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TP4 - ASSIGMENT 3

3.5

1 - $$\tilde{\psi}(k) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dx \, e^{-ikx} \, \psi(x)$$

$$\psi(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dx \, e^{+ikx} \, \tilde{\psi}(k)$$

A) FOURIER TRANSFORM OF

$$\psi(x) = (\pi w_0^2)^{-\frac{1}{4}} \exp\left(i p_0 \frac{x}{\hbar} - \frac{(x-x_0)^2}{2 w_0^2}\right)$$

$$x_0, p_0, w_0 \in \mathbb{R}$$

1 → $$\tilde{\psi}(k) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dx \, \frac{\mathrm{Exp}(-ikx)\, \mathrm{Exp}\left(i p_0 \frac{x}{\hbar} - \frac{(x-x_0)^2}{2 w_0^2}\right)}{\sqrt[4]{\pi w_0^2}}$$

$$= \frac{1}{\sqrt{2\pi} \cdot \sqrt[4]{\pi w_0^2}} \int_{\mathbb{R}} dx \; \mathrm{Exp}(-ikx)\, \mathrm{Exp}\left(\frac{i p_0}{\hbar} x\right) \mathrm{Exp}\left(-\frac{(x-x_0)^2}{2 w_0^2}\right)$$

MULTIPLY BY : → $\dfrac{\mathrm{Exp}(-ikx_0)}{\mathrm{Exp}(-ikx_0)} \cdot \dfrac{\mathrm{Exp}(i p_0/\hbar \, x_0)}{\mathrm{Exp}(i p_0/\hbar \, x_0)}$

$$= \underbrace{\frac{\text{Exp}(-ikx_0)\cdot\text{Exp}(ip_0/\hbar\, x_0)}{\sqrt{2\pi}\sqrt[4]{\pi\omega_0^2}}}_{\text{CALL IT } W \text{ FOR NOW}} \int_{\mathbb{R}} dx\, \text{Exp}(ik(x-x_0))\,\text{Exp}\left(\frac{ip_0}{\hbar}(x-x_0)\right) \text{Exp}\left(-\frac{(x-x_0)^2}{2\omega_0^2}\right)$$

$\rightarrow \quad x' = x - x_0 \qquad dx' = dx$

$$= W \int_{\mathbb{R}} dx'\, \text{Exp}(ikx')\,\text{Exp}\left(\frac{ip_0}{\hbar}x'\right)\text{Exp}\left(-\frac{x'^2}{2\omega_0^2}\right)$$

$$= \underbrace{W \int_{\mathbb{R}} dx'\, \text{Exp}\left(-\underbrace{\boxed{\frac{1}{2\omega_0^2}}}_{\rightarrow a} x'^2 + \underbrace{\boxed{i\left(k + \frac{p_0}{\hbar}\right)}}_{\rightarrow b} x'\right)}$$

$\rightarrow$ KNOWN GAUSSIAN INTEGRAL IN THE FORM:

$$\int_{-\infty}^{+\infty} e^{-ax^2+bx}\,dx = e^{\frac{b^2}{4a}}\sqrt{\frac{\pi}{a}}$$

$$= W\, \text{Exp}\left(\frac{\overset{\rightarrow -1}{i^2}\,(k + p_0/\hbar)^2}{\frac{2}{\omega_0^2}}\right)\sqrt{2\pi\omega_0^2}$$

$$= \frac{\text{Exp}(-ikx_0)\,\text{Exp}\left(i\frac{p_0}{\hbar}x_0\right)\text{Exp}\left(-\frac{\omega_0^2}{2}\left(k+\frac{p_0}{\hbar}\right)^2\right)\sqrt{2\pi\omega_0^2}}{\sqrt{2\pi}\sqrt[4]{\pi\omega_0^2}}$$

$$= \mathrm{EXP}\left(i\left(\frac{p_0}{\hbar} - k\right)x_0 - \frac{\omega_0^2}{2}\left(k + \frac{p_0}{\hbar}\right)^2 \right) \quad \frac{\omega_0}{\sqrt[4]{\pi \omega_0^2}}$$

$$\Rightarrow \boxed{\tilde{\psi}(k) = e^{i\left(\frac{p_0}{\hbar} - k\right)x_0} \cdot e^{-\frac{\omega_0^2}{2}\left(k + \frac{p_0}{\hbar}\right)^2} \cdot \frac{\omega_0}{\sqrt[4]{\pi \omega_0^2}}}$$

