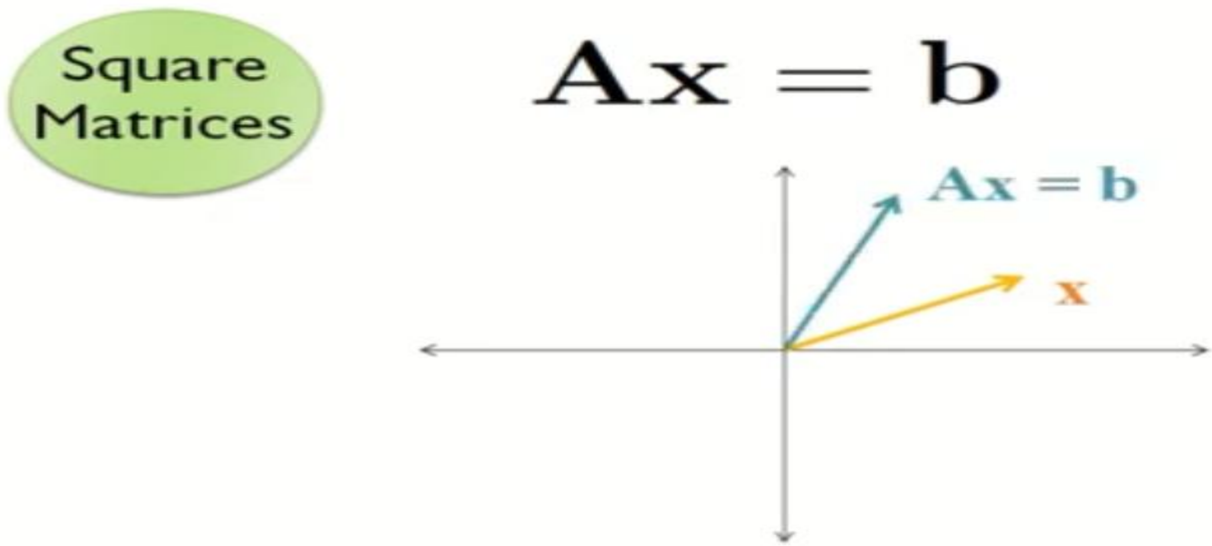
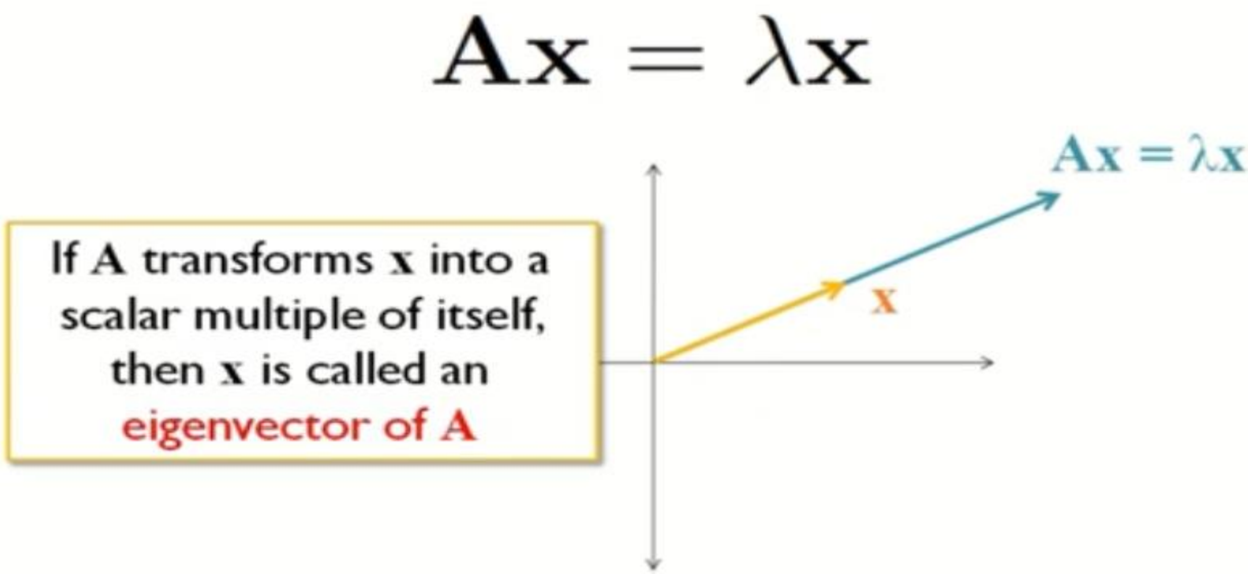
# MOODLE III: EIGENVALUES AND VECTORS



$$\mathbf{A}\underbrace{\boxed{\mathbf{x}}}_{\text{Eigenvector}} = \underbrace{\boxed{\lambda}}_{\text{Eigenvalue}}\mathbf{x}$$

