AI in Mathematics
Lecture 1

Bar-Ilan University
Nebius Academy | Stevens Institute of Technology

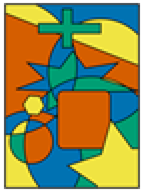
March 18, 2025

Computers in Mathematics

The Pioneering Era (Mid-20th Century)

Computers in Mathematics

The Four Color Theorem

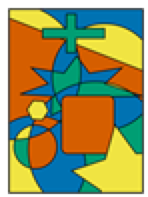


Computers in Mathematics

Software and globalization

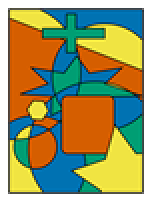
Computers in Mathematics

Formal Proof Assistants



Computers in Mathematics

AI and LLMs everywhere



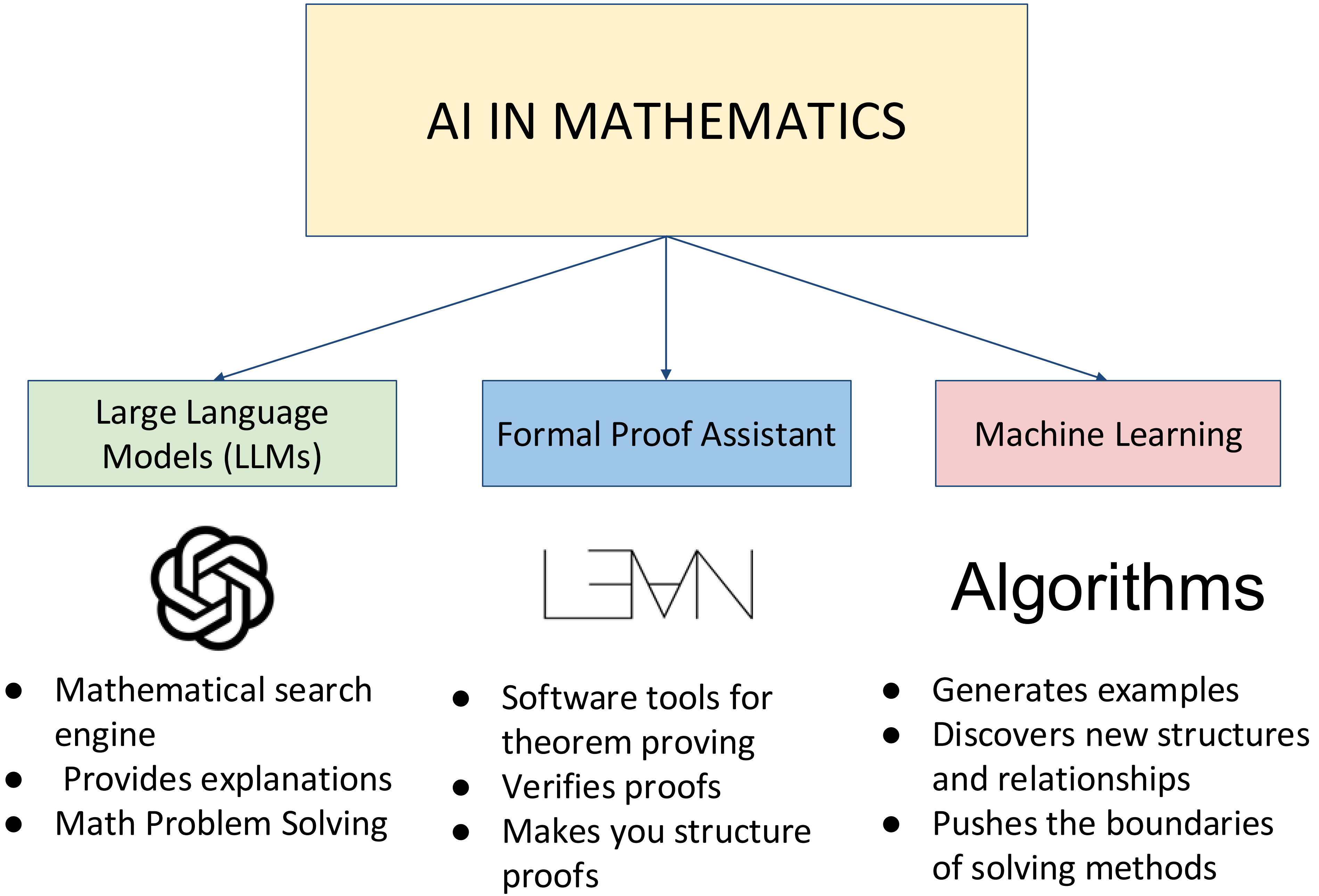
LaTeX

LEAN

LLaMA

MATLAB

arXiv


AI IN MATHEMATICS
Large Language Models (LLMs)
Formal Proof Assistant
Machine Learning
LEAN
Algorithms
● Mathematical search engine
● Provides explanations
● Math Problem Solving
● Software tools for theorem proving
● Verifies proofs
● Makes you structure proofs
● Generates examples
● Discovers new structures and relationships
● Pushes the boundaries of solving methods

Large Language Models (LLMs)

Examples:

Predict the next “word” based on previously generated tokens and the given context.

