Standard Valid Argument Forms

Modus Ponens

$$A \longrightarrow B$$
$$A$$
$$\therefore B$$

Elimination

a.
$$A \vee B$$
$$\sim B$$
$$\therefore A$$

b.
$$A \vee B$$
$$\sim A$$
$$\therefore B$$

Modus Tollens

$$A \longrightarrow B$$
$$\sim B$$
$$\therefore \sim A$$

Transitivity

$$A \longrightarrow B$$
$$B \longrightarrow C$$
$$\therefore A \longrightarrow C$$

Generalization

a.
$$A$$
$$\therefore A \vee B$$

b.
$$B$$
$$\therefore A \vee B$$

Proof by division into cases

$$A \vee B$$
$$A \longrightarrow C$$
$$B \longrightarrow C$$
$$\therefore C$$

Specialization

a.
$$A \wedge B$$
$$\therefore A$$

b.
$$A \wedge B$$
$$\therefore B$$

Conjunction

$$A$$
$$B$$
$$\therefore A \wedge B$$

Contradiction

$$\sim A \longrightarrow \mathbf{c}$$
$$\therefore A$$

Universal Modus Ponens

$$\forall\, x, P(x) \longrightarrow Q(x)$$
$$P(a)$$
$$\therefore Q(a)$$

Universal Instantiation

$$\forall\, x \in D, P(x)$$
$$a \in D$$
$$\therefore P(a)$$

Universal Modus Tollens

$$\forall\, x, P(x) \longrightarrow Q(x)$$
$$\sim Q(a)$$
$$\therefore \sim P(a)$$

Universal Transitivity

$$\forall\, x, P(x) \longrightarrow Q(x)$$
$$\forall\, x, Q(x) \longrightarrow R(x)$$
$$\therefore \forall\, x, P(x) \longrightarrow R(x)$$

U. Converse Error(not valid)

$$\forall\, x, P(x) \longrightarrow Q(x)$$
$$Q(a)$$
$$\therefore P(a)$$

U. Inverse Error (not valid)

$$\forall\, x, P(x) \longrightarrow Q(x)$$
$$\sim P(a)$$
$$\therefore \sim Q(a)$$

Chapter 4: Definitions and Theorems

Definitions: Even and odd integers
n is an even integer $\Leftrightarrow$ $n = 2k$ for some integer k.
n is an odd integer $\Leftrightarrow$ $n = 2k + 1$ for some integer k.

Definitions: Rational numbers ($\mathbb{Q}$) & Irrational numbers ($\mathcal{S}$)
r is rational $\Leftrightarrow$ $r = \frac{a}{b}$ for some integers a & b where $b \neq 0$.
s is irrational $\Leftrightarrow$ s is a real # and s is not rational.

Definitions: Prime and composite numbers
Suppose $n \in \mathbb{Z}$ & $n > 1$. Then
n is prime $\Leftrightarrow$ $\forall\, r, s \in \mathbb{Z}^+$, if $n = rs$ then either $r = 1$ or $s = 1$.
n is composite $\Leftrightarrow$ $\exists\, r, s \in \mathbb{Z}^+$ such that $n = rs$ & neither $r = 1$ nor $s = 1$.

Definition: "divides", "is a divisor of", "is a factor of", "is divisible by", & "is a multiple of"
If n and d are integers and $d \neq 0$, then
$\boldsymbol{d}$ **divides** $\boldsymbol{n}$, denoted by $\boldsymbol{d \mid n}$ $\Leftrightarrow$ $n = dk$ for some integer k

We can also express "$\boldsymbol{d}$ **divides** $\boldsymbol{n}$" $\equiv$ "$\boldsymbol{d \mid n}$" as: "$\boldsymbol{d}$ **is a divisor of** $\boldsymbol{n}$" or "$\boldsymbol{n}$ **is divisible by** $\boldsymbol{d}$" or
"$\boldsymbol{d}$ **is a factor of** $\boldsymbol{n}$" or "$\boldsymbol{n}$ **is a multiple of** $\boldsymbol{d}$"

$$\boldsymbol{d \nmid n} \quad \equiv \quad \sim(d \mid n) \quad \Leftrightarrow \quad \frac{n}{d} \notin \mathbb{Z}$$

Theorem (Quotient Remainder Theorem (QRT)): Given any integer n and a positive integer d, there exists unique integers q and r such that $n = dq + r$ and $0 \leq r < d$.
The integer q is denoted by n div d i.e. $(n \text{ div } d) = q$ &
the integer r is denoted by $n \bmod d$ i.e. $(n \bmod d) = r$
Corollary of QRT: For all integers n and $d > 0$, if $(n \bmod d) \neq 0$, then $d \nmid n$.

Theorem: The sums, differences, & products of integers are integers i.e. $\mathbb{Z}$ is closed under $+, -, \times$.

Zero Product Property: The product of any two nonzero real numbers is nonzero.

Theorem 4.3.1: The sum of any two rational #s is a rational #.
Theorem 4.7.1: The sum of any rational # and any irrational # is an irrational #.

Theorem 4.7.2: Every integer is either even or odd but not both.

Theorem 4.7.3: $\forall\, n \in \mathbb{Z}$, if n^2 is even then n is even.

Theorem 4.8.1: $\sqrt{2}$ is irrational.

Theorem 4.4.2: For all integers $n > 1$, the **standard prime factorization** of n exists and is unique.

Theorem 4.4.4: The set of all prime numbers is infinite.

4.1-4.2: Method of Direct Proof

In order to prove a universal non-conditional statement:

$$\text{“}\forall\, x \in D, Q(x)\text{”}$$

1. Let x be any element of D.
2. Show that $Q(x)$ is true.

In order to prove a universal conditional statement:

$$\text{“}\forall\, x \in D, P(x) \longrightarrow Q(x)\text{”}$$

1. Let x be any element of D such that $P(x)$ is true.
2. Show that $Q(x)$ is true.

4.6: (Indirect) Method of Contradiction

In order to prove any statement “q”

1. Suppose that the negation “$\sim q$” is true.
2. Prove that this assumption leads to a “contradiction”.
3. Conclude that our assumption is false and “q” is true.

In order to prove $q \equiv$“$\forall\, x \in D, Q(x)$”

1. Suppose that the negation “$\exists\, x \in D$ such that $\sim Q(x)$” is true.
2. Prove that this assumption leads to a “contradiction”.
3. Conclude that our assumption is false and “q” is true.

In order to prove $q \equiv$“$\forall\, x \in D, P(x) \longrightarrow Q(x)$”

1. Suppose that the negation
   “$\exists\, x \in D$ such that $P(x) \wedge \sim Q(x)$” is true.
2. Prove that this assumption leads to a “contradiction”.
3. Conclude that our assumption is false and “q” is true.

4.6: (Indirect) Method of Contraposition

In order to prove a universal conditional statement:

$$\text{“}\forall\, x \in D,\ P(x) \longrightarrow Q(x)\text{”}$$
$$\equiv \text{“}\forall\, x \in D, \sim Q(x) \longrightarrow \sim P(x)\text{”} \text{ (Contrapositive)}$$

1. Let x be <u>any</u> element of D such that $\sim Q(x)$ is true.
2. Prove “$\sim P(x)$ is true”.
3. Conclude that the contrapositive and thus the given statement is true.

<u>Note</u>: Proving a given universal **conditional** statement by the Method of Contraposition is the same as proving its contrapositive using the Method of Direct Proof.