$\begin{array}{c} \begin{array}{ccc} -1 & -1 & -2 \end{array} \\ \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & -2 & -2 \end{array}\right] \end{array}$$

$$\boxed{\begin{aligned} x + y &= 2 \\ -2y &= -2 \end{aligned}}$$

Backward Substitution

$$y = 1$$
$$x = 2 - y = 1$$

- Example: 1:

$$
\begin{aligned}
x - 2y + z &= 0 \\
2y - 8z &= 8 \\
-4x + 5y + 9z &= -9
\end{aligned}
$$

$$\mathbf{Ax = B}$$

Augmented Matrix

$$
\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ -4 & 5 & 9 \end{bmatrix}
\begin{bmatrix} x \\ y \\ z \end{bmatrix}
=
\begin{bmatrix} 0 \\ 8 \\ -9 \end{bmatrix}
\Rightarrow
\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right]
$$

pivot

$$
\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right]
$$

pivot

$$
\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right]
\quad \begin{array}{c} row1 \times 4 \\ \text{and add} \end{array} \quad
\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right]
$$

$$
\begin{array}{c} (row1) \times 4 \\ \text{and add} \end{array}
\sim
\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{array}\right]
$$

Equivalent System

$$
\sim
\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{array}\right]
\div 2 \Rightarrow \sim
\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{array}\right]
$$

$$
\begin{array}{c} (row2) \times 3 \\ \text{and add} \end{array}
\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]
$$

End of Forward Elimination

$$
\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]
\begin{array}{l}
\Rightarrow x - 2y + z = 0 \;\rightarrow\; x = 29 \\
\Rightarrow y - 4z = 4 \;\rightarrow\; y = 16 \\
\Rightarrow z = 3
\end{array}
$$

Backward Substitution

The Echelon Form:

1. Leading entries move to the right
2. Elements below leading entries = 0
3. Leading entries = 1
4. Zero rows at the bottom

$$\begin{bmatrix} \boxed{1} & -2 & 1 & 0 \\ 0 & \boxed{1} & -4 & 4 \\ 0 & 0 & \boxed{1} & 3 \end{bmatrix}$$

Example 2:

$$\begin{aligned} x-2y+z&=0 \\ -4x+5y+9z&=-9 \\ 5x-7y-8z&=9 \\ 2y-8z&=8 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ -4 & 5 & 9 & -9 \\ 5 & -7 & -8 & 9 \\ 0 & 2 & -8 & 8 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ -4 & 5 & 9 & -9 \\ 5 & -7 & -8 & 9 \\ 0 & 2 & -8 & 8 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & -3 & 13 & -9 \\ 0 & 3 & -13 & 9 \\ 0 & 2 & -8 & 8 \end{array}\right]$$

$(row1)\times 4 \;+\; (row2)$

$(row1)\times -5 \;+\; (row3)$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & -3 & 13 & -9 \\ 0 & 3 & -13 & 9 \\ 0 & 2 & -8 & 8 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & -3 & 13 & -9 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & -8 & 8 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & -3 & 13 & -9 \\ 0 & 2 & -8 & 8 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & -3 & 13 & -9 \\ 0 & 2 & -8 & 8 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Just to make calculations easier

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\boxed{z=3}$ $\boxed{y=16}$ $\boxed{x=29}$

- **Example 3:**

$$x - 2y + z = 0$$
$$-4x + 5y + 9z = -9$$
$$5x - 7y - 8z = 9$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ -4 & 5 & 9 & -9 \\ 5 & -7 & -8 & 9 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ -4 & 5 & 9 & -9 \\ 5 & -7 & -8 & 9 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & -3 & 13 & -9 \\ 0 & 3 & -13 & 9 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & -3 & 13 & -9 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Infinite Number of Solutions

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & -3 & 13 & -9 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\boxed{\begin{aligned} -3y+13z&=-9 \\ -3y&=-9-13t \\ y&=3+\frac{13}{3}t \end{aligned}}$$

$\boxed{z=t}$

FREE VARIABLE

$$\boxed{\begin{aligned} x-2y+z&=0 \\ x&=2(3+\frac{13}{3}t)-t \\ x&=6+\frac{23}{3}t \end{aligned}}$$

- Some possible solutions:

$$z=t \qquad x=6+\frac{23}{3}t \qquad y=3+\frac{13}{3}t \quad \xrightarrow[t=6]{\text{For example}} \quad z=6 \qquad x=6+\frac{23}{3}(6)=52 \qquad y=3+\frac{13}{3}(6)=29$$

$$\xrightarrow{t=0} \quad z=0 \qquad x=6 \qquad y=3$$

- ***How to write the solution in vector form:***

$$z=t \qquad x=6+\frac{23}{3}t \qquad y=3+\frac{13}{3}t$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t\begin{bmatrix} 23/3 \\ 13/3 \\ 1 \end{bmatrix} + \begin{bmatrix} 6 \\ 3 \\ 0 \end{bmatrix} \quad \text{for } t \in \mathbb{R}$$

