[Performance of a Vertical Fin Array in a Naturally Convective Environment]

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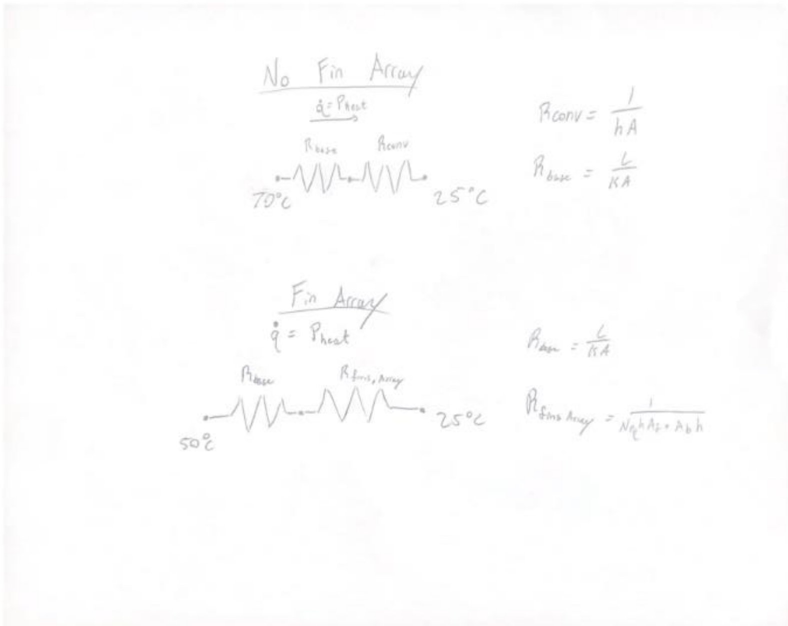
Project Description

The project consists of designing of a finned heat sink able to decrease the temperature of a small heater of $40mm \times 40mm$ area from 70℃ to 50℃ under natural convection (no fans). The heat sink must be designed using Ice9 Flex TPU from TCPoly. The room air temperature is set at 25℃ and the heater, with the heat sink, is mounted vertically on the wall. The heater can be assumed to be a thin film.

Assumptions

1. Natural Convection
2. Rectangular Fins
3. T_{film} = 50℃ with no heat exchanger, T_{film} = 37.5℃ with heat exchanger
4. $T_{surface}$ = 70℃ with no heat exchanger, $T_{surface}$ = 47.5℃ with heat exchanger
5. Thin film
6. Constant Thermal Properties
7. Symmetrical Isothermal plates

System Thermal Resistance Network



Analysis

The heat sink design typically utilizes rectangular fins or pin fins (elongated individual structures), available in various orientations and sizes. For this project, rectangular fins were chosen due to their simplicity and effectiveness. Given the vertical orientation and the limited footprint available, rectangular fins were deemed the most suitable for cooling the heater.

To accurately determine the heater power, fin efficiency, and fin effectiveness, specific design constants needed to be selected. These include the base thickness of the fin array, fin length, width, thickness, number of fins, and fin spacing. Maintaining consistency in these parameters simplifies the calculations required to achieve the project's objectives.

Although these design choices contribute to meeting the project's goals, additional calculations and research were necessary to fully understand the thermal conditions being addressed. In scenarios without a heat exchanger, standard equations for the Rayleigh number and Nusselt number for a vertical plate can be applied, as shown below.

$$Ra_L = \frac{g\beta(T_s - T_\infty)L^3}{\upsilon\alpha}$$

$$Nu_L = \{0.825 + \frac{0.387 Ra_L^{\frac{1}{6}}}{\left[1 + \left(\frac{0.492}{\mathrm{Pr}}\right)^{\frac{9}{16}}\right]^{\frac{8}{27}}}\}^2$$

After determining the Rayleigh number and Nusselt number, the value for the Convective Heat Transfer Coefficient symbolized by *h* can be determined through the equation below.

