TP4 - ASSIGMENT 2 DUE 21/04/23

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2.3

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

a) GIVEN MATRIXES $\hat{A}$, $\hat{B}$, DEFINE A COMMUTATION AS:

$$[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$$

- $$[\hat{\sigma}_x, \hat{\sigma}_y] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} - \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

$$= \begin{pmatrix} i+i & 0 \\ 0 & -i-i \end{pmatrix} = 2i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- $$[\hat{\sigma}_z, \hat{\sigma}_x] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} = (-1) \cdot 2 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$-1 = i^2$

$$= 2i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

- $$[\hat{\sigma}_y, \hat{\sigma}_z] = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 2i \\ 2i & 0 \end{pmatrix} = 2i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 2i \hat{\sigma}_x$$

B) CONSIDER $\hat{H} = \mu B \hat{\sigma}_x$, WITH $\mu = e\hbar/(2m_e) \sim 0.5\, eV/G$

HEISENBERG EQ OF MOTION: $\frac{d}{dt}\hat{A}(t) = \left(\frac{i}{\hbar}\right)[\hat{H}, \hat{A}(t)]$ (1)

WITH i.c: $\hat{A}(0) = \hat{A}$, DERIVE THE TIME EVOLUTION FOR

$\hat{\sigma}_z(t)$

$$(1) \to \frac{d}{dt}\hat{A}(t) = \left(\frac{i}{\hbar}\right)\left[\mu B \hat{\sigma}_x, \hat{A}(t)\right]$$

$$\underset{\text{LINEARITY}}{=} \underbrace{\left(\frac{i}{\hbar}\right) \mu B}_{\frac{\omega}{2} = \frac{\mu B}{\hbar}} \left[\frac{2i\,\hat{\sigma}_x}{2i}, \hat{A}(t)\right]$$

$$= \frac{\omega}{4}\left[\left[\hat{\sigma}_y, \hat{\sigma}_z\right], \hat{A}\right]$$

• $2i\,\hat{\sigma}_x = [\hat{\sigma}_y, \hat{\sigma}_z]$ •

$$\underset{\text{JACOBI IDENTITY}}{=} \frac{\omega}{4}\left(-\left[\left[\hat{\sigma}_z, \hat{A}\right]\hat{\sigma}_y\right] - \left[\left[\hat{A}, \hat{\sigma}_z\right]\hat{\sigma}_y\right]\right)$$

$$\left[\left[\hat{\sigma}_y, \hat{\sigma}_z\right]\hat{A}\right] = -\left[\left[\hat{\sigma}_z, \hat{A}\right]\hat{\sigma}_y\right] - \left[\left[\hat{A}, \hat{\sigma}_z\right]\hat{\sigma}_y\right]$$

• EXPAND COMMUTATOR:

$$= \frac{\omega}{4}\left(-\left[\hat{\sigma}_z \hat{A} - \hat{A}\hat{\sigma}_z , \hat{\sigma}_y\right] - \left[\hat{A}\hat{\sigma}_z - \hat{\sigma}_z\hat{A} , \hat{\sigma}_y\right] \right)$$

$$= \frac{\omega}{4}\left(-\cancel{\left[\hat{\sigma}_z \hat{A} , \hat{\sigma}_y\right]} + \cancel{\left[\hat{A}\hat{\sigma}_z , \hat{\sigma}_y\right]} - \cancel{\left[\hat{A}\hat{\sigma}_z , \hat{\sigma}_y\right]} + \cancel{\left[\hat{\sigma}_z\hat{A} , \hat{\sigma}_y\right]} \right)$$

$$\frac{d}{dt}\hat{A}(t) = 0 \,!$$

(c) $\omega = \frac{2\mu B}{\hbar}$ AND $\nu = \frac{\omega}{2\pi}$

IF $B = 1\ G$

$$\Rightarrow \nu = \frac{\omega}{2\pi} = \frac{\cancel{2}\mu B}{\cancel{2}\pi\hbar} = \frac{\mu}{\pi\hbar} \cdot 1\ G$$

$$= \frac{e\cancel{\hbar}\ 1G}{2m_e\pi\cancel{\hbar}} = \frac{1}{2\pi}\frac{e}{m_e} 1\ G$$

$$\mu = \frac{e\hbar}{2m_e}$$

$$\Rightarrow \nu = \frac{1G}{2\pi} \cdot \frac{1{,}602 \cdot 10^{-19}\ C}{9.109 \cdot 10^{-31}\ kg} = 2.799 \cdot 10^{10}\ \frac{C \cdot G}{kg}$$

