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http://www.ist.uni-stuttgart.de/education/courses/OptimalControl/

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Homework exercise 1

Solution of finite horizon open loop optimal control problems via nonlinear programming

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Consider the following optimal control problem

$$\min \quad J = \int_0^{t_f} x(t)^\top x(t) + \alpha u^2(t)\, dt \tag{1a}$$

$$\text{s.t.} \quad \dot{x}(t) = \underbrace{\begin{bmatrix} 0 & -1 \\ -1 & 2 \end{bmatrix}}_{=:A} x(t) + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{=:B} u(t) \tag{1b}$$

with $\alpha > 0$ and $x(0) = x_0 \in \mathbb{R}^2$. We want to obtain an approximate solution of this problem by transforming the infinite dimensional optimal control problem into a finite dimensional nonlinear program via discretization. In the following we consider the time discretization $t_k := kh$, $h = \frac{t_f}{N}$, $N > 0$, and assume that the control input $u(t)$ is piecewise constant on the equidistant time intervals $[t_k, t_{k+1})$.

a) Discretize the cost functional (1a) by using an appropriate approximation method for the integral, e.g. the rectangle method.

b) Utilize the general representation of the solution of a linear differential equation given by

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)} Bu(\tau) d\tau$$

in order to verify that the discretization of (1b) is given by

$$x_{k+1} = (e^{Ah})\; x_k + (\int_0^h e^{A\tau} B d\tau)\; u_k$$

with $x_k = x(t_k)$, $u_k = u(t_k)$.

c) Utilize the Euler approximation of the derivative

$$\dot{x}(t_k) \approx \frac{x(t_{k+1}) - x(t_k)}{h}$$

to derive another discretization of (1b). How are the discretizations from b) and c) related?

In the following we consider the discretization of the optimal control problem (1) given by

$$\begin{aligned} \min \quad & \sum_{k=0}^{N} g(x_k, u_k) \\ \text{s.t.} \quad & x_{k+1} = A_D x_k + B_D u_k \end{aligned} \tag{2}$$

where g, A_D and B_D were determined in a) - c).

d) Define a suitable optimization variable y to write the discretized problem (2) as a quadratic program of the form

$$\begin{aligned} \min_{y} \quad & \frac{1}{2} y^\top H y + f^\top y \\ \text{s.t.} \quad & A_{ineq} y \leq b_{ineq} \\ & A_{eq} y = b_{eq}. \end{aligned} \tag{3}$$

e) Derive the necessary conditions at an optimum of (3) and argue whether or not they are also sufficient. Give the optimal solution y^* in terms of H, f, A_{eq}, b_{eq}, A_{ineq}, b_{ineq}.

f) Let $t_f = 5$, $\alpha = 0.2$, $N = 50$ and $x(0) = \begin{bmatrix} 1 & -2 \end{bmatrix}^\top$. Solve the quadratic program (3) numerically with the help of MATLAB by solving the KKT-conditions from e) as well as by using the function `quadprog`. Compare the results of both discretization methods from b) and c) and provide plots of x_k and u_k for both cases.

g) How does a change in the parameter α affect the solution and how can you explain this behavior? Provide plots of x_k and u_k for $\alpha = 0.001$, $\alpha = 0.2$ and $\alpha = 10$.

h) Simulate the continuous-time system (1b) applying the piecewise continuous optimal input obtained in f) when using the exact discretization. How are the closed loop solutions of the continuous-time system and the discrete-time system related? Provide a plot of the error between the two solutions at the discretization time points t_k.