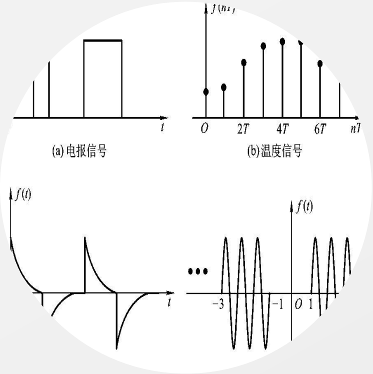
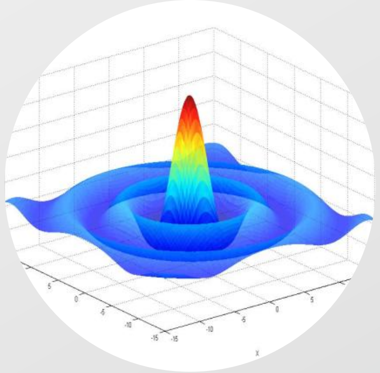
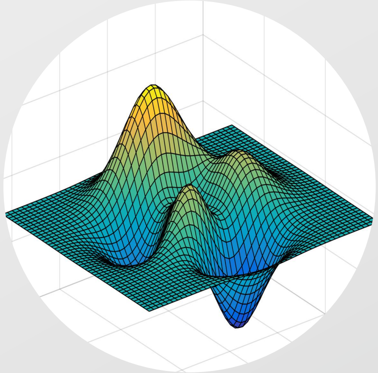
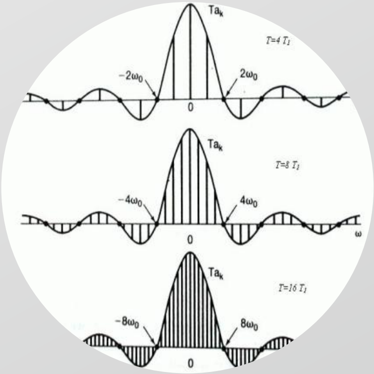




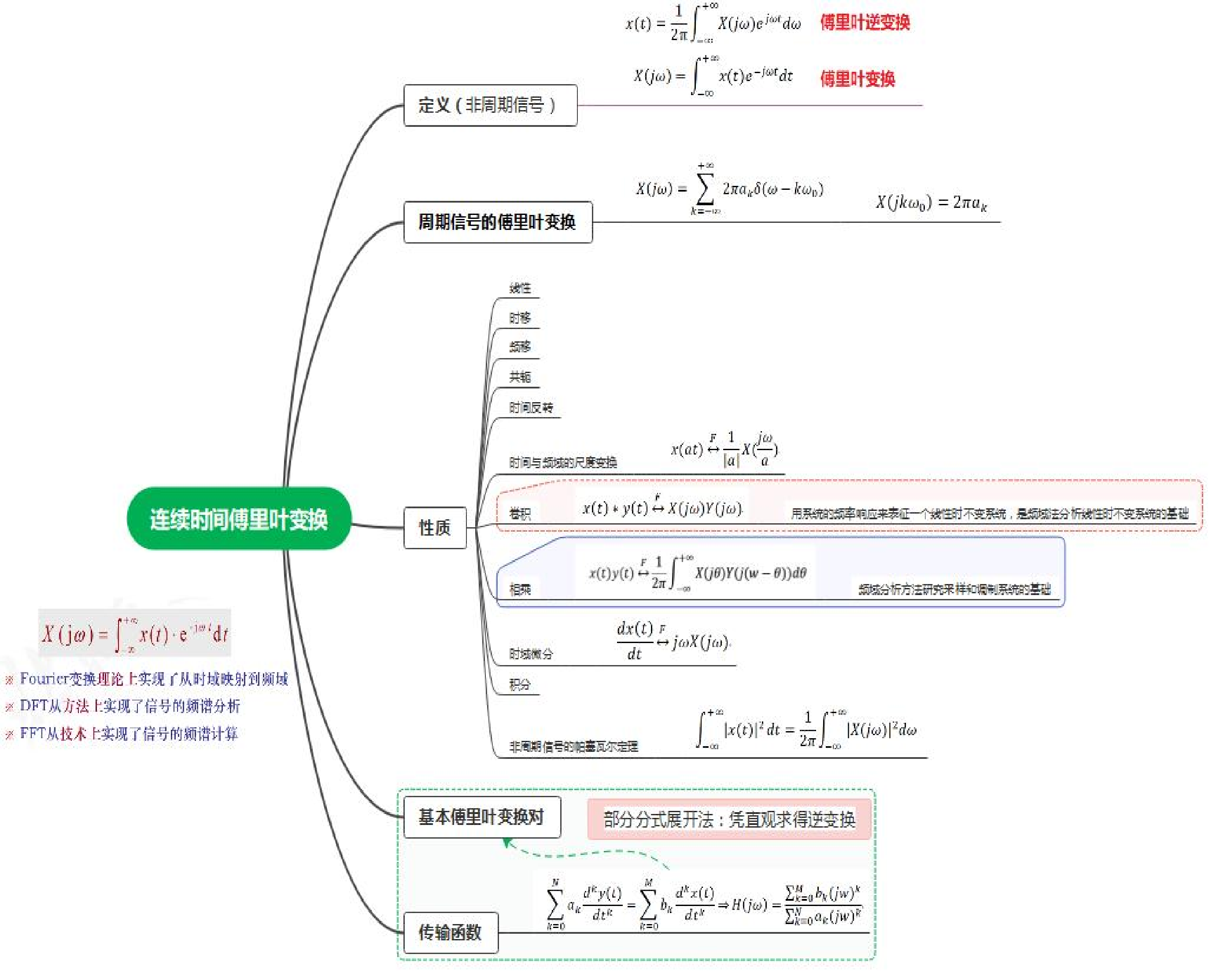




Lab4 The Continuous-Time Fourier Transform
连续时间傅里叶变换

主讲老师：王小静
办公地点：一教131

Objectives



- CT Fourier Transform
  （Periodic or Aperiodic）
- Properties of CTFT & Basic Fourier Transform Pairs
- Application of CTFT

Contents

Catalogue

CT Fourier Transform—CTFT

chapter3 → chapter4、5

Periodic Signals—Fourier Series

Aperiodic Signals—Fourier Transform

Fourier Transform Pair

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}d\omega$$ → ***inverse Fourier Transform, Synthesis Equation***

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt$$ → ***Fourier Transform or Fourier Integral Analysis Equation***

$X(j\omega)$ is the function of ω, which is called the spectrum of $x(t)$.
Note: ω is continuous between $-\infty$ and $+\infty$.

DTFS

$$x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n} = \sum_{k=<N>} a_k e^{jk(2\pi/N)n}$$ → *Synthesis Equation*

$$a_k = \frac{1}{N} \sum_{n=<N>} x[n]\, e^{-jk\omega_0 n} = \frac{1}{N} \sum_{n=<N>} x[n]\, e^{-jk(2\pi/N)n}$$ → *Analysis Equation*

$\{a_k\}$ are often called the Fourier series coefficients or the **spectral coefficients**.

There is N values of k, that is a_k repeat periodiclly with period N.

Representation of Aperiodic Signals: CT Fourier Transform

$$a_k = \frac{1}{T}\int_T \tilde{x}(t)e^{-jk\omega_0 t}dt$$

$T \to \infty$

$$a_k = \frac{1}{T}\int_{-\infty}^{+\infty} \boldsymbol{x(t)}e^{-jk\omega_0 t}dt$$

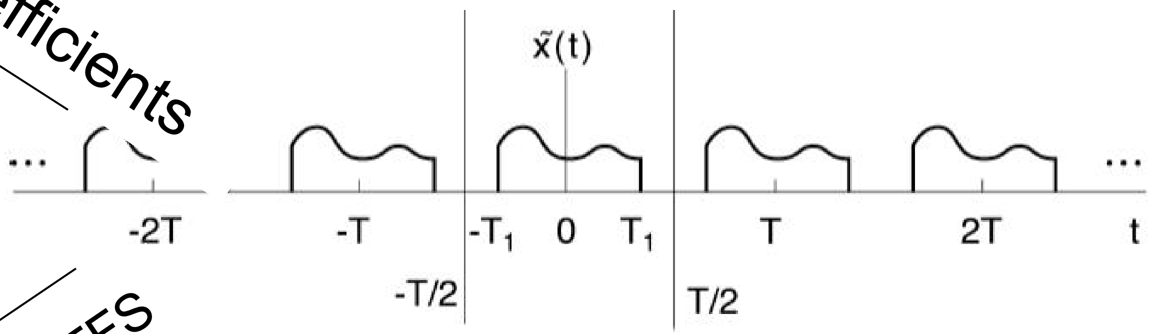
Define the envelope $X(j\omega)$ of Ta_k as

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt$$

$\omega = k\omega_0$ $\omega_0 \to 0$

$$a_k = \frac{1}{T}X(jk\omega_0)$$

Fourier series coefficients



Synthesis of CTFS

$$\tilde{x}(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} \frac{1}{T}X(jk\omega_0)e^{jk\omega_0 t} = \frac{1}{2\pi}\sum_{k=-\infty}^{+\infty} X(jk\omega_0)e^{jk\omega_0 t}\omega_0$$

$T \to \infty$ $\omega_0 \to 0$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}d\omega$$

Ex.3.5

$$x(t)=\begin{cases}1, & |t|<T_1\\ 0, & T_1\le |t|<T\end{cases}$$

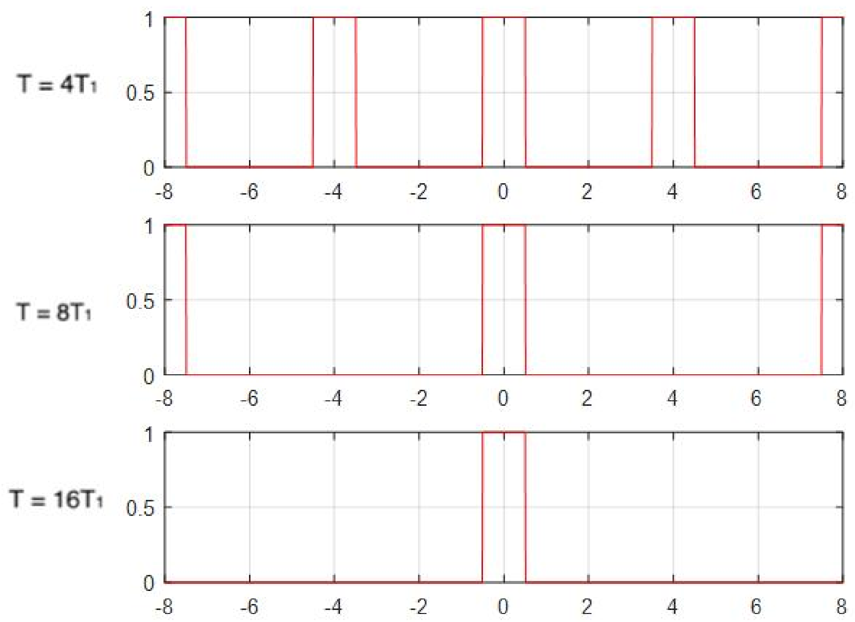
$\Updownarrow$

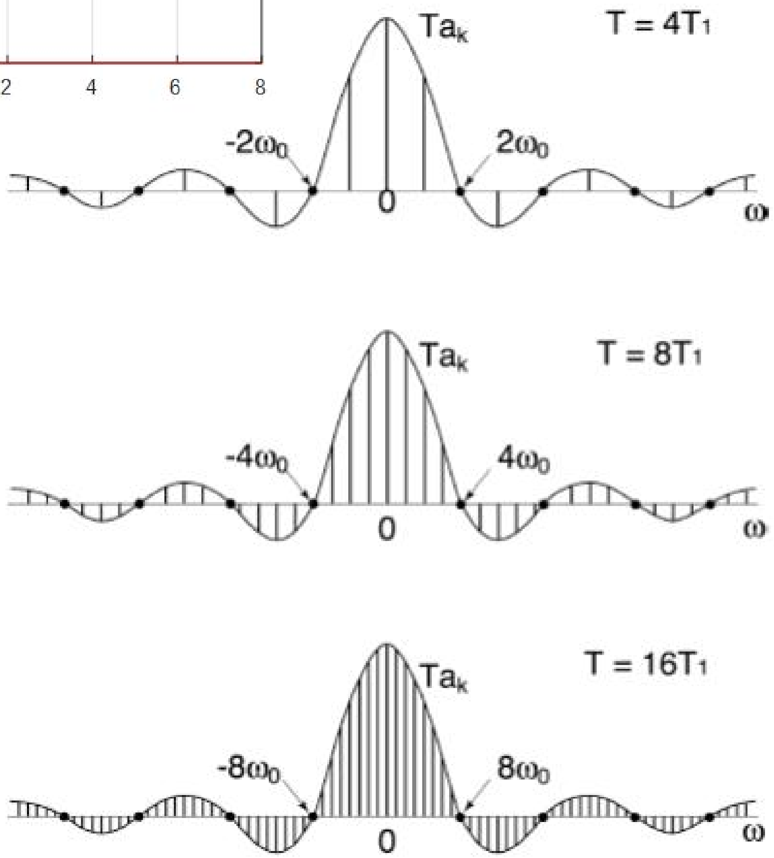
$$a_k=\frac{2\sin(k\omega_0 T_1)}{k\omega_0 T}\qquad \omega_0=\frac{2\pi}{T}$$

