The Response of LTI systems to Complex Exponentials

$x(t) = e^{st}$ → LTI Systems $h(t)/h[n]$ → $y(t)=H(s)e^{st}$

$x[n] = z^n$ → LTI Systems $h(t)/h[n]$ → $y[n]=H(z)z^n$

The response of LTI Systems to Compex Exponentials *is the* Compex Exponentials *multiply a complex constant.*

For CT LTI System： Compex Exponentials e^{st} *is its eigenfunction*. The constant H(s) for a specific value of s is then the eigenvalue

$$H(s) = \int_{-\infty}^{+\infty} h(\tau)\, e^{-s\tau} d\tau$$

For DT LTI System： Compex Exponentials z^n *is its eigenfunction*， *The constant* $H(Z)$ *for a specific value of* z *is then the eigenv*

$$H(z) = \sum_{-\infty}^{+\infty} h[k] z^{-k}$$

$$x(t) = e^{st} \longrightarrow \boxed{h(t)} \longrightarrow y(t) = \int_{-\infty}^{+\infty} h(\tau) e^{s(t-\tau)} d\tau$$

$$= \left[\int_{-\infty}^{+\infty} h(\tau) e^{-s\tau} d\tau\right] e^{st}$$

$$= H(s) e^{st}$$

$$H(s) = \int_{-\infty}^{+\infty} h(\tau)\, e^{-s\tau} d\tau$$

when s is arbitrary complex number, H(s) is System Function.
when s=jw， H(s) is H(jw), that we named Frequency Response.

$$x[n] = z^n \longrightarrow \boxed{h[n]} \longrightarrow y[n] = \sum_{k=-\infty}^{+\infty} h[k] z^{n-k}$$

$$= z^n \sum_{k=-\infty}^{+\infty} h[k] z^{-k}$$

$$= H(z) z^n$$

$$H(z) = \sum_{-\infty}^{+\infty} h[k] z^{-k}$$

when z is arbitrary complex number, H(z) is System Function.
when z=ejw， H(z) is H(ejw), that we named Frequency Response.

Frequency Response of CT LTI Systems

$$x(t) = e^{j\omega t} \longrightarrow \boxed{\begin{array}{c}\text{CT LTI Systems} \\ (h(t)]) \end{array}} \longrightarrow y(t) = H(j\omega)e^{j\omega t}$$

The response of LTI Systems to Compex Exponentials is the Compex Exponentials multiply a complex constant.

Frequency Response Formula: $$H(j\omega) = \int_{-\infty}^{+\infty} h(t)e^{-j\omega t}dt$$

$H(j\omega)$ is the Fourier transform of $h(t)$

Convolution property of Fourier transform $y(t) = x(t) * h(t)$ → $Y(jw) = X(jw)\,H(j\omega)$

CT LTI Systems characterized by Differential Equations

Frequency Response

$$\sum_{k=0}^{K} a_k \frac{d^k y(t)}{dt^k} = \sum_{m=0}^{M} b_m \frac{d^m x(t)}{dt^m} \Rightarrow H(jw) = \frac{Y(jw)}{X(jw)} = \frac{\sum_{m=0}^{M} b_m (jw)^m}{\sum_{k=0}^{K} a_k (jw)^k}$$

Prove

$$\mathcal{F}\left[\sum_{k=0}^{K} a_k \frac{d^k y(t)}{dt^k}\right] = \sum_{k=0}^{K} \mathcal{F}\left[a_k \frac{d^k y(t)}{dt^k}\right] = \sum_{k=0}^{K} a_k (j\omega)^k Y(j\omega)$$

$$\mathcal{F}\left[\sum_{m=0}^{M} b_m \frac{d^m x(t)}{dt^m}\right] = \sum_{m=0}^{M} \mathcal{F}\left[b_m \frac{d^m x(t)}{dt^m}\right] = \sum_{m=0}^{M} b_m (j\omega)^m X(j\omega)$$

Differentiation in Time Property

$$\sum_{k=0}^{K} a_k \frac{d^k y(t)}{dt^k} = \sum_{m=0}^{M} b_m \frac{d^m x(t)}{dt^m}$$

+ Initial rest

coefficient vectors : **A=[a_K, a_{k-1},… a_1,a_0] ,B=[b_M , b_{M-1}… b_1,b_0]**.