Recent developments:

- Non-linear reasoning
- Usage of Agents

Formal Proof Assistants

Examples:

Check out: https://adam.math.hhu.de

Verify mathematical proofs with absolute rigor.

- Reduce the need in human reviews
- Generate datasets of proofs
- Allow simpler collaboration

IMO Participation

AI won “silver” medal on International Math Olympiad

- Solved the **hardest** (for humans) problem of the Olympiad.
- Wrote proofs in Lean and scored **28/42** points (just **one point shy** of a gold medal).
- **Three days** to find the solution.
- **Didn’t solve combinatorics.** Apparently, AI is not a fan of this field!

AlphaProof solved

Problem 1. Determine all real numbers α such that, for every positive integer n, the integer

$$\lfloor \alpha \rfloor + \lfloor 2\alpha \rfloor + \cdots + \lfloor n\alpha \rfloor$$

is a multiple of n. (Note that $\lfloor z \rfloor$ denotes the greatest integer less than or equal to z. For example, $\lfloor -\pi \rfloor = -4$ and $\lfloor 2 \rfloor = \lfloor 2.9 \rfloor = 2$.)

Problem 2. Determine all pairs (a, b) of positive integers for which there exist positive integers g and N such that

$$\gcd(a^n + b, b^n + a) = g$$

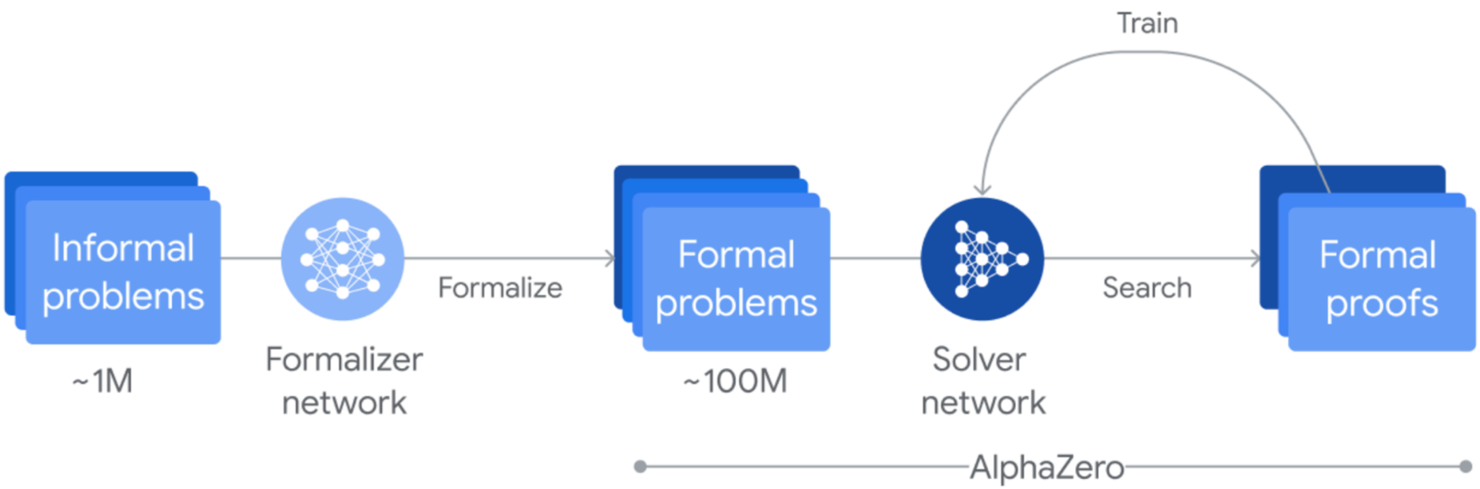
holds for all integers $n \geqslant N$. (Note that $\gcd(x, y)$ denotes the greatest common divisor of integers x and y.)

Problem 6. Let $\mathbb{Q}$ be the set of rational numbers. A function $f\colon \mathbb{Q} \to \mathbb{Q}$ is called *aquaesulian* if the following property holds: for every $x, y \in \mathbb{Q}$,

$$f(x + f(y)) = f(x) + y \qquad \text{or} \qquad f(f(x) + y) = x + f(y).$$

Show that there exists an integer c such that for any aquaesulian function f there are at most c different rational numbers of the form $f(r) + f(-r)$ for some rational number r, and find the smallest possible value of c.

AlphaProof



Train
Informal problems
~1M
Formalizer network
Formalize
Formal problems
~100M
Solver network
Search
Formal proofs
AlphaZero

Problem 1

Problem 1. Determine all real numbers α such that, for every positive integer n, the integer

$$\lfloor \alpha \rfloor + \lfloor 2\alpha \rfloor + \cdots + \lfloor n\alpha \rfloor$$

is a multiple of n. (Note that $\lfloor z \rfloor$ denotes the greatest integer less than or equal to z. For example, $\lfloor -\pi \rfloor = -4$ and $\lfloor 2 \rfloor = \lfloor 2.9 \rfloor = 2$.)