$$= 2.799 \cdot 10^{6}\ \frac{C \cdot T}{kg}$$

$$\frac{\cancel{A} \cdot \cancel{s} \cdot \cancel{kg} \cdot \cancel{A^{-1}} \cdot s^{\cancel{-2}}}{\cancel{kg}}$$

$$= 2.799 \cdot 10^{6}\ \frac{1}{s}$$

0.29

D

e

$\boxed{2.4}$

$$L = \frac{m_e}{2}|\dot{\vec{x}}|^2 + \frac{e^2}{|\vec{x}|} - \frac{e}{c}\vec{A}(\vec{x})\cdot\dot{\vec{x}}$$

WITH $\vec{A} = \frac{1}{2}(\vec{B} \times \vec{x})$

(A) FROM LAGRANGIAN $\rightarrow$ EULER LAGRANGE:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x}$$

$$\rightarrow \frac{d}{dt}\left(m_e \dot{\vec{x}} - \frac{e}{c}\vec{A}(\vec{x})\right) = \left(-\frac{e^2}{\vec{x}^2} - \frac{e}{c}\dot{\vec{x}}\frac{\partial}{\partial \vec{x}}\vec{A}(\vec{x})\right)$$

$$\rightarrow m_e \ddot{\vec{x}} - \frac{e}{c}\left(\frac{d}{dt}\left(\vec{A}(\vec{x})\right)\right) = -\frac{e^2}{\vec{x}^2} - \frac{e}{c}\dot{\vec{x}}\frac{\partial}{\partial \vec{x}}\vec{A}(\vec{x})$$

CHAIN RULE:

- $\frac{d\vec{A}}{dt} = \underbrace{\frac{\partial \vec{x}}{\partial t}}_{\dot{\vec{x}}}\underbrace{\frac{\partial \vec{A}}{\partial \vec{x}}}_{\partial_i \vec{A}} + \frac{\partial \vec{A}}{\partial t}$

- $\partial_j \vec{A}(x)$

$$\Rightarrow \quad m_e \ddot{\vec{x}} - \frac{e}{c}\left(\dot{\vec{x}}\, \partial_i \vec{A} + \frac{\partial \vec{A}}{\partial t} \right) = -\frac{e^2}{\vec{x}^2} - \frac{e}{c}\, \dot{\vec{x}}\, \partial_j \vec{A}(x)$$

$$\Rightarrow \quad \underbrace{m_e \ddot{\vec{x}}}_{F} = \frac{e}{c} \underbrace{\dot{\vec{x}}\left(\partial_i \vec{A} - \partial_j \vec{A} \right)}_{\rightarrow (\dot{\vec{x}} \times B)} + \frac{e}{c}\frac{\partial \vec{A}}{\partial t} - \frac{e^2}{\vec{x}^2}$$

$$\Rightarrow \quad \vec{F} = \frac{e}{c}\left(\dot{\vec{x}} \times \vec{B} \right) + \frac{e}{c}\frac{\partial \vec{A}}{\partial t} - \frac{e^2}{\vec{x}^2}$$

IF WE ASSUME $\frac{e}{|\vec{x}|} = \phi$ SCALAR POTENTIAL, THEN

$$\underbrace{\frac{\partial}{\partial \vec{x}} \phi}_{\nabla \phi} = -e \frac{1}{\vec{x}^2}$$

$$\Rightarrow \quad \vec{F} = \frac{e}{c}\left(\dot{\vec{x}} \times \vec{B} \right) + \frac{e}{c}\frac{\partial \vec{A}}{\partial t} + e \nabla \phi$$

$c = 1$

$$\vec{E} = \left(\frac{\partial \vec{A}}{\partial t} + \nabla \phi \right)$$

$$= e \left(\vec{E} + \dot{\vec{x}} \times \vec{B} \right)$$

B) INTRODUCE $x = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = r \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}$

AND $\vec{B} = (0, 0, B)$

REWRITE $\vec{A} = \frac{1}{2}(\vec{B} \times \vec{x}) = \frac{1}{2}\begin{pmatrix} 0 \\ 0 \\ B \end{pmatrix} \times \begin{pmatrix} r\sin\theta\cos\phi \\ r\sin\theta\sin\phi \\ r\cos\theta \end{pmatrix}$

$$= \frac{1}{2} B r \sin\theta \begin{pmatrix} -\sin\phi \\ \cos\phi \end{pmatrix}$$

$$\Rightarrow L = \frac{m_e}{2}|\dot{\vec{x}}|^2 + \frac{e^2}{|\vec{x}|} - \frac{e}{c}\vec{A}(x)\cdot\dot{\vec{x}}$$