Laplace Transformation

$$\sum_{k=0}^{K} a_k s^k Y(s) = \sum_{m=0}^{M} b_m s^m X(s)$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{b_M s^M + b_{M-1} s^{M-1} + \dots + b_1 s^1 + b_0}{a_K s^K + a_{K-1} s^{K-1} + \dots + a_1 s^1 + a_0}$$

Transfer Function/
System Function

$s = jw$

$$\sum_{k=0}^{K} a_k \frac{d^k y(t)}{dt^k} = \sum_{m=0}^{M} b_m \frac{d^m x(t)}{dt^m} \Rightarrow H(jw) = \frac{Y(jw)}{X(jw)} = \frac{\sum_{m=0}^{M} b_m (jw)^m}{\sum_{k=0}^{K} a_k (jw)^k}$$

Frequency Response

$$\sum_{k=0}^{K} a_k \frac{d^k y(t)}{dt^k} = \sum_{m=0}^{M} b_m \frac{d^m x(t)}{dt^m} \Rightarrow H(jw) = \frac{Y(jw)}{X(jw)} = \frac{\sum_{m=0}^{M} b_m (jw)^m}{\sum_{k=0}^{K} a_k (jw)^k}$$

Syntax 1

$$\sum_{k=0}^{K} a_k \frac{d^k y(t)}{dt^k} = \sum_{m=0}^{M} b_m \frac{d^m x(t)}{dt^m}$$

y=lsim(B,A,x,t); simulates the response of the system to the input signal specified by the vectors x and t.

Syntax 2

$$\sum_{k=0}^{K} a_k \frac{d^k y(t)}{dt^k} = \sum_{m=0}^{M} b_m \frac{d^m x(t)}{dt^m}$$

H = freqs(B,A,W) $\longrightarrow H(jw)$, computes the frequency response of this system.

Syntax 3

$$\sum_{k=0}^{K} a_k \frac{d^k y(t)}{dt^k} = \sum_{m=0}^{M} b_m \frac{d^m x(t)}{dt^m}$$

residue

$$\boxed{H(j\omega) = \frac{A_1}{j\omega - p_1} + \frac{A_2}{j\omega - p_2} + \cdots + \frac{A_N}{j\omega - p_N}} \longrightarrow h(t)$$

freqs(Frequency Response of CT LTI Systems)

```
>> help freqs
```

freqs Laplace-transform (s-domain) frequency response.

H = freqs(B,A,W) returns the complex frequency response vector H of the filter B/A:

$$H(s) = \frac{B(s)}{A(s)} = \frac{b(1)s^n + b(2)s^{n-1} + \cdots + b(n+1)}{a(1)s^m + a(2)s^{m-1} + \cdots + a(m+1)}$$

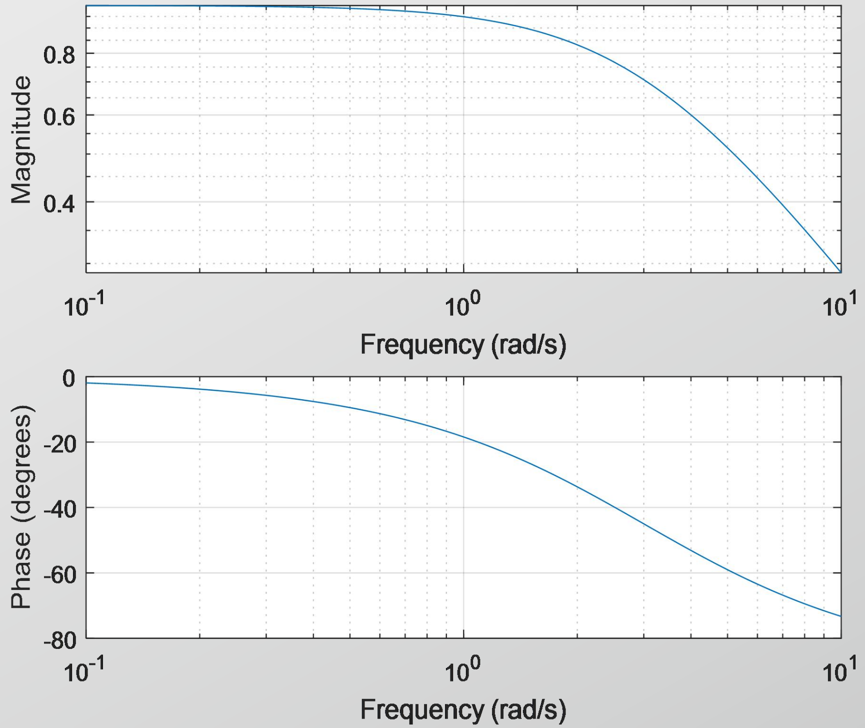
given the numerator and denominator coefficients in vectors B and A. The frequency response is evaluated **at the points specified in vector W (in rad/s)**. The magnitude and phase can be graphed by calling freqs(B,A,W) with no output arguments.