Example 4:

$$x-2y+z=0$$
$$-4x+5y+9z=-9$$
$$5x-7y-8z=1$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ -4 & 5 & 9 & -9 \\ 5 & -7 & -8 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ -4 & 5 & 9 & -9 \\ 5 & -7 & -8 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & -3 & 13 & -9 \\ 0 & 3 & -13 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & -3 & 13 & -9 \\ 0 & 0 & 0 & -8 \end{array}\right]$$

NO SOLUTION
(Inconsistent System)

$0=-8$???

- The different types of solutions:

$$\begin{array}{c} \begin{array}{cccc} x & y & z & \end{array} \\ \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right] \end{array} \qquad \begin{array}{c} \begin{array}{cccc} x & y & z & \end{array} \\ \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & -3 & 13 & -9 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \qquad \begin{array}{c} \begin{array}{cccc} x & y & z & \end{array} \\ \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & -3 & 13 & -9 \\ 0 & 0 & 0 & \boxed{-8} \end{array}\right] \end{array}$$

Unique Solution — Unknowns = Pivots

Infinite Number of Solutions — Unknowns > Pivots

NO Solution — Pivot in Last col. of augA

Rank and Types of Solutions

- **The Rank of a Matrix**
  - Definition:

$$\rho(\mathbf{A})$$

- What you probably know already ➔ The size of the largest non-zero sub-determinant
- What's new ➔ The number of pivots in the Echelon form

> We want to classify the types of solutions based on the relation between $\rho(\mathbf{A})$ and $\rho(\mathrm{aug}\mathbf{A})$

Unique Solution	Infinite Number of Solutions	NO Solution
$\begin{array}{cccc} x & y & z & \\ \end{array}$ $\left[\begin{array}{ccc\|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$	$\begin{array}{cccc} x & y & z & \\ \end{array}$ $\left[\begin{array}{ccc\|c} 1 & -2 & 1 & 0 \\ 0 & -3 & 13 & -9 \\ 0 & 0 & 0 & 0 \end{array}\right]$	$\begin{array}{cccc} x & y & z & \\ \end{array}$ $\left[\begin{array}{ccc\|c} 1 & -2 & 1 & 0 \\ 0 & -3 & 13 & -9 \\ 0 & 0 & 0 & \boxed{-8} \end{array}\right]$
$\rho(\mathbf{A}) = n$	$\rho(\mathbf{A}) < n$	$\rho(\text{aug}\mathbf{A}) \neq \rho(\mathbf{A})$

$\rho(\text{aug}\mathbf{A}) = \rho(\mathbf{A})$

Types of Solutions

Consistent System

$\rho(\text{aug}\mathbf{A}) = \rho(\mathbf{A})$

- **Unique Solution**: $\rho(\mathbf{A}) = n$
- **Infinite Number of Solutions**: $\rho(\mathbf{A}) < n$

Inconsistent System

$\rho(\text{aug}\mathbf{A}) \neq \rho(\mathbf{A})$

- **NO Solution**

n ➔ number of unknowns

Review Questions

- **Which of these statements are true?**
  - A system of four equations and three unknowns can never have a unique solution
  - If the augmented matrix of a system of linear equations has a zero row at the bottom then it has an infinite number of solutions
  - A system of linear equations has no solution if the rank of the augmented matrix is less than that of the coefficient matrix

Gauss-Jordan Elimination, Matrix Inversion

Gauss-Jordan Elimination

Gauss Elimination

1. Forward Elimination
2. Backward Substitution

Echelon Form
(EF or Row-EF "REF")

Gauss-Jordan Elimination

1. Forward Elimination
2. Backward Elimination

Reduced Row-Echelon Form
(RREF)

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

- Remember this?

$$
\begin{aligned}
x - 2y + z &= 0 \\
2y - 8z &= 8 \\
-4x + 5y + 9z &= -9
\end{aligned}
$$

Echelon form →

$$
\left[\begin{array}{ccc|c}
1 & -2 & 1 & 0 \\
0 & 1 & -4 & 4 \\
0 & 0 & 1 & 3
\end{array}\right]
$$

