Arbitrage Pricing Theory (APT) Models & Multifactor Models

Nathan NDJOLI - nathan.ndjoli1@gmail.com

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The Capital Asset Pricing Model (CAPM)

The Security Market Line (SML) is a graphical representation that illustrates the relationship between the expected return of assets and their systematic, non-diversifiable risk, which is measured by beta (β). This line is a key concept in the Capital Asset Pricing Model (CAPM) and serves to show how the risk of an asset is compensated by its expected return. Below is a diagram illustrating the SML and its components, with the expected return on the y-axis and the beta on the x-axis.

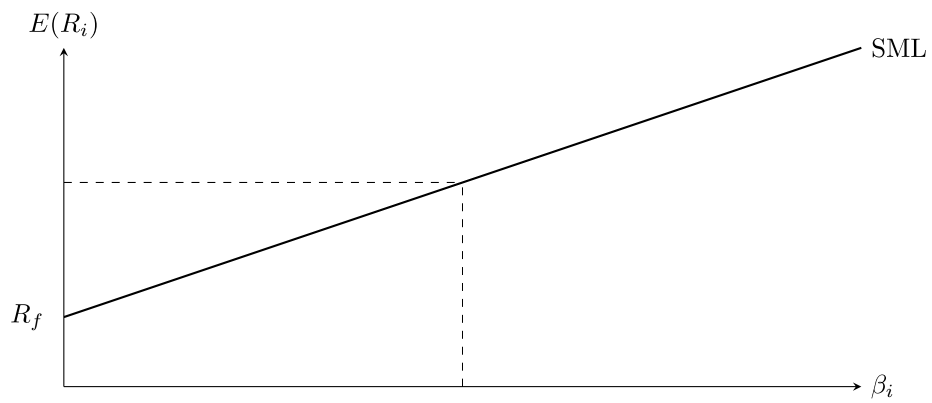

Figure 1: Security Market Line

The diagram uses the following notations: R_i represents the return of asset i, $E(R_i)$ is the expected return of asset i, R_f denotes the risk-free rate, $E(R_M)$ is the expected market return, and β_i indicates the non-diversifiable risk of asset i.

The return of asset i can be expressed as:

$$R_i = \alpha_i + \beta_i R_M + \epsilon_i$$

Where β_i is calculated using:

$$\beta_i = \frac{Cov(R_i, R_M)}{Var(R_M)} = \frac{\sigma_{i,M}}{\sigma_M^2}$$

Beta quantifies the asset's sensitivity to market movements.

The analytical expression of the SML is:

$$E(R_i) = R_f + \beta_i(E(R_M) - R_f)$$

Which can be rearranged to show the excess return:

$$E(R_i) - R_f = \beta_i(E(R_M) - R_f)$$

In the context of an efficient portfolio, the beta of the portfolio (β_P) can be derived as follows:

$$\beta_P = w\beta_M + (1 - w)\beta_f = w$$

Where w represents the weight of the market portfolio in the overall portfolio. Additionally, the variance of the portfolio (σ_P^2) is given by:

$$\sigma_P^2 = w^2\sigma_M^2 = \beta_P^2\sigma_M^2$$

Thus, beta can also be expressed in terms of standard deviations:

$$\beta_P = \frac{\sigma_P}{\sigma_M}$$

From this, we derive the Capital Market Line (CML), which relates the expected return of the portfolio ($E(R_P)$) to its total risk (σ_P):

$$E(R_P) = R_f + \frac{\sigma_P}{\sigma_M}(E(R_M) - R_f)$$

This equation highlights that the risk of an efficient portfolio is entirely non-diversifiable risk:

$$\sigma_P = \beta_P\sigma_M$$

1 Criticisms of the CAPM

The Capital Asset Pricing Model (CAPM) has faced numerous criticisms, particularly concerning the assumptions it makes about market conditions and investor behavior. One major criticism of the CAPM pertains to its assumption that returns are normally distributed. In real-world markets, asset returns often deviate from normality, exhibiting skewness and kurtosis. This leads to return distributions that can have fat tails or other anomalies, which the CAPM does not account for. This discrepancy between the assumed and actual distribution of returns can lead to inaccuracies in the model's predictions and its practical applications.

Another significant criticism is the assumption of homogeneous investor expectations. The CAPM posits that all investors share the same expectations regarding future returns and risks, an assumption that is far from realistic. In reality, investors have diverse risk preferences, investment horizons, and expectations based on their unique information sets and personal circumstances. This heterogeneity among investors means that their actions and reactions to market events can be vastly different, thereby challenging the uniformity assumed by the CAPM.