- How do we find the eigenvalues and eigenvectors of $\mathbf{A}$?
  - ***Finding the eigenvalues & eigenvectors***

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$$

$$\mathbf{A}\mathbf{x} - \lambda\mathbf{x} = \mathbf{0}$$

$$\mathbf{A}\mathbf{x} - \lambda\mathbf{I}\mathbf{x} = \mathbf{0}$$

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$$

*Homogeneous system*
- Trivial (zero) solution
- Infinite number of solutions

- Coefficient matrix $(\mathbf{A} - \lambda\mathbf{I})$ should be ***singular (non-invertible)***
- $|\mathbf{A} - \lambda\mathbf{I}| = 0$ → ***Determinant = 0***

1. To find the eigenvalues of A:

$$|\mathbf{A} - \lambda\mathbf{I}| = 0 \quad \Rightarrow \quad \textit{Characteristic Equation}$$

2. To find the eigenvectors of each $\lambda$

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$$

- **Example 1:**
  - Find the eigenvalues and eigenvectors of the matrix:

$$\mathbf{A} = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$$

- Step 1:

$$\underbrace{|\mathbf{A} - \lambda\mathbf{I}|}_{\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}} = \begin{vmatrix} 1-\lambda & 6 \\ 5 & 2-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(2-\lambda) - (6)(5) = 0$$

$$\lambda^2 - 3\lambda - 28 = 0$$

$$(\lambda - 7)(\lambda + 4) = 0$$

- The eigenvalues of $\mathbf{A}$ are $\lambda = 7$ and $\lambda = -4$
- Step 2: For $\lambda = 7$, solve:

$$(\mathbf{A} - 7\mathbf{I})\mathbf{x} = \mathbf{0}$$

$$\left[\begin{array}{cc|c} 1 - 7 & 6 & 0 \\ 5 & 2 - 7 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} -6 & 6 & 0 \\ 5 & -5 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad x_1 - x_2 = 0$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

All scalar multiples of this vector are eigenvectors of $\lambda = 7$

Eigenspace of $\lambda = 7$

$$\begin{bmatrix}1 & 6\\5 & 2\end{bmatrix}\begin{bmatrix}2\\2\end{bmatrix} = \begin{bmatrix}14\\14\end{bmatrix} = 7\begin{bmatrix}2\\2\end{bmatrix}$$

a multiple of $\begin{bmatrix}1\\1\end{bmatrix}$

$$\boxed{\mathbf{Ax} = 7\mathbf{x}}$$

For $\lambda = -4$, solve:

$$(\mathbf{A} + 4\mathbf{I})\mathbf{x} = \mathbf{0}$$

$$\left[\begin{array}{cc|c} 1+4 & 6 & 0\\ 5 & 2+4 & 0\end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 5 & 6 & 0\\ 0 & 0 & 0\end{array}\right] \quad 5x_1 + 6x_2 = 0$$

$$\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}-\frac{6}{5}x_2\\x_2\end{bmatrix} = x_2\begin{bmatrix}-\frac{6}{5}\\1\end{bmatrix}$$

All scalar multiples of this vector are eigenvectors of $\lambda = -4$

$$\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ -5 \end{bmatrix} = \begin{bmatrix} -24 \\ 20 \end{bmatrix} = -4 \begin{bmatrix} 6 \\ -5 \end{bmatrix}$$

a multiple of $\begin{bmatrix} -\frac{6}{5} \\ 1 \end{bmatrix}$

$\boxed{\mathbf{Ax} = -4\mathbf{x}}$

| | | |
|---|---|---|
| ***Eigenvalues*** | $\lambda = 7$ | $\lambda = -4$ |
| ***Basis of the eigenspace*** | $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ | $\begin{bmatrix} -\frac{6}{5} \\ 1 \end{bmatrix}$ |