Solution ideas:

- Even integers satisfy the condition.
- Consider α as a sum of integer k and $0 \le \epsilon < 1$.
- Notice that we can take k out of the brackets.
- Consider case of even and odd k separately and prove that only even integers satisfy.

Problem 2

Problem 2. Determine all pairs (a, b) of positive integers for which there exist positive integers g and N such that

$$\gcd(a^n + b,\, b^n + a) = g$$

holds for all integers $n \geqslant N$. (Note that $\gcd(x, y)$ denotes the greatest common divisor of integers x and y.)

Let again $K = ab + 1$, which is coprime to both a and b. By Euler's theorem, for $n = k \cdot \varphi(K) - 1$ we have

$$a(a^n + b) = (a^k)^{\varphi(K)} + ab \equiv 1 + ab = K \equiv 0 \pmod{K},$$

so $K \mid a^n + b$ and similarly $K \mid b^n + a$. Hence K is a common divisor of $a^n + b$ and $b^n + a$. If n is sufficiently large, the greatest common divisor is supposed to be g, so $K \mid g$.

Now, for sufficiently large $n = k \cdot \varphi(K)$ we have $K \mid g \mid a^n + b$ and therefore

$$0 \equiv a^n + b = (a^k)^{\varphi(K)} + b \equiv 1 + b \pmod{K},$$

so $K = ab + 1 \mid b + 1$ and similarly $ab + 1 \mid a + 1$, which is possible only for $a = b = 1$,

Problem 6

Problem 6. Let $\mathbb{Q}$ be the set of rational numbers. A function $f\colon \mathbb{Q} \to \mathbb{Q}$ is called *aquaesulian* if the following property holds: for every $x, y \in \mathbb{Q}$,

$$f(x + f(y)) = f(x) + y \qquad \text{or} \qquad f(f(x) + y) = x + f(y).$$

Show that there exists an integer c such that for any aquaesulian function f there are at most c different rational numbers of the form $f(r) + f(-r)$ for some rational number r, and find the smallest possible value of c.

Solution ideas:

- Example $f(x) = \lfloor x \rfloor - \{x\}$ gives $c \geq 2$.
- Denote $g(x) = f(x) + f(-x)$.
- We want to show that it can not have more than 1 nonzero value.
- Show that f is a bijection with $f\big(-f(-x)\big) = x$.
- Derive the result for $g(x)$ knowing that $f^{-1}(x) = -f(-x)$.

AlphaGeometry solved

Problem 4. Let ABC be a triangle with $AB < AC < BC$. Let the incentre and incircle of triangle ABC be I and ω, respectively. Let X be the point on line BC different from C such that the line through X parallel to AC is tangent to ω. Similarly, let Y be the point on line BC different from B such that the line through Y parallel to AB is tangent to ω. Let AI intersect the circumcircle of triangle ABC again at $P \neq A$. Let K and L be the midpoints of AC and AB, respectively.

Prove that $\angle KIL + \angle YPX = 180°$.

in 19 seconds…

Much stronger than AlphaProof:
Solves almost all geometry problems from IMO this century.

AlphaGeometry solved

Problem 4. Let ABC be a triangle with $AB < AC < BC$. Let the incentre and incircle of triangle ABC be I and ω, respectively. Let X be the point on line BC different from C such that the line through X parallel to AC is tangent to ω. Similarly, let Y be the point on line BC different from B such that the line through Y parallel to AB is tangent to ω. Let AI intersect the circumcircle of triangle ABC again at $P \neq A$. Let K and L be the midpoints of AC and AB, respectively.

