Analyse eines Forschungsthemas
Stochastic Shortest Paths

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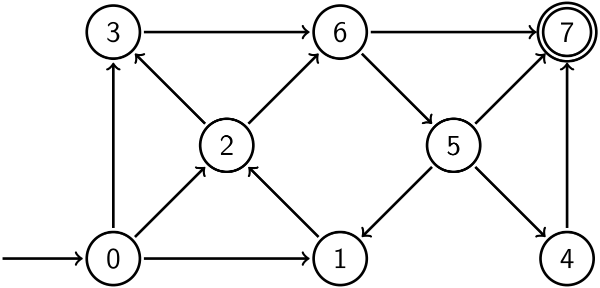
Section 1

Introduction

- The *simplest* shortest path problem



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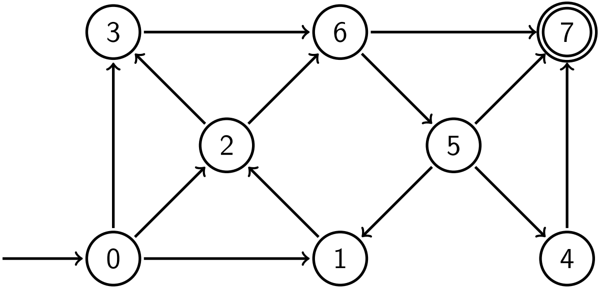

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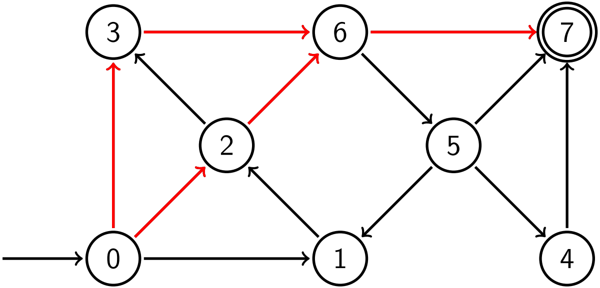

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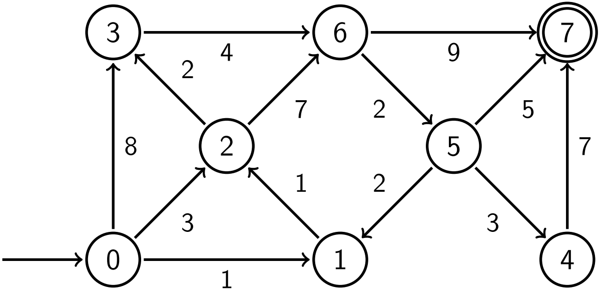

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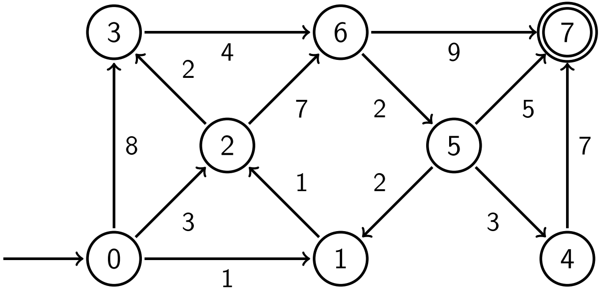

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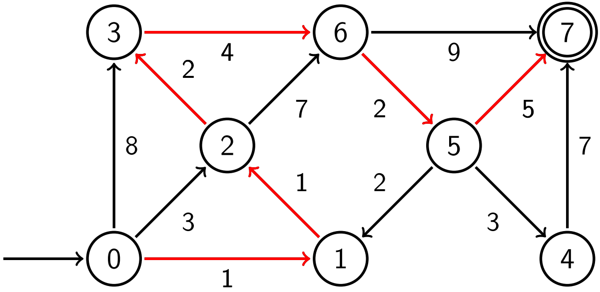

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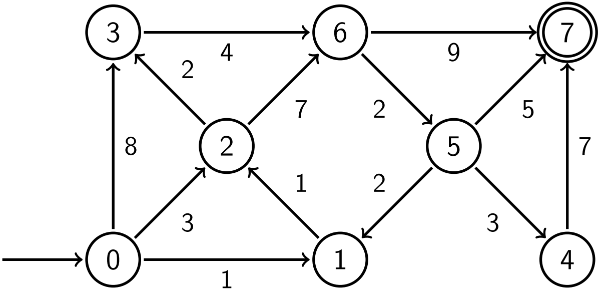