# Special Cases and Properties of Eigenvalues & Eigenvectors

## Properties of Eigenvalues & Eigenvectors

- Diagonal and Triangular Matrices
  - The eigenvalues are the ***diagonal elements***

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix} \qquad \lambda = 1,\ 5,\ 2$$

WHY?

$$\begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & 5-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{bmatrix} \qquad |\mathbf{A} - \lambda\mathbf{I}| = 0$$

$$(1-\lambda)(5-\lambda)(2-\lambda) = 0$$

$$\lambda = 1, 5, 2$$

ANOTHER

$$\begin{bmatrix} 2 & 0 & 0 \\ 7 & -5 & 0 \\ 6 & 8 & -1 \end{bmatrix} \qquad \lambda = 2, \text{-}5, \text{-}1$$

Lower Triangular

WHY?

$$\begin{bmatrix} \boxed{2-\lambda} & 0 & 0 \\ 7 & \boxed{-5-\lambda} & 0 \\ 6 & 8 & \boxed{-1-\lambda} \end{bmatrix} \qquad |\mathbf{A} - \lambda\mathbf{I}| = 0$$

$$(2\text{-}\lambda)(\text{-}5\text{-}\lambda)(\text{-}1\text{-}\lambda)=0$$

ANOTHER

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & 3 \\ 0 & 0 & 6 \end{bmatrix} \qquad \lambda = 1, 0, 6$$

Upper Triangular

ANOTHER

$$\begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix} \qquad \lambda = 2, 4$$

Upper Triangular

- Non-invertible matrices

$\mathbf{A}^{-1}$ **Does NOT exist** $\Longleftrightarrow$ $\lambda = 0$

$$|\mathbf{A} - \lambda\mathbf{I}| = |\mathbf{A} - 0\mathbf{I}| = 0$$

$$|\mathbf{A}| = 0$$

$$\boxed{\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|}\operatorname{adj}(\mathbf{A})}$$

$$|\mathbf{A}| = 0$$

- Transpose of a matrix

**A** and $\mathbf{A}^{\mathsf{T}}$ have the same eigenvalues

$$\boxed{|\mathbf{A} - \lambda\mathbf{I}| = 0}$$
$$|(\mathbf{A} - \lambda\mathbf{I})^T| = 0$$
$$|\mathbf{A}^T - \lambda\mathbf{I}^T| = 0$$
$$\boxed{|\mathbf{A}^T - \lambda\mathbf{I}| = 0}$$

$$|\mathbf{B}| = |\mathbf{B}^T|$$

Same characteristic equation…
Same eigenvalues

- Inverse of a matrix

| | Eigenvalues | Eigenvectors |
|---|---|---|
| $\mathbf{A} \rightarrow$ | $\lambda_1, \lambda_2, \lambda_3, \dots$ | $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots$ |
| $\mathbf{A}^{-1} \rightarrow$ | $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}, \dots$ | $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots$ |

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$$
$$\mathbf{A}^{-1}\mathbf{A}\mathbf{x} = \lambda\mathbf{A}^{-1}\mathbf{x}$$
$$\mathbf{I}\mathbf{x} = \lambda\,\mathbf{A}^{-1}\mathbf{x}$$
$$\boxed{\mathbf{A}^{-1}\mathbf{x} = \frac{1}{\lambda}\mathbf{x}}$$

- Powers of a matrix

| | Eigenvalues | Eigenvectors |
|---|---|---|
| $\mathbf{A} \rightarrow$ | $\lambda_1,\ \lambda_2,\ \lambda_3,\ \ldots$ | $\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3,\ \ldots$ |
| $\mathbf{A}^n \rightarrow$ | $(\lambda_1)^n,(\lambda_2)^n,(\lambda_3)^n\ \ldots$ | $\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3,\ \ldots$ |

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$$
$$\mathbf{A}\mathbf{A}\mathbf{x} = \mathbf{A}(\lambda\mathbf{x})$$
$$\mathbf{A}^2\mathbf{x} = \lambda\cancel{\mathbf{A}\mathbf{x}} \rightarrow \lambda\mathbf{x}$$
$$\mathbf{A}^2\mathbf{x} = \lambda^2\mathbf{x}$$

***Repeating:*** $\boxed{\mathbf{A}^n\mathbf{x} = \lambda^n\mathbf{x}}$

FROM

- **Example 2:**

$$\mathbf{A} = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \longrightarrow \lambda = 7, -4$$

$$\mathbf{A}^{-1} \longrightarrow \lambda = 1/7, -1/4$$

$$\mathbf{A}^{T} \longrightarrow \lambda = 7, -4$$

$$\mathbf{A}^{2} \longrightarrow \lambda = 49, 16$$

- What if the roots of the characteristic equation were **complex**?