[H,W] = freqs(B,A) automatically picks a **set of 200 frequencies** W on which the frequency response is computed.

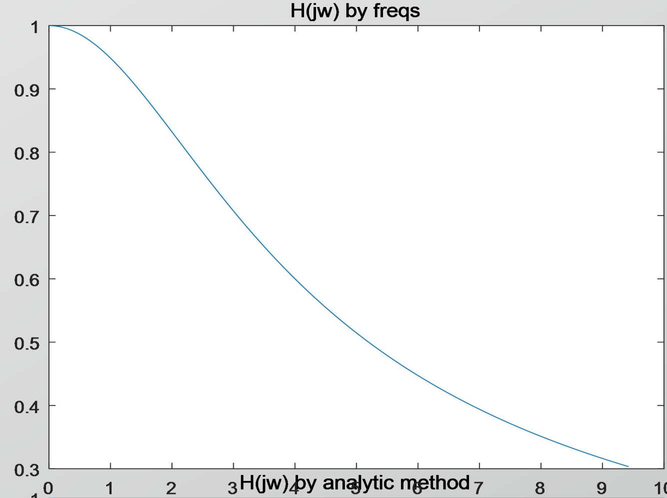
A stable LTI system is characterized by the following differential equation, please plot its frequency response.

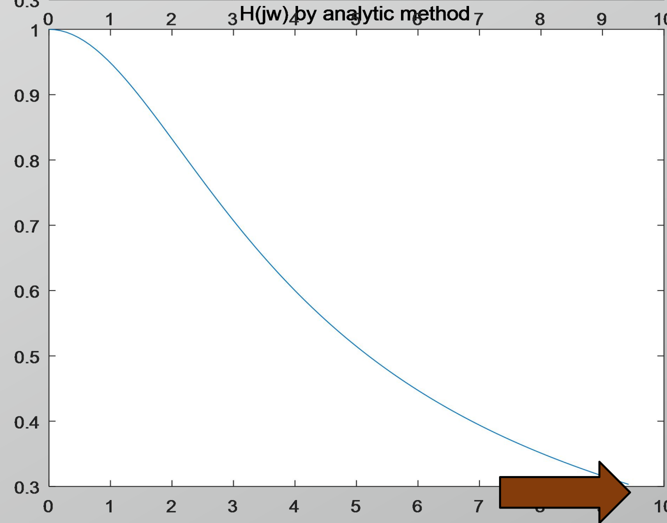
$$\frac{dy(t)}{dt} + 3y(t) = 3x(t)$$

```
 a=[1 3];
 b=3;
%automatically picks a set of  W
 freqs(b,a)

%Specify vector W (in rad/s)
 w=linspace(0,3*pi);
 H=freqs(b,a,w);
 figure
 plot(w,abs(H)),title('H(jw) by freqs')


figure
plot(w,abs(3./(3+1j*w))),title('H(jw) by
analytic method')
```



loglog(x,y)