SWITCH TO POLAR:

$$\vec{x} \equiv \vec{q} = r\hat{r} \rightarrow \dot{\vec{x}} \equiv \dot{\vec{q}} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + r\dot{\phi}\sin\theta\,\hat{\phi}$$

$$|\vec{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

$$\Rightarrow L = \frac{m_e}{2}\left(\cancel{\sqrt{\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2\sin^2\theta}}\right)^{\cancel{2}} + \frac{e^2}{\cancel{\sqrt{r^{\cancel{2}}}}} - \frac{e}{c}\begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix}$$

$$\Rightarrow L = \frac{m_e}{2}(\text{''}) + \frac{e^2}{r} - \frac{eBr}{2c}\sin\theta\begin{pmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{pmatrix}\begin{pmatrix} \star \end{pmatrix}$$

$$= \frac{m_e}{2}(\text{''}) + \frac{e^2}{r} - \frac{eBr\sin\theta}{2}\left(r\dot{\phi}\sin\theta \right)$$

$$\left[\begin{pmatrix}-\sin\phi\\ \cos\phi\\ 0\end{pmatrix}\begin{pmatrix}\dot r\sin\theta\cos\phi+r\dot\theta\cos\theta\cos\phi-r\dot\phi\sin\theta\sin\phi\\ \dot r\sin\theta\sin\phi+r\dot\theta\cos\theta\sin\phi+r\dot\phi\sin\phi\cos\phi\\ \end{pmatrix}\right.$$

$$\Rightarrow -\cancel{\dot r\sin\theta\sin\phi}\cos\phi-\cancel{r\dot\theta\cos\theta\sin\phi\cos\phi}+r\dot\phi\sin\theta\sin^2\phi$$

$$\left.+\cancel{\dot r\sin\theta\sin\phi\cos\phi}+\cancel{r\dot\theta\cos\theta\sin\phi}\cos\phi+r\dot\phi\sin\theta\cos^2\phi\right]$$

$$\downarrow$$

$$r\dot\phi\sin\theta$$

$$\Rightarrow L=\frac{m_e}{2}\left(\dot r^2+r^2\dot\theta^2+r^2\sin^2\theta\,\dot\phi^2\right)+\frac{e^2}{r}-\frac{eB}{2c}r^2\dot\phi\sin^2\theta$$

$\boxed{c}$ REMEMBER THAT $p_i=\dfrac{\partial L}{\partial\dot q_i}$

ASSOCIATED MOMENTUMS:

$$p_r=\frac{\partial L}{\partial\dot r}=m_e\dot r,\qquad p_\phi=m_e r^2\sin^2\theta\,\dot\phi-\frac{eB}{2c}r^2\sin^2\theta$$

$$p_\theta=\frac{\partial L}{\partial\dot\theta}=m_e r^2\dot\theta,\qquad =m_e r^2\sin^2\theta\left(\dot\phi-\frac{eB}{2cm_e}\right)$$

$$\Rightarrow\ \dot r=\frac{p_r}{m_e},\quad \dot\theta=\frac{p_\theta}{m_e r^2},\quad \dot\phi=\frac{p_\phi}{m_e r^2\sin^2\theta}+\frac{eB}{2cm_e}$$

$$=\frac{2c\,p_\phi+r^2\sin^2\theta\,eB}{2cm_e r^2\sin^2\theta}$$

REMINDER THAT THE HAMILTONIAN IS:

$$H = p_i \dot{q}_i(p_i, q_i) - L\left(q_i, \dot{q}_i(p_i, q_i)\right)$$

$$= \frac{p_r^2}{m_e} + \frac{p_\theta^2}{m_e r^2} + \frac{p_\phi^2}{m_e r^2 \sin^2\theta} + \frac{p_\phi e B}{2 c m_e}$$