The CAPM also assumes that investors can borrow and lend unlimited amounts at the risk-free rate. This assumption is problematic because, in the real world, the conditions for borrowing and lending are not symmetric. Borrowing rates are typically higher than lending rates, reflecting the risk premium demanded by lenders. Moreover, access to borrowing is often constrained by factors such as creditworthiness and market conditions, which the CAPM does not consider. These limitations on borrowing can significantly impact an investor's ability to leverage their portfolio as the model assumes.

Roll's critique presents another profound challenge to the CAPM. According to this critique, the CAPM is inherently untestable because it requires the true market portfolio, encompassing all investable assets, to be observed. In practice, such a comprehensive market portfolio is impossible to construct or observe. Empirical tests of the CAPM often use stock market indices as proxies for the market portfolio, but these indices represent only a subset of all possible investments. Consequently, the validity of empirical tests of the CAPM is questionable, as they do not account for the full scope of the market portfolio.

Arbitrage Pricing Theory (APT) Models

The Arbitrage Pricing Theory (APT) relies on the existence of common risk factors. The return on asset i can be expressed as:

$$R_i = \alpha_i + \beta_1 F_1 + \beta_2 F_2 + \ldots + \beta_k F_k + \epsilon_i$$

Where:

$$E(F_1) = E(F_2) = \ldots = E(F_k) = E(\epsilon_i) = 0$$

And:

$$Cov(\epsilon_i, \epsilon_j) = Cov(F_k, F_l) = Cov(F_k, \epsilon_i) = 0$$

The APT can be depicted through a factor model, which illustrates how different factors influence asset returns. Below is an example:

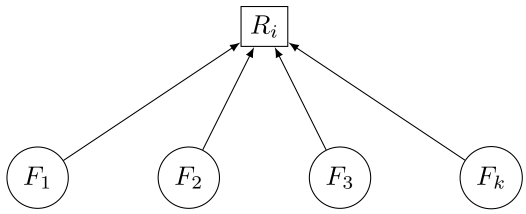

Figure 2: Multifactor Model illustrating influences on asset returns

Consider an example with a single risk factor. Suppose two risky assets with expected returns $E(R_A) = 10\%$ and $E(R_B) = 8\%$. The sensitivities of these assets to the risk factor are $\beta_A = 0.8$ and $\beta_B = 0.6$. We seek α_0 and α_1 such that $E(R_i) = \alpha_0 + \alpha_1\beta_i$. Given the expected returns and betas, we can set up the following system of equations to solve for α_0 and α_1:

$$0.10 = \alpha_0 + 0.8\alpha_1$$

$$0.08 = \alpha_0 + 0.6\alpha_1$$

Solving this system, we find:

$$\alpha_1 = \frac{0.10 - 0.08}{0.8 - 0.6} = \frac{0.02}{0.2} = 0.1$$

And:

$$\alpha_0 = 0.10 - 0.8 \times 0.1 = 0.10 - 0.08 = 0.02$$

Thus, the expected return equation becomes:

$$E(R_i) = 0.02 + 0.1\beta_i$$

Arbitrage opportunities arise when there are mispricings in the market that allow for risk-free profits. In the context of APT, consider two portfolios.

First case: Consider a portfolio C with $E(R_C) = 11\%$ and $\beta_C = 0.7$.

Second case: Consider a portfolio C' with $E(R_{C'}) = 9\%$ and $\beta_{C'} = 0.7$.

Comparing these portfolios to the expected return equation $E(R_i) = 0.02+0.1\beta_i$, we find that for $\beta_C = 0.7$:

$$E(R_C) = 0.02 + 0.1 \times 0.7 = 0.09$$

Since $E(R_C) = 11\%$ is greater than 9%, it suggests an arbitrage opportunity where portfolio C is overpriced. Conversely, if $E(R_{C'}) = 9\%$, it is correctly priced according to the model.

The results of the APT show that, under the assumption of orthogonality:

$$E(R_i) = \alpha_0 + \sum_{k=1}^{K} \alpha_k \beta_{i,k}$$

In the presence of a risk-free asset with a return R_f, we can write:

$$E(R_i) = R_f + \sum_{k=1}^{K} \alpha_k \beta_{i,k}$$

This generalization allows for multiple factors to explain asset returns, providing a more comprehensive model compared to CAPM, which only considers market risk.

The selection of factors in APT is crucial and can include macroeconomic variables, industry-specific factors, and other relevant variables. Each factor α_k represents the risk premium associated with that factor. Investors are compensated for bearing systematic risk associated with each factor. Compared to the CAPM, which is a single-factor model focusing solely on market risk, the APT considers multiple factors. The CAPM is expressed as:

$$E(R_i) = R_f + \beta_i(E(R_M) - R_f)$$

with $\alpha_M = E(R_M) - R_f$. This highlights that the CAPM is a single-factor model focusing solely on market risk, whereas APT considers multiple factors.