$\Downarrow$ T_1 keep fixed

$$X(j\omega)=\ Ta_k=\left.\frac{2\sin(\omega T_1)}{\omega}\right|_{\omega=k\omega_0}$$

Envelope Function





$\left(\omega_0=\dfrac{2\pi}{T}\right)$

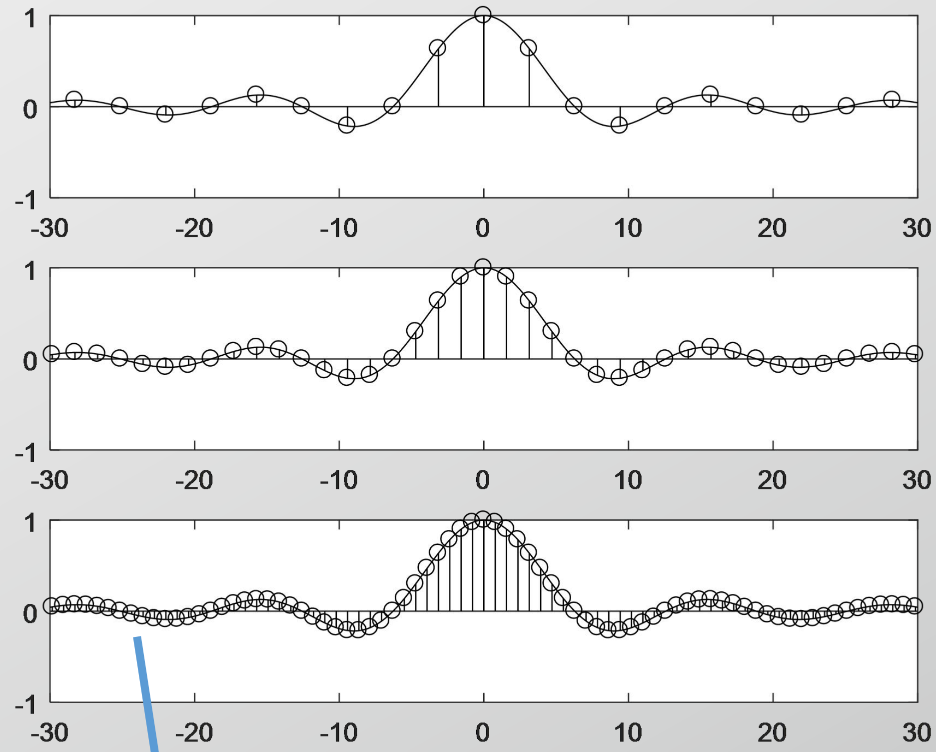
Become denser in ω as T increases

```
clear, close all
T1=0.5;
w=-30:0.01:30;
y=2*sin(w.*T1)./w;

N1=4;
w0_1=2*pi/N1/T1;
k1=fix(30/w0_1);
w1=-k1*w0_1:w0_1:k1*w0_1;
y1=2*sin(w1.*T1)./w1;
figure
subplot(3,1,1),plot(w,y,'-k'),hold on,plot(0,1,'-k'),hold on,
stem(w1,y1,'ko');hold on,stem(0,1,'ko');hold off

N2=8;
w0_2=2*pi/N2/T1;
k2=fix(30/w0_2);
w2=-k2*w0_2:w0_2:k2*w0_2;
y2=2*sin(w2.*T1)./w2;
subplot(3,1,2),plot(w,y,'-k'),hold on,plot(0,1,'-k'),hold on,
stem(w2,y2,'ko');hold on,stem(0,1,'ko');hold off
```

$$X(j\omega) = Ta_k = \left.\frac{2\sin(\omega T_1)}{\omega}\right|_{\omega = k\omega_0}$$



Subplot this figure by yourself; Write comments.

CT Fourier Transform of Aperiodic Signals —CTFT

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}d\omega \longrightarrow$$

inverse Fourier Transform,
Synthesis Equation

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt \longrightarrow$$

Fourier Transform or Fourier Integral
Analysis Equation

$X(j\omega)$ is the function of ω, which is called the spectrum of $x(t)$.
Note: ω is continuous between $-\infty$ and $+\infty$.

Convergece of Aperiodic Signals'CTFT

Does the periodic signal meet the conditions?

1: Finite Energy: $\int_{-\infty}^{+\infty} |x(t)|^2\, dt < \infty$

2: Dirichlet conditions

1): x(t) absolutely integrable $\int_{-\infty}^{+\infty} |x(t)|\, dt < \infty$

2): finite number of maxima and minima within any finite interval

3): finite number of discontinuities with finite values within any finite interval

Fourier Transforms of CT Periodic Signals

The Fourier transforms of periodic signals with Fourier series coefficients $\{a_k\}$ can be interpreted as a train of impulse occuring at the harmonically related frequencies and for which the area of the impulse at the kth harmoic frequency $k\omega_0$ is 2π times the kth Fourier series coefficient a_k.

$$X(jw) = \sum_{k=-\infty}^{+\infty} 2\pi a_k \delta(w - kw_0)$$

$$\omega_0 = \frac{2\pi}{T}$$

Fourier Transforms of CT Periodic Signals

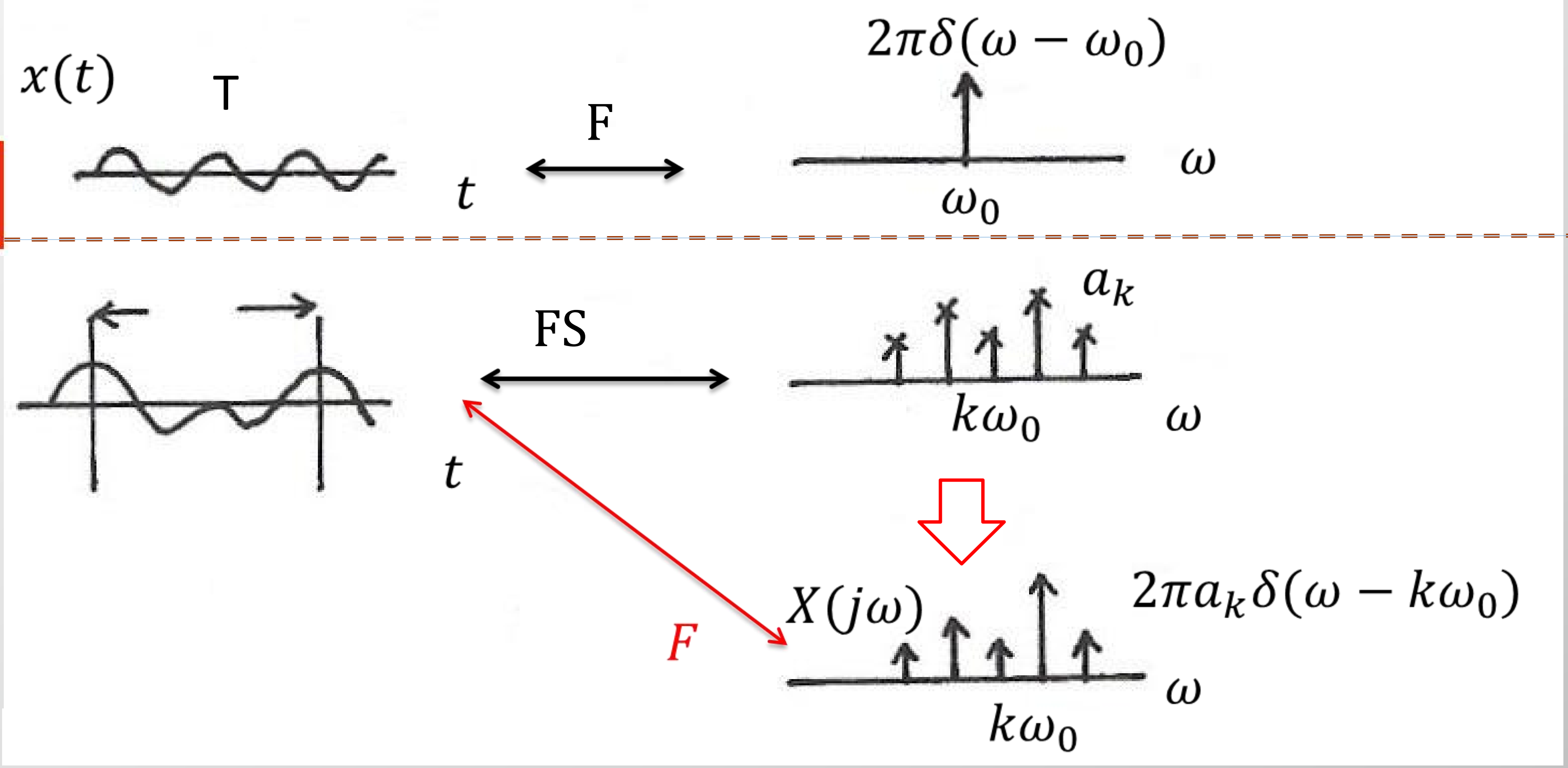
$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jkw_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t} \longrightarrow \textit{Synthesis Equation}$$

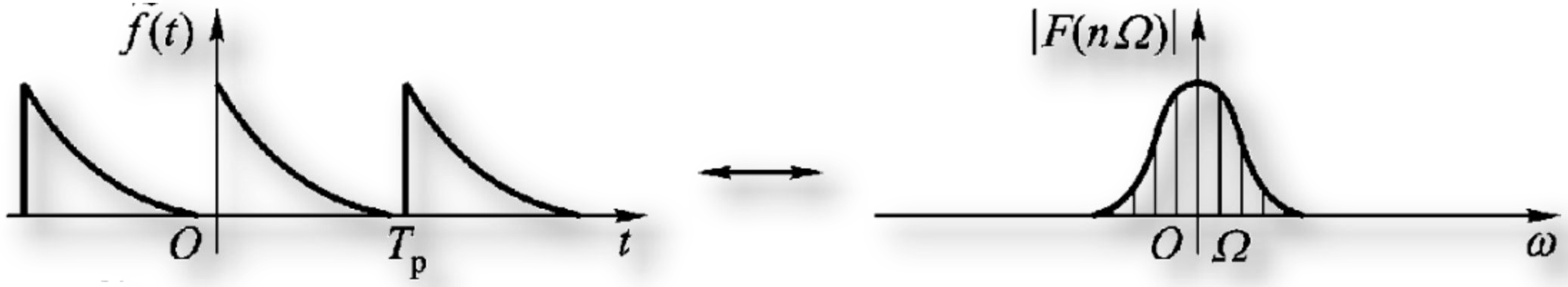
$$a_k = \frac{1}{T}\int_T x(t)e^{-jkw_0 t}dt = \frac{1}{T}\int_T x(t)e^{-jk(2\pi/T)}dt \longrightarrow \textit{Analysis Equation}$$