$$h = \frac{Nu_L k}{L_c}$$

However, these equations are not applicable for the system with the heat exchanger attached to the heater. Through investigation in research papers online in conjunction with the use of the textbook, the proper equations to calculate Rayleigh's number and Nusselt number in the case of a finned heat exchanger for parallel plate convection. These Equations can be seen below.

$$Ra_S = \frac{g\beta(T_s - T_\infty)S^3}{\upsilon\alpha}$$

$$Nu_S = [\frac{C_1}{\left(\frac{Ra_S S}{L}\right)^2} + \frac{C_2}{\left(\frac{Ra_S S}{L}\right)^{\frac{1}{2}}}]^{-1/2}$$

TABLE 9.4 Heat transfer parameters for free convection between vertical parallel plates

Surface Condition	C_1	C_2	S_{opt}	S_{max}/S_{opt}
Symmetric isothermal plates ($T_{s,1} = T_{s,2}$)	576	2.87	$2.71\,(Ra_S/S^3L)^{-1/4}$	1.71
Symmetric isoflux plates ($q''_{s,1} = q''_{s,2}$)	48	2.51	$2.12(Ra^*_S/S^4L)^{-1/5}$	4.77
Isothermal/adiabatic plates ($T_{s,1}, q''_{s,2} = 0$)	144	2.87	$2.15\,(Ra_S/S^3L)^{-1/4}$	1.71
Isoflux/adiabatic plates ($q''_{s,1} = q''_{s,2} = 0$)	24	2.51	$1.69(Ra^*_S/S^4L)^{-1/5}$	4.77

Following finding the Rayleigh number and then the Nusselt number, the Convective Heat Transfer Coefficient can be calculated using the same equation for *h* as in the scenario without a heat exchanger. The calculations performed above will be instrumental in the following sections, particularly in calculating the heater power, which forms the basis of the design.

Heater Power

The power of the heater can be calculated using the Rayliegh number, Nusselt number, and convective heat transfer coefficient for the system with and without a heat exchanger. The convective resistance (R_{conv}) can be calculated using the following equation based on the convective heat transfer coefficient.

$$R_{conv} = \frac{1}{h * A_{conv}}$$

The above equation only applies to the system without a heat exchanger; numerous aspects of the finned system must be taken into account in order to appropriately determine the resistance of the finned system. Below is a presentation of the fin-specific resistance equation analysis.

$$R_{t,fins} = \frac{1}{\eta_0 h A_t}$$

In addition to determining Rconv, the following equation must be used to get the base resistance (Rbase).

$$R_{base} = \frac{L_{base}}{k * A}$$

Once the values of the two resistances have been determined, the system's total resistance can be ascertained. Convective heat transfer and base resistance can be added to determine the total resistance.

$$R_{tot} = R_{conv} + R_{base}$$

The total resistance and temperature change (ΔT) data can be used to calculate the heater's power in systems with or without a finned heat exchanger. The necessary equation is provided below.

$$P_{heat} = \frac{\Delta T}{R_{tot}}$$

Heat Sink Design

Finding fin efficiency, total surface efficiency, and characteristic length was the focus of the following phase. We can now replace the heat flow value into the array's heat flow equation after figuring out the heat flow.

$$P_{heat} = N * \eta_f * A_f * h * \theta_b + h * A_b * \theta_b$$

After substituting the known values into the equations, we can also substitute the equations for the area of the fin (A_f) and the efficiency of the fin (η_f) to acquire the equation in terms of the characteristic length (L_c).

$$A_f = 2 * w * L_c$$

$$\eta_f = \frac{\tanh(mL_c)}{mL_c}$$

$$m = \sqrt{\frac{2 * h}{k * t}}$$

Thus we can solve the equation for the characteristic length (L_c)

$$L_c = \frac{\tanh^{-1}\left(\frac{P_{heat} - h * A_b * \theta_b}{N * 2 * w * \theta_b * h}\right)}{m}$$