VERIFY BY NORMALIZING:

$$\int_{\mathbb{R}} dk \, |\psi(k)|^2 = 1 \quad \Rightarrow \quad \int_{\mathbb{R}} \psi(k) \cdot \psi^*(k) \, dk = 1$$

$e^0 = 1$

$$\Rightarrow \int_{\mathbb{R}} e^{i\left(\frac{p_0}{\hbar} - k\right)x_0} \cdot e^{-\frac{\omega_0^2}{2}\left(k + \frac{p_0}{\hbar}\right)^2} \cdot \frac{\omega_0}{\sqrt[4]{\pi \omega_0^2}} \cdot e^{-i\left(\frac{p_0}{\hbar} - k\right)x_0} \cdot e^{-\frac{\omega_0^2}{2}\left(k + \frac{p_0}{\hbar}\right)^2} \cdot \frac{\omega_0}{\sqrt[4]{\pi \omega_0^2}}$$

$$= \frac{\omega_0}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-\underbrace{\omega_0^2\left(k + \frac{p_0}{\hbar}\right)^2}} \, dk$$

$\Rightarrow$ GAUSSIAN: $\int_{\mathbb{R}} e^{-a(x+b)^2} = \sqrt{\frac{\pi}{a}}$

$$= \frac{\cancel{\omega_0}}{\cancel{\sqrt{\pi}}} \sqrt{\frac{\cancel{\pi}}{\cancel{\omega_0^2}}} = 1 \quad \checkmark\checkmark$$

$\boxed{B}$ SHOW $\langle \hat{x} \rangle = \langle \psi | \hat{x} \psi \rangle = x_0$

$\langle \hat{p} \rangle = \langle \psi | \hat{p} \psi \rangle = p_0$

WITH $(\hat{p}\psi)(x) = -i\hbar \frac{d}{dx} \psi(x)$

$(\hat{x}\psi)(x) = x\psi(x)$ AND

$$\langle \psi | \phi \rangle = \int_{\mathbb{R}} \overline{\psi(x)}\, \phi(x)\, dx$$

(1) $$\langle \psi(x) | (\hat{x}\psi)(x) \rangle = \int_{\mathbb{R}} \overline{\psi(x)} \cdot (\hat{x}\psi)(x)\, dx$$

$$= \int_{\mathbb{R}} \overline{\psi}(x)\, \psi(x)\, x \quad dx$$

$$= \int_{\mathbb{R}} (\pi w_0^2)^{-\frac{1}{4}} \cdot \mathrm{EXP}\left(-i\, p_0 \frac{x}{\hbar}\right) \mathrm{Ex}\left(-\frac{(x-x_0)^2}{2w_0^2}\right) .$$

EXP(0) = 1

$$(\pi w_0^2)^{-\frac{1}{4}} \cdot \mathrm{EXP}\left(i\, p_0 \frac{x}{\hbar}\right) \mathrm{Ex}\left(-\frac{(x-x_0)^2}{2w_0^2}\right) . \quad x$$

$$= \frac{1}{\sqrt{\pi w_0^2}} \int_{\mathbb{R}} \mathrm{EXP}\left(-\frac{\cancel{2}(x-x_0)^2}{\cancel{2}w_0^2}\right) x \; dx$$

$$x' = x - x_0 \quad \rightarrow \quad x = x' + x_0$$

$$dx = dx'$$

$$\Rightarrow \quad = \frac{1}{\sqrt{\pi \omega_0^2}} \int_{\mathbb{R}} \mathrm{EXP}\left(-\frac{1}{\omega_0^2}(x')^2\right)(x' + x_0)\, dx$$

$$= \frac{1}{\sqrt{\pi \omega_0^2}} \left(\cancel{\int_{-\infty}^{\infty} \underbrace{e^{-\frac{1}{\omega^2}(x')^2}}_{\text{EVEN F}} \underbrace{(x')}_{\text{ODD F}} dx} \rightarrow 0 + x_0 \underbrace{\int_{-\infty}^{\infty} e^{-\frac{1}{\omega^2}(x')^2} dx}_{} \right)$$