4.1 Tutorial: `freqs`

A stable LTI system is completely characterized in terms of its frequency response, $H(j\omega)$. If $X(j\omega)$ is the CTFT of the system input, then

$$Y(j\omega) = H(j\omega)\,X(j\omega)$$

Consider the first-order differential equation

$$\frac{dy(t)}{dt} + 3\,y(t) = 3\,x(t)$$

which describes the input-output relationship of a causal, stable LTI system. The frequency response of this system is

$$H(j\omega) = \frac{3}{3 + j\omega}\,. \tag{4.5}$$

If no output argument is supplied to **`freqs(b,a)`**, the magnitude and phase of $H(j\omega)$ will be automatically plotted. Executing the commands

```
>> a = [1 3];
>> b = 3;
>> freqs(b,a)
```

Frequency Response & Impulse Response

Frequency Response Formula: $H(j\omega) = \int_{-\infty}^{+\infty} h(t) e^{-j\omega t} dt$ ***$H(j\omega)$ is the Fourier transform of $h(t)$***

Frequency Response $\xLeftrightarrow{\text{Fourier Transform}}$ **Impulse Response**

Textbook Table 4.2

Syntax 3

$$\sum_{k=0}^{K} a_k \frac{d^k y(t)}{dt^k} = \sum_{m=0}^{M} b_m \frac{d^m x(t)}{dt^m}$$

$$\longrightarrow \quad H(jw) = \frac{Y(jw)}{X(jw)} = \frac{\sum_{m=0}^{M} b_m (jw)^m}{\sum_{k=0}^{K} a_k (jw)^k}$$

residue

$$\boxed{H(j\omega) = \frac{A_1}{j\omega - p_1} + \frac{A_2}{j\omega - p_2} + \cdots + \frac{A_N}{j\omega - p_N}}$$

Basic Fourier Transform Pairs

$$\longleftarrow$$

$\mathrm{h}(t)$

$$H(j\omega)=\frac{\sum_{k=0}^{M} b_k(jw)^k}{\sum_{k=0}^{N} a_k(jw)^k}=\frac{b_M(jw)^M+b_{M-1}(jw)^{M-1}+\cdots b_1(jw)^1+b_0}{a_N(jw)^N+a_{N-1}(jw)^{N-1}+\cdots a_1(jw)^1+a_0}$$

Partial Fraction Expansion (No identical poles):

$$\boxed{H(j\omega)=\frac{b_M(j\omega-z_1)(j\omega-z_2)\ldots(j\omega-z_M)}{a_N(j\omega-p_1)(j\omega-p_2)\ldots(j\omega-p_N)}=\frac{A_1}{j\omega-p_1}+\frac{A_2}{j\omega-p_2}+\cdots+\frac{A_N}{j\omega-p_N}}$$

Partial Fraction Expansion (with identical poles):

$$H(j\omega)=\frac{b_M(j\omega-z_1)(j\omega-z_2)\ldots(j\omega-z_M)}{a_N(j\omega-p_1)^{k_1}(j\omega-p_2)^{k_2}\ldots(j\omega-p_n)^{k_n}}$$
$$=\frac{A_{1,1}}{(j\omega-p_1)^{k_1}}+\frac{A_{1,2}}{(j\omega-p_1)^{k_1-1}}+\cdots\frac{A_{1,k_1}}{(j\omega-p_1)}+\cdots+\frac{A_{n,1}}{(j\omega-p_n)^{k_n}}+\frac{A_{n,2}}{(j\omega-p_n)^{k_n-1}}+\cdots\frac{A_{n,k_n}}{(j\omega-p_n)}$$

Basic Fourier Transform Pairs

$$e^{-at}u(t), Re(a) > 0 \overset{F}{\Leftrightarrow} \frac{1}{a+jw}$$

$$te^{-at}u(t), Re(a) > 0 \overset{F}{\Leftrightarrow} \frac{1}{(a+jw)^2}$$

$$H(jw) = \frac{1}{3+jw} \quad \Rightarrow \quad h(t) = e^{-3t}u(t)$$

$$H(jw) = \frac{1}{(3+jw)^2} \quad \Rightarrow \quad h(t) = te^{-3t}u(t)$$

If $H(j\omega) = \frac{1}{3+jw} + \frac{1}{(3+jw)^2} \quad h(t) =$?

residue

```
>> help residue
 residue Partial-fraction expansion (residues).
```

[R,P,K] = residue(B,A) finds the residues, poles and direct term of a partial fraction expansion of the ratio of two polynomials B(s)/A(s).