Fourier Series coefficients of CT Periodic Signals

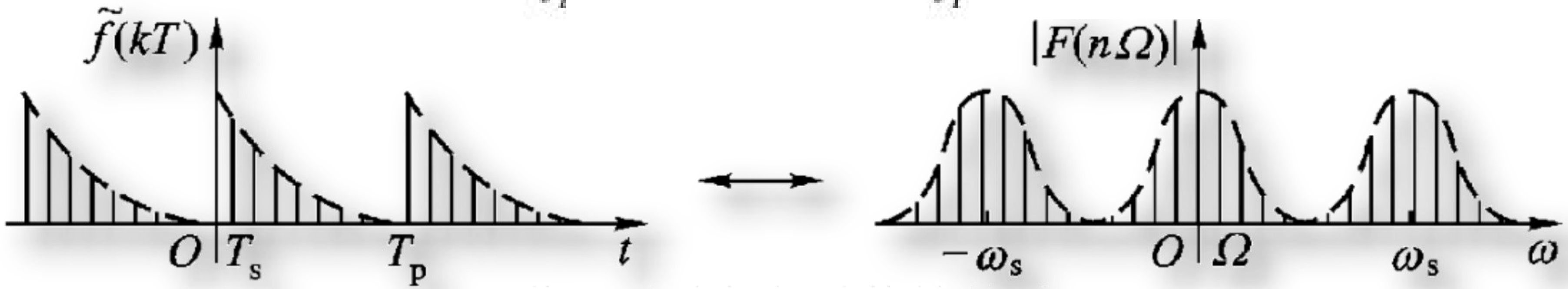
$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} \longleftrightarrow X(j\omega) = \sum_{k=-\infty}^{+\infty} 2\pi a_k \delta(\omega - k\omega_0)$$

$$\boxed{e^{j\omega_0 t} \longleftrightarrow 2\pi\delta(\omega - \omega_0)}$$

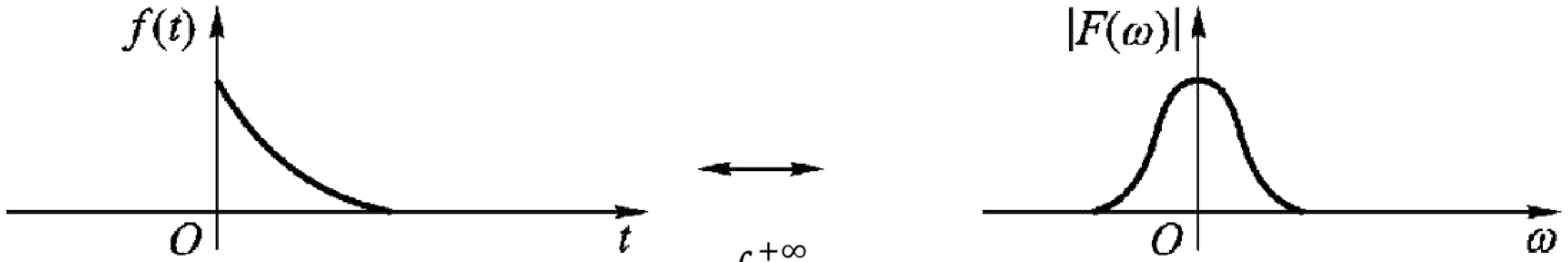




$$a_k = \frac{1}{T}\int_T x(t)e^{-jkw_0 t}dt = \frac{1}{T}\int_T x(t)e^{-jk(2\pi/T)}dt$$



$$a_k = \frac{1}{N}\sum_{n=<N>} x[n]\, e^{-jk\omega_0 n} = \frac{1}{N}\sum_{n=<N>} x[n]\, e^{-jk(2\pi/N)n}$$

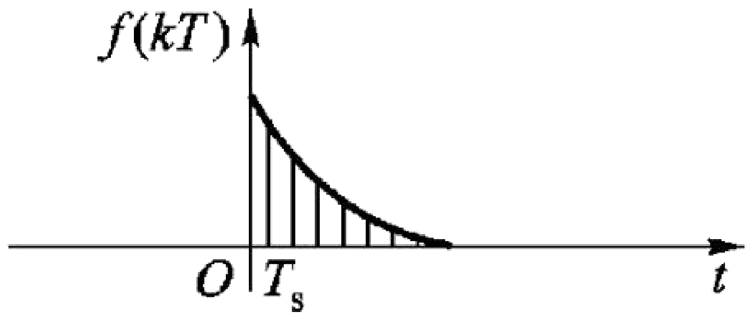


$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt$$

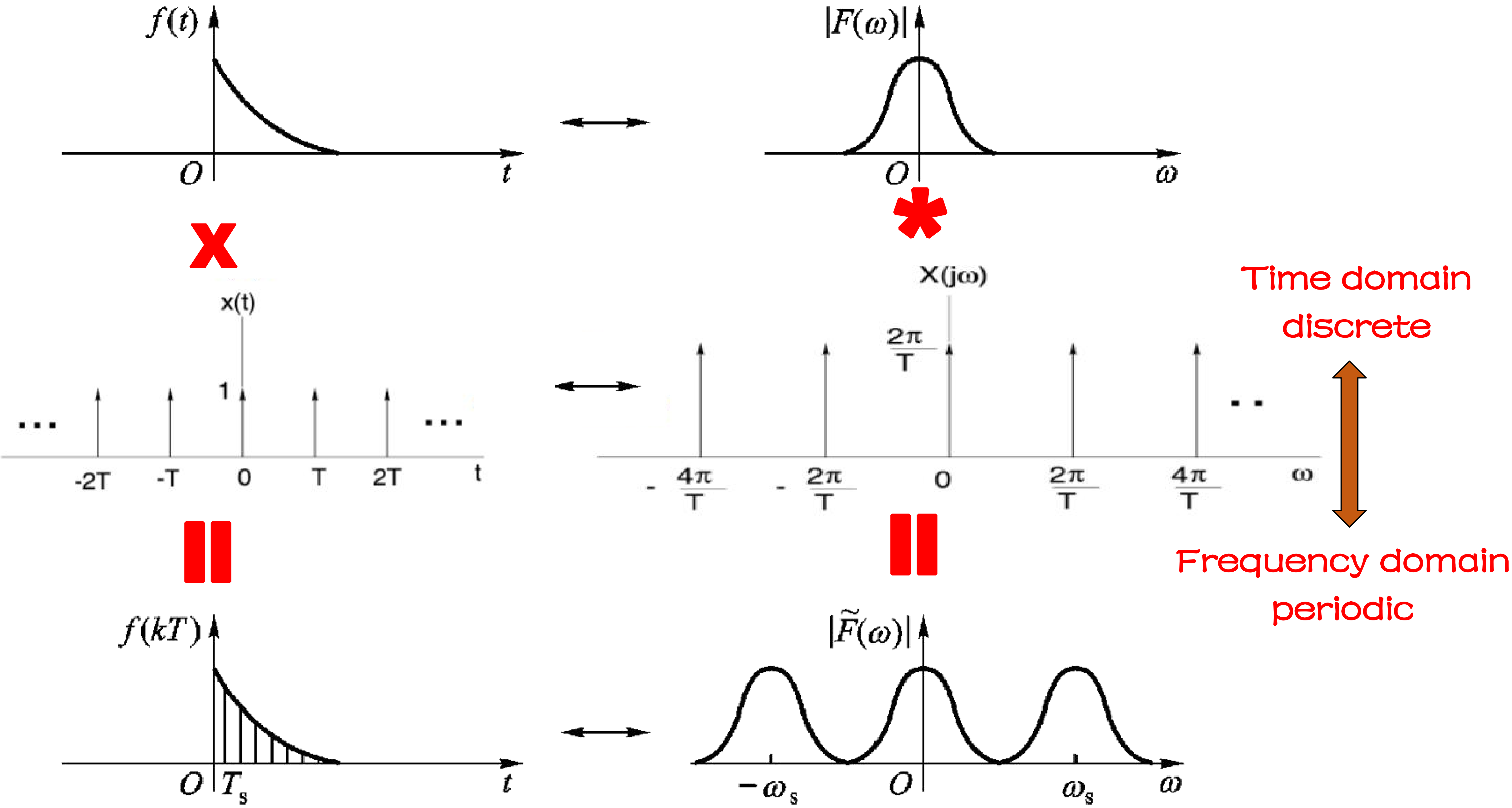
Time domain periodic

$\updownarrow$

Frequency domain discrete



$\longleftrightarrow$?

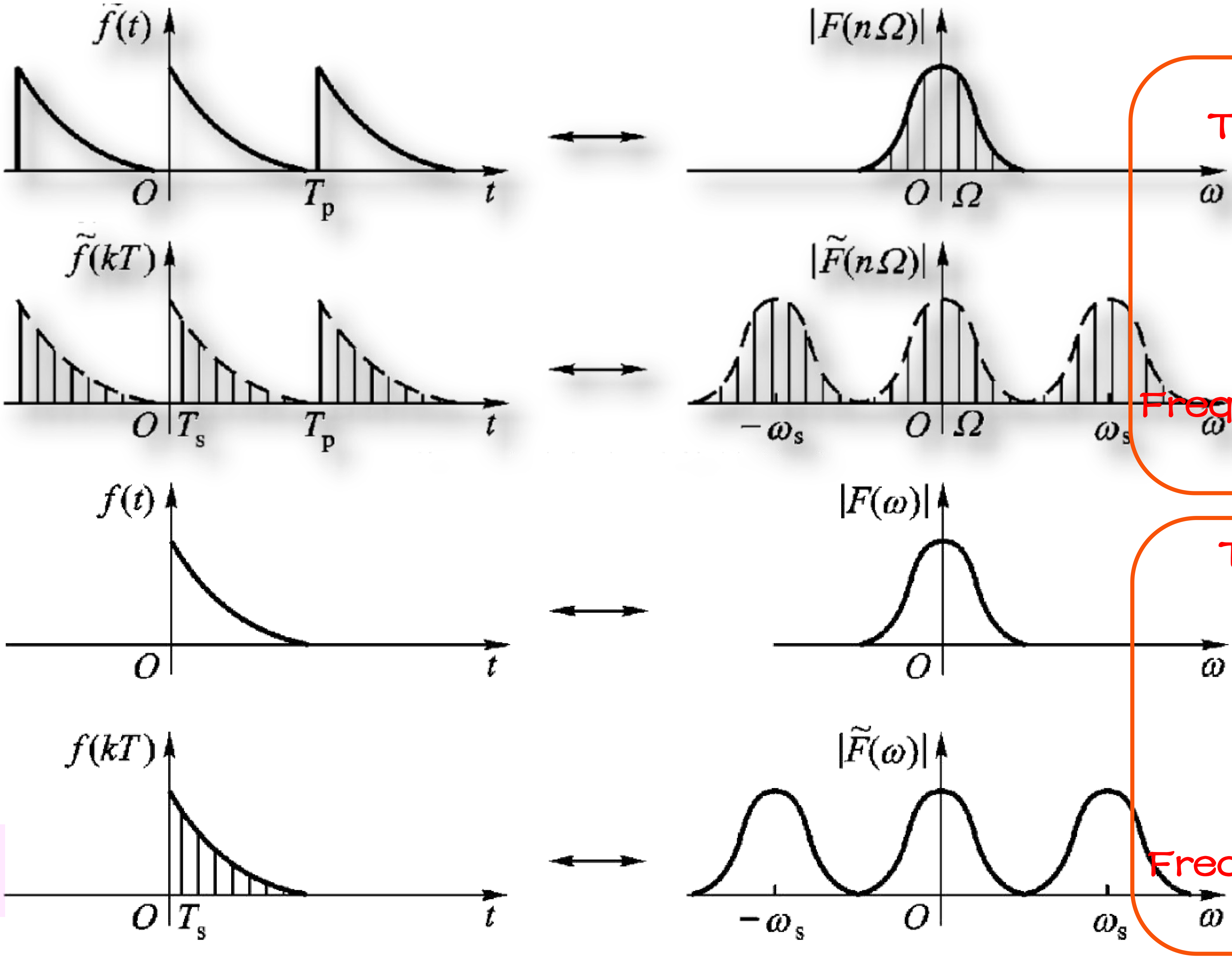

f(t)
O
t
|F(ω)|
O
ω
x(t)
1
-2T
-T
0
T
2T
t
X(jω)
2π/T
-4π/T
-2π/T
0
2π/T
4π/T
ω
Time domain discrete
Frequency domain periodic
f(kT)
O
T_s
t
|F̃(ω)|
-ω_s
O
ω_s
ω