$\Downarrow$

ODD F EVEN

A ± INTERVAL = 0

GAUSSIAN IN THE FORM

$$\int_{-\infty}^{\infty} e^{-a x^2} dx = \sqrt{\frac{\pi}{a}}$$

$$= \omega_0 \sqrt{\pi}$$

$$\Rightarrow \quad \langle \hat{x} \rangle = \frac{x_0 \cancel{\omega_0 \sqrt{\pi}}}{\cancel{\omega_0 \sqrt{\pi}}} = x_0$$

(2) $\langle \hat{p} \rangle = \langle \psi(x) | (\hat{p}\psi)(x) \rangle$

$$= \int_{\mathbb{R}} dx \, \overline{\psi(x)} \left(\hat{p} \psi \right)(x) = \int_{\mathbb{R}} dx \, \overline{\psi}(x) \left(-i\hbar \frac{d}{dx} \psi(x) \right)$$

$$\frac{d}{dx}\psi(x) = \frac{d}{dx} \left(\pi w_0^2\right)^{-\frac{1}{4}} \cdot \mathrm{EXP}\left(i p_0 \frac{x}{\hbar} \right) \mathrm{EXP}\left(-\frac{(x-x_0)^2}{2w_0^2} \right)$$

$$= \left(\pi w_0^2\right)^{-\frac{1}{4}} \frac{d}{dx} \mathrm{EXP}\left(i p_0 \frac{x}{\hbar} - \frac{(x^2 + x_0^2 - 2x_0 x)}{2w_0^2} \right)$$

$$= \left(\pi w_0^2\right)^{-\frac{1}{4}} \frac{d}{dx} \mathrm{EXP}\left(\frac{2w_0^2 i p_0 x - \hbar x^2 - \hbar x_0^2 + 2 x_0 \hbar x}{2 w_0^2 \hbar} \right)$$

$$= \left(\pi w_0^2\right)^{-\frac{1}{4}} \frac{d}{dx} \mathrm{EXP}\left(-\frac{1}{2w_0^2} x^2 + \frac{(w_0^2 i p_0 + x_0 \hbar) x}{w_0^2 \hbar} - \frac{x_0^2}{2 w_0^2} \right)$$

$$= \left(\pi w_0^2\right)^{-\frac{1}{4}} \mathrm{EXP}\left(i p_0 \frac{x}{\hbar} \right) \mathrm{EXP}\left(-\frac{(x-x_0)^2}{2w_0^2} \right) \left(-\frac{x}{w_0^2} + \frac{i w_0^2 p_0 + x_0 \hbar}{w_0^2 \hbar} \right)$$

$$\Rightarrow \langle \hat{p} \rangle = \int_{\mathbb{R}} dx \left(\pi w_0^2\right)^{-\frac{1}{4}} \cdot \mathrm{EXP}\left(-i p_0 \frac{x}{\hbar} \right) \mathrm{EXP}\left(-\frac{(x-x_0)^2}{2w_0^2} \right)$$

e^0

$$(-i\hbar)\left(\pi w_0^2\right)^{-\frac{1}{4}} \mathrm{EXP}\left(i p_0 \frac{x}{\hbar} \right) \mathrm{EXP}\left(-\frac{(x-x_0)^2}{2w_0^2} \right) \left(-\frac{x}{w_0^2} + \frac{i w_0^2 p_0 + x_0 \hbar}{w_0^2 \hbar} \right)$$

$$= \left(\pi\omega_0^2\right)^{-\frac{1}{2}}(-i\hbar)\int_{\mathbb{R}} dx \left(\mathrm{Exp}\left(-\frac{(x-x_0)^2}{\omega_0^2}\right)\left(-\frac{x}{\omega_0^2} + \frac{i\omega_0^2 p_0 + x_0\hbar}{\omega_0^2\hbar}\right)\right)$$