If there are no multiple roots,

```
   B(s)       R(1)       R(2)             R(n)
   ----  =  -------- + -------- + ... + -------- + K(s)
   A(s)     s - P(1)   s - P(2)         s - P(n)
```

Vectors B and A specify the coefficients of the numerator andenominator polynomials in descending powers of s. The residues are returned in the column vector R, the pole locations in column vector P, and the direct terms in row vector K. The number of poles is n = length(A)-1 = length(R) = length(P). The direct term coefficient vector is empty if length(B) < length(A), otherwise length(K) = length(B)-length(A)+1.

If P(j) = ... = P(j+m-1) is a pole of multplicity m, then the expansion includes terms of the form

```
       R(j)        R(j+1)                R(j+m-1)
     -------- + ------------   + ... + ------------
     s - P(j)   (s - P(j))^2           (s - P(j))^m
```

[B,A] = residue(R,P,K), with 3 input arguments and 2 output arguments, converts the partial fraction expansion back to the polynomials with coefficients in B and A.

Syntax： [R,P,K] = residue(B,A) ; [B,A] = residue(R,P,K)

```
B(s)       R(1)       R(2)             R(n)
----  =  -------- + -------- + ... + -------- + K(s)
A(s)      s - P(1)   s - P(2)         s - P(n)
```

$s = jw$

Example：

$$\frac{d^2y(t)}{dt^2} + 4\frac{dy}{dt} + 3\mathrm{y(t)} = \frac{dx(t)}{dt} + 2x(t)$$

$$H(jw) = \frac{(jw)+2}{(jw)^2+4(jw)+3}$$

```
b=[1 2];
a=[1 4 3];
[r p k]=residue(b,a)
```

```
r =              p =           k =
    0.5000          -3
    0.5000          -1              []
```

$$H(jw) = \frac{0.5}{(jw)+3} + \frac{0.5}{(jw)+1}$$

$$\mathrm{h(t)} = 0.5e^{-3t}u(t) + 0.5e^{-t}u(t)$$

```
[B A]=residue(r,p,k)
```

```
B =              A =
    1    2           1    4    3
```

$$\frac{d^3y(t)}{dt^3} + 7\frac{d^2y(t)}{dt^2} + 16\frac{dy(t)}{dt} + 12y(t) = 3\frac{d^2x(t)}{dt^2} + 10\frac{dx(t)}{dt} + 5x(t).$$

```
b=[3 10 5];
a=[1 7 16 12];
[r p k]=residue(b,a)
```

```
r =              p =              k =
   2.0000          -3.0000
   1.0000          -2.0000           []
  -3.0000          -2.0000
```

?

?

```
[B A]=residue(r,p,k)
```

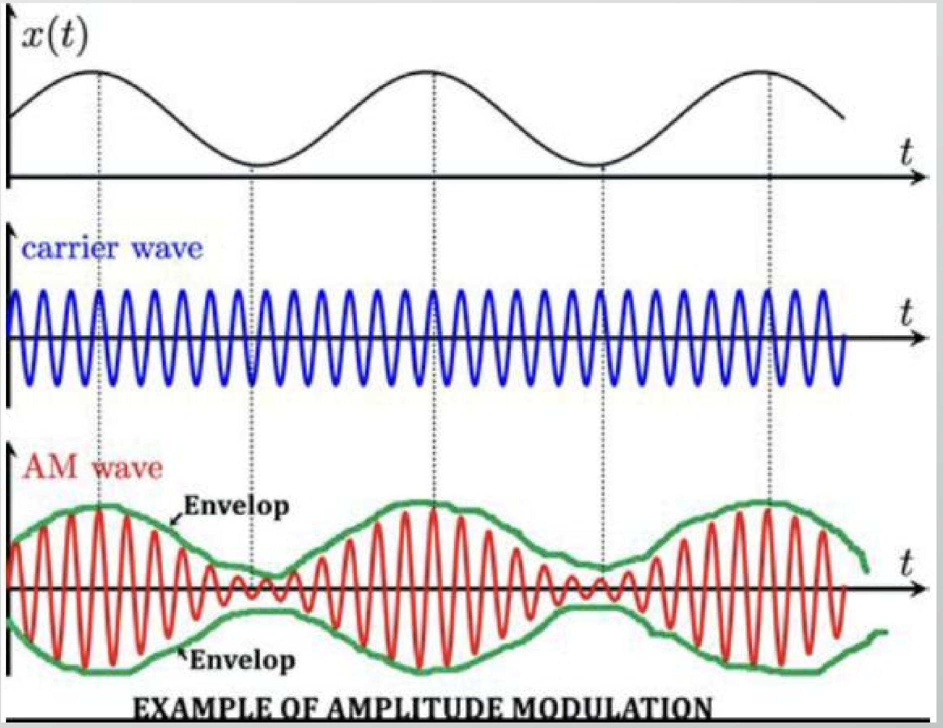
?