L_c defined as:

$$L_c = L + 0.5 * t$$

Knowing the thickness, we can now find the necessary length of the fins (L)

$$L = L_c - (0.5 * t)$$

Knowing L_c we can now find η_f, A_f, A_t

Fin effectiveness, which is the ratio of heat transferred with fins to heat transmitted without fins, is significant because it serves as an additional metric to assess how well a heat exchanger is designed. Below is the fin effectiveness equation.

$$\varepsilon_f = \frac{q_f}{q_{f,max}} = \eta_f$$

q_f defined as:

$$q_f = \eta_f * h * A_f * \theta_b$$

$q_{f,max}$ defined as:

$$q_{f,max} = h * A_f * \theta_b$$

The total surface efficiency, which enables the calculation of the overall surface resistance and, eventually, the heater power, is another helpful indicator of the overall heat exchanger efficiency. The general surface efficiency equation is shown below.

$$\eta_o = 1 - \frac{N * A_f}{A_t}(1 - \eta_f)$$

A_f defined as:

$$A_f = 2 * L * W$$

A_t defined as:

$$A_t = N * A_f + A_b$$

The formula for A_b is shown below. A_b is the total area of the base exposed to convection, and N is the number of fins.

$$A_b = H * W - N * t * W$$

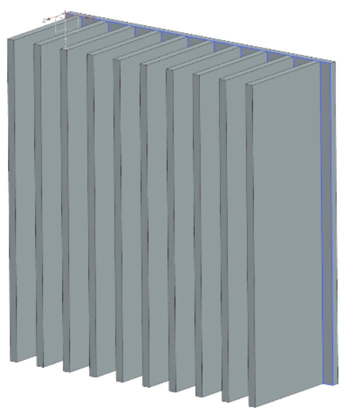
Selected Heat Sink Design

Parameters	Magnitude
Base Thickness of the Fin Array, L_{base} (m)	0.001
Fin Length, L (m)	.0135
Fin Width, w (m)	0.04
Fin Thickness, t (m)	0.001
Number of Fins, N (-)	10
Characteristic Length of the Heater Surface (No Heat Sink), L (m)	0.040
Rayleigh Number (No Heat Sink), Ra (-)	174597
Nusselt Number (No Heat Sink), Nu (-)	10.66
Convective Heat Transfer Coefficient (No Heat Sink), h (W/m^2K)	7.25
Heater Power, P_{heat} (W)	0.5216
Fin Spacing, s (m)	0.003
Characteristic Length of the Heat Sink, L (m)	0.014
Rayleigh Number (With Heat Sink), Ra (-)	61.82
Nusselt Number (With Heat Sink), Nu (-)	0.15
Convective Heat Transfer Coefficient (With Heat Sink), h (W/m^2K)	1.7
Single Fin Cross-sectional Area, $A_{c,b}$ (m^2)	.00004
Base Area of the Heat Sink, A_b (m^2)	0.0016
Single Fin Surface Area, $A_{f,}$ (m^2)	.00112
Total Area of Heat Transfer, A_t (m^2)	.0124
Single Fin Efficiency, η_f (-)	97.34%

Overall Surface Efficiency of the Fin Array, η_o (-)	97.51%
Thermal Resistance due to Base of Fin Array, $R_{t,base}$ (K/W)	0.078125
Thermal Resistance due to Fin Array, $R_{t,fins}$ (K/W)	48.645
Re-calculated Heater Power, P_{heat} (W)	.5131

Table 1 – Calculated data for designed heat sink

CAD FILE 2



Conclusion

The design and analysis of the finned heat sink, vertically oriented and under free convection, has demonstrated the effective cooling capabilities under the stringent design specifications. The design shown achieved the explicit goal of reducing the heater's temperature reduction from 70°C to 50°C. This effort was achieved through setting design specifications then solving for the determined length of the fins based on those specifications. For the system including 10 fins that span the entire length of the heater and with a thickness of 1mm, the fins must branch out 13.5mm from the base. Such fins will reliably meet the required temperature difference predicated that the heat generation of the heat is consistently held at 0.5216 watts. Each fin is highly efficient boasting more than 97% efficiency, with the overall body performing with an efficiency of 97.5%. The calculations performed provided accurate answers that fell comfortably within reason, and that any engineer would continue forward to production confidently with.