CTFS:

DTFS:

CTFT:

DTFT:

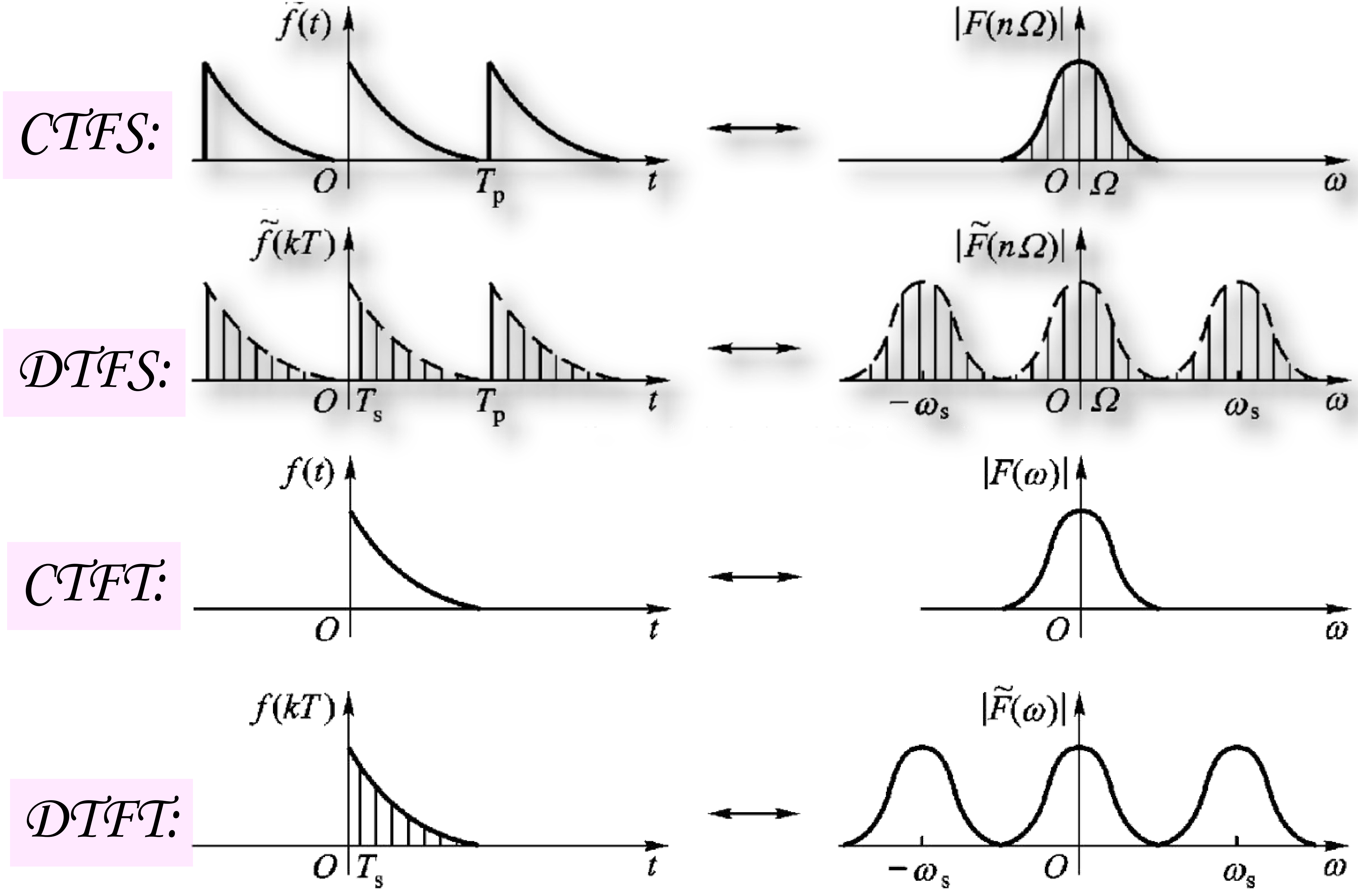


Time domain periodic

Frequency domain discrete

Time domain discrete

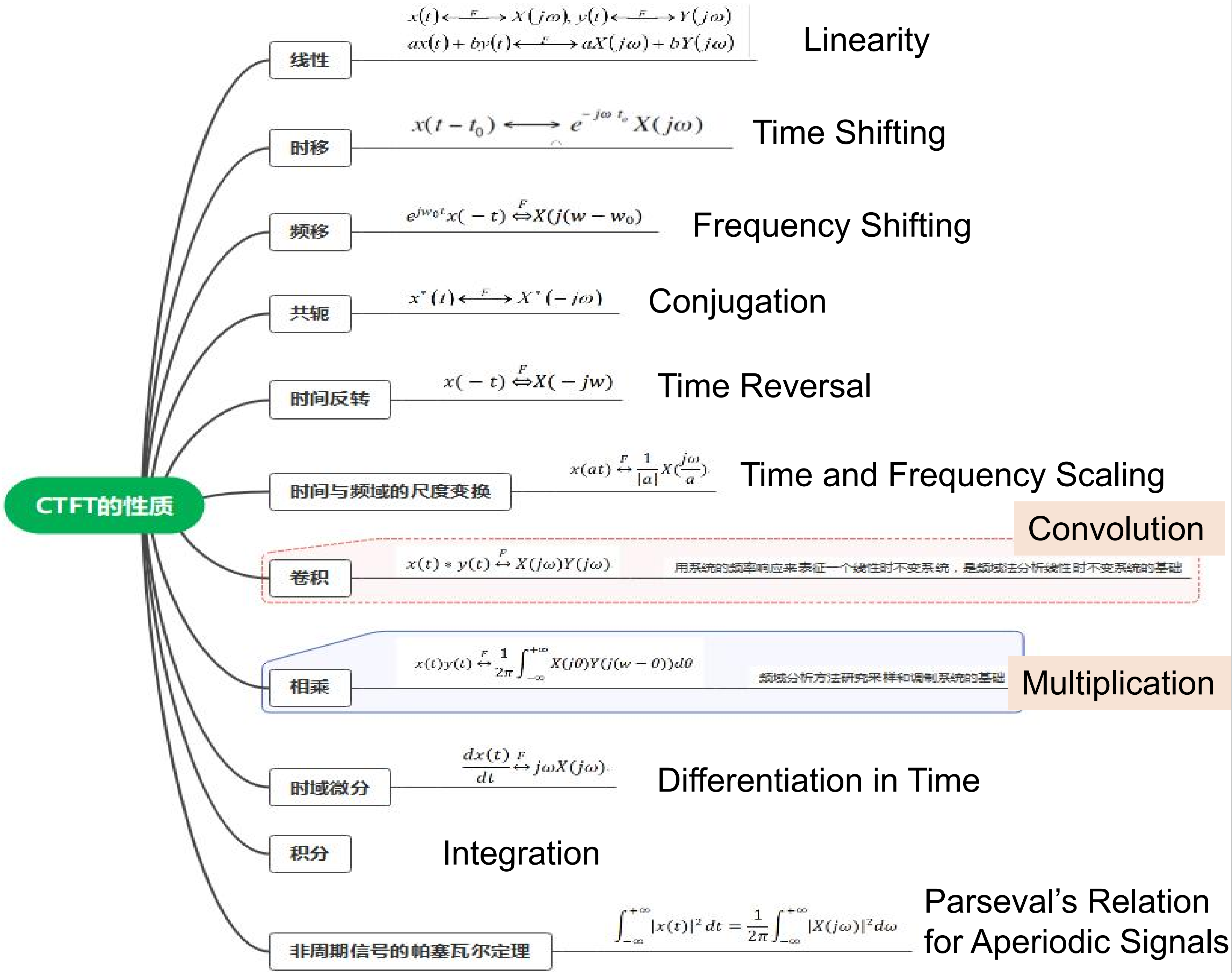
Frequency domain periodic



时域周期化相当于频域离散化

时域离散化相当于频域周期化

Properties of the CTFT



Basic Fourier Transform Pairs

TABLE 4.2 BASIC FOURIER TRANSFORM PAIRS

Signal	Fourier transform	Fourier series coefficients (if periodic)
$\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\omega - k\omega_0)$	a_k
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$	$a_1 = 1$ $a_k = 0$, otherwise
$\cos \omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$	$a_1 = a_{-1} = \frac{1}{2}$ $a_k = 0$, otherwise
$\sin \omega_0 t$	$\frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$	$a_1 = -a_{-1} = \frac{1}{2j}$ $a_k = 0$, otherwise
$x(t) = 1$	$2\pi\,\delta(\omega)$	$a_0 = 1$, $a_k = 0$, $k \neq 0$ (this is the Fourier series representation for any choice of $T > 0$)
Periodic square wave $x(t) = \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & T_1 < \lvert t\rvert \leq \frac{T}{2} \end{cases}$ and $x(t + T) = x(t)$	$\sum_{k=-\infty}^{+\infty} \frac{2 \sin k\omega_0 T_1}{k} \delta(\omega - k\omega_0)$	$\frac{\omega_0 T_1}{\pi} \operatorname{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right) = \frac{\sin k\omega_0 T_1}{k\pi}$
$\sum_{n=-\infty}^{+\infty} \delta(t - nT)$	$\frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$	$a_k = \frac{1}{T}$ for all k
$x(t) \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & \lvert t\rvert > T_1 \end{cases}$	$\frac{2 \sin \omega T_1}{\omega}$	—
$\frac{\sin Wt}{\pi t}$	$X(j\omega) = \begin{cases} 1, & \lvert\omega\rvert < W \\ 0, & \lvert\omega\rvert > W \end{cases}$	—
$\delta(t)$	1	—
$u(t)$	$\frac{1}{j\omega} + \pi\,\delta(\omega)$	—
$\delta(t - t_0)$	$e^{-j\omega t_0}$	—
$e^{-at}u(t)$, $\mathcal{R}e\{a\} > 0$	$\frac{1}{a + j\omega}$	—
$te^{-at}u(t)$, $\mathcal{R}e\{a\} > 0$	$\frac{1}{(a + j\omega)^2}$	—
$\frac{t^{n-1}}{(n-1)!} e^{-at}u(t)$, $\mathcal{R}e\{a\} > 0$	$\frac{1}{(a + j\omega)^n}$	—

Textbook 4.2

Calculate CTFT via Matlab

$$X(j\omega)=\int_{-\infty}^{+\infty}x(t)e^{-j\omega t}dt$$

$X(j\omega)$ is the function of ω, which is called the spectrum of $x(t)$.
Note: ω is continuous between $-\infty$ and $+\infty$.

According to the equation, plot its figure

- sample �
- calculate each $X(j\omega)$ at a specific ω
- plot(w, X(jw))

CTFT:

$$\mathbf{X}(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

$$= \lim_{\tau\to 0} \tau \sum_{n=-\infty}^{\infty} x(n\tau)e^{-j\omega\tau n} \qquad t = n\tau$$

Similarity

DTFS:

$$a_k = \frac{1}{N}\sum_{n=0}^{N-1} \mathrm{x}[\mathrm{n}]e^{-jk\left(\frac{2\pi}{N}\right)n}$$

Can we use FFT to calculate CTFT ?