4.5 Impulse Responses of Differential Equations by Partial Fraction Expansion

In this exercise, you will learn how to find analytic expressions for the impulse responses of stable LTI systems whose inputs and outputs satisfy linear constant-coefficient differential equations. The frequency response of systems of this form can be written as ratios of polynomials in $(j\omega)$. MATLAB represents these polynomials as a vector of coefficients of the polynomial in decreasing powers of the dependent variable, $j\omega$. For example, the polynomial $G(j\omega) = 4(j\omega)^3 - 5(j\omega)^2 + 2(j\omega) - 7$ would be represented in MATLAB by the vector `G=[4 -5 2 -7]`. MATLAB contains several functions for manipulating polynomials in this format. One very useful function is `residue`, which computes the partial fraction expansion of a function consisting of a ratio of polynomials. In this exercise, you will learn to convert the differential equation relating the input and output of a stable, continuous-time, LTI system into vectors representing the polynomials appearing in the numerator and denominator of the frequency response. Then, you will use `residue` to process the frequency response so that the impulse response may be easily determined from the partial fraction expansion.

Application of CTFT——Amplitude Modulation

- In telecommunications, modulation is the process of conveying a message signal, for example a digital bit stream or an analog audio signal, inside another signal that can be physically transmitted.
- Amplitude modulation(AM) is a modulation technique used in electronic communication.In amplitude modulation，the amplitude(signal strength) of the carrier wave is varied in proportion to the waveform being transmitted.This technique contrasts with frequency modulation， in which the frequency of the carrier signal is varied， and phase modulation， in which its phase is varied.



- The purpose of amplitude modulation is to achieve amplification and transmission of slowly changing signals, and then extract useful signals from the amplified modulation waves through demodulation. So the amplitude modulation process is multiplying the original signal with the carrier signal, which equivalent to the spectrum "shifting" process. The purpose of demodulation is to restore the modulated signal.
- Multiplying the amplitude modulation wave with the original carrier signal again, the spectrum will be "shifted" again. When a low-pass filter is used to filter out components with frequencies greater than fm, the spectrum of the original signal can be reproduced. The difference from the original spectrum is that the amplitude is half of the original, which can be compensated by amplification. This process is called synchronous demodulation, which refers to the signal being demodulated having the same frequency and phase as the carrier signal being modulated.

Multiplication Property of Fourier Transform

$$x(t)y(t) \overset{F}{\leftrightarrow} \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\theta)Y(j(w-\theta))d\theta$$

Modulation

Signal m(t) , let $x(t) = m(t)\cos(2\pi f_0 t)$

Multiplication

$$x(t)y(t) \overset{F}{\leftrightarrow} \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\theta)Y(j(w-\theta))d\theta$$

$$\begin{aligned} X(jw) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} M(j\theta)\, C\big(\mathrm{j}(\mathrm{w}-\theta)\big)\mathrm{d}\theta \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} M(j\theta)\,[\pi\delta(w-\theta-2\pi f_0)+\pi\delta(w-\theta+2\pi f_0)]\mathrm{d}\theta \\ &= \frac{1}{2}\mathrm{M}\big(\mathrm{j}(\mathrm{w}-2\pi f_0)\big)+\frac{1}{2}\mathrm{M}\big(\mathrm{j}(\mathrm{w}+2\pi f_0)\big) \end{aligned}$$

multiply the original signal with the carrier signal, its spectrum is shifted to $2\pi f_0$