Corrections and Reflections

There are slight discrepancies within the data below much of this can be attributed to slight rounding errors or minute differences in the heat flow calculations that were carried through multiple equations. Some of the disparages in the data can be found in the heat flow calculations as well as the resistance calculations.

Below we will determine the percent difference in the heat flow when calculated ideally, vs when calculated based on the parameters given:

$$Percent\ Difference = \frac{A - B}{0.5(A + B)} * 100$$

$$\%\ diff_{heat\ flow} = \frac{0.5216 - 0.5131}{0.5(.5216 + .5131)} * 100 = 1.642\%$$

This percent error is only 1.642%, typically a good rule of thumb is that any percent error less than 10% is negligible, which our answer comfortably falls below.

This is one of the larger outliers in our data which has just been shown to be a negligible outcome, thus with fair certainty I can state that our calculations are correct and that the designed heat sink will function as desired.

References

1. WileyPlus (through LMS): Fundamental of Heat and Mass Transfer, F. P. Incropera, D. P. Dewitt, T. L. Bergman, and A. S. Lavine, Wiley & Sons, 8th Edition (2018).

Technical Contributions

Team Member	Contribution
James Mclean	Calculations (with fins and design), Conclusion, Analysis Thermal Resistance Network
Nicholas Walker	Analysis, CAD, Calculations (without fins)

Table 2 – work contributions

Appendix

Sample Calculation

Calculations Without Fins

Given: k= 8 W/mK, c_p =1300 J/kgK, p = 1400/kg/m^3, given coefficients

$$Ra_L = \frac{\left(\frac{9.81m}{s^2}\right)\left(\frac{1}{323K}\right)(70℃ - 25℃)0.04^3}{(1.91 * 10^{-5}m^2/sec)(2.47 * 10^{-5}m^2/sec)} = 1.74597 * 10^5$$

Raliegh number for free convection on the heater

$$Pr = \frac{1.91 * 10^{-5}m^2/sec}{2.47 * 10^{-5}m^2/sec} = 0.77328$$

Prandtl number given the temperature

$$Nu_L = \left\{0.825 + \frac{0.387(1.74597 * 10^5)^{\frac{1}{6}}}{\left[1 + \left(\frac{0.492}{0.77328}\right)^{\frac{9}{16}}\right]^{\frac{8}{27}}}\right\}^2 = 10.66$$

Calculated Nusselt Number from Pr and Ra for free convection of the vertical plate

$$h = \frac{(10.66)\left(\frac{0.0272W}{m * K}\right)}{0.04m} = 7.25\frac{W}{m^2}K$$

Open convection coefficient calculated through the determined Nu value, below are the calculated resistance of the system to determine the overall heat transfer through the system.

$$R_{conv} = \frac{1}{\left(\frac{7.252W}{m^2K}\right) * (0.04m * 0.04m)} = 86.182\frac{K}{W}$$

$$R_{base} = \frac{(0.001m)}{(8W/mK) * (0.04m * 0.04m)} = 0.078125\frac{K}{W}$$

$$R_{tot} = 86.182\frac{K}{W} + 0.078125\frac{K}{W} = 86.26\frac{K}{W}$$

$$P_{heat} = \frac{\Delta T}{R_{tot}} = \frac{45℃}{86.677\frac{K}{W}} = 0.5216W$$

This heat transfer value will be treated as a constant regardless of if the fin array is being applied. By determining the heat transfer value and the necessary change in temperature, 25 K, we can find the necessary length of the fin given that there are 10 of them, they are 1mm thick, and 40mm wide.