Application of APT and Multifactor Models

Arbitrage Pricing Theory (APT) and multifactor models play a crucial role in portfolio management by providing a comprehensive framework for assessing risk and return characteristics across various assets. These models incorporate both macroeconomic factors, such as GDP growth and inflation, and microeconomic factors, such as company earnings, to give a holistic view of asset returns.

APT and multifactor models are used to capture the impact of these diverse factors on asset returns, enabling investors to make informed decisions based on a broader set of information. For instance, the return on an asset can be influenced by several macroeconomic factors, including economic growth rates, interest rates, and inflation, as well as microeconomic factors specific to the company, such as revenue growth, profit margins, and market share.

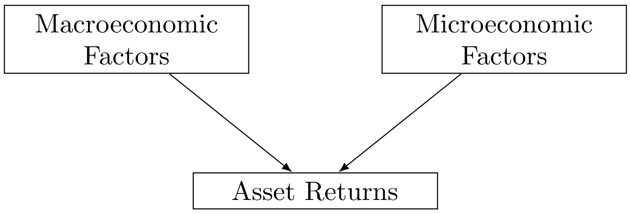

Figure 3: Influence of macro and microeconomic factors on asset returns

The Fama & French Model

The Fama and French three-factor model is particularly influential in understanding risk and return in equity markets. This model extends the Capital Asset Pricing Model (CAPM) by adding two additional factors to the market risk factor: size and value. The Fama and French model, formulated in 1992 and 1996, can be expressed as:

$$E(R_i) - R_f = \beta_M(E(R_M) - R_f) + \beta_{SMB}(E(S) - E(B)) + \beta_{HML}(E(H) - E(L))$$

In this equation, SMB (Small Minus Big) captures the size premium, reflecting the tendency for smaller companies to outperform larger ones. HML (High Minus Low) captures the value premium, indicating that companies with high book-to-market ratios tend to outperform those with low book-to-market ratios.

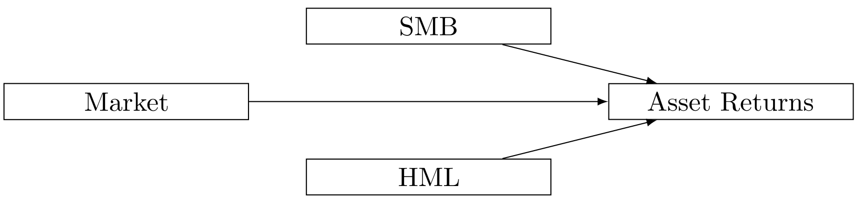

Figure 4: Fama and French three-factor model

The Carhart Model

The Carhart four-factor model, developed in 1997, builds on the Fama and French model by incorporating a momentum factor. This factor captures the tendency for stocks that have performed well in the past to continue performing well in the future, and for stocks that have performed poorly to continue underperforming. The Carhart model is expressed as:

$$E(R_i)-R_f = \beta_M(E(R_M)-R_f)+\beta_{SMB}(E(S)-E(B))+\beta_{HML}(E(H)-E(L))+\beta_{UMD}(E(U)-E(D))$$

Here, UMD (Up Minus Down) captures the momentum premium. This extension makes the model more robust by considering the performance trend of stocks over time.

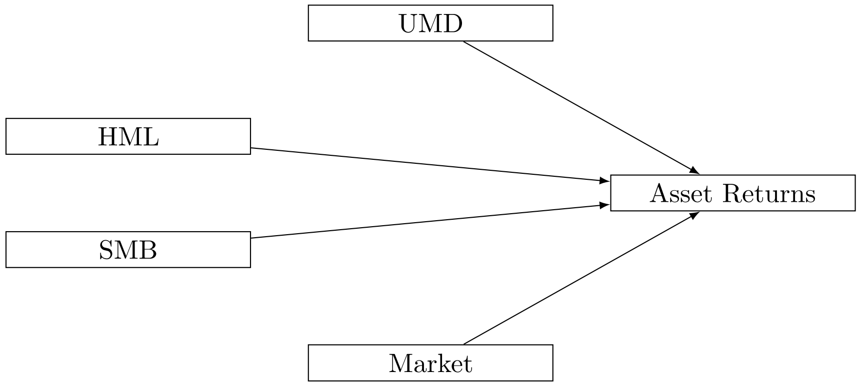

Figure 5: Carhart four-factor model

The application of these models in portfolio management enables investors to better understand the sources of risk and return in their portfolios. By incorporating multiple factors, these models provide a more nuanced view of asset performance, allowing for more effective risk management and investment strategies. They help investors to identify and capitalize on the various risk premiums present in the market, thereby enhancing their ability to achieve superior risk-adjusted returns.

Overall, APT and multifactor models offer a sophisticated approach to analyzing asset returns by considering a broad array of factors that influence market performance. They move beyond the single-factor approach of the CAPM, providing a richer framework for understanding and managing investment risks and opportunities.