$$= \left(\pi\omega_0^2\right)^{-\frac{1}{2}}(-i\hbar)\left(\underbrace{\int_{\mathbb{R}} dx\ \mathrm{Exp}\left(-\frac{(x-x_0)^2}{\omega_0^2}\right)\left(-\frac{x}{\omega_0^2}\right)}_{1} + \int_{\mathbb{R}} dx\ \mathrm{Exp}\left(-\frac{(x-x_0)^2}{\omega_0^2}\right)\left(\frac{i\omega_0^2 p_0 + x_0\hbar}{\omega_0^2\hbar}\right)\right)$$

$$x' = x - x_0 \quad \Rightarrow \quad x = x' + x_0$$
$$\Rightarrow \quad dx = dx'$$

$$= \int_{\mathbb{R}} dx'\ \mathrm{Exp}\left(-\frac{1}{\omega_0^2}x'^2\right)\left(-\frac{x'}{\omega_0} - \frac{x_0}{\omega_0}\right)$$

LIKE BEFORE

$$= -\frac{1}{\omega_0^2}\cancelto{0}{\int_{\mathbb{R}} dx'\ \underbrace{\mathrm{Exp}\left(-\frac{1}{\omega_0^2}x'^2\right)}_{\text{EVEN}}\underbrace{x'}_{\text{ODD}}} - \frac{x_0}{\omega_0}\underbrace{\int_{\mathbb{R}} dx'\ e^{-\frac{1}{\omega_0^2}x'^2}}$$

$\Rightarrow$ ODD f on $\int_{-\infty}^{\infty} = 0$

GAUSSIAN:

$$\int_{-\infty}^{\infty} dx\ e^{-\alpha x^2} = \sqrt{\frac{\pi}{\alpha}}$$
$$= \sqrt{\pi\omega_0^2}$$

$$= -\frac{x_0\cancel{\omega_0}\sqrt{\pi}}{\omega_0^{\cancel{2}}} = -\frac{\sqrt{\pi}\,x_0}{\omega_0}$$

$$= \left(\pi\omega_0^2\right)^{-\frac{1}{2}}(-i\hbar)\left(-\sqrt{\pi}\,x_0 + \int_{\mathbb{R}} dx'\ \mathrm{Exp}\left(-\frac{(x')^2}{\omega_0^2}\right)\left(\frac{i\omega_0^2 p_0 + x_0\hbar}{\omega_0^2\hbar}\right)\right)$$

$$= (\pi \omega_0^2)^{-\frac{1}{2}} (-i\hbar) \left(-\frac{\sqrt{\pi}\, x_0}{\omega_0} + \left(\frac{i\omega_0^2 p_0 + x_0 \hbar}{\omega_0^{\cancel{2}} \hbar} \right) \underbrace{\int_{\mathbb{R}} dx' \, \mathrm{Exp}\left(-\frac{1}{\omega_0^2} x'^2 \right)}_{= \cancel{\omega_0} \sqrt{\pi}} \right)$$

$$= \frac{(-i\hbar)}{\omega_0 \cancel{\sqrt{\pi}}} \left(-\frac{\cancel{\sqrt{\pi}}\, X_0}{\omega_0} + \frac{\cancel{\sqrt{\pi}}\, i\omega_0^2 p_0 + \cancel{\sqrt{\pi}}\, X_0 \hbar}{\omega_0 \hbar} \right)$$

$$= \left(\frac{-i\cancel{\hbar}}{\omega_0} \right) \left(\frac{\cancel{-\hbar x_0} + i\omega_0^2 p_0 + \cancel{x_0 \hbar}}{\omega_0 \cancel{\hbar}} \right)$$

$$= \frac{\overbrace{-i\,i}^{\to 1}\, \cancel{\omega_0^2}}{\cancel{\omega_0^2}} p_0 = p_0$$

c) $\phi(x) = (\pi w_0^2)^{-\frac{1}{4}} e^{i p_1 \frac{x}{\hbar} - (x-x_1)^2/(2w_0^2)}$

$$= (\pi w_0^2)^{-\frac{1}{4}} \operatorname{EXP}\left(i p_1 \frac{x}{\hbar}\right) \operatorname{EXP}\left(-\frac{(x-x_1)^2}{2w_0^2}\right)$$