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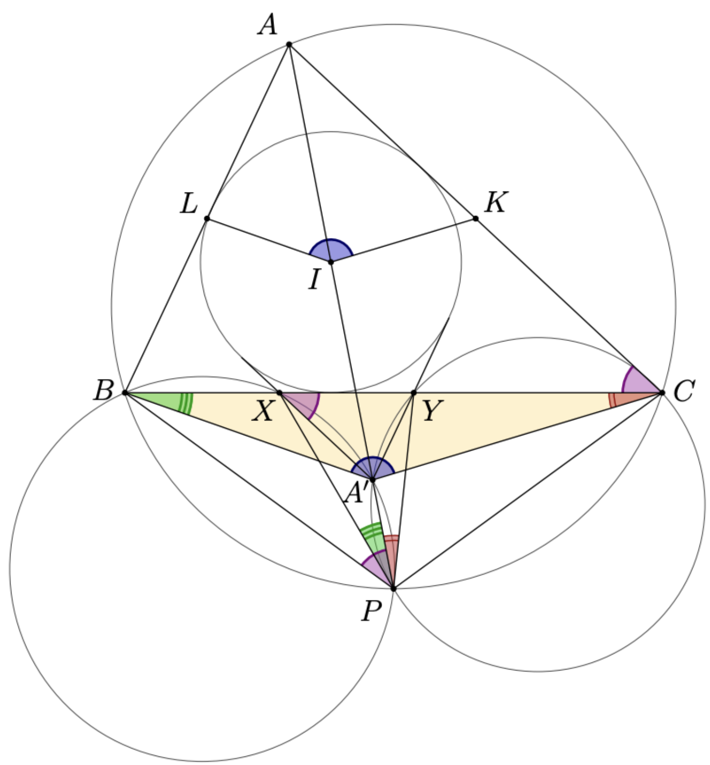
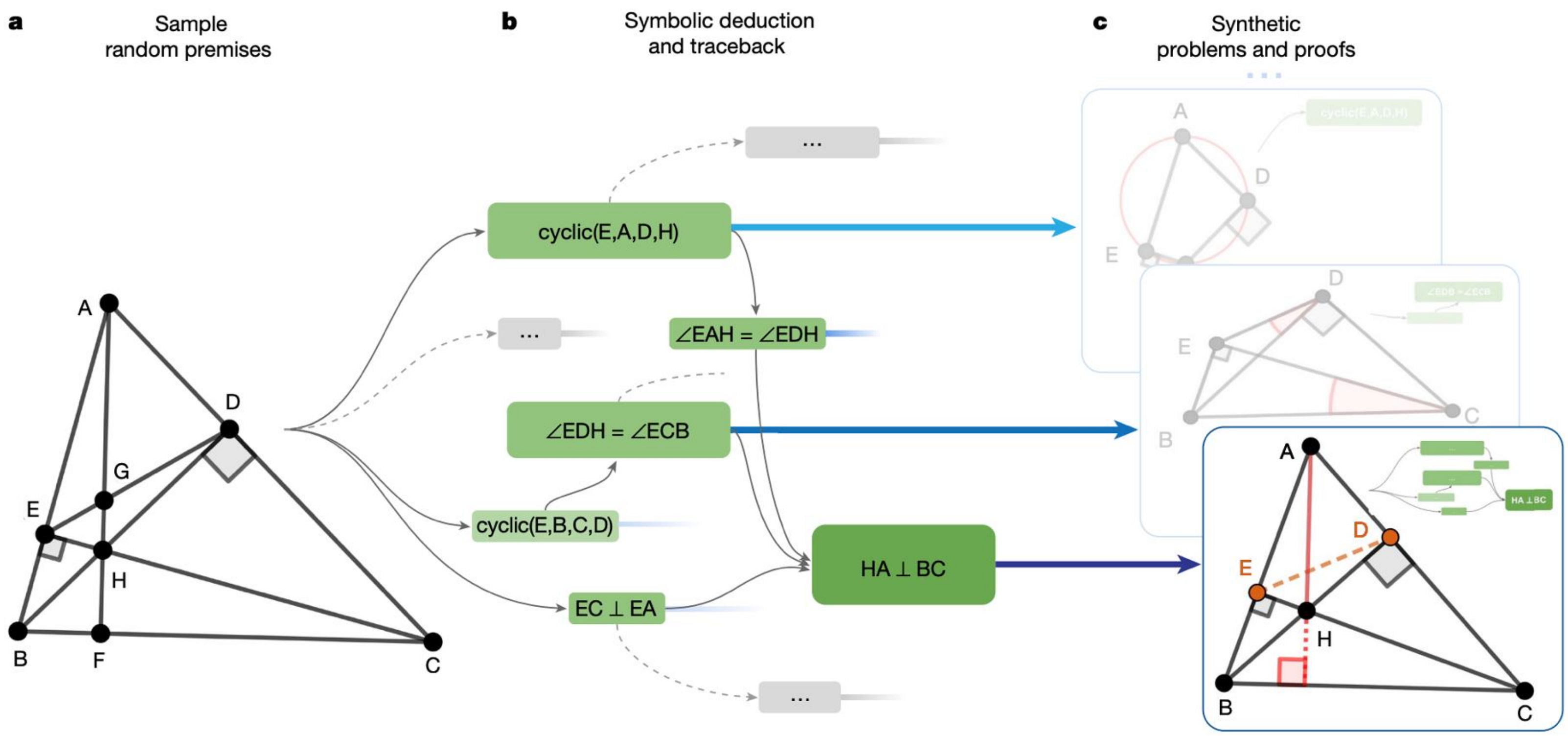
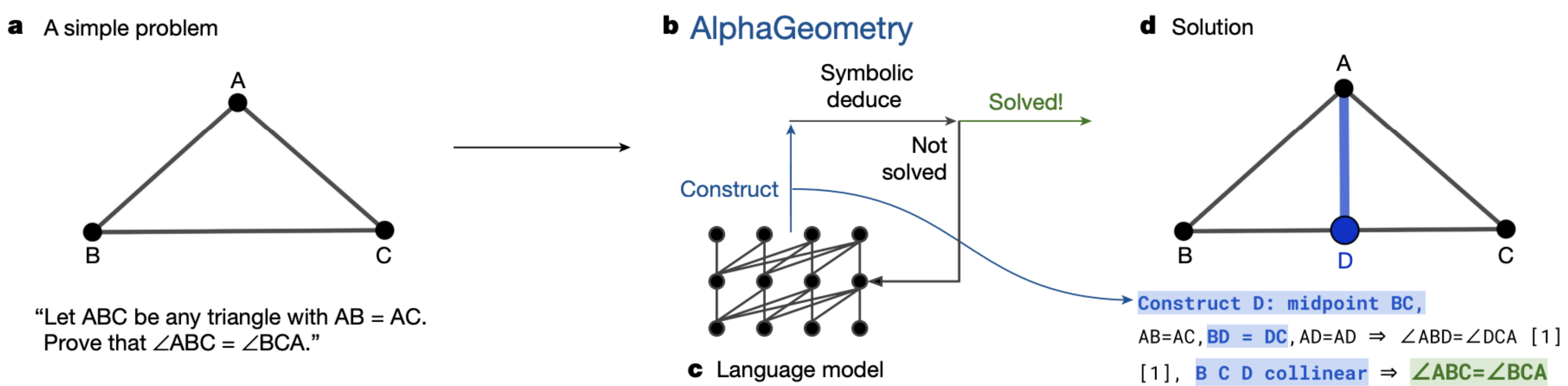