- We get COMPLEX eigenvalues & eigenvectors

$$\lambda^2 + 9 = 0 \Rightarrow \lambda = \pm i3$$

$$\lambda^2 + \lambda + 1 = 0 \Rightarrow \lambda = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

**Example 3:** Find the eigenvalues and eigenvectors

$$\mathbf{A} = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}$$

$$|\mathbf{A} - \lambda \mathbf{I}| = \begin{vmatrix} 3-\lambda & -2 \\ 4 & -1-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)(-1-\lambda) + 8 = 0$$

$$\lambda^2 - 2\lambda + 5 = 0$$

$$\boxed{\lambda = 1 + 2i} \longleftrightarrow \boxed{\lambda = 1 - 2i}$$

Complex Conjugates

The eigenvector elements are also complex conjugates

$$\boxed{\lambda = 1 + 2i} \longrightarrow (\mathbf{A} - (1+2i)\mathbf{I})\mathbf{x} = \mathbf{0}$$

$$\begin{bmatrix} 3-(1+2i) & -2 \\ 4 & -1-(1+2i) \end{bmatrix}$$

$$\sim \begin{bmatrix} 2+2i & -2 \\ 4 & -2+2i \end{bmatrix}$$

$$\sim \begin{bmatrix} 2+2i & -2 \\ 0 & 0 \end{bmatrix} \quad \begin{array}{r} (2+2i)x_1 - 2x_2 = 0 \\ x_2 = (1+i)x_1 \end{array}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ (1+i)x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 1+i \end{bmatrix}$$

IF

$$\boxed{\lambda = 1 + 2i} \qquad\qquad \boxed{\lambda = 1 - 2i}$$

$$\begin{bmatrix} 1 \\ 1+i \end{bmatrix} \quad \longrightarrow \quad \begin{bmatrix} 1 \\ 1-i \end{bmatrix}$$

- **Examples:**
- Is $\begin{bmatrix}1\\1\\1\end{bmatrix}$ an eigenvector of $\begin{bmatrix}1 & 2 & 3\\3 & 2 & 1\\1 & 3 & 2\end{bmatrix}$ ?
- Is $\begin{bmatrix}1\\2\end{bmatrix}$ an eigenvector of $\begin{bmatrix}5 & 1\\-2 & 3\end{bmatrix}$ ?
- Is 3 an eigenvalue of $\begin{bmatrix}5 & 1\\-2 & 3\end{bmatrix}$ ?
- ***Answer:***

$$\begin{bmatrix}1 & 2 & 3\\3 & 2 & 1\\1 & 3 & 2\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}6\\6\\6\end{bmatrix} = 6\begin{bmatrix}1\\1\\1\end{bmatrix}$$

- It is an eigenvector because the matrix transforms it into a scalar multiple of itself (and its eigenvalue is 6)

- **NOTE:**
  - **When the sum of each row of the matrix is equal to some constant value k, then we know that one of its eigenvectors is an all-ones vector and its corresponding eigenvalue is k**

$$\begin{bmatrix} 1 & 8 \\ 5 & 4 \end{bmatrix} \begin{matrix} \rightarrow \text{Sum} = 1 + 8 = 9 \\ \rightarrow \text{Sum} = 5 + 4 = 9 \end{matrix}$$

$$\boxed{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \lambda = 9}$$

- ***Answer:***

$$\begin{bmatrix} 5 & 1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 2 \end{bmatrix} \neq c \begin{bmatrix} 7 \\ 4 \end{bmatrix}$$

- It is not an eigenvector

- ***Answer:***

$$|\mathbf{A} - \lambda \mathbf{I}| = \begin{vmatrix} 1-3 & -1 \\ -6 & 0-3 \end{vmatrix} = \begin{vmatrix} -2 & -1 \\ -6 & -3 \end{vmatrix}$$

$$= (-2)(-3) - (-1)(-6) = 6 - 6 = 0$$

Therefore 3 is an eigenvalue because it satisfies $|\mathbf{A} - \lambda \mathbf{I}| = 0$