Basic Fourier Transform Pairs $\cos(w_0 t) \overset{F}{\Leftrightarrow} \pi\delta(w-w_0)+\pi\delta(w+w_0)$

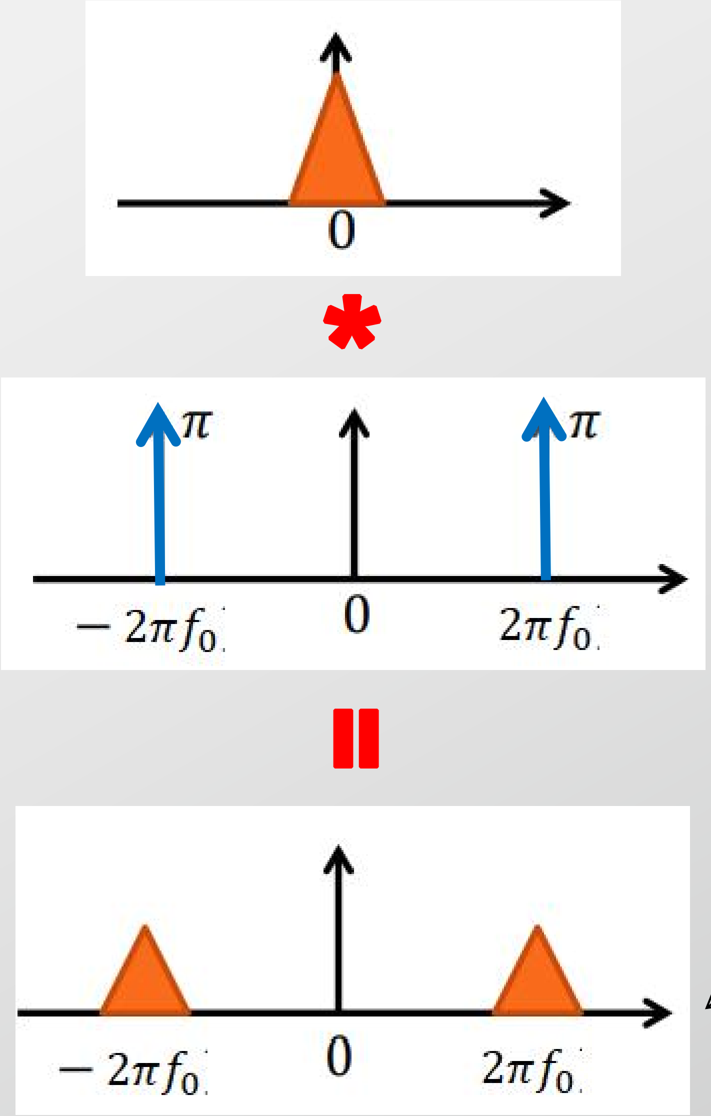
Modulation——Spectrum shifting

$x(t) = m(t)\cos(2\pi f_0 t)$

$M(jw)$

$C(jw)$

$X(jw)$



multiply the original signal with the carrier signal, its spectrum is shifted to $2\pi f_0$

Demodulation

Multiply the amplitude modulation wave with the original carrier signal again

$$y(t) = m(t)cos(2\pi f_0 t)cos(2\pi f_0 t)$$

$$cos(A)cos(A) = \frac{cos(2A) + 1}{2}$$

Multiplication

$$x(t)y(t) \overset{F}{\leftrightarrow} \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\theta)Y(j(w-\theta))d\theta$$

$$Y(jw) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} M(j\theta)\,(\frac{D(j(w-\theta))}{2})d_\theta$$

$$= \frac{1}{4}M(j(w - 4\pi f_0 t) + \frac{1}{4}M(j(w + 4\pi f_0 t) + \frac{1}{2}M(jw)$$