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$$

CTFT:

If the signal x(t) is equal to zero for t < 0 and t >T,

$$\int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt = \int_{0}^{T} x(t) e^{-j\omega t} dt$$

Let $T = N\tau, \tau \to 0$

$$\approx \sum_{n=0}^{N-1} x(n\tau)\, e^{-j\omega n\tau} \tau$$

$w_k = k\frac{2\pi}{T} \quad T = N\tau$

$$X(jk\frac{2\pi}{T}) \approx \sum_{n=0}^{N-1} x(n\tau) e^{-jk\frac{2\pi}{T}n\tau} \tau = \sum_{n=0}^{N-1} x(n\tau) e^{-jk\frac{2\pi}{N\tau}n\tau} \tau = \tau \sum_{n=0}^{N-1} x(n\tau) e^{-jk\frac{2\pi}{N}n} \rightarrow X = \tau * fft(x)$$

DTFS:

$$a_k = \frac{1}{N} \sum_{k=0}^{N-1} x[n] e^{-jk\frac{2\pi}{N}n} \rightarrow a = \frac{1}{N} * fft(x)$$

$$X(j\omega)=\int_{-\infty}^{+\infty}x(t)e^{-j\omega t}dt$$

$$X(jk\frac{2\pi}{T}) \approx \sum_{n=0}^{N-1} x(n\tau)e^{-jk\frac{2\pi}{T}n\tau}\,\tau = \sum_{n=0}^{N-1} x(n\tau)e^{-jk\frac{2\pi}{N\tau}n\tau}\,\tau = \tau\sum_{n=0}^{N-1}x(n\tau)e^{-jk\frac{2\pi}{N}n}$$

$$X=\tau*fft(x)$$

$$X(k+1)=\tau\sum_{n=0}^{N-1}x(n\tau)\,e^{-j\omega_k n\tau}\approx X(j\omega_k),0\le k<\mathrm{N}$$

$$w_k = k\frac{2\pi}{T}$$

$$T=N\tau$$

$X=\tau*fft(x)$ $\quad\omega_k=\frac{2\pi k}{N\tau}$ $\quad$ 即$\omega_k = 0:\frac{2\pi}{N\tau}:\frac{2\pi(N-1)}{N\tau}$

$$X = \tau * fft(x) \quad \omega_k = \frac{2\pi k}{N\tau} \quad 即 \omega_k = 0:\frac{2\pi}{N\tau}:\frac{2\pi(N-1)}{N\tau}$$

Shift zero-frequency component to center of spectrum

$$X = fftshift(\tau * fft(x))$$

>> help fftshift
fftshift Shift zero-frequency component to center of spectrum. For vectors, fftshift(X) swaps the left and right halves of X.

$$0,1,2,3,4,5,6,7 \xRightarrow{\text{fftshift}} 4,5,6,7,0,1,2,3$$

$$0,1,2,3,4,5,6,7,8 \xRightarrow{\text{fftshift}} 5,6,7,8,0,1,2,3,4,$$

N is even ， $\omega_k = -\frac{\pi}{\tau} + (0:N-1) * \frac{2\pi}{N\tau})$

$\Updownarrow$

$$\omega_k = \begin{cases} \frac{2\pi k}{N\tau}, 0 \le k \le \frac{N}{2}-1 \\ \omega_{k-N} = \frac{2\pi k}{N\tau} - \frac{2\pi}{\tau}, \quad \frac{N}{2} \le k < N \end{cases}$$ N is even

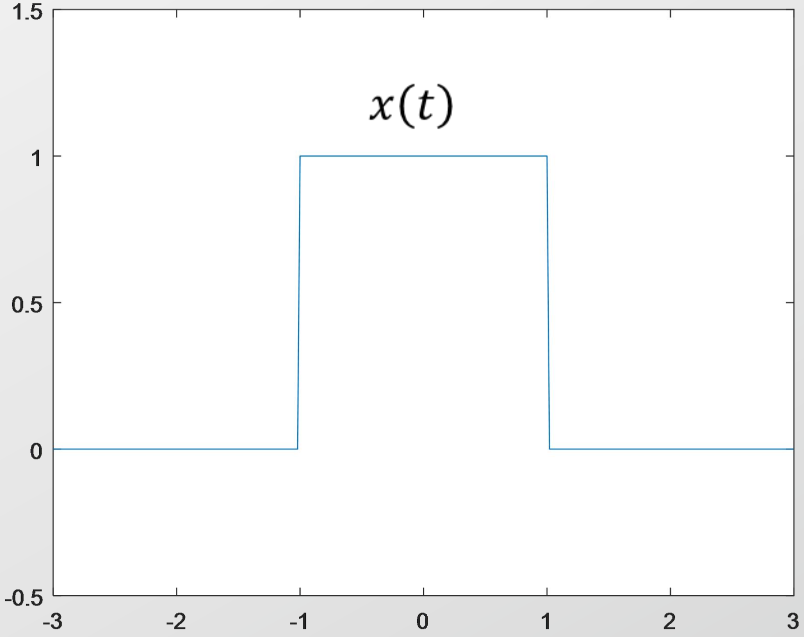
N is odd ， $\omega_k = \frac{(1-N)\pi}{N\tau} + (0:N-1) * \frac{2\pi}{N\tau})$

$\Updownarrow$

$$\omega_k = \begin{cases} \frac{2\pi k}{N\tau}, 0 \le k \le \frac{N-1}{2} \\ \omega_{k-N} = \frac{2\pi k}{N\tau} - \frac{2\pi}{\tau}, \frac{N-1}{2} + 1 \le k < N \end{cases}$$ N is odd

Calculate CTFT by fft and fftshift

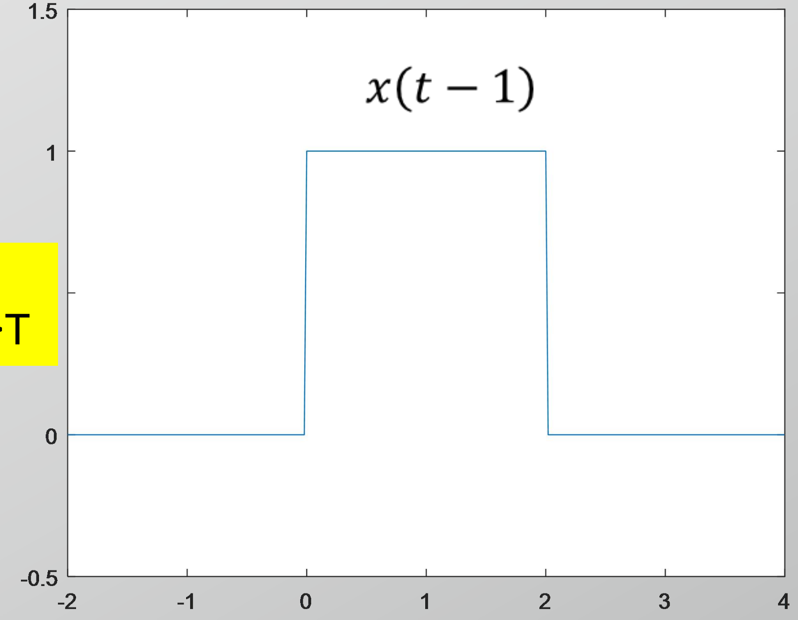
$$x(t) = \begin{cases} 1\,, & |t| < T_1 \\ 0, & T_1 \le |t| < T \end{cases} \longrightarrow X(j\omega) = \frac{2\sin(\omega T_1)}{\omega}$$

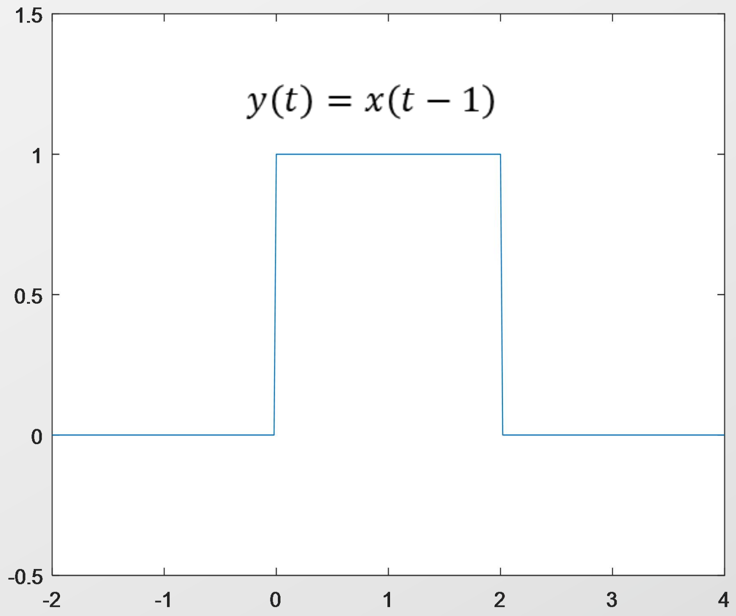


$$X = fft\mathrm{shift}(\tau * fft(x))$$

need

x(t) is equal to zero for t < 0 and t >T

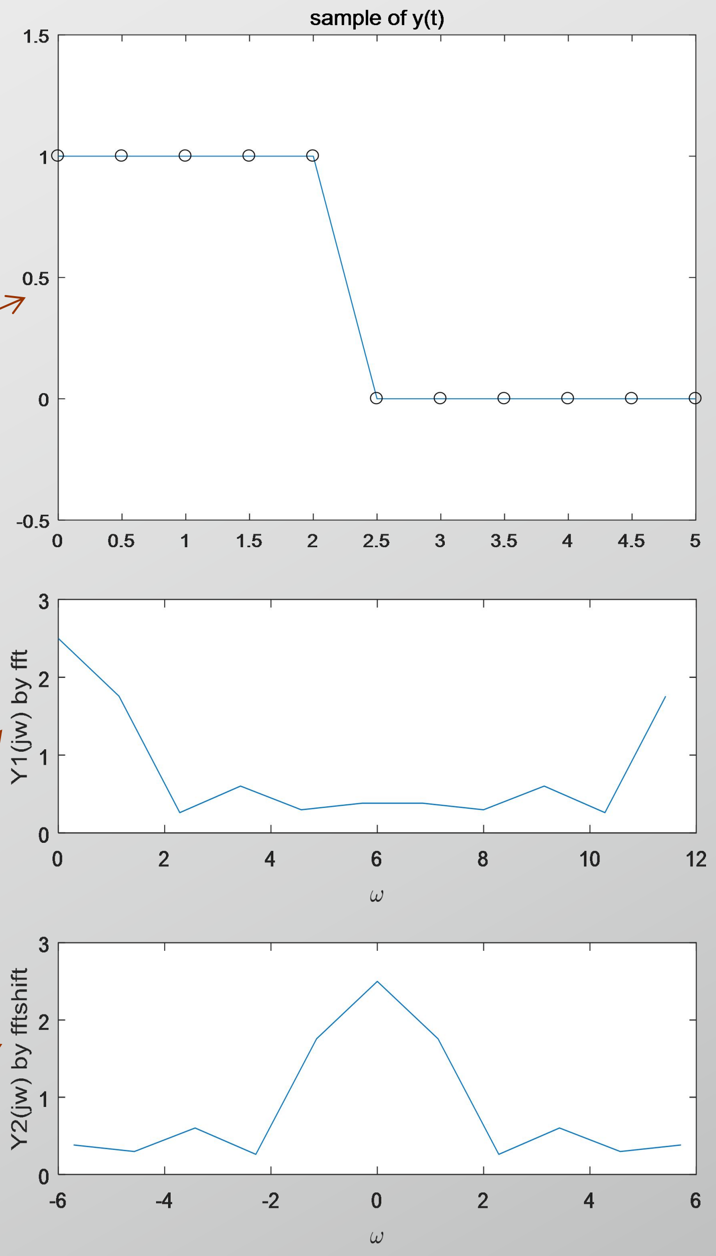




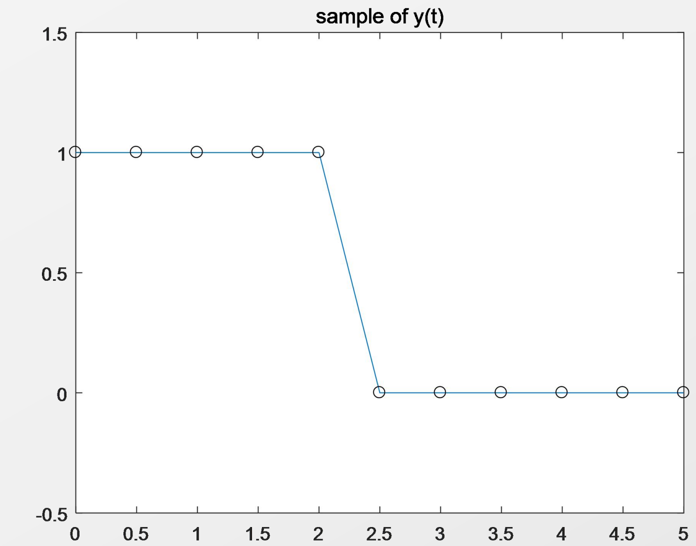
now, y(t) is equal to zero for t < 0 and t >T

