COMP90038 Algorithms and Complexity

Priority Queues, Heaps and Heapsort

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Heaps and Priority Queues

The heap is a very useful data structure for priority queues, used in many algorithms.

A priority queue is a set (or pool) of elements.

An element is injected into the priority queue together with a priority (often the key value itself) and elements are ejected according to priority.

We think of the heap as a partially ordered binary tree.

Since it can easily be maintained as a complete tree, the standard implementation uses an array to represent the tree.

The Priority Queue

As an abstract data type, the priority queue supports the following operations on a "pool" of elements (ordered by some linear order):

- find an item with maximal priority
- insert a new item with associated priority
- test whether a priority queue is empty
- eject the largest element

Other operations may be relevant, for example:

- replace the maximal item with some new item
- construct a priority queue from a list of items
- join two priority queues

Stacks and Queues as Priority Queues

Special instances are obtained when we use time for priority:

- If “large” means “late” we obtain the stack.
- If “large” means “early” we obtain the queue.

Some Uses of Priority Queues

- Job scheduling done by your operating system. The OS will usually have a notion of "importance" of different jobs.
- (Discrete event) simulation of complex systems (like traffic, or weather). Here priorities are typically event times.
- Numerical computations involving floating point numbers. Here priorities are measures of computational "error".

Many sophisticated algorithms make essential use of priority queues (Huffman encoding and many shortest-path algorithms, for example).

Possible Implementations of the Priority Queue

Assume priority $=$ key.

	$\textsc{Inject}(e)$	$\textsc{Eject}()$
Unsorted array or list		
Sorted array or list		
Heap	$O(\log n)$	$O(\log n)$

How is this accomplished?

The Heap

A heap is a complete binary tree which satisfies the heap condition:

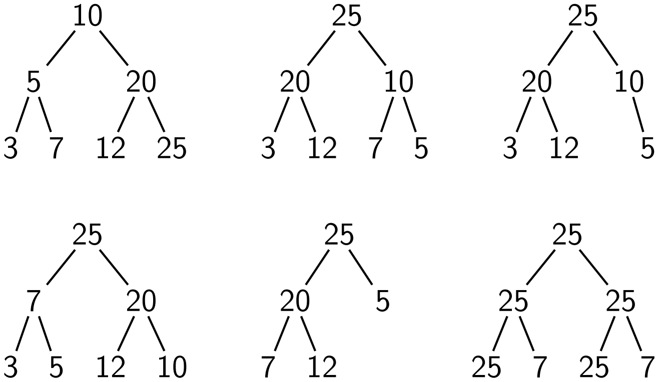
Each child has a priority (key) which is no greater than its parent's.

This guarantees the the root of the tree is a maximal element.

(Sometimes we talk about this as a max-heap—one can equally well have min-heaps, in which each child is no smaller than its parent.)

Heaps and Non-Heaps

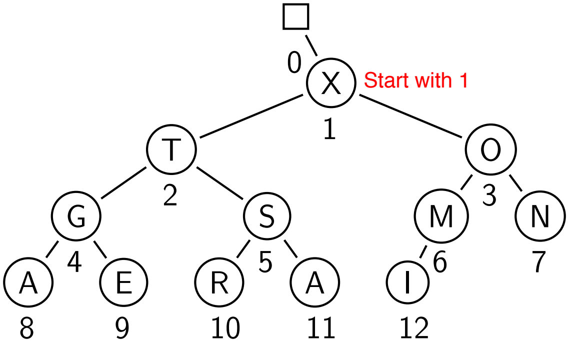
Which of these are heaps? 2 marks in exam



Heaps as Arrays

We can utilise the completeness of the tree and place its elements in level-order in an array H.

Note that the children of node i will be nodes $2i$ and $2i + 1$.