## Matrix Diagonalization

$$\mathrm{A} = \mathrm{P}\underbrace{\boxed{\mathrm{D}}}_{\text{Diagonal Matrix}}\mathrm{P}^{-1}$$

- ***Applications:***
  - Solving sets of linear differential equations
  - Computing $A^k$

- ***Definition:***

**Similar matrices have the same eigenvalues**

- If $\mathbf{A}$ and $\mathbf{B}$ are similar:

$$\mathrm{A} = \mathrm{PBP}^{-1} \quad \& \quad \mathrm{B} = \mathrm{QAQ}^{-1}$$

***HOW?***

- Assuming $\mathbf{A}$ and $\mathbf{B}$ are similar i.e. $\mathbf{A} = \mathbf{PBP}^{-1}$

$$\boxed{|\mathbf{A} - \lambda\mathbf{I}|} = |\mathbf{PBP}^{-1} - \lambda\mathbf{I}|$$

$$= |\mathbf{PBP}^{-1} - \lambda\mathbf{PP}^{-1}|$$

$$= |\mathbf{P}(\mathbf{B} - \lambda\mathbf{I})\mathbf{P}^{-1}|$$

$$|\mathbf{P}^{-1}| = \frac{1}{|\mathbf{P}|} \qquad = \cancel{|\mathbf{P}|}.|(\mathbf{B} - \lambda\mathbf{I})|.\cancel{|\mathbf{P}^{-1}|}$$

$$= \boxed{|\mathbf{B} - \lambda\mathbf{I}|}$$

A and B have the same characteristic equation → Same λ

- Our goal is to find a ***diagonal matrix similar to*** $\mathbf{A}$, so that we can write

$$\boxed{\mathbf{A} = \mathbf{PDP}^{-1}}$$

→ **Diagonalization**

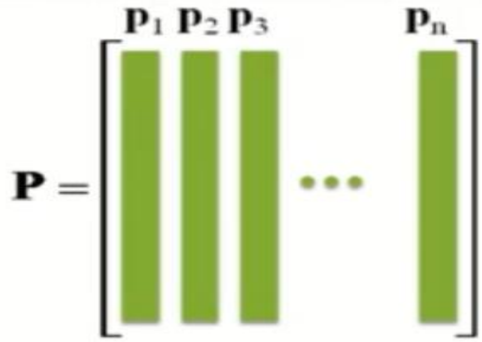
- Since similar matrices have the SAME eigenvalues

$$\mathbf{D} = \begin{bmatrix} \lambda_1 & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & 0 & \dots & 0 \\ 0 & 0 & \lambda_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_n \end{bmatrix}$$

Eigenvalues of $\mathbf{A}$

# What about $\mathbf{P}$ ?

$$\mathbf{A} = \mathbf{PDP}^{-1}$$
$$\mathbf{AP} = \mathbf{PD}$$

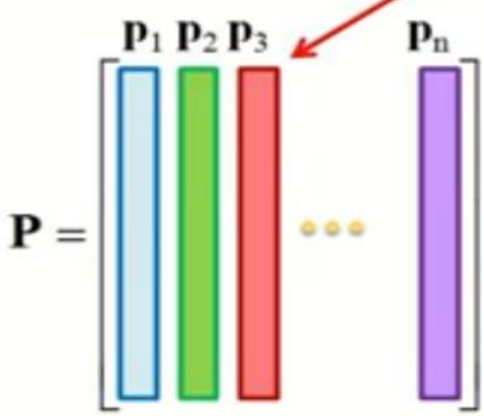



$$[\mathbf{Ap}_1 \;\; \mathbf{Ap}_2 \;\; \ldots \mathbf{Ap}_n]$$

$$[\mathbf{Ap}_1 \;\; \mathbf{Ap}_2 \;\; \ldots \mathbf{Ap}_n] = [\lambda_1\mathbf{p}_1 \;\; \lambda_2\mathbf{p}_2 \;\; \ldots \lambda_n\mathbf{p}_n]$$

The columns of P are the eigenvectors!

$$\mathbf{A} = \mathbf{PDP}^{-1}$$




$$\mathbf{D} = \begin{bmatrix} \boxed{\lambda_1} & 0 & 0 & \cdots & 0 \\ 0 & \boxed{\lambda_2} & 0 & \cdots & 0 \\ 0 & 0 & \boxed{\lambda_3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \boxed{\lambda_n} \end{bmatrix}$$