AlphaGeometry



AlphaGeometry



AlphaProof didn’t solve

Problem 3. Let $a_1, a_2, a_3, \ldots$ be an infinite sequence of positive integers, and let N be a positive integer. Suppose that, for each $n > N$, a_n is equal to the number of times a_{n-1} appears in the list $a_1, a_2, \ldots, a_{n-1}$.

Prove that at least one of the sequences $a_1, a_3, a_5, \ldots$ and $a_2, a_4, a_6, \ldots$ is eventually periodic.

(An infinite sequence $b_1, b_2, b_3, \ldots$ is *eventually periodic* if there exist positive integers p and M such that $b_{m+p} = b_m$ for all $m \geqslant M$.)

Problem 5. Turbo the snail plays a game on a board with 2024 rows and 2023 columns. There are hidden monsters in 2022 of the cells. Initially, Turbo does not know where any of the monsters are, but he knows that there is exactly one monster in each row except the first row and the last row, and that each column contains at most one monster.

Turbo makes a series of attempts to go from the first row to the last row. On each attempt, he chooses to start on any cell in the first row, then repeatedly moves to an adjacent cell sharing a common side. (He is allowed to return to a previously visited cell.) If he reaches a cell with a monster, his attempt ends and he is transported back to the first row to start a new attempt. The monsters do not move, and Turbo remembers whether or not each cell he has visited contains a monster. If he reaches any cell in the last row, his attempt ends and the game is over.

Determine the minimum value of n for which Turbo has a strategy that guarantees reaching the last row on the n^{th} attempt or earlier, regardless of the locations of the monsters.

Is it too lazy to read so many words?)

Problem 3

Problem 3. Let $a_1, a_2, a_3, \ldots$ be an infinite sequence of positive integers, and let N be a positive integer. Suppose that, for each $n > N$, a_n is equal to the number of times a_{n-1} appears in the list $a_1, a_2, \ldots, a_{n-1}$.

Prove that at least one of the sequences $a_1, a_3, a_5, \ldots$ and $a_2, a_4, a_6, \ldots$ is eventually periodic.

(An infinite sequence $b_1, b_2, b_3, \ldots$ is *eventually periodic* if there exist positive integers p and M such that $b_{m+p} = b_m$ for all $m \geqslant M$.)

Ideas:

1. At least one integer appears infinitely often.
2. Only finitely many integers appear infinitely often.
3. Classify numbers and show the sequence eventually alternates between “big” and “small.”
4. Analyze the “small” terms to show eventual periodicity.

Problem 5

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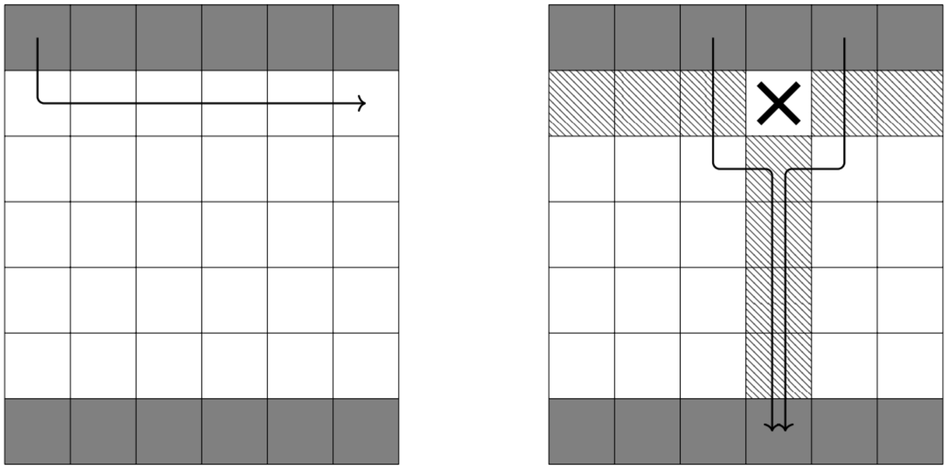
Determine the minimum value of n for which Turbo has a strategy that guarantees reaching the last row on the n^{th} attempt or earlier, regardless of the locations of the monsters.

Answer: n = 3



AI and Mathematics

And now we will look at some examples how machine learning solves real mathematical problems

AI and Matrix Multiplication

Consider multiplying two matrices:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

How many multiplications are needed in the classical approach?

AI and Matrix Multiplication

We can do 7!!!!