Heat Sink Design Calculations

$$Ra_S = \frac{\left(\frac{9.81m}{s^2}\right)\left(\frac{1}{310.5K}\right)(50-25)0.0033^3}{\left(1.91 * \frac{10^{-5}m^2}{s}\right)\left(2.47 * \frac{10^{-5}m^2}{s}\right)} = 61.82$$

Raliegh number as defined by the necessary change in temperature and characteristic lengths

$$Nu_S = \left[\frac{576}{\left(\frac{(61.82)(0.00333)}{(0.04m)}\right)^2} + \frac{2.87}{\left(\frac{(61.82)(0.00333m)}{(0.04m)}\right)^{\frac{1}{2}}}\right]^{-\frac{1}{2}} = 0.20846$$

Nusselt number calculated by the various characteristic lengths and Raliegh number.

$$h = \frac{(0.20846)(0.0272 * \frac{W}{m * K})}{0.0033m} = 1.7\frac{W}{m^2K}$$

Calculated convective heat transfer coefficient value from Nusselt number

$$m = \sqrt{\frac{2 * h}{k * t}} = \sqrt{\frac{2 * 1.7 * \frac{W}{m^2K}}{8W/mK * 0.001m}} = 20.615m^{-1}$$

This m value will be used to within the equation to find the required efficiency of each fin

$$A_b = A - N(t * w) = .0016m^2 - 10(.040m * .001m) = .0012m^2$$

This is the area of the base exposed to convection, calculated by the total area of the plate minus the area taken up by the fins.

$$P_{heat} = N * \eta_f * A_f * h * \theta_b + h * A_b * \theta_b$$

This is the equation for the heat flow through the fin array

$$A_f = 2 * w * L_c = 2 * 0.040m * L_c$$

Utilizing the equation for P_{heat} above, the equation for A_f , and the values determined in previous portions we can substitute the values in and solve for L_c. This L_c will determine the necessary length of the fins given that there will be 10 of them, their thickness will be 1mm and width of 40mm.

$$.5192W = 10 * \eta_f * (2 * .040m * L_c) * \left(1.7\frac{W}{m^2}K\right)(25K) + \left(1.7\frac{W}{m^2}K\right)(.0012m^2)(25K)$$

$$\eta_f = \frac{\tanh\,(mL_c)}{mL_c} = \frac{\tanh(20.615m^{-1} * L_c)}{20.615m^{-1}}$$

Substituting the equation for fin efficiency in terms of L_c we can now solve for the characteristic length.

$$L_c = \frac{\tanh^{-1}\left(\frac{P_{heat} - h * A_b * \theta_b}{N * 2 * w * \theta_b * h}\right)}{m} = \frac{\tanh^{-1}(.2838)}{20.615} = 0.01415m$$

Utilizing the determined L_c we can now find the necessary length for our design specifications.

$$L = L_c - \frac{t}{2} = 0.01415m - .0005m = 0.01365m$$

With L_c we can now find the efficiency, various area relations, the efficiency of the fin, and the overall effectiveness.

$$\eta_f = \frac{\tanh\,(mL_c)}{mL_c} = \frac{\tanh(20.615m^{-1} * .01415m)}{20.615m^{-1} * 0.01415m} = .97437 = 97.34\%$$

Below are the various area calculations as well as the effectiveness of a singular fin, these values will help determine the overall resistance of the array.

$$A_f = 2 * w * L_c = 2 * 0.04m * 0.0140m = .00112m^2$$

$$A_{c-b} = w * t = .040m * .001m = .0004m\text{^}2$$

$$q_f = \eta_f * h * A_f * \theta_b = 0.9734 * 1.7 * \frac{W}{m^2K} * .00112m^2 * 25K = 0.04633W$$

$$q_{f,max} = h * A_f * \theta_b = 1.7 * \frac{W}{m^2K} * .0012m^2 * 25K = .0467$$

$$\varepsilon_f = \frac{q_f}{q_{f,max}} = \frac{0.04633W}{0.0467W} = 0.9734 = \eta_f$$

Calculations With Fins

$$R_{t,o} = \frac{\theta_b}{P_{heat}} = 48.151\frac{K}{W}$$

The above value is the idealized resistance based on the desired change in temperature and heat flow.