T>2, that is $N\tau > 2$

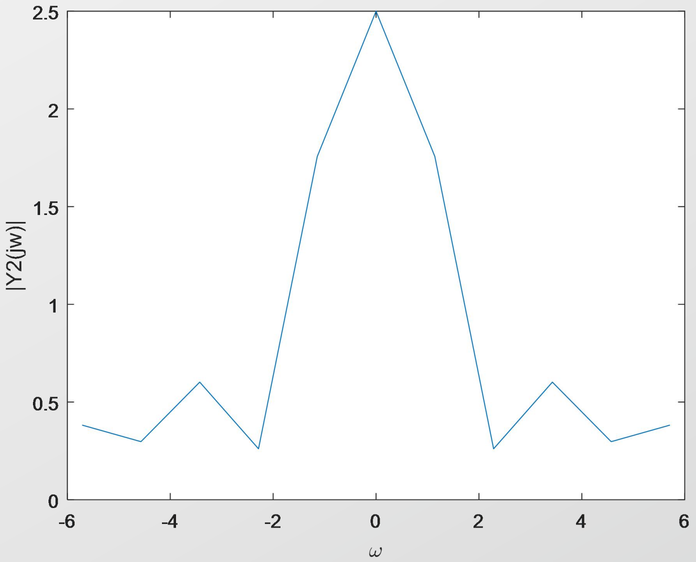
```
tau=0.5;
N=11; % let N be odd to get symmetrical result,N*tau=T>2
t=0:tau:(N-1)*tau;
y=[ones(1,fix(2/tau)+1),zeros(1,N-1-fix(2/tau))]; %sample y(t)
plot(t,y),hold on, scatter(t,y,'ko'),ylim([-0.5 1.5]);title('sample of y(t)')
hold off；
%caculate Y1(jw) by fft ， Y2(jw) by fftshift
Y1=tau*fft(y);
Y2=fftshift(tau*fft(y));
w1=0:2*pi/N/tau:2*pi*(N-1)/N/tau;
w2=(1-N)*pi/N/tau+(0:N-1)*2*pi/N/tau;
figure
subplot(2,1,1),plot(w1,abs(Y1)),xlabel('\omega'),ylabel('Y1(jw) by fft');
subplot(2,1,2),plot(w2,abs(Y2));xlabel('\omega'),ylabel('Y2(jw) by fftshift');
```

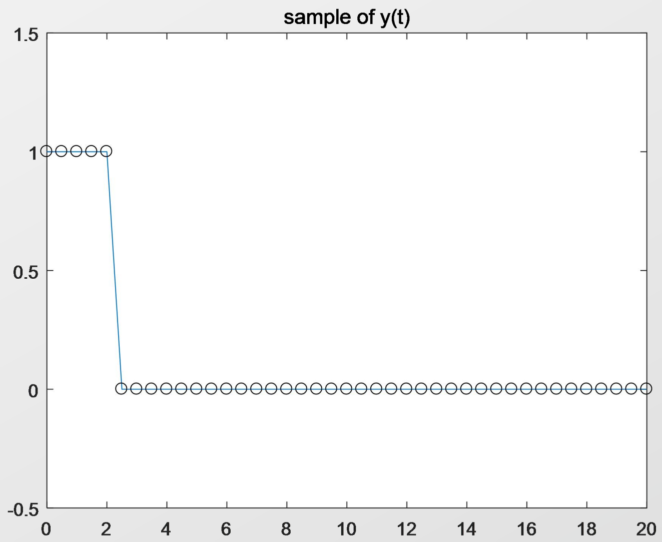


What happen if we change the value of tau and N ?

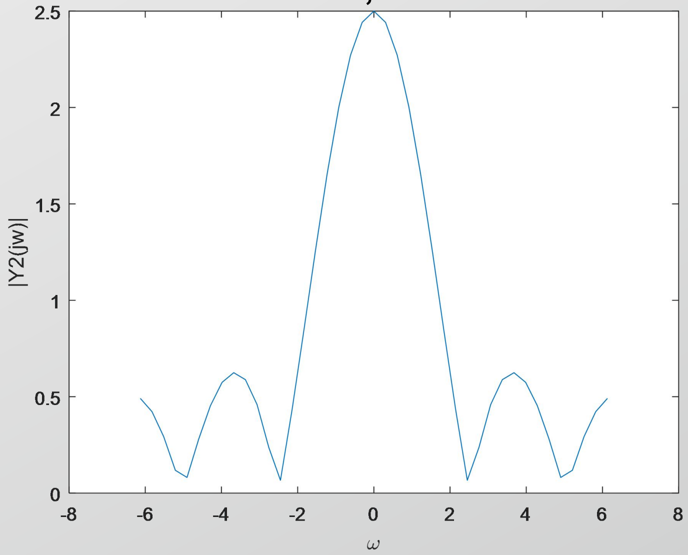


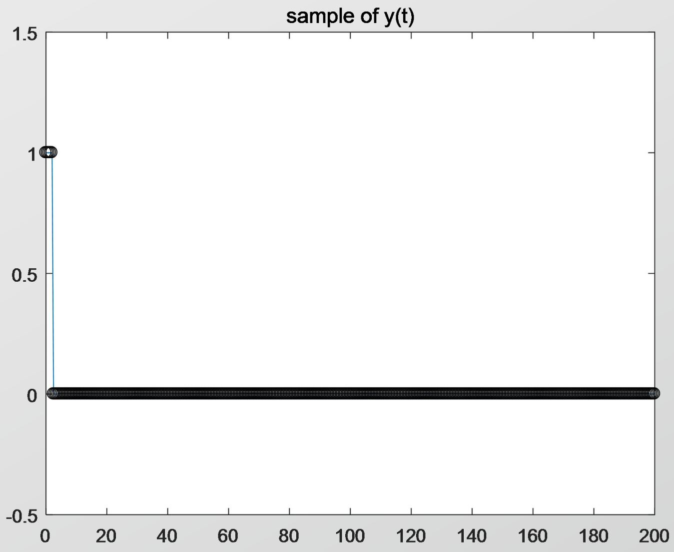
tau=0.5; N=11



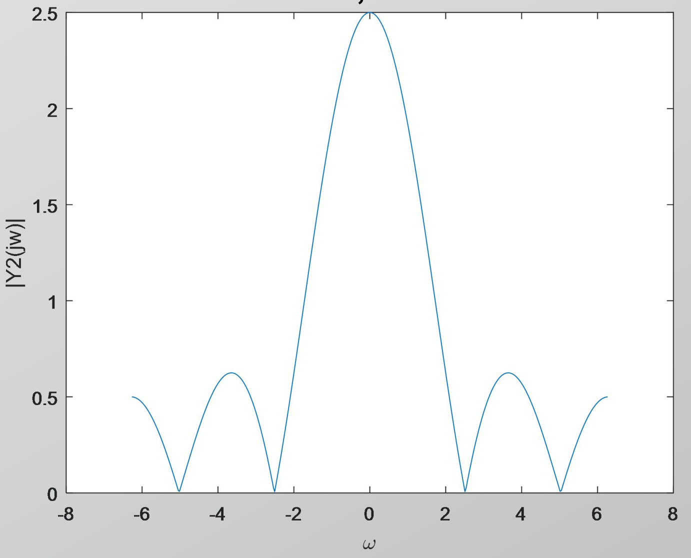


tau=0.5; N=41

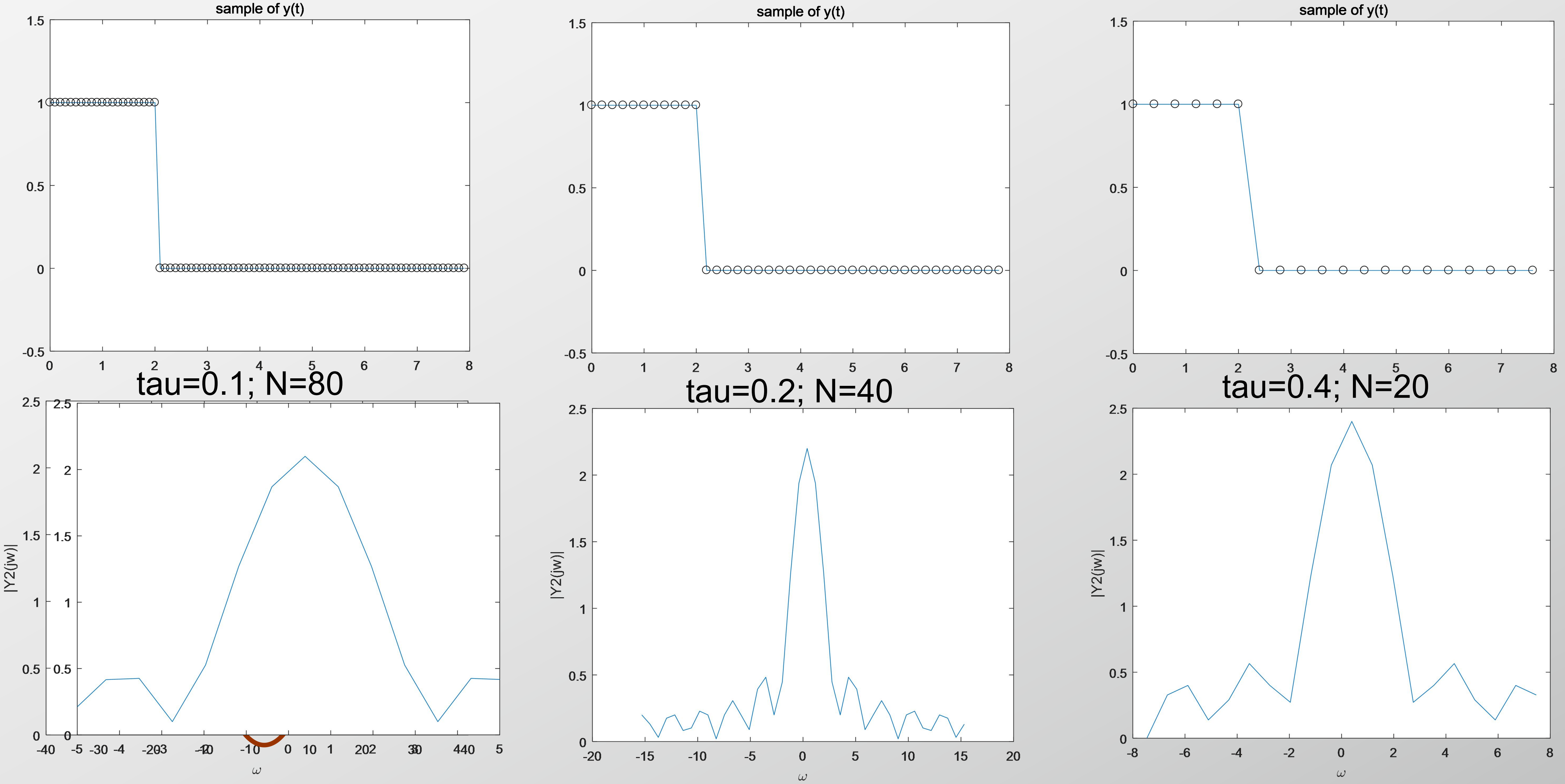




tau=0.5; N=401



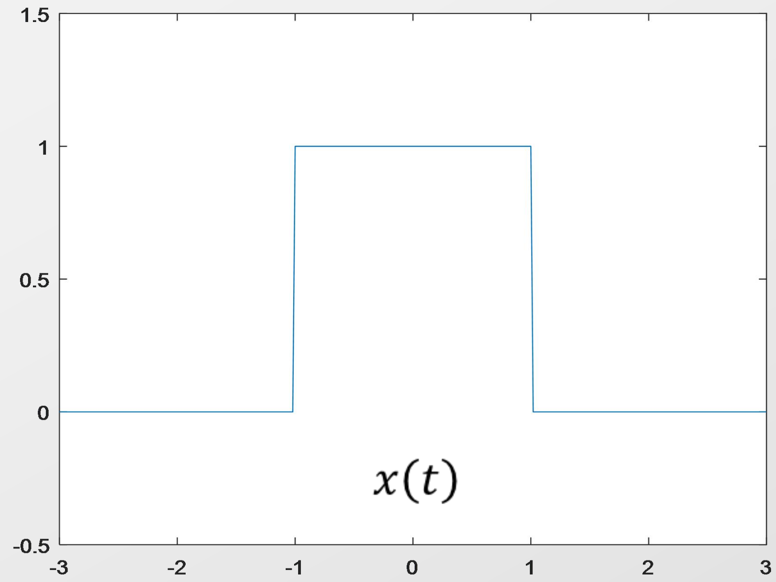
$$w_k = k\frac{2\pi}{T} \quad T = N\tau$$