$$- \frac{\cancel{m_e}}{2}\left(\frac{p_r^2}{m_e^{\cancel{2}}} + \frac{\cancel{r^2} p_\theta^2}{m_e^{\cancel{2}} r^{\cancel{4}\,2}} + \frac{p_\phi^2 \cancel{r^2 \sin^2\theta}}{m_e^{\cancel{2}} r^{\cancel{4}\,2} \sin^{\cancel{4}\,2}\theta} + \frac{e^2 B^2 r^2\sin^2\theta}{4 c^2 m_e^{\cancel{2}}} + \frac{\cancel{2} p_\phi e^2 B^2 \cancel{r^2\sin^2\theta}}{\cancel{2} c\, m_e^{\cancel{2}} \cancel{r^2 \sin^2\theta}} \right)$$

$$- \frac{e^2}{r} + \frac{eB}{2c} \cancel{r^2} \cancel{\sin^2\theta} \left(\frac{2c p_\phi + r^2 \sin^2\theta\, e B}{2 c m_e \cancel{r^2} \cancel{\sin^2\theta}} \right)$$

$$H = \frac{1}{2m_e}\left(p_r^2 + \frac{1}{r^2} p_\theta^2 + \frac{1}{r^2 \sin^2\theta} p_\phi^2 \right) + \cancel{\frac{p_\phi e B}{2 c m_e}} - \frac{e^2 B^2 r^2 \sin^2\theta}{8 c^2 m_e}$$

$$- \cancel{\frac{p_\phi e^2 B^2}{2 c m_e}} - \frac{e^2}{r} + \frac{p_\phi e B}{2 c m_e} + \frac{r^2 \sin^2\theta\, e^2 B^2}{4 c^2 m_e}$$

$$\frac{\cancel{2} r^2 \sin^2 e^2 B^2 \cancel{- e^2 B^2 r^2 \sin^2}}{8 c^2 m_e}$$

$$\Rightarrow H = \frac{1}{2m_e}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2 \sin^2\theta} \right) - \frac{e^2}{r} + \frac{p_\phi e B}{2 c m_e} + \frac{r^2 \sin^2\theta\, e^2 B^2}{8 c^2 m_e}$$

P_ϕ IS A CONSERVED QUANTITY SINCE THE LAGRANGIAN DOESN DEPEND ON ϕ :

FROM E - L $\quad \frac{d}{dt}\underbrace{\boxed{\frac{\partial L}{\partial \dot{\phi}}}}_{P_\phi} = \underbrace{\boxed{\frac{\partial L}{\partial \phi}}}_{0}$

$\frac{d}{dt} P_\phi = 0$ ONLY IF $P_\phi = C_{onstant}$

H IS CONSERVED IF THE SYSTEM IS TIME INVARIANT: H DOESN'T EXPLICITLY DEPEND ON TIME.

SINCE $\frac{\partial H}{\partial t} = 0 \Rightarrow H$ IS A CONSERVED QUANTITY

D) r = BOHR RADIUS, $p_\phi = \hbar$, $B \lesssim 10^5$ G $\rightsquigarrow 10^{15}$ G

$r \to 10^{-11}$ m, $p_\phi \to 10^{-16}$ eV·s, 10^5 G $\to 10$ T, 10^{15} G $\to 10^{11}$ T

m_e = MASS ELECT. $\to 10^{-31}$ kg

e = CHARGE E. $\to 10^{-19}$ C

c = LIGHT SPEED $\to 10^8$

$0 \leq \sin \leq 1$ [ASSUME IT MAX $\to 1$]

$$H = \frac{p_r^2}{10^{-31}} + \frac{p_\theta^2}{10^{-53}} + \frac{\cancel{10^{-32}}}{10^{-\cancel{53}\,21}} + \frac{10^{-\cancel{38}\,-27}}{10^{\cancel{-11}}} +$$

$$+ \frac{10^{-\cancel{19}} \cdot \cancel{10^{-16}}\,10\ 10^{-4}}{10^{-\cancel{31}\,-12}\ 10^{\cancel{8}\,7}} \quad \frac{1}{10^{11}} + \frac{10^{-\cancel{38}\,-7}\ \cancel{10^{2}} \cdot 10^{-22}}{10^{-\cancel{31}} \cdot 10^{\cancel{16}\,14}} \to \frac{10^{-29}}{10^{14}} \to 1/10^{43}$$

$$H = \frac{1}{2m_e}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2\sin^2\theta}\right) - \frac{e^2}{r} + \frac{p_\phi e B}{2cm_e} + \frac{r^2\sin^2 e^2 B^2}{8c^2 m_e}$$