Backward Elimination ↑

$$
\left[\begin{array}{ccc|c}
1 & -2 & 1 & 0 \\
0 & 1 & -4 & 4 \\
0 & 0 & 1 & 3
\end{array}\right]
\quad
\begin{array}{l}
4R_3+R_2 \rightarrow R_2 \\
\text{and} \\
-R_3+R_1 \rightarrow R_1
\end{array}
\quad
\left[\begin{array}{ccc|c}
1 & -2 & 0 & -3 \\
0 & 1 & 0 & 16 \\
0 & 0 & 1 & 3
\end{array}\right]
\quad 2R_2 + R_1 \rightarrow R_1
$$

$$
\left[\begin{array}{ccc|c}
1 & 0 & 0 & 29 \\
0 & 1 & 0 & 16 \\
0 & 0 & 1 & 3
\end{array}\right]
\quad
\begin{array}{l}
\boxed{x = 29} \\
\boxed{y = 16} \\
\boxed{z = 3}
\end{array}
$$

- # Reduced Row Echelon Form
- The reduced row-echelon form (RREF) is the same as the echelon form but with zeros below **AND ABOVE** the leading entries.

1. Leading entries move to the right
2. Elements **above and below** leading entries = 0
3. Leading entries = 1
4. Zero rows at the bottom

RREF = EF + elements above leading entries = 0

- **Which of these is in the RREF?**

$$\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{bmatrix} \checkmark \qquad \begin{bmatrix} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \checkmark$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \boxed{2} & 5 \\ 0 & 0 & 1 & -1 & 0 & 6 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \times$$

- **NOTE: the RREF is "unique"** → لا تعني و لا علاقة لها بـ Unique solution
  - **A system can have more than one EF but it has only one RREF**

$$x+y=2$$
$$x-y=0$$

$$\begin{bmatrix}1 & 1 & | & 2\\ 1 & -1 & | & 0\end{bmatrix} \sim \begin{bmatrix}1 & 1 & | & 2\\ 0 & -2 & | & -2\end{bmatrix} \sim \underbrace{\begin{bmatrix}1 & 1 & | & 2\\ 0 & 1 & | & 1\end{bmatrix}}_{\text{EF}} \qquad \underbrace{\begin{bmatrix}1 & 0 & | & 1\\ 0 & 1 & | & 1\end{bmatrix}}_{\text{RREF}}$$

$$\begin{bmatrix}1 & -1 & | & 0\\ 1 & 1 & | & 2\end{bmatrix} \sim \begin{bmatrix}1 & -1 & | & 0\\ 0 & 2 & | & 2\end{bmatrix} \sim \underbrace{\begin{bmatrix}1 & -1 & | & 0\\ 0 & 1 & | & 1\end{bmatrix}}_{\text{EF}} \qquad \begin{bmatrix}1 & 0 & | & 1\\ 0 & 1 & | & 1\end{bmatrix}$$

- **Example: Use the Gauss-Jordan method to solve the following system**

$$x + 2y + 4z = 8$$
$$2x + 4y + 6z = 8$$
$$3x + 6y + 9z = 12$$

$$\left[\begin{array}{ccc|c}1 & 2 & 4 & 8\\ 2 & 4 & 6 & 8\\ 3 & 6 & 9 & 12\end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & 8 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 4 & 8 \\ 0 & 0 & -2 & -8 \\ 0 & 0 & -3 & -12 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 4 & 8 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & -3 & -12 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 4 & 8 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & -8 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & -8 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x + 2y = -8$
$x = -8 - 2y$
$z = 4$

y is a free variable

$x = -8 - 2y = -8 - 2t$
$y = t$
$\boxed{z = 4}$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -8 \\ 0 \\ 4 \end{bmatrix}$$

Matrix Inversion

- **Matrix Inverse**

$$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|}\operatorname{adj}(\mathbf{A})$$

  - Instead of using this definition, we will use the **Gauss-Jordan elimination** method

- **How can we use the Gauss-Jordan method to find the inverse of a given matrix?**
  - **Note that only square matrices have an inverse*, so assume that A is a square matrix**
  - **We know that:**

$$AA^{-1} = I$$

  - **Now compare that to:**

$$Ax = b$$

$$\mathbf{Ax = b}$$

- *To solve this:*

$$\text{aug}\mathbf{A} = [\mathbf{A}|\mathbf{b}]$$

- *If it has a unique solution and A is **square**:*

$$[\mathbf{A}|\mathbf{b}] \sim [\mathbf{I}|\mathbf{x}]$$

Like this one $\rightarrow$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$Ax = b \;\Downarrow\; [A|b] \sim [I|x]$$

$$AA^{-1} = I \;\Downarrow\; [A|I] \sim [I|A^{-1}]$$

- **Use Gauss-Jordan elimination to find the inverse of :**

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 1 \\ 2 & -2 & 1 \end{bmatrix}$$