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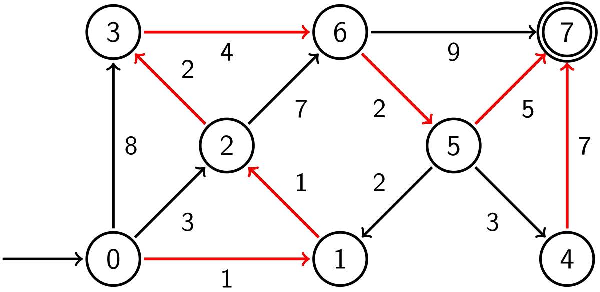

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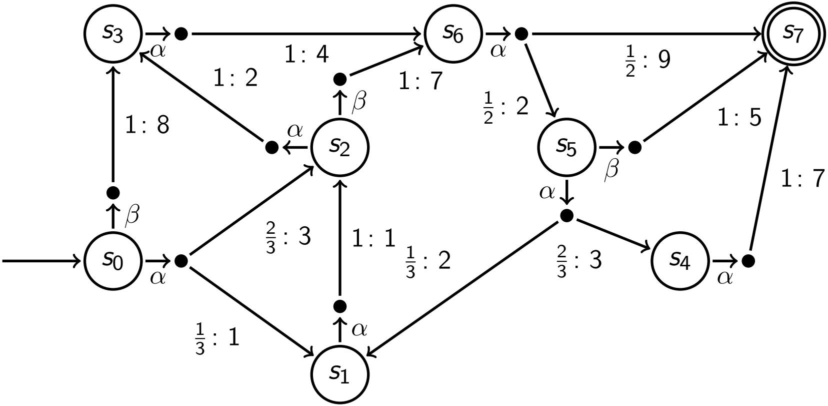

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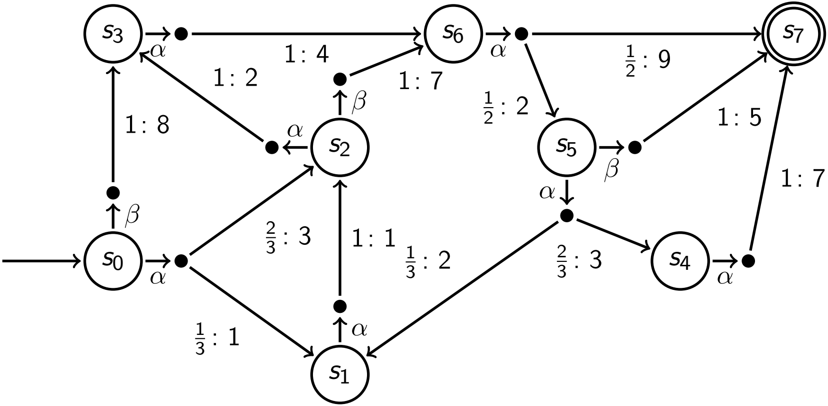

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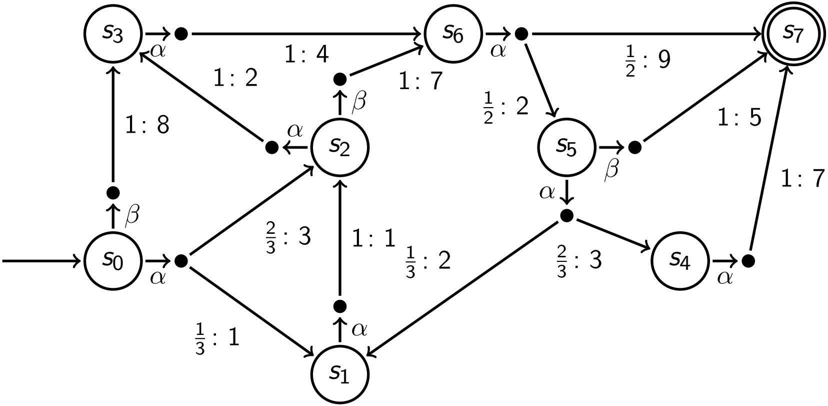

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accumulated weights until reaching a goal

$$\oplus F : \mathrm{Paths}(\mathcal{M}^{\mathfrak{S}}) \to \mathbb{Q} :$$

$$\oplus F(\pi) = \begin{cases} wgt(\hat{\pi}), & \hat{\pi} \text{ is shortest prefix of } \pi \text{ s.t. } \mathrm{last}(\hat{\pi}) \in F \\ \infty, & \text{otherwise, i.e. } \pi \not\models \Diamond F \end{cases}$$

Section 2

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- The following can be solved in polynomial time:
  - Check: $\mathbb{E}^{inf}_{\mathcal{M},s}(\oplus \mathrm{goal}) > -\infty$?
  - Compute $\mathbb{E}^{inf}_{\mathcal{M},s}$ if it is finite

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  3. For each $s \in \mathcal{E} \setminus \{s_0\}$ and $\beta \in \mathrm{Act}_{\mathcal{M}}(s) \setminus \{\alpha_s\}$ let us replace (s, β) by (s_0, β) where
     - $P_{\mathcal{N}}(s_0, \beta, u) := P_{\mathcal{M}}(s, \beta, u)$
     - $wgt_{\mathcal{N}}(s_0, \beta) + wgt(s, s_0) = wgt_{\mathcal{M}}(s, \beta)$

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- bounded EC: There exists an upper bound and a lower bound

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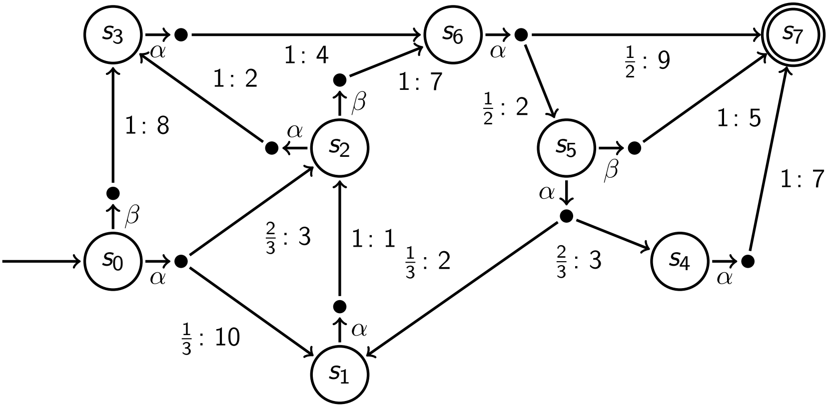

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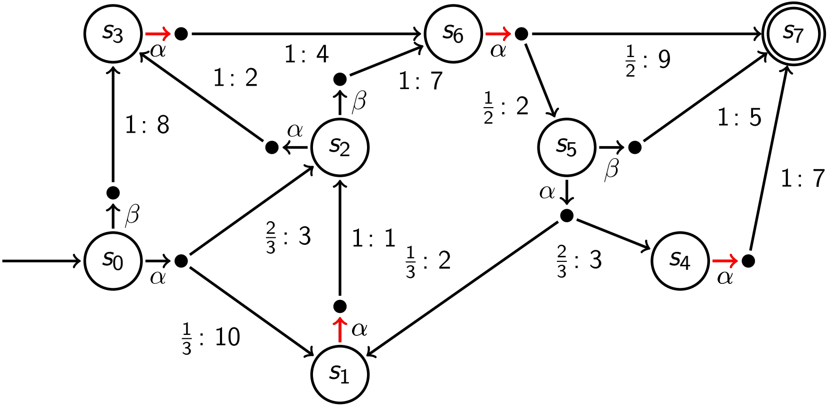

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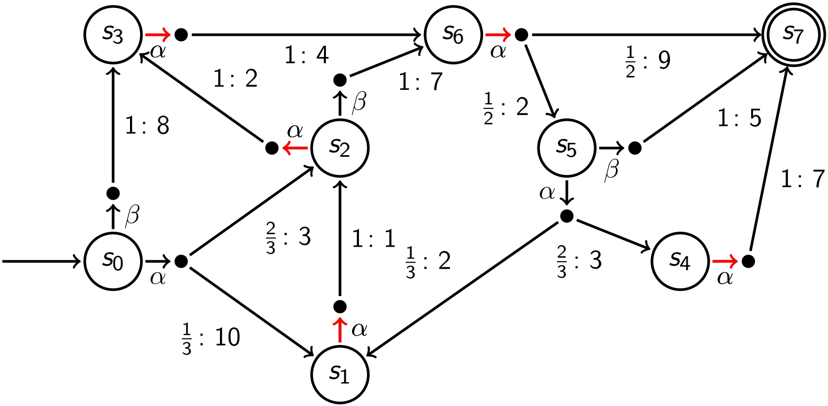

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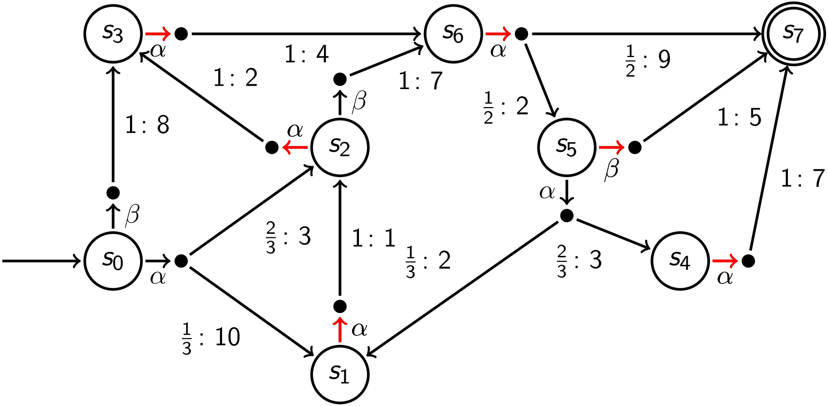

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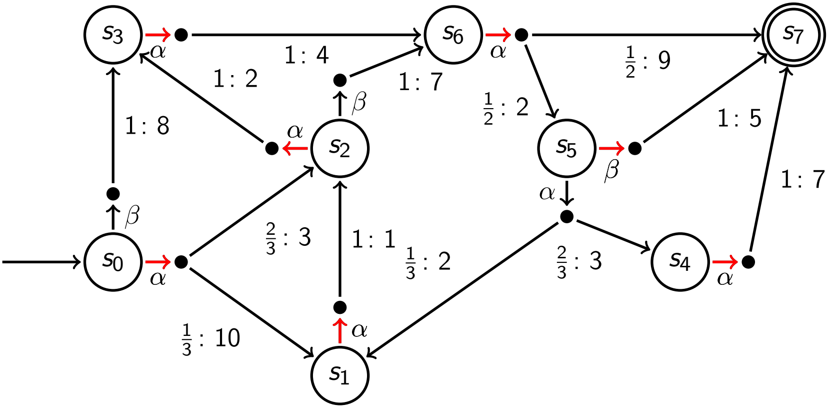

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Section 3

Different variants of the stochastic shortest path problem

conditional expected accumulated weights

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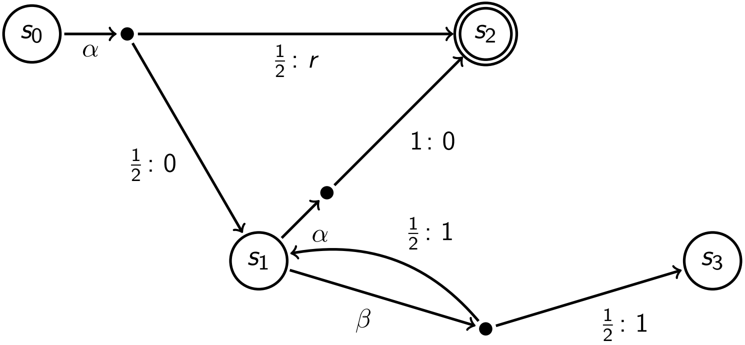
conditional expected accumulated reward



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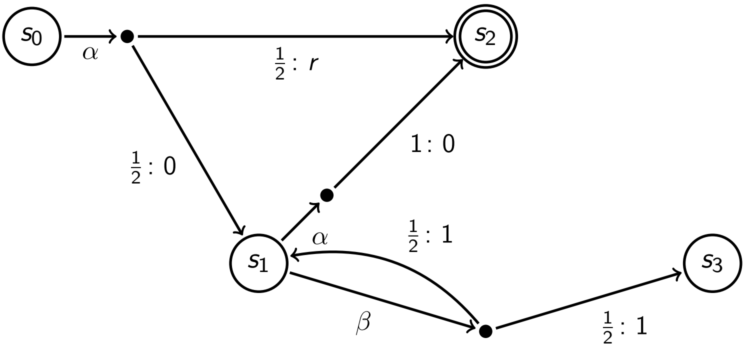

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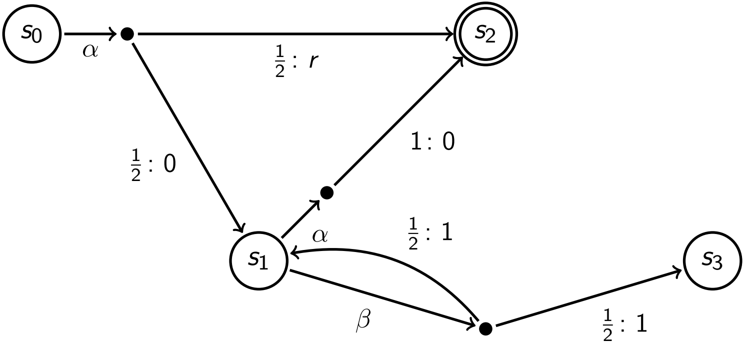

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$\mathcal{N}$ has no *critical* scheduler $\mathfrak{S}$: $\mathbb{Pr}^{\mathfrak{S}}(\Diamond fail) = 1$ and there is a reachable positive $\mathfrak{S}$-cycle.

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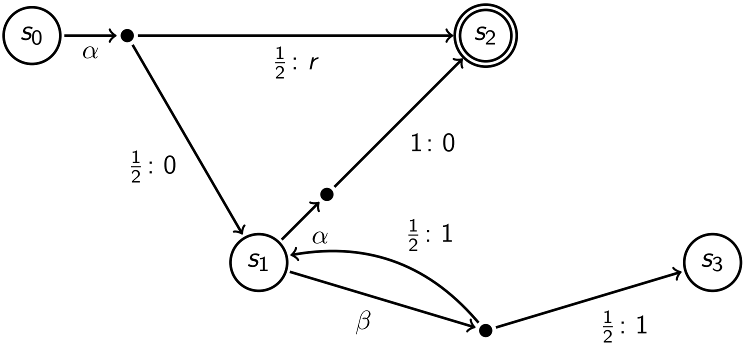
s_0
α
$\frac{1}{2}: r$
s_2
$\frac{1}{2}: 0$
$1: 0$
α
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s_1
s_3
β
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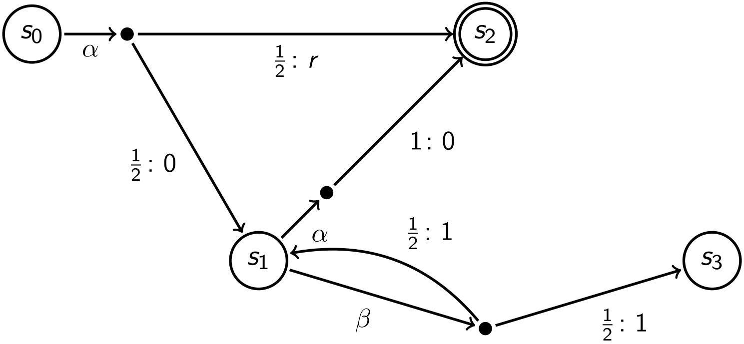

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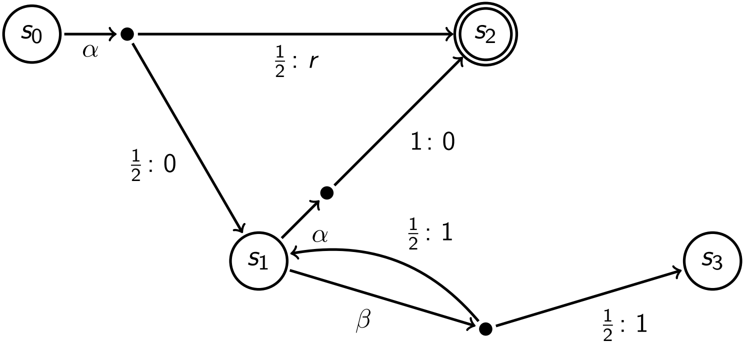

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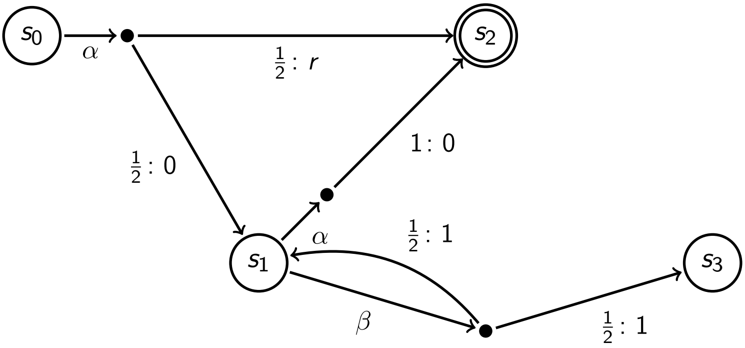

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Section 4

Keep an eye on the variance

Variance-penalized expectation

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- The threshold problem, i.e. checking $\mathbb{VPE}^{max} \geq t$ is decidable in **NExpTime** and is known to be **ExpTime-hard**.

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- ...There is an optimal deterministic scheduler, computable in **ExpTime**

Tools

- Storm model checker
- Prism model checker