CALCULATE $\langle \psi | \phi \rangle = \int_{\mathbb{R}} \overline{\psi}\, \phi \, dx$

$$\left(-\frac{(x-x_1)^2 + (x-x_0)^2}{2w_0^2}\right)$$

$$\Downarrow$$

$$= (\pi w_0^2)^{-\frac{1}{2}} \int_{\mathbb{R}} dx \operatorname{EXP}\left(i \frac{x}{\hbar}(p_1 - p_0)\right) \operatorname{EXP}\left(-\frac{2x^2 - 2(x_0+x_1)x + x_1^2 + x_0^2}{2w_0^2}\right)$$

$$\frac{2i w_0^2 (p_1 - p_0) x - 2\hbar x^2 + 2\hbar (x_0 + x_1) x - (x_1^2 + x_0^2)\hbar}{2\hbar w_0^2}$$

$$= (\pi w_0^2)^{-\frac{1}{2}} \int_{\mathbb{R}} dx \operatorname{EXP}\left(\frac{-2\hbar x^2 + 2\left[i w_0^2 (p_1 - p_0) + \hbar (x_1 + x_0)\right] x - (x_1^2 + x_0^2)\hbar}{2\hbar w_0^2}\right)$$

$$= (\pi w_0^2)^{-\frac{1}{2}} \operatorname{EXP}\left(-\frac{(x_1^2 + x_0^2)}{2w_0^2}\right) \int_{\mathbb{R}} dx \operatorname{EXP}\left[-\overbrace{\boxed{\frac{1}{w_0^2}}}^{a} x^2 + \overbrace{\boxed{\frac{\left(i w_0^2 (p_1 - p_0) + \hbar (x_1 + x_0)\right)}{\hbar w_0^2}}}^{b} x\right]$$

GAUSSIAN IN THE FORM $\int_{-\infty}^{\infty} e^{-ax^2 + bx} dx = e^{\frac{b^2}{4a}} \sqrt{\frac{\pi}{a}} \longrightarrow$

$$\rightarrow \left[\mathrm{Exp}\left(\frac{\left(i\omega_0^2(p_1-p_0)+\hbar(x_1+x_0)\right)^2}{\hbar^2\omega_0^{\cancel{4}\,2}} \cdot \frac{\cancel{\omega_0^2}}{4} \right) \omega_0\sqrt{\pi} \right]$$

$$= \frac{\cancel{\omega_0\sqrt{\pi}}}{\cancel{\omega_0\sqrt{\pi}}}\ \mathrm{Exp}\left(-\frac{(x_1^2+x_0^2)}{2\omega_0^2} + \frac{1}{4\hbar^2\omega_0^2}\underbrace{\left(i\omega_0^2(p_1-p_0)+\hbar(x_1+x_0)\right)^2}\right)$$

$$\rightarrow\ -\omega_0^4(p_1-p_0)^2+\hbar^2(x_1+x_0)^2+2i\hbar\,\omega_0^2(p_1-p_0)(x_1+x_0)$$

$$= \mathrm{Exp}\left[\frac{-2\hbar^2x_1^2-2\hbar^2x_0^2-\omega_0^4(p_1-p_0)^2+\hbar^2(x_1+x_0)^2+i\hbar\omega_0^2(p_1-p_0)(x_1+x_0)}{4\omega_0^2\hbar^2} \right]$$

$$= \mathrm{Exp}-\left[\frac{\overbrace{2\hbar^2x_1^2+2\hbar^2x_0^2-\hbar^2(x_1+x_0)^2}^{\hbar^2(2x_1^2+2x_0^2-x_1^2-x_0^2-2x_0x_1)\,=\,\hbar^2(x_1^2+x_0^2-2x_0x_1)\,=\,\hbar^2(x_1-x_0)^2}+\omega_0^4(p_1-p_0)^2-2i\hbar\omega_0^2(p_1-p_0)(x_1+x_0)}{4\omega_0^2\hbar^2} \right]$$

$$= \mathrm{Exp}-\left[\frac{\hbar^2(x_1-x_0)^2+\omega_0^4(p_1-p_0)^2-2i\hbar\omega_0^2(p_1-p_0)(x_1+x_0)}{4\omega_0^2\hbar^2} \right]$$