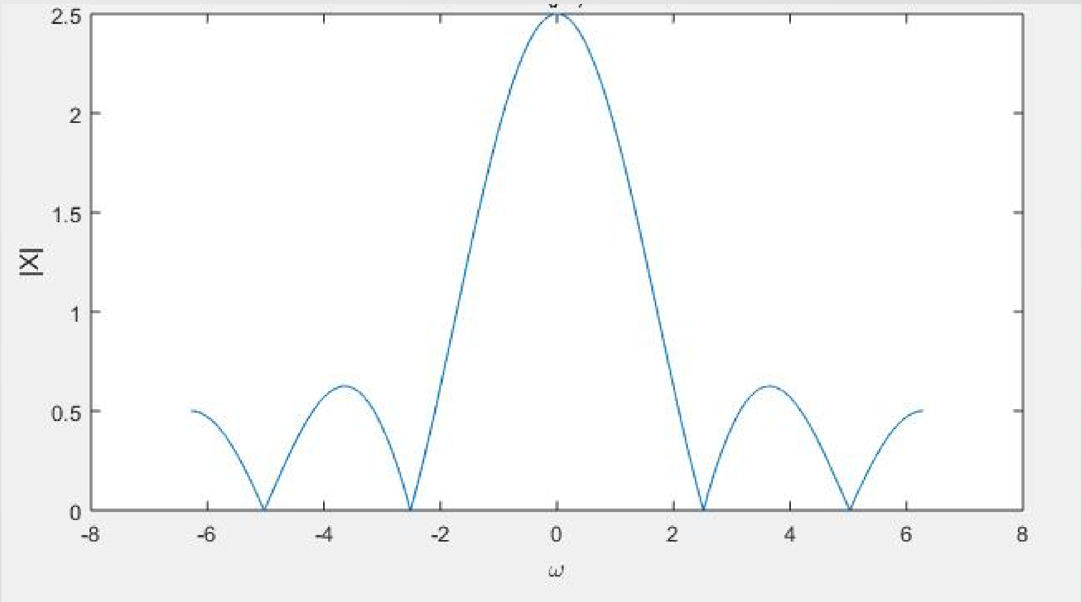


$$w_k = k\frac{2\pi}{T} \quad T = N\tau$$

Analytical calculation according to CTFT equation

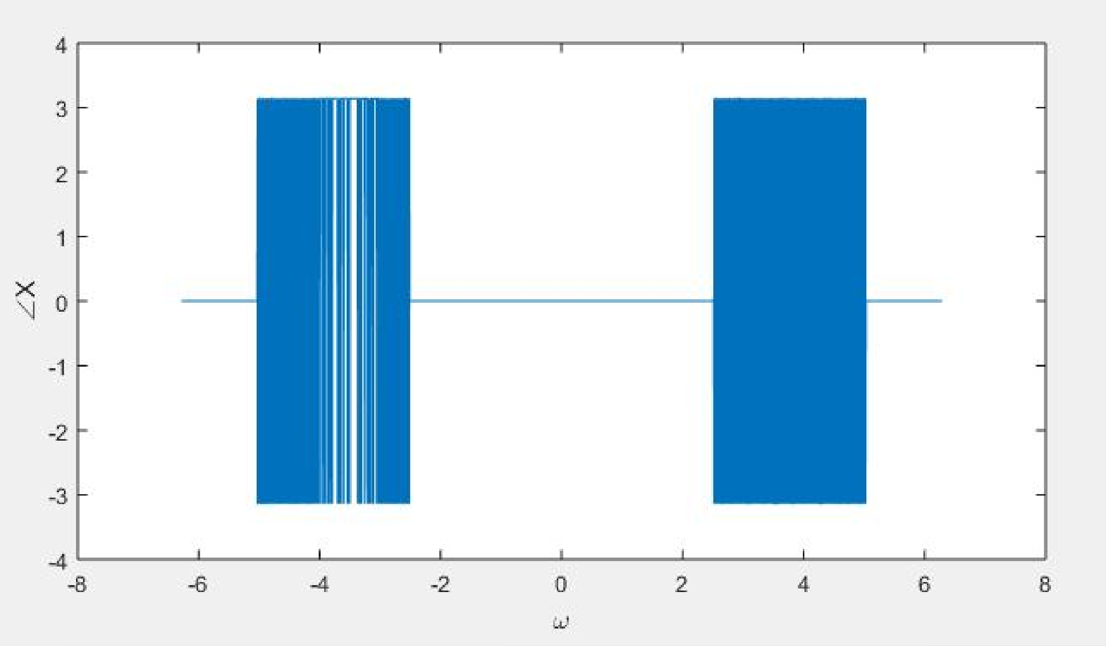
$$x(t)=\begin{cases}1, & |t|<T_1 \\ 0, & T_1\le |t|<T\end{cases} \longrightarrow X(j\omega)=\frac{2\sin(\omega T_1)}{\omega}$$



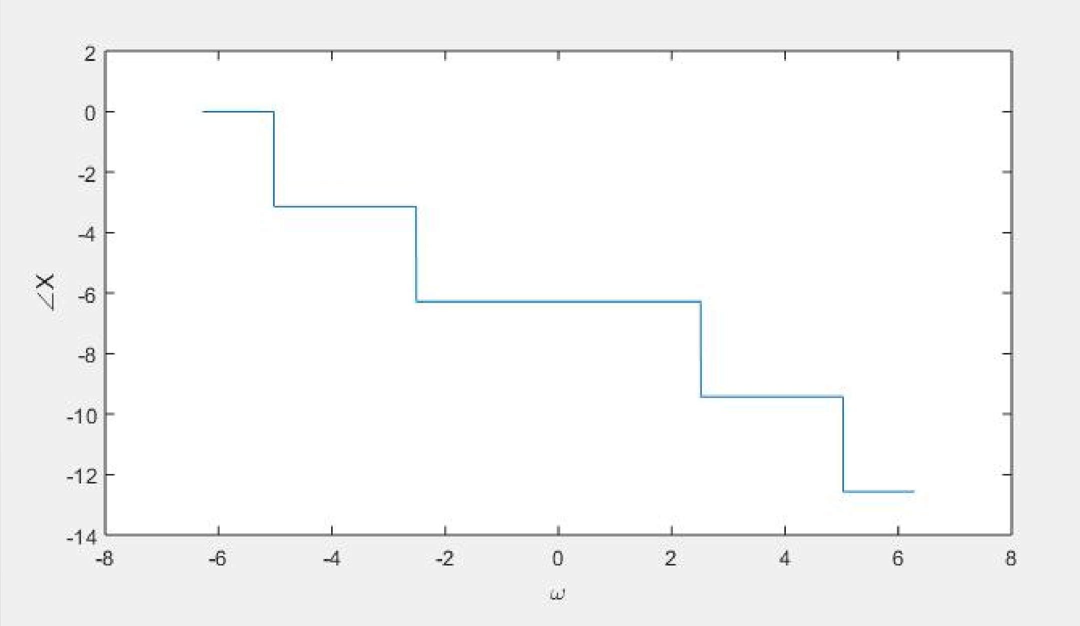


The is the sprctrum of x(t) drawn by Matlab when tau=0.5，N=40001

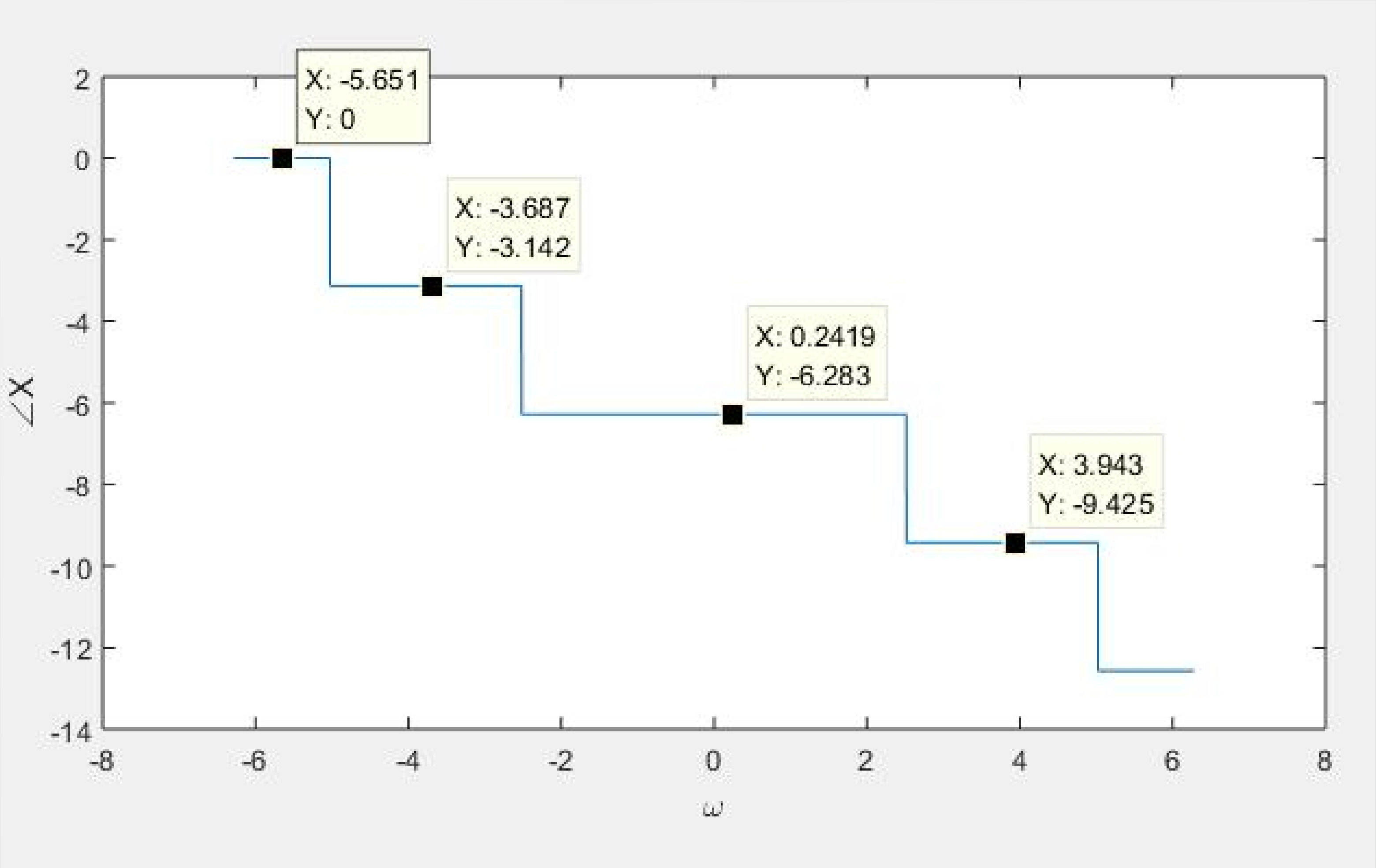
Is it absolutely right?
shape: sinc
value?
angle?