$$\eta_0 = 1 - \frac{N * A_f}{A_t}\left(1 - \eta_f\right) = 1 - \frac{10 * .0012m^2}{.0124}(1 - .97437) = .97519 = 97.51\%$$

Using the determined efficiency of the fins and the various area relations we can find the overall efficiency which in turn can be used to find the resistance of the overall fin array, as seen below.

$$R_{t,fins} = \frac{1}{\eta_0 h A_t} = \frac{1}{(0.9751)(1.04 * \frac{W}{m^2K})(0.025m^2)} = 48.645\frac{K}{W}$$

$$R_{base} = \frac{L_{base}}{k * A} = \frac{(0.001m)}{(8W/mK) * (0.04m * 0.040m)} = 0.78125\frac{K}{W}$$

The above equation for R_{base} is so small relative to the resistance of the fins that it could be considered negligible, however we will carry the value through our calculations.

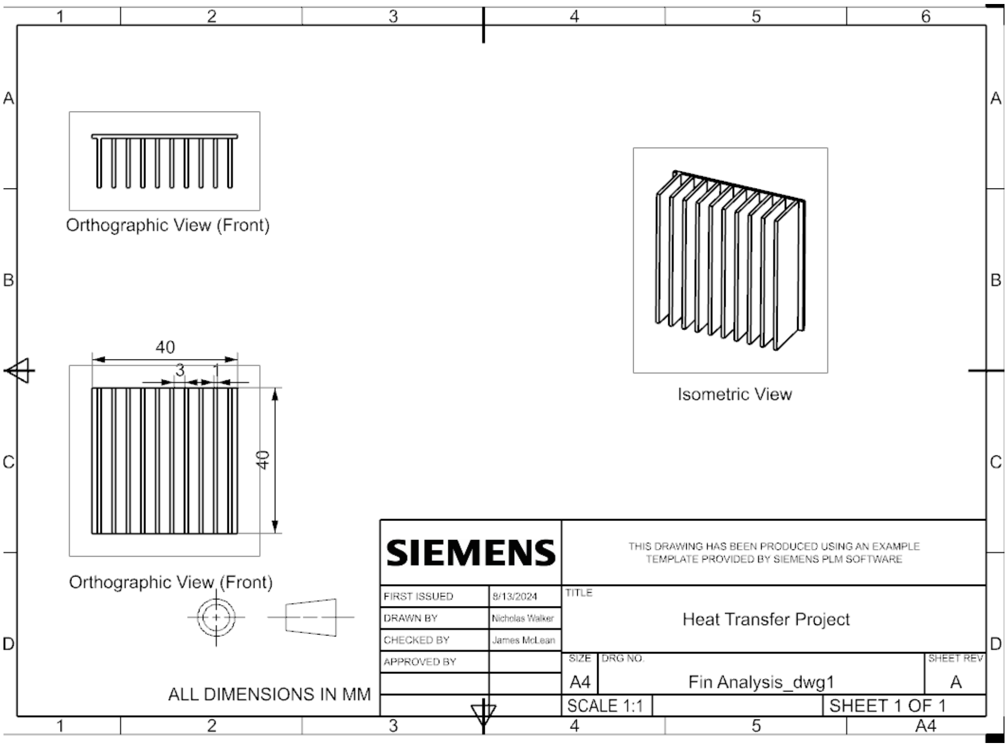
$$R_{tot} = R_{t,fins} + R_{base} = 48.645\frac{K}{W} + .078125\frac{K}{W} = 48.7233\frac{K}{W}$$

The determined R_{tot} is extremely close to the idealized Resistance determined at the beginning of this section, thus we can be certain that our calculations and our design specifications we accurate.

$$P_{heat} = \frac{\Delta T}{R_{tot}} = \frac{25K}{48.7233\frac{K}{W}} = .51310W$$

This is the determined heat transfer through the fin array given the change in temperature and the determined resistance. This value is extremely close to the predetermined heat transfer value and the difference can be assumed to be negligible and caused by slight rounding in the calculations.

Heat Sink Drawing



Link to Excel Spreadsheet:

[Heat Sink Calculations]