$\boxed{D}$ WE KNOW THE OPERATORS $\hat{x}$, $\hat{p}$ ARE SELF-ADJOINT:

THAT IS: $\begin{cases} \hat{x} = \hat{x}^+ \\ \hat{p} = \hat{p}^+ \end{cases}$

TO PROVE THIS:

FOR $\hat{x}$: ADJOINT IS SUCH THAT

$$\langle \phi | \hat{x} \psi \rangle = \langle \hat{x}^+ \phi | \psi \rangle$$

LHS:

$$\langle \phi | \hat{x} \psi \rangle = \int \phi^* x \psi \, dx$$

RHS:

$$\langle \hat{x}^+ \phi | \psi \rangle = \int (\hat{x}^+ \phi)^* \psi$$

$$= \int (\hat{x}^+)^* \phi^* \psi \, dx$$

EIGENV. OF $\hat{x} \to \hat{x}^+$ ARE REAL: $x^* = x$

$$= \int x \phi^* \psi \, dx$$

$$\Rightarrow \hat{x} = \hat{x}^+$$

FOR $\hat{p}$:

$$\langle \phi | \hat{p} \psi \rangle = \langle \hat{p}^{\dagger} \phi | \psi \rangle$$

LHS: $$\langle \phi | \hat{p} \psi \rangle = \int \phi^{*} (-i\hbar) \frac{\partial \psi}{\partial x} \, dx$$

RHS: $$\langle \hat{p}^{\dagger} \phi | \psi \rangle = \int (\hat{p}^{\dagger} \phi)^{*} \psi \, dx$$

$$= \int \left(i\hbar \frac{\partial \phi^{*}}{\partial x} \right) \psi \, dx$$

$$= i\hbar \int \frac{\partial \phi^{*}}{\partial x} \psi \, dx$$

BY PARTS:

$$= i\hbar \left(\cancel{\left[\phi^{*} \psi \right]_{-\infty}^{\infty}} - \int \phi^{*} \left(\frac{\partial \psi}{\partial x} \right) dx \right)$$

IGNORE BOUND. COND.

$$= \int \phi^{*} (-i\hbar) \frac{\partial \psi}{\partial x} \, dx \iff \hat{p} = \hat{p}^{\dagger}$$

THEN, WHEN DEALING WITH COMPOSITE VECTORS:

- $\hat{X}\hat{p} + \hat{p}\hat{x} \xrightarrow{\text{ADJOINT}} (\hat{X}\hat{p} + \hat{p}\hat{x})^{\dagger} = (\hat{X}\hat{p})^{\dagger} + (\hat{p}\hat{x})^{\dagger} =$

$= \hat{p}^{\dagger}\hat{x}^{\dagger} + \hat{x}^{\dagger}\hat{p}^{\dagger}$ BUT $\hat{x} = \hat{x}^{\dagger}$, $\hat{p} = \hat{p}^{\dagger}$

$\Rightarrow \quad = \hat{p}\hat{x} + \hat{x}\hat{p}$

WITH A TEST FUNCTION:

$$(\hat{p}\hat{x} + \hat{x}\hat{p})\psi(x) = -i\hbar\frac{\partial}{\partial x}(x\cdot\psi(x)) + x(-i\hbar)\frac{\partial}{\partial x}\psi(x)$$

$$= -i\hbar\left(\psi(x) + x\frac{\partial}{\partial x}\psi(x) + x\frac{\partial}{\partial x}\psi(x)\right)$$

$$= -i\hbar\left(\psi(x) + 2x\frac{\partial}{\partial x}\psi(x)\right)$$

$$\Rightarrow \quad \hat{p}\hat{x} + \hat{x}\hat{p} = -i\hbar\left(1 + 2x\frac{\partial}{\partial x}\right)$$