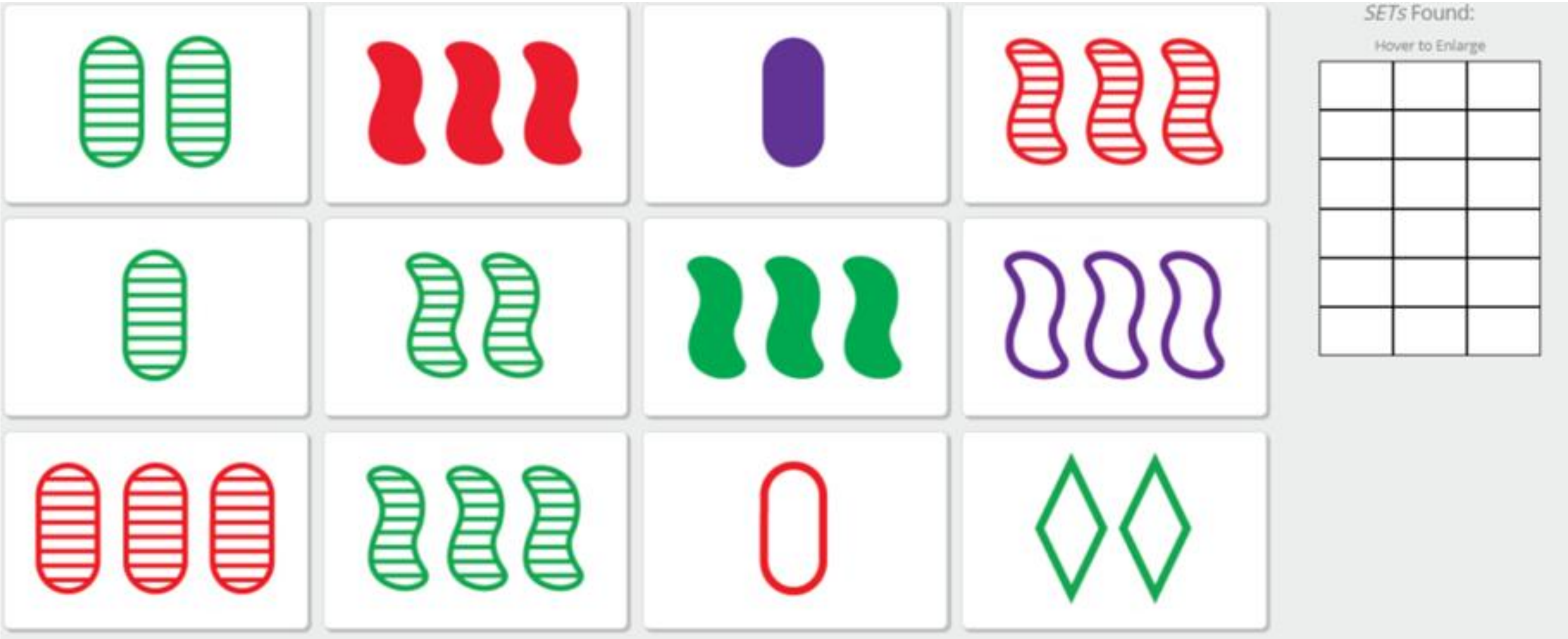
$$M_1 = (a_{11} + a_{22}) \cdot (b_{11} + b_{22}), \quad M_2 = (a_{21} + a_{22}) \cdot b_{11}$$
$$M_3 = a_{11} \cdot (b_{12} - b_{22}), \quad M_4 = a_{22} \cdot (b_{21} - b_{11})$$
$$M_5 = (a_{11} + a_{12}) \cdot b_{22}, \quad M_6 = (a_{21} - a_{11}) \cdot (b_{11} + b_{12})$$
$$M_7 = (a_{12} - a_{22}) \cdot (b_{21} + b_{22})$$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} M_1 + M_4 - M_5 + M_7 & M_3 + M_5 \\ M_2 + M_4 & M_1 - M_2 + M_3 + M_6 \end{pmatrix}.$$

Alpha Tensor results

Size (n, m, p)	Best method known	Best rank known	AlphaTensor rank Modular	AlphaTensor rank Standard
(2, 2, 2)	(Strassen, 1969)[2]	7	7	7
(3, 3, 3)	(Laderman, 1976)[15]	23	23	23
(4, 4, 4)	(Strassen, 1969)[2] $(2, 2, 2) \otimes (2, 2, 2)$	49	47	49
(5, 5, 5)	(3, 5, 5) + (2, 5, 5)	98	96	98
(2, 2, 3)	(2, 2, 2) + (2, 2, 1)	11	11	11
(2, 2, 4)	(2, 2, 2) + (2, 2, 2)	14	14	14
(2, 2, 5)	(2, 2, 2) + (2, 2, 3)	18	18	18
(2, 3, 3)	(Hopcroft and Kerr, 1971)[16]	15	15	15
(2, 3, 4)	(Hopcroft and Kerr, 1971)[16]	20	20	20
(2, 3, 5)	(Hopcroft and Kerr, 1971)[16]	25	25	25
(2, 4, 4)	(Hopcroft and Kerr, 1971)[16]	26	26	26
(2, 4, 5)	(Hopcroft and Kerr, 1971)[16]	33	33	33
(2, 5, 5)	(Hopcroft and Kerr, 1971)[16]	40	40	40
(3, 3, 4)	(Smirnov, 2013)[18]	29	29	29
(3, 3, 5)	(Smirnov, 2013)[18]	36	36	36
(3, 4, 4)	(Smirnov, 2013)[18]	38	38	38
(3, 4, 5)	(Smirnov, 2013)[18]	48	47	47
(3, 5, 5)	(Sedoglavic and Smirnov, 2021)[19]	58	58	58
(4, 4, 5)	(4, 4, 2) + (4, 4, 3)	64	63	63
(4, 5, 5)	$(2, 5, 5) \otimes (2, 1, 1)$	80	76	76

