Math 263 : Discrete Mathematics Towson University
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HW-4.4-4.5

1. (16 *points*) Provide a direct proof of the following statement using QRT.

$\forall$ integers n, if $3|n$, then $3 \nmid (n^2 - 2)$.

setup + definitions

Proof

Let $n \in \mathbb{Z}$ such that $3 | n$

We need to prove that $3 \nmid (n^2-2)$

By the definition of "|"

$n = 3k$ for some $k \in \mathbb{Z}$

Then $n^2 - 2 = (3k)^2 - 2$

$= 9k^2 - 2$

$9k^2-2$ div $3 = 3k^2-1$

$9k^2-2$ mod $3 = 1$

QRT

Then, take $9k^2-2$ as $\underline{n}$ for QRT Corollary

3 as $\underline{d} > 0$

$9k^2-2 = 3(q) + (r)$, where $q = 3k^2-1$, $r = 1$

Saying we can use QRT

$n^2-2, 3, q$ & $r \in \mathbb{Z}$ because $9, k, 2, 3 \in \mathbb{Z}$ & sums, products, & differences of ints are ints.

Further $0 \le r < 3$, $1 < 3$

$3k^2-1$ mod $3 = r = 1 \neq 0$

The conclusion QRT lets us make

By corollary of QRT, $3 \nmid (n^2-2)$.

$a \to b$

$\sim a \wedge \sim b$

2. (32 *points*) If you were writing the proof of the following statement, for each of the methods below, write the beginning statement (Let…) and what you will need to prove. Then state which method is best suited for proving the statement and prove it using that method.

$\forall$ integers n, if $6 \mid (n-3)$ then $3 \nmid (n-2)$.

Method of Contradiction (8 *points*)

Suppose the negation, i.e.

"$\exists$ an int n such that $6 \mid (n-3)$ & $3 \mid (n-2)$." is true

We need to prove that this leads to a contradiction,

Method of Contraposition (5 *points*)

Let n be any int such that $3 \mid (n-2)$.

We need to prove that $6 \nmid (n-3)$.

Method of Direct Proof (4 *points*)

Let n be any integer such that "6 divides $n-3$" is true.

We need to prove that $3 \nmid n-2$.

Proof using the Method of Contradiction (15 *points*)

Suppose the negation, ie

"$\exists$ an int n such that $6 \mid (n-3)$ & $3 \mid (n-2)$" is true.

We need to prove that this leads to a contradiction.

By the definition of "$\mid$"

$(n-3) = 6k$ for some $k \in \mathbb{Z}$

$(n-2) = 3r$ for some $r \in \mathbb{Z}$

$3r - 1 = 6k$ by substitution

$3r - 6k = 1$

$3(r-2k) = 1$

$r - 2k = \frac{1}{3}$

The LHS $\in \mathbb{Z}$ since $r, 2, k \in \mathbb{Z}$ & differences & products of ints are ints,

Thus the RHS $\in \mathbb{Z}$, but we know $\frac{1}{3} \notin \mathbb{Z}$.

This is a contradiction.

Thus our assumption is false & the given statement is true.

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3. (24 *points*) Provide a direct proof of the following statement using Proof by Division into Cases.

$$\forall \text{ integers } n,\ (n^2 \bmod 3) \text{ is } 0 \text{ or } 1.$$

Direct Proof

Let n be any int.

We need to prove $(n^2 \bmod 3)$ is 0 or 1.

Using QRT w/ n as n & d as $3>0$.

$\exists$ unique $q, r \in \mathbb{Z}$ such that $n = 3q + r$ & $0 \le r < 3$. $r = 0, 1, 2$

So, $n = 3q$ or $n = 3q+1$ or $n = 3q+2$

Case 1: $n = 3q$

$n^2 = (3q)^2$

$= 9q^2$

$= 3(k)$ where $k = 3q^2$ & $k \in \mathbb{Z}$ because $3, q \in \mathbb{Z}$ & products of ints are ints.

$n^2 \bmod 3 = 0$

Case 2: $n = 3q+1$

$n^2 = (3q+1)^2$

$= 9q^2 + 6q + 1$

$= 3(k) + 1$ where $k = 3q^2 + 2q$ & $k \in \mathbb{Z}$ because $3, q, 2 \in \mathbb{Z}$ & sums & products of ints are ints

$n^2 \bmod 3 = 1$

Case 3: $n = 3q+2$

$n^2 = (3q+2)^2$

$= 9q^2 + 12q + 4$

$= 3(k) + 1$ where $k = 3q^2 + 4q + 1$ & $k \in \mathbb{Z}$ because $3, q, 4, 1 \in \mathbb{Z}$ & sums & products of ints are ints

$n^2 \bmod 3 = 1$

The 3 cases are the only possibilities & in each case $(n^2 \bmod 3) = 0$ or 1.

Thus, $(n^2 \bmod 3) = 0$ or 1.

4. (20 *points*) Consider an integer $n > 1$ with the following standard prime factorization:

$$n = 2^a \cdot 3^b \cdot 5^c \quad \text{where} \quad a, b, c \in \mathbb{Z}^+$$

a) (4 *points*) Write the definition of the statement, "$14 \mid n$" i.e. "14 divides n".

$14 \mid n = 14k$ for some $k \in \mathbb{Z}$

b) (16 *points*) Prove the following statement

$$\forall\, a, b, c \in \mathbb{Z}^+, 14 \nmid (2^a \cdot 3^b \cdot 5^c)$$

using Method of Contradiction using the uniqueness of "standard prime factorization" by first replacing 14 by its standard prime factorization (SPF) in the equation obtained in part a).

Proof by the Method of Contradiction:

Suppose the negation of the statement, i.e.

"$\exists\, a,b,c \in \mathbb{Z}^+$ such that $14 \mid 2^a \cdot 3^b \cdot 5^c$" is true

We need to prove that this leads to a contradiction

By definition of "$\mid$"

$(2^a 3^b 5^c) = 14k$ for some $k \in \mathbb{Z}$

$= 2^1 7^1 k$

First $k \in \mathbb{Z}^+$ since LHS > 0 & $14 > 0$

So, RHS is an int > 1. Then, the SPF of the RHS contains some power of the prime # 7,

This means the SPF of LHS also contains some power of 7. But it doesn't.

This is a contradiction.

Thus our assumption is false & the given statement is true.