- $\hat{x}\hat{p}^m\hat{x} \xrightarrow{\text{ADJOINT}} (\hat{x}\hat{p}^m\hat{x})^{\dagger} =$

$= (\hat{p}^m\hat{x})^{\dagger}\hat{x}^{\dagger} = \hat{x}^{\dagger}(\hat{p}^m)^{\dagger}\hat{x}^{\dagger}$

ASSUME $m = 3$: $\hat{p}^3 = (\hat{p}\hat{p}\hat{p})^\dagger = \hat{p}^\dagger \hat{p}^\dagger \hat{p}^\dagger = \hat{p}\hat{p}\hat{p} = \hat{p}^3$

$\Rightarrow$ ADJOINT TO ITSELF: $(\hat{x}\hat{p}_m\hat{x})^\dagger = \hat{x}\hat{p}^m\hat{x}$

$$= x\left(\frac{\partial^m}{\partial x^m} x\right)$$

$\downarrow$

WITH TEST FUNCTION:

$$(\hat{x}\hat{p}_m\hat{x})\,\psi(x) = -i\hbar\, x \frac{\partial^m}{\partial x^m}(x\,\psi(x))$$

$\downarrow$

LEIBNIZ RULE:

$$(fg)^{(m)} = \sum_{k=0}^{m}\binom{m}{k} f^{(m-k)} g^{(k)}$$

$$\frac{\partial^{m-1}}{\partial x^{m-1}}\left(\frac{\partial}{\partial x}(x\,\psi(x))\right) = \frac{\partial^{m-1}}{\partial x^{m-1}}\left(\psi(x) + x\frac{\partial}{\partial x}\psi(x)\right)$$

$$= \frac{\partial^{m-2}}{\partial x^{m-2}}\left(\frac{\partial}{\partial x}\psi(x) + \frac{\partial}{\partial x}\psi(x) + x\frac{\partial^2}{\partial x^2}\psi(x)\right)$$

$$= \frac{\partial^{m-3}}{\partial x^{m-3}}\left(2\frac{\partial^2}{\partial x^2}\psi(x) + \frac{\partial^2}{\partial x^2}\psi(x) + x\frac{\partial^3}{\partial x^3}\psi(x)\right)$$

...

$$= \left[(m)\,\partial_{(m-1)}\psi(x) + x\,\partial_m\psi(x)\right]$$

$$\Rightarrow \left(\hat{x}\,\hat{p}^m\,\hat{x}\right) = -i\hbar\left(m\,\partial_{m-1} + x\,\partial_m\right)$$

- $$(\hat{x}\,\hat{p})^m = \hat{x}^m\,\hat{p}^m \longrightarrow \left(\hat{x}^m\,\hat{p}^m\right)^+ \longrightarrow \hat{p}^m\,\hat{x}^m$$

$\downarrow$ WITH TEST FUNCTION:

ψ

$$\left(\hat{p}^m\hat{x}^m\right)\psi(x) = -i\hbar\,\frac{\partial^m}{\partial x^m}\left(x^m\,\psi(x)\right)$$

$$= \frac{\partial^{m-1}}{\partial x^{m-1}}\left(m\,x^{m-1}\,\psi(x) + x^m\,\frac{\partial}{\partial x}\psi(x)\right)$$

$$= \frac{\partial^{m-2}}{\partial x^{m-2}}\left(m\left((m-1)\,x^{m-2}\,\psi(x) + x^{m-1}\,\frac{\partial}{\partial x}\psi(x)\right) + m\,x^{m-1}\,\frac{\partial}{\partial x}\psi(x) + x^m\,\frac{\partial^2}{\partial x^2}\psi(x)\right)$$

$$= \partial_{x^{m-2}}\left(m(m-1)\,x^{m-2}\,\psi + m\,x^{m-1}\,\partial_x\psi + m\,x^{m-1}\,\partial_x\psi + x^m\,\partial_{x^2}\psi\right)$$

$$\longrightarrow \quad m!\,x^m\,\psi + 2m\,x^{m-1}\,\frac{\partial\psi}{\partial x^{(m-1)}} + x^m\,\frac{\partial}{\partial x^m}\psi\,)$$

$$\Rightarrow (\hat{p}\,\hat{x})^m = \left(m!\,x^m + 2m\,x^{m-1}\,\partial_{x^{(m-1)}} + x^m\,\partial_{x^m}\right)$$

e) COMPUTE VARIANCE FOR $\psi(x)$

- VARIANCE OF $\hat{x}$:

$$\Delta \hat{x} = \sqrt{\langle (\hat{x} - \langle \hat{x} \rangle)^2 \rangle}$$