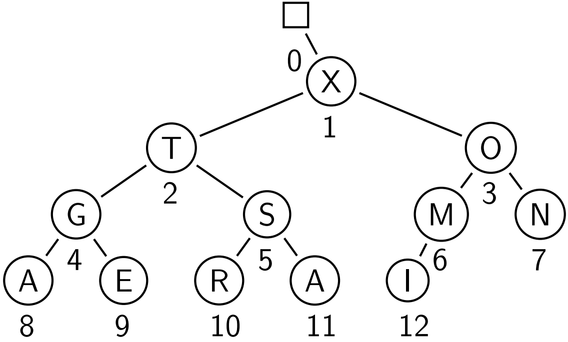


H:		X	T	O	G	S	M	N	A	E	R	A	I
	0	1	2	3	4	5	6	7	8	9	10	11	12

Heaps as Arrays

This way, the heap condition is very simple:

For all $i \in \{0, 1, \ldots, n\}$, we must have $H[i] \leq H[i/2]$.



H:	0	1	2	3	4	5	6	7	8	9	10	11	12
		X	T	O	G	S	M	N	A	E	R	A	I

Properties of the Heap

The root of the tree $H[1]$ holds a maximal item; the cost of $\textsc{Eject}$ is $O(1)$ plus time to restore the heap.

The height of the heap is $\lfloor \log_2 n \rfloor$.

Each subtree is also a heap.

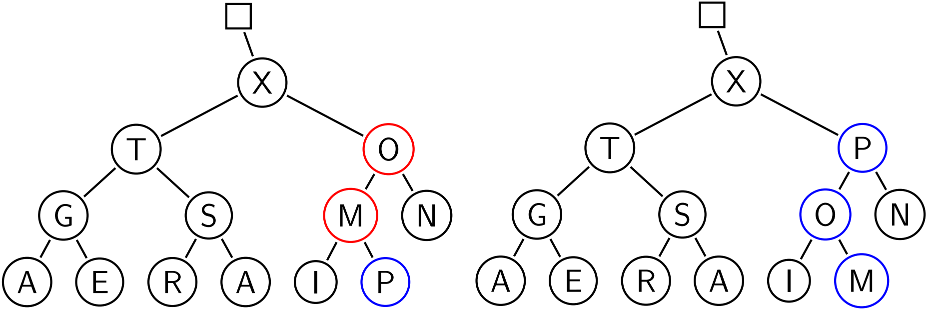
The children of node i are $2i$ and $2i + 1$.

The nodes which happen to be parents are in array positions 1 to $\lfloor n/2 \rfloor$.

It is easier to understand the heap operations if we think of the heap as a tree.

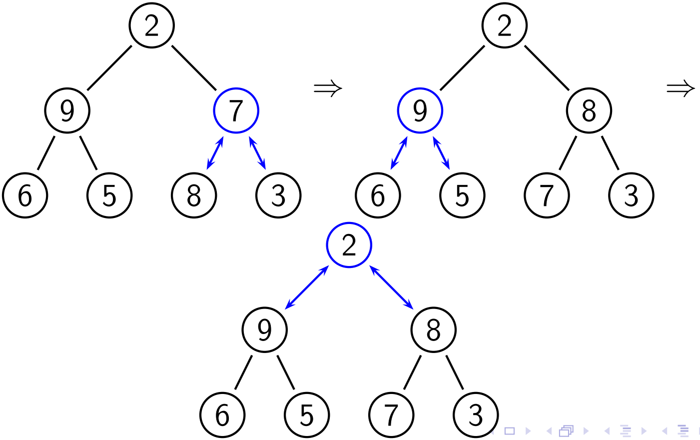
Injecting a New Item

Place the new item at the end; then let it “climb up”, repeatedly swapping with parents that are smaller:



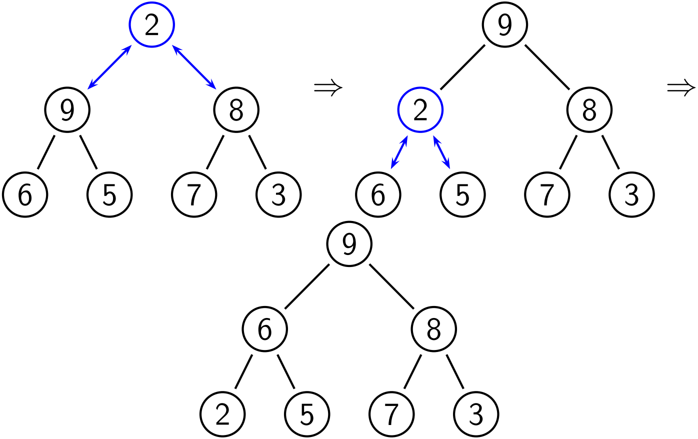
Building a Heap Bottom-Up

To construct a heap from an arbitrary set of elements, we can just use the inject operation repeatedly. The construction cost will be $n \log n$. But there is a better way:



Building a Heap Bottom-Up: Sifting Down

Whenever a parent is found to be out of order, let it "sift down" until both children are smaller:



Algorithm to Turn $H[1..n]$ into a Heap, Bottom-Up

```
for i ← ⌊n/2⌋ downto 1 do
    k ← i
    v ← H[k]
    heap ← False
    while not heap and 2 × k ≤ n do
        j ← 2 × k
        if j < n then
            if H[j] < H[j + 1] then
                j ← j + 1
        if v ≥ H[j] then
            heap ← True
        else
            H[k] ← H[j]
            k ← j
    H[k] ← v
```

Analysis of Bottom-Up Heap Creation

For simplicity, assume the heap is a full binary tree: $n = 2^{h+1} - 1$. Here is an upper bound on the number of "down-sifts" needed (consider the root to be at level h, so leaves are at level 0): Use the slide, the textbook is different here

$$\sum_{i=1}^{h} \sum_{\text{nodes at level } h-i} i \quad = \quad \sum_{i=1}^{h} i \cdot 2^{h-i} \quad = \quad 2^{h+1} - h - 2$$

The last equation is easily proved by mathematical induction.

Note that $2^{h+1} - h - 2 < n$, so we perform at most a linear number of down-sift operations. Each down-sift is preceded by two key comparisons, so the number of comparisons is also linear.

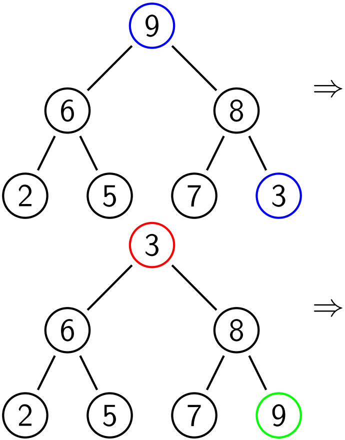
Hence we have a linear-time algorithm for heap creation.

Ejecting a Maximal Element from a Heap

Here the idea is to swap the root with the last item z in the heap, and then let z "sift down" to its proper place.

After this, the last element (here shown in green) is no longer considered part of the heap, that is, n is decremented.

Clearly ejection is $O(\log n)$.



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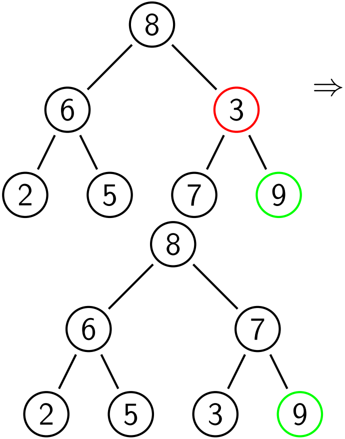

Exercise: Build and Then Deplete a Heap

First build a heap from the items S, O, R, T, I, N, G.

Then repeatedly eject the largest, placing it at the end of the heap.
✎

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Anything interesting to notice about the tree that used to hold a heap?

Heapsort

Heapsort is a $\Theta(n \log n)$ sorting algorithm, based on the idea from this exercise.

Given unsorted array $H[1..n]$:

- Step 1 Turn H into a heap.
- Step 2 Apply the eject operation $n - 1$ times.

Properties of Heapsort

On average slower than quicksort, but stronger performance guarantee.

Truly in-place.

Not stable.

Coming Up Next

We will look at the “transform and conquer” paradigm.