Basic Fourier Transform Pairs

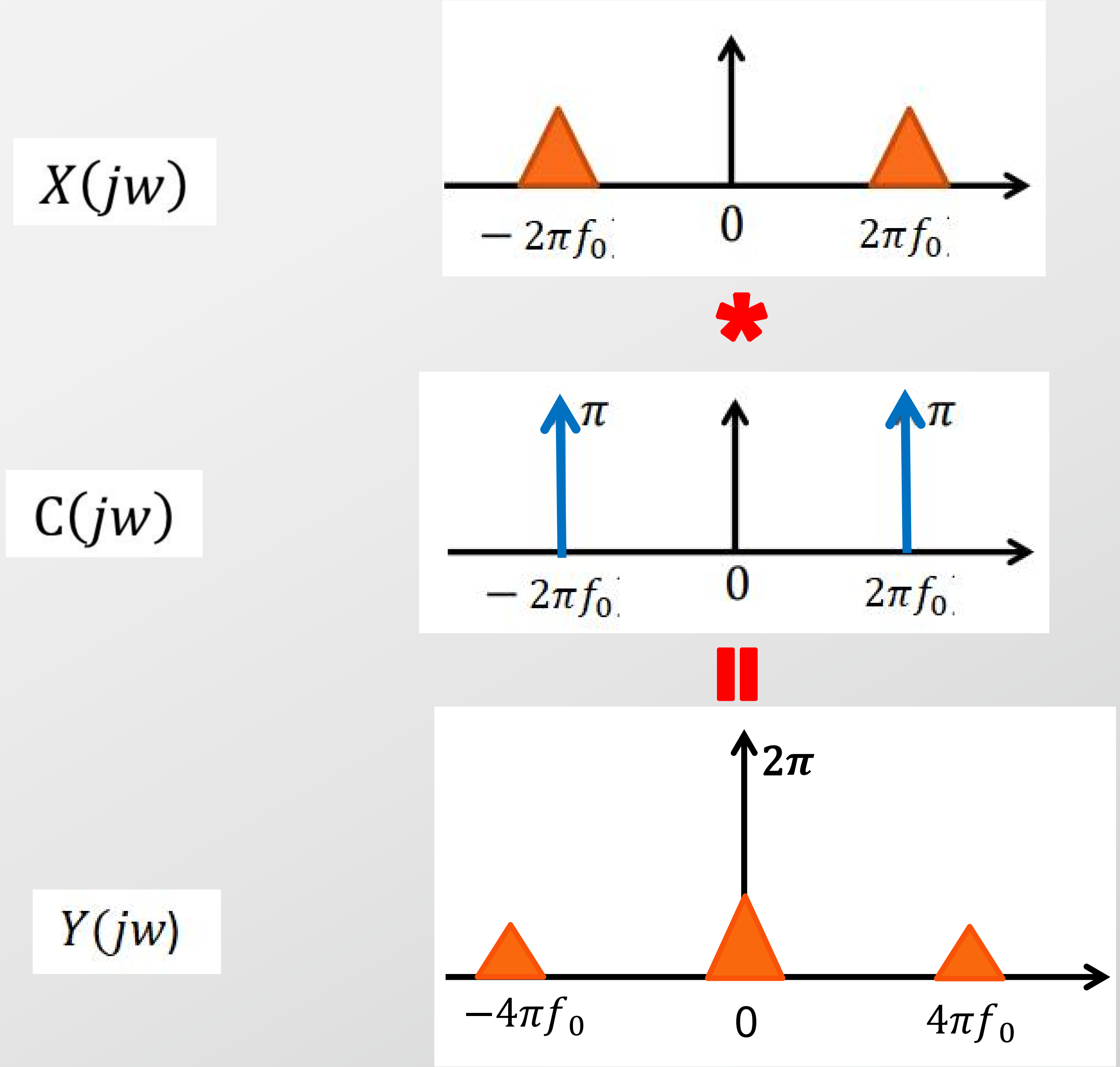
$$cos(w_0 t) \overset{F}{\Leftrightarrow} \pi\delta(w - w_0) + \pi\delta(w + w_0)$$

$$1 \overset{F}{\Leftrightarrow} 2\pi\delta(w)$$

Multiply the amplitude modulation wave with the original carrier signal again, the spectrum will be "shifted" again.the spectrum of the original signal can be found in low frequency. The difference from the original spectrum is that the amplitude is half of the original, which can be compensated by amplification.