$$[\mathbf{A}|\mathbf{I}] = \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 1 & 0 \\ 2 & -2 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 1 & 0 \\ 2 & -2 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 2 & 2 & 1 & 1 & 0 \\ 0 & -6 & -1 & -2 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 2 & 2 & 1 & 1 & 0 \\ 0 & -6 & -1 & -2 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0.5 & 0.5 & 0 \\ 0 & -6 & -1 & -2 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0.5 & 0.5 & 0 \\ 0 & 0 & 5 & 1 & 3 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0.5 & 0.5 & 0 \\ 0 & 0 & 1 & 0.2 & 0.6 & 0.2 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0.5 & 0.5 & 0 \\ 0 & 0 & 1 & 0.2 & 0.6 & 0.2 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 0.8 & -0.6 & -0.2 \\ 0 & 1 & 0 & 0.3 & -0.1 & -0.2 \\ 0 & 0 & 1 & 0.2 & 0.6 & 0.2 \end{array}\right]$$

$$\sim \left[\begin{array}{c|c} \underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{\mathbf{I}} & \underbrace{\begin{matrix} 0.2 & -0.4 & 0.2 \\ 0.3 & -0.1 & -0.2 \\ 0.2 & 0.6 & 0.2 \end{matrix}}_{\mathbf{A}^{-1}} \end{array}\right]$$

- Find out whether the given matrix is invertible, and use the Gauss-Jordan elimination to find the inverse if it is.

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & 1 \\ -2 & 4 & 5 \\ 3 & -4 & -4 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ -2 & 4 & 5 & 0 & 1 & 0 \\ 3 & -4 & -4 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 4 & 7 & 2 & 1 & 0 \\ 0 & -4 & -7 & -3 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 4 & 7 & 2 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \end{array}\right]$$

NO inverse

- **NOTE:**

$$(\mathbf{A}^T)^{-1} = (\mathbf{A}^{-1})^T$$

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$$
$$(\mathbf{A}\mathbf{A}^{-1})^T = \mathbf{I}^T$$
$$(\mathbf{A}^{-1})^T \mathbf{A}^T = \mathbf{I}$$
$$(\mathbf{A}^{-1})^T = (\mathbf{A}^T)^{-1}$$

Homogeneous System of Equations

Homogeneous System

$$\mathrm{Ax} = 0$$

- A homogeneous system has either:
  - **A Unique Solution → Zero Solution "Trivial Solution"**
  - **An Infinite Number of Solutions** (including the zero solution)

Example:

$$x + y - z = 0$$
$$x - 2y - 3z = 0$$
$$5x + y + z = 0$$

- **Example: Solve the following system of equations:**

$$x_1 - x_2 + 3x_3 = 0$$
$$2x_1 + x_2 + 3x_3 = 0$$

- *Homogeneous system*

$$\left[\begin{array}{ccc|c} 1 & -1 & 3 & 0 \\ 2 & 1 & 3 & 0 \end{array}\right] \quad -2R_1 + R_2 \to R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 3 & 0 \\ 0 & 3 & -3 & 0 \end{array}\right] \quad 1/3 \times R_3 \to R_3$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 3 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

$$\begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & \end{array} \\ \left[\begin{array}{ccc|c} \boxed{1} & -1 & 3 & 0 \\ 0 & \boxed{1} & -1 & 0 \end{array}\right] \end{array}$$

$$x_3 = t \qquad t \in \mathbb{R}$$
$$x_2 = x_3 \quad \to \quad x_2 = t$$
$$x_1 = x_2 - 3x_3 = t - 3t \quad \to \quad x_1 = -2t$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \quad t \in \mathbb{R}$$

- ***For a homogeneous system***

$$\mathbf{A}\mathbf{x} = \mathbf{0}$$

the **rank** of the coefficient matrix is always equal to that of the augmented matrix

$$\rho(\text{aug}\mathbf{A}) = \rho(\mathbf{A})$$

$$\rho(\mathbf{A}) = n \qquad\qquad \rho(\mathbf{A}) < n$$

$\rho(\mathbf{A}) = n$	$\rho(\mathbf{A}) < n$
Unique Solution “ZERO solution”	**Infinite Number of Solutions**

Vector Equation

- We learned how to write a system of linear equations as a matrix equation

$$a_{11}x + a_{12}y + a_{13}z = b_1$$
$$a_{21}x + a_{22}y + a_{23}z = b_2$$
$$a_{31}x + a_{32}y + a_{33}z = b_3$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Matrix Equation

$$x \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} + y \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} + z \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Vector Equation

$$x\mathbf{a}_1 + y\mathbf{a}_2 + z\mathbf{a}_3 = \mathbf{b}$$

$$
\begin{aligned}
x_1 + 2x_2 &= 7 \\
-2x_1 + 5x_2 &= 4 \\
-5x_1 + 6x_2 &= -3
\end{aligned}
$$

$$
x_1 \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}
$$