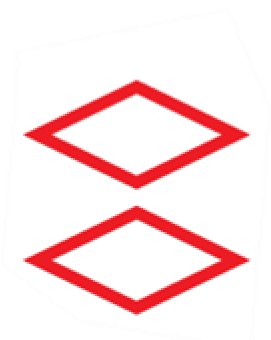
The SET Game



The SET Game

What mathematical structure do the cards correspond to?

Each card represents a point in $\mathbb{Z}_3^4$ (a four-dimensional vector space over $\mathbb{Z}_3$). A "set" is a triple of cards whose sum in $\mathbb{Z}_3^4$ is the zero vector.



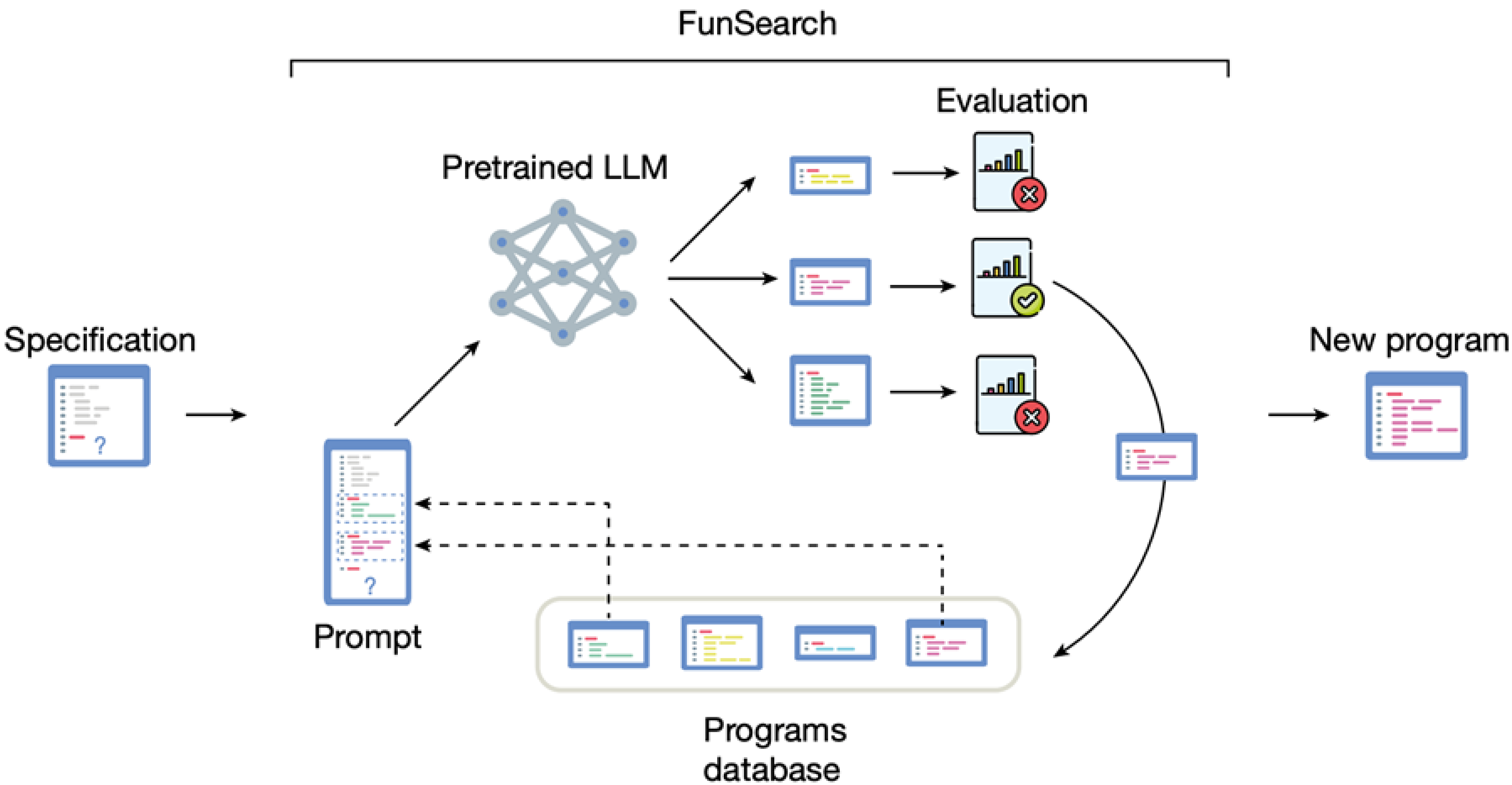
(diamond, 2, red,open)

AI improved the results!

Find the largest possible subset of $\mathbb{Z}_3^n$, such that sum of any triplet doesn't equal to zero.

n	3	4	5	6	7	8
Best known	9	20	45	112	236	496
FunSearch	9	20	45	112	236	512

How AI Helped in the Capset Problem?



Lyapunov Functions

A Lyapunov function is a function associated with an ordinary differential equation (ODE):

$$\dot{x} = g(x), \quad x \in \mathbb{R}^n.$$

Definition: A function $V : \mathbb{R}^n \to \mathbb{R}$ is called a **Lyapunov function** for the system if:

- $V(x) > 0$ for all $x \neq 0$,
- $V(0) = 0$,
- $\dot{V}(x) = \langle \nabla V(x), \dot{x} \rangle \leq 0$.

Why are they important?

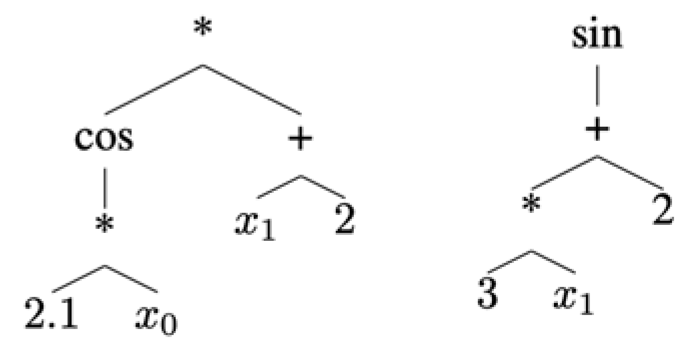
Lyapunov function $\Leftrightarrow$ Stable system

Lyapunov Functions

Even writing system of equation in a machine readable text is challenging:

$$\begin{cases} \dot{x}_0 = \cos(2.1x_0)(x_1 + 2) \\ \dot{x}_1 = \sin(3x_1 + 2) \end{cases}$$

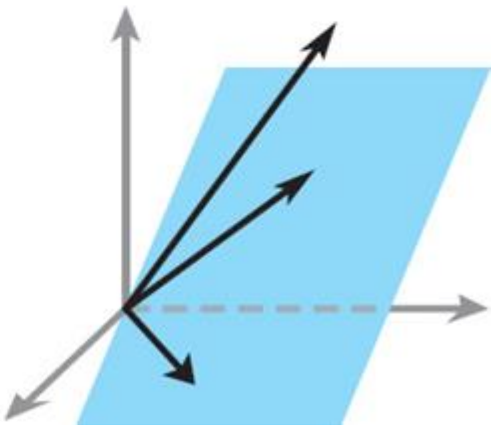
is represented as



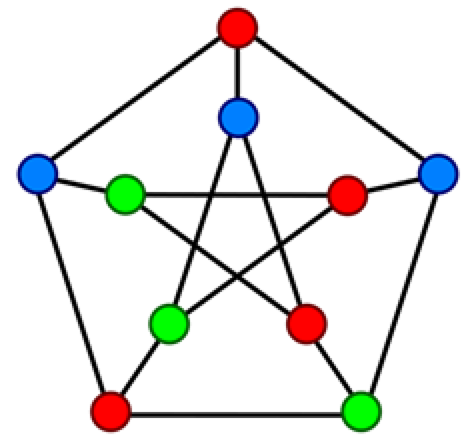
Lyapunov Functions

AI found Lyapunov functions on a test dataset **84%** of the time, while Master’s students in relative field only solved **9.3%** of those equations.

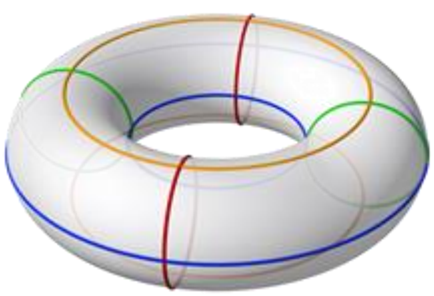
Mathematical Data

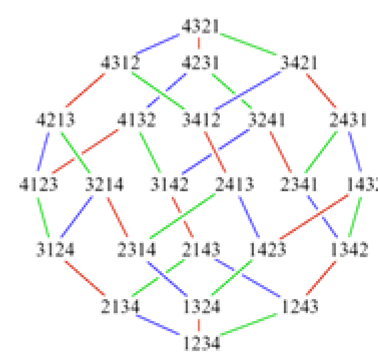


$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$



$$f(x) = a_{\mathrm{n}} x^{\mathrm{n}} + \ldots + a_1 x + a_0$$





About This Course

1 week: Intro

2 weeks: Data Science in Mathematics

2 weeks: Deep Learning in Mathematics

3 weeks: Math as an NLP problem (LLMs etc.)

3 weeks: Reinforcement Learning (RL) in Math

1 week: Advanced AI topics

1 week: Project Presentations

Thank you!