FROM

- **Example 1:**

$$\mathbf{A} = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \qquad \begin{array}{c|c} \lambda = 7 & \lambda = -4 \\ \begin{bmatrix} 1 \\ 1 \end{bmatrix} & \begin{bmatrix} -\frac{6}{5} \\ 1 \end{bmatrix} \end{array}$$

$$\mathbf{A} = \mathbf{PDP}^{-1} = \begin{bmatrix} 1 & -\frac{6}{5} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 7 & 0 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} 1 & -\frac{6}{5} \\ 1 & 1 \end{bmatrix}^{-1}$$

- **Matrix Powers using Diagonalization:**

$$\mathbf{A}^k = \mathbf{PD}^k\mathbf{P}^{-1}$$

*Proof:*

$$\mathbf{A} = \mathbf{PDP}^{-1}$$
$$\mathbf{A}^2 = \mathbf{PDP}^{-1}\mathbf{PDP}^{-1}$$
$$\mathbf{A}^2 = \mathbf{PDIDP}^{-1}$$
$$\mathbf{A}^2 = \mathbf{PD}^2\mathbf{P}^{-1}$$

Repeating this process we get : $\mathbf{A}^k = \mathbf{PD}^k\mathbf{P}^{-1}$

- **Q:** ***Can ANY matrix be diagonalized?***
- **A:** ***NO!***

$$\underset{n x n}{\mathbf{A}} = \mathbf{PDP}^{-1}$$

- $\mathbf{P}$ must be invertible
- The columns of $\mathbf{P}$ are the eigenvectors
- $\mathbf{A}$ must have $n$ independent eigenvectors

- **Condition for Diagonalization:**

> An $n$x$n$ matrix can be diagonalized if it has $n$ independent eigenvectors

- ***Theorem:***
  - Eigenvectors of a matrix corresponding to different (unequal) eigenvalues are independent.
    - $\mathbf{A}_{nxn}$ has $n$ different eigenvalues ➔ ***Diagonalizable***
    - $\mathbf{A}_{nxn}$ has less than $n$ different eigenvalues ➔ ***Might be diagonalizable*** (check the repeated eigenvalues)

- **Example 2:** If possible, diagonalize the matrix:

$$\mathbf{A} = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

- ***First, what are the eigenvalues?***

$$|\mathbf{A} - \lambda \mathbf{I}| = \begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix} = 0$$

$$(\lambda - 1)^2(\lambda - 5) = 0$$

$\lambda = 1$ $\lambda = 5$

Multiplicity= 2

FOR

$$\lambda = 1$$

$$(\mathbf{A} - \mathbf{I})\mathbf{x} = \mathbf{0} \qquad x_1 + 2x_2 + x_3 = 0$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 2 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \boxed{\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}} + x_3 \boxed{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}$$

**Diagonalizable!**

FOR

$$\lambda = 5$$

$$(\mathbf{A} - 5\mathbf{I})\mathbf{x} = \mathbf{0} \qquad x_1 = x_3 \quad x_2 = x_3$$

$$\left[\begin{array}{ccc|c} -3 & 2 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 2 & -3 & 0 \end{array}\right] \cdots \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \boxed{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}$$

THEN

$$\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$$

$$= \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}^{-1}$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/4 & 1/2 & 1/4 \\ 1/4 & 1/2 & -3/4 \\ 1/4 & -1/2 & 1/4 \end{bmatrix}$$

$$\mathbf{A}^5 = \mathbf{P}\mathbf{D}^5\mathbf{P}^{-1}$$

$$= \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 5^5 & 0 & 0 \\ 0 & 1^5 & 0 \\ 0 & 0 & 1^5 \end{bmatrix} \begin{bmatrix} 1/4 & 1/2 & 1/4 \\ 1/4 & 1/2 & -3/4 \\ 1/4 & -1/2 & 1/4 \end{bmatrix}$$

$$= \begin{bmatrix} 782 & 1562 & 781 \\ 781 & 1563 & 781 \\ 781 & 1562 & 782 \end{bmatrix}$$