$p_r^2 \to 10^{31}$, $\frac{p_\theta^2}{r^2} \to 10^{53}$, $\frac{p_\phi^2}{r^2\sin^2\theta} \to 10^{21}$, $\frac{e^2}{r} \to 10^{-27}$, $\frac{p_\phi e B}{2cm_e} \to 10^{-11}$ (1), $\frac{r^2\sin^2 e^2 B^2}{8c^2 m_e} \to 10^{-43}$

LAST TERM IS SMALLER BY DIFFERENT MAGNITUDES, AS SUCH WE CAN CONSIDER IT NEGLIGIBLE

IN A NEWTON STAR, THE LAST TWO POLYNOMIALS WOULD BE IN THE MAGNITUDE OF:

(1) $\frac{1}{10^{11}} \cdot 10^{10} \rightarrow 10^{-1}$

(2) $\frac{1}{10^{43}} \, 10^{21} \rightarrow 10^{-22}$

STILL NEGLIGIBLE COMPARATIVELY TO THE FIRST TERMS

e) $H = \frac{1}{2m_e}\left(P_r^2 + \frac{P_\theta^2}{r^2} + \frac{P_\phi^2}{r^2 \sin^2\theta}\right) - \frac{e^2}{r} + \frac{P_\phi e B}{2 c m_e}$

CONSIDER AN ORBIT SUCH THAT $\theta = \pi/2 \Rightarrow \sin^2\left(\frac{\pi}{2}\right) = 1$

$P_\theta = 0$

SOMMERFELD $P_\phi = m_\phi \hbar$, $E = H$

- $$E = \frac{1}{2m_e}\left(P_r^2 + \frac{m_\phi^2 \hbar^2}{r^2}\right) - \frac{e^2}{r} + \frac{eB}{2cm_e} m_\phi \hbar$$

$$\Rightarrow P_r = \sqrt{2 m_e E - \frac{m_\phi^2 \hbar^2}{r^2} + \frac{2m_e e^2}{r} - \frac{eB m_\phi \hbar}{c}}$$

- EVALUATE SOMMERFIELD CONDITIONS:

$$2\int_{r_-}^{r_+} P_r \, dr = m_r h \quad (1)$$

- FROM CONDITION $P_r = 0$ WE FIND:

$$\frac{c 2m E r^2 - c m_\phi^2 \hbar^2 + c 2m_e e^2 r - eB m_\phi \hbar r^2}{\cancel{c} r^2} = 0$$

$$r^2\left(2cmE - eBm_\phi \hbar\right) + \left(2cme^2\right)r - cm_\phi^2\hbar^2 = 0$$

$$r = \frac{-b \pm \sqrt{b^2-4ac}}{2A}$$

$$= \frac{-\overset{A}{2cme^2} \pm \sqrt{\overset{B}{4c^2m^2e^4 + 4cm_\phi^2\hbar^2\left(2cmE - eBm_\phi\hbar\right)}}}{2(\underset{C}{2cmE - eBm\phi\hbar})}$$

Now (1) turns into

$$n_r h = \sqrt{-8m_eE}\int_{r_-}^{r_+}\frac{1}{r}\sqrt{(r_+-r)(r-r_-)}\,dr = \frac{\pi}{2}\left(\sqrt{r_+}-\sqrt{r_-}\right)^2$$

$$\Rightarrow\ m_r h = \frac{\pi}{2}\left(\underbrace{r_+ + r_-}_{\frac{-A+B}{C}+\frac{-A-B}{C}=\frac{-2A}{C}} - \underbrace{\sqrt{r_+}\sqrt{r_-}}_{\sqrt{\frac{(-A+B)}{C}\frac{(-A-B)}{C}} = \sqrt{\frac{A^2-B^2}{C^2}}}\right)$$

$$\Rightarrow\ \frac{2m_r h}{\pi} = \frac{-2A-\sqrt{A^2-B^2}}{C}$$

$$\frac{2 m_r h}{\pi} = \frac{4c m_e e^2 - \sqrt{\cancel{4c^2 m^2 e^4} - \cancel{4c^2 m^2 e^4} + 4c m_\phi^2 \hbar^2 \left(2c m E - eB m_\phi \hbar\right)}}{2\left(2c m E - eB m_\phi \hbar\right)}$$

$$\Rightarrow \quad E = -\frac{m_e c^2}{2}\left(\frac{e^2}{\hbar c}\right)^2 \frac{1}{(m_r + m_\phi)^2} + \frac{eB\hbar m_\phi}{2 m_e c}$$