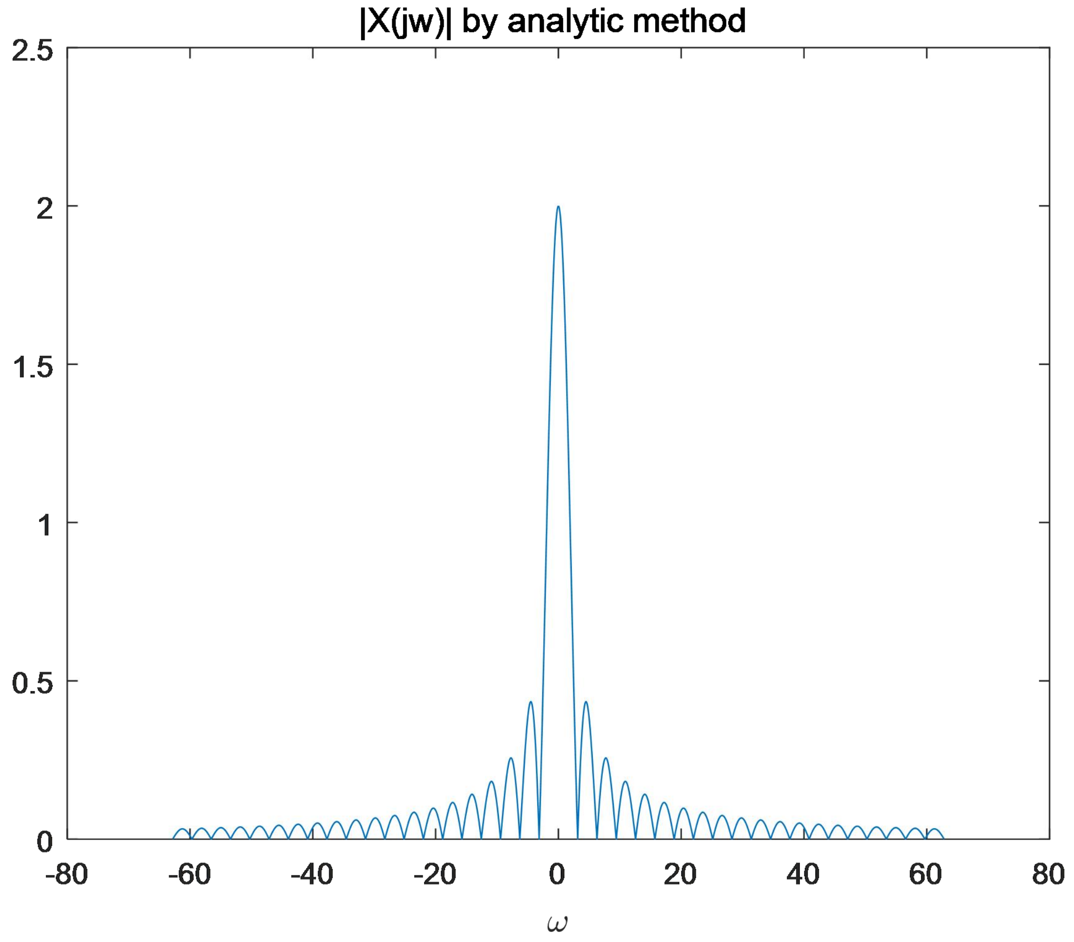

```
plot(w2,angle(X)),xlabel('\omega'),ylabel('\angleX');
```

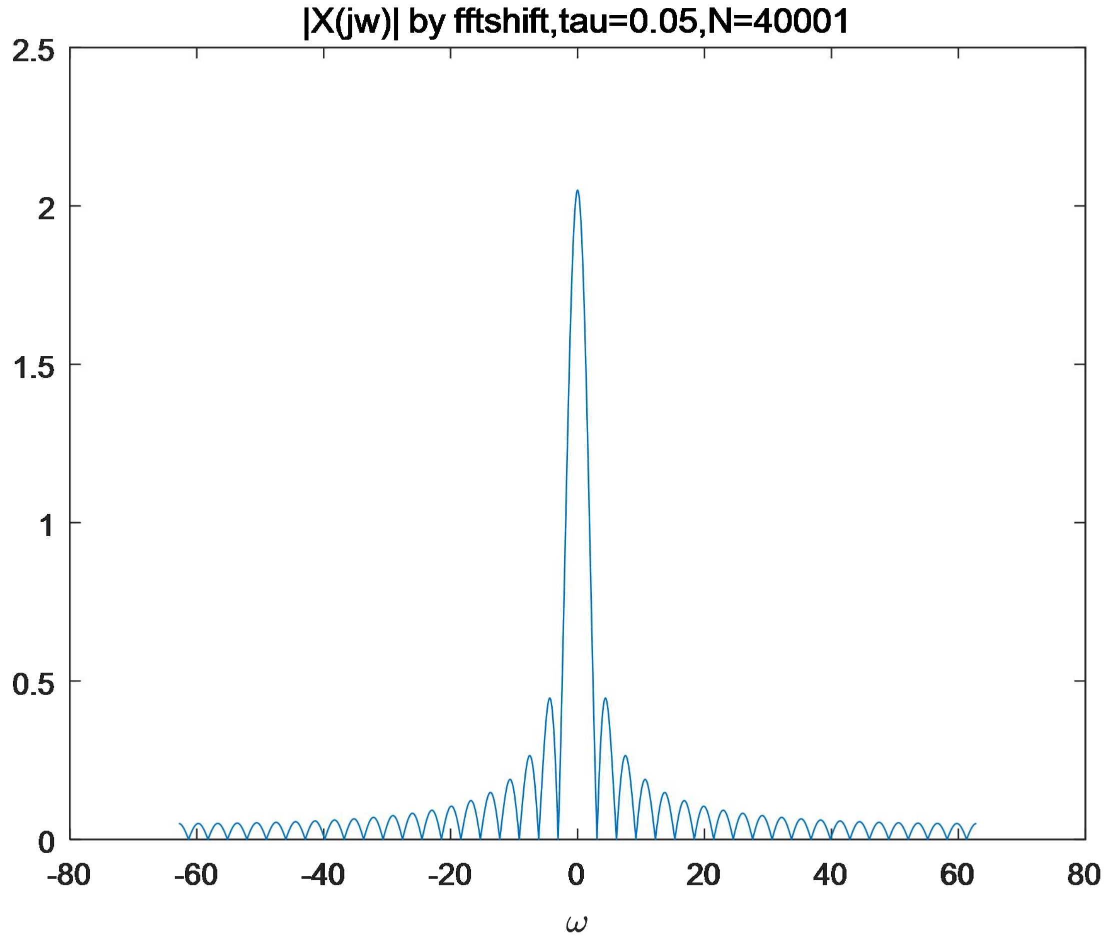



```
plot(w2,unwrap(angle(X))),xlabel('\omega'),ylabel('\angleX');
```


X: -5.651
Y: 0
X: -3.687
Y: -3.142
X: 0.2419
Y: -6.283
X: 3.943
Y: -9.425
∠X
ω
2
0
-2
-4
-6
-8
-10
-12
-14
-8
-6
-4
-2
0
2
4
6
8

理论值为0


|X(jw)| by analytic method
2.5
2
1.5
1
0.5
0
-80
-60
-40
-20
0
20
40
60
80
ω


|X(jw)| by fftshift,tau=0.05,N=40001
2.5
2
1.5
1
0.5
0
-80
-60
-40
-20
0
20
40
60
80
ω

■ 4.2 Numerical Approximation to the Continuous-Time Fourier Transform

Basic Problems

(a). Find an analytic expression for the CTFT of $x(t) = e^{-2|t|}$. You may find it helpful to think of $x(t) = g(t) + g(-t)$, where $g(t) = e^{-2t}u(t)$.

(b). Create a vector containing samples of the signal $y(t) = x(t-5)$ for $\tau = 0.01$ and $T = 10$ over the range `t=[0:tau:T-tau]`. Since $x(t)$ is effectively zero for $|t| > 5$, you can calculate the CTFT of the signal $y(t) = x(t-5)$ from the above analysis using $N = T/\tau$. Your vector `y` should have length N.

(c). Calculate samples $Y(j\omega_k)$ by typing `Y=fftshift(tau*fft(y))`.

(d). Construct a vector `w` of frequency samples that correspond to the values stored in the vector `Y` as follows

```
>> w = -(pi/tau)+(0:N-1)*(2*pi/(N*tau));
```

(e). Since $y(t)$ is related to $x(t)$ through a time shift, the CTFT $X(j\omega)$ is related to $Y(j\omega)$ by a linear phase term of the form $e^{j5\omega}$. Using the frequency vector `w`, compute samples of $X(j\omega)$ directly from `Y`, storing the result in the vector `X`.

Lab4 Assignments a

- Assignments：4.2,
- Preparation ： prepare the lesson 4.6 in lab-book

Tips :

1

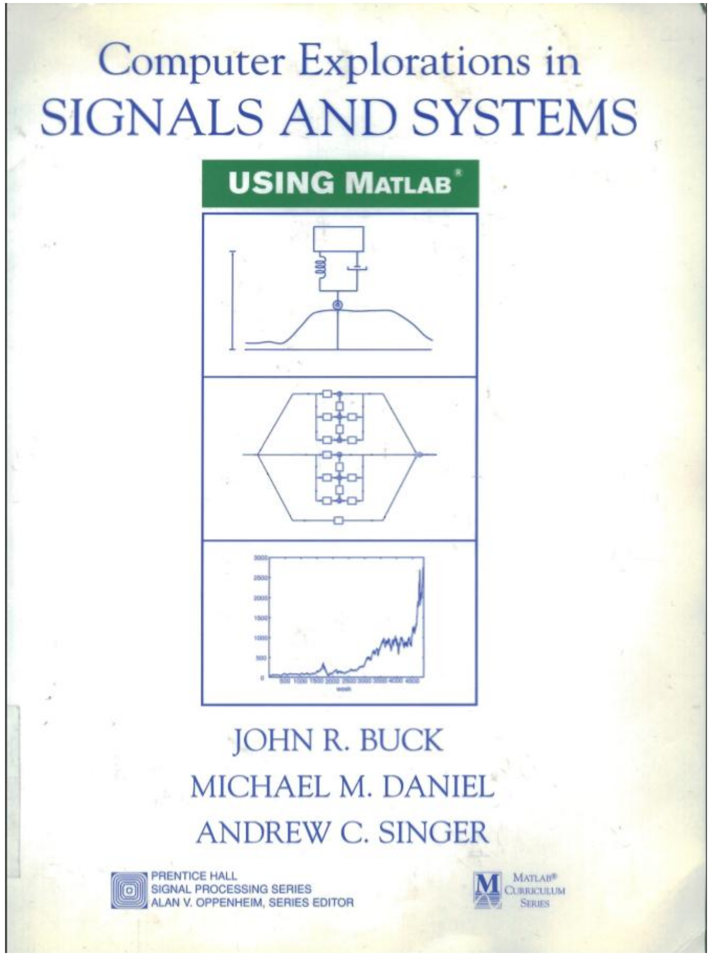
$$T = N\tau, X = fftshift(\tau * fft(x))$$

N is even ， $\omega_k = -\frac{\pi}{\tau} + (0{:}N-1) * \frac{2\pi}{N\tau})$

N is odd ， $\omega_k = \frac{(1-N)\pi}{N\tau} + (0{:}N-1) * \frac{2\pi}{N\tau})$

2

$$x(t-t_0) \overset{F}{\leftrightarrow} e^{-jwt_0} X(j\omega)$$



One useful Matlab function: unwrap

unwrap

Syntax：Q = unwrap(P) ;

unwraps the radian phase angles in a vector P. Whenever the jump between consecutive angles is greater than or equal to π radians, unwrap shifts the angles by adding multiples of ±2π until the jump is less than π.

Matlab中unwrap函数内容详解