## *How to use diagonalization to solve a set of linear first order differential equations*

## Decoupling Systems

- Let's say we have two variables $x(t)$ and $y(t)$ that are functions of time $t$

| Decoupling | Separation of Variables |
|---|---|
| $\frac{dx}{dt} = 3x + y$, $\frac{dy}{dt} = x - 4y$ | $\frac{dx}{dt} = -2x$, $\frac{dy}{dt} = y$ |
| **Coupled System** | **Decoupled System** |

$$\frac{dx_1}{dt} = ax_1 + bx_2$$
$$\frac{dx_2}{dt} = cx_1 + dx_2$$

$$\begin{bmatrix} dx_1/dt \\ dx_2/dt \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
$$\mathbf{X'} \quad = \quad \mathbf{A} \quad \mathbf{X}$$

$$\mathbf{X'} = \mathbf{A}\mathbf{X} \qquad \boxed{\mathbf{A} = \mathbf{PDP}^{-1}}$$

$$\mathbf{X'} = \mathbf{PDP}^{-1}\mathbf{X}$$
$$\mathbf{P}^{-1}\mathbf{X'} = \mathbf{P}^{-1}\mathbf{PDP}^{-1}\mathbf{X}$$
$$\mathbf{P}^{-1}\mathbf{X'} = \mathbf{D}\mathbf{P}^{-1}\mathbf{X}$$
$$\mathbf{Y'} = \mathbf{DY} \Longrightarrow \text{Decoupled System}$$

$$\mathbf{Y'} = \mathbf{DY}$$

$$\begin{bmatrix} dy_1/dt \\ dy_2/dt \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\frac{dy_1}{dt} = \lambda_1 y_1 \qquad\qquad y_1(t) = c_1 e^{\lambda_1 t}$$

$$\frac{dy_2}{dt} = \lambda_2 y_2 \qquad\qquad y_2(t) = c_1 e^{\lambda_1 t}$$

- Now we need the solution in terms of $\mathbf{X}$

$$\mathbf{P}^{-1}\mathbf{X} = \mathbf{Y} \quad \Rightarrow \quad \mathbf{X} = \mathbf{P}\mathbf{Y}$$

$$\begin{aligned} \mathbf{X} \;&=\; \quad\mathbf{P} \qquad\quad \mathbf{Y} \\ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\ &= \begin{bmatrix} p_{11}y_1 + p_{12}y_2 \\ p_{21}y_1 + p_{22}y_2 \end{bmatrix} \\ &= y_1 \begin{bmatrix} p_{11} \\ p_{21} \end{bmatrix} + y_2 \begin{bmatrix} p_{12} \\ p_{22} \end{bmatrix} \\ &= c_1 e^{\lambda_1 t}\mathbf{p}_1 + c_2 e^{\lambda_2 t}\mathbf{p}_2 \end{aligned}$$

$$\boxed{\mathbf{X} = \sum_{i=1}^{n} c_i e^{\lambda_i t}\mathbf{p}_i}$$

- **Example:**

$$\begin{aligned} x_1' &= x_1 - x_2 - x_3 \\ x_2' &= x_1 + 3x_2 + x_3 \\ x_3' &= -3x_1 + x_2 - x_3 \end{aligned}$$

$$\boxed{\begin{aligned} x_1(0) &= 0 \\ x_2(0) &= -1 \\ x_3(0) &= 10 \end{aligned}}$$

ANSWER

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & 1 & -1 \end{bmatrix}$$

$$|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} 1-\lambda & -1 & -1 \\ 1 & 3-\lambda & 1 \\ -3 & 1 & -1-\lambda \end{vmatrix} = 0$$

$$(\lambda-2)(\lambda+2)(\lambda-3) = 0$$

$$\boxed{\lambda = 2, -2, 3}$$

$$\begin{array}{cccc} & (\mathbf{A} - \lambda\mathbf{I}) & & \text{Eigenvector} \\ \lambda = 2: & \begin{bmatrix} -1 & -1 & -1 \\ 1 & 1 & 1 \\ -3 & 1 & -3 \end{bmatrix} & \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \\ \lambda = -2: & \begin{bmatrix} 3 & -1 & -1 \\ 1 & 5 & 1 \\ -3 & 1 & 1 \end{bmatrix} & \rightarrow \begin{bmatrix} 1 & 0 & -1/4 \\ 0 & 1 & 1/4 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} \\ \lambda = 3: & \begin{bmatrix} -2 & -1 & -1 \\ 1 & 0 & 1 \\ -3 & 1 & -4 \end{bmatrix} & \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \end{array}$$

$$\mathbf{X} = \sum_{i=1}^{3} c_i e^{\lambda_i t} \mathbf{p}_i \qquad \begin{array}{ccc} \lambda = 2 & \lambda = -2 & \lambda = 3 \\ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} & \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} & \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \end{array}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = c_1 e^{2t} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} + c_3 e^{3t} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$\boxed{x_1(t) = -c_1 e^{2t} + c_2 e^{-2t} - c_3 e^{3t}}$$

$$\boxed{x_2(t) = -c_2 e^{-2t} + c_3 e^{3t}}$$

$$\boxed{x_3(t) = c_1 e^{2t} + 4c_2 e^{-2t} + c_3 e^{3t}}$$

$$\begin{aligned} x_1(t) &= -c_1 e^{2t} + c_2 e^{-2t} - c_3 e^{3t} \\ x_2(t) &= -c_2 e^{-2t} + c_3 e^{3t} \\ x_3(t) &= c_1 e^{2t} + 4c_2 e^{-2t} + c_3 e^{3t} \end{aligned} \qquad \boxed{\begin{aligned} x_1(0) &= 0 \\ x_2(0) &= -1 \\ x_3(0) &= 10 \end{aligned}}$$

$$\begin{aligned} x_1(0) &= 0 = -c_1 + c_2 - c_3 \\ x_2(0) &= -1 = -c_2 + c_3 \\ x_3(0) &= 10 = c_1 + 4c_2 + c_3 \end{aligned}$$

$$\left.\begin{aligned} -c_1 + c_2 - c_3 &= 0 \\ -c_2 + c_3 &= -1 \\ c_1 + 4c_2 + c_3 &= 10 \end{aligned}\right\} \begin{aligned} c_1 &= 1 \\ c_2 &= 2 \\ c_3 &= 1 \end{aligned}$$

$$\boxed{\begin{aligned} x_1(t) &= -e^{2t} + 2e^{-2t} - e^{3t} \\ x_2(t) &= -2e^{-2t} + e^{3t} \\ x_3(t) &= e^{2t} + 8e^{-2t} + e^{3t} \end{aligned}}$$

**NOTE**

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \overset{1}{\cancel{c_1}} e^{2t} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} + c_3 e^{3t} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \checkmark$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \overset{-1}{\cancel{c_1}} e^{2t} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} + c_3 e^{3t} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \checkmark$$

## The Cayley-Hamilton Theorem

"Every matrix satisfies its own characteristic equation"

$$f(\lambda) = |\mathbf{A} - \lambda\mathbf{I}| = 0 \longrightarrow \text{Function of } \lambda$$

$$f(\mathbf{A}) = \mathbf{0} \longrightarrow \text{Function of A}$$

$$\mathbf{A} = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$$

$$\lambda^2 - 3\lambda - 28 = 0$$

$$\mathbf{A}^2 - 3\mathbf{A} - 28\mathbf{I} = \mathbf{0}$$

- To get $\mathbf{A}^2$:

$$\mathbf{A}^2 = 3\mathbf{A} + 28\mathbf{I}$$

$$= 3\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} + \begin{bmatrix} 28 & 0 \\ 0 & 28 \end{bmatrix} = \begin{bmatrix} 31 & 18 \\ 15 & 34 \end{bmatrix}$$

- To get $\mathbf{A}^{-1}$ : Multiply by $\mathbf{A}^{-1}$

$$\mathbf{A} - 3\mathbf{I} - 28\mathbf{A}^{-1} = \mathbf{0}$$

$$\mathbf{A}^{-1} = \frac{1}{28}(\mathbf{A} - 3\mathbf{I})$$

$$\mathbf{A}^{-1} = \frac{1}{28}\left(\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\right) = \frac{1}{28}\begin{bmatrix} -2 & 6 \\ 5 & -1 \end{bmatrix}$$

- To get $\mathbf{A}^3$:

$$\mathbf{A}^3 = 3\underbrace{\mathbf{A}^2}_{\boxed{3\mathbf{A} + 28\mathbf{I}}} + 28\mathbf{A}$$

$$= 3[3\mathbf{A} + 28\mathbf{I}] + 28\mathbf{I}$$

$$= 37\mathbf{A} + 84\mathbf{I} = \begin{bmatrix} 121 & 222 \\ 185 & 158 \end{bmatrix}$$

➔ **Higher powers can be obtained recursively**