FOR $\langle \hat{x} \rangle = \langle \psi | \hat{x} \psi \rangle =$ WE FOUND IT BEFORE EQUAL TO x_0:

$$\Delta \hat{x} = \sqrt{\langle (\hat{x} - x_0)^2 \rangle}$$

$$\langle (\hat{x} - x_0)^2 \rangle = \langle \psi | (\hat{x} - x_0)^2 \psi \rangle$$

$$= \int_{\mathbb{R}} \overline{\psi} \cdot (\hat{x} - x_0)^2 \psi \, dx$$

$$= \int_{\mathbb{R}} \overline{\psi} \left(\hat{x}^2 + x_0^2 - 2 x_0 \hat{x} \right) \psi$$

$$= \int_{\mathbb{R}} \overline{\psi} \left(x^2 + x_0^2 - 2 x x_0 \right) \psi$$

$\psi \cdot \overline{\psi} =$ FOUND BEFORE;

$$= \frac{1}{\sqrt{\pi \omega_0^2}} \int_{\mathbb{R}} \operatorname{EXP}\left(-\frac{\cancel{2}(x-x_0)^2}{\cancel{2}\omega_0^2}\right) (x-x_0)^2 \, dx$$

$$(x - x_0) = x' \quad dx = dx'$$

$$\Rightarrow \quad \frac{1}{\sqrt{\pi \omega_0^2}} \int_{\mathbb{R}} \operatorname{EXP}\left(-\frac{1}{\omega_0^2} x'^2\right) x'^2 \, dx'$$

SOLVED BEFORE BY PARTS:

$$\Rightarrow \; = \frac{1}{\sqrt{\pi \omega_0^2}} \frac{\sqrt{\pi \omega_0^6}}{2} = \frac{1}{2} \frac{\omega_0^{\cancel{3}\,2}}{\cancel{\omega_0}} = \frac{1}{2}\omega_0^2$$

$$\Rightarrow \quad \Delta \hat{X} = \sqrt{\frac{\omega_0^2}{2}} = \frac{|\omega_0|}{\sqrt{2}}$$

- VARIANCE OF $\hat{p}$:

$$\Delta \hat{p} = \sqrt{\langle (\hat{p} - \underbrace{\langle \hat{p} \rangle}_{p_0})^2 \rangle} = \sqrt{\langle (\hat{p} - p_0)^2 \rangle}$$

$$\langle (\hat{p} - p_0)^2 \rangle = \langle \psi \,|\, (\hat{p} - p_0)^2 \psi \rangle$$

$$= \int_{\mathbb{R}} \overline{\psi} \cdot (\hat{p} - p_0)^2 \psi \, dx$$

$$= \int_{\mathbb{R}} \bar{\Psi} \left(\hat{p}^2 + p_0^2 - 2 p_0 \hat{p} \right) \psi \, dx$$

INTEGRATING BY PARTS AGAIN, WE FIND THAT

$$\langle (\hat{p} - p_0)^2 \rangle = \frac{\hbar^2}{2 \omega_0^2}$$

$$\Rightarrow \quad \Delta \hat{p} = \frac{\hbar}{|\omega_0| \sqrt{2}}$$

F)

6)

$$\hat{H} = \frac{1}{2m}\hat{p}^2 = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}$$

SOLVE TIME DEPENDENT SCHRÖDINGER EQUATION:

$$i\hbar \frac{d}{dt}\psi_t(x) = \hat{H}\,\psi_t(x) \quad , \quad \psi_0(x) = \psi(x) \quad [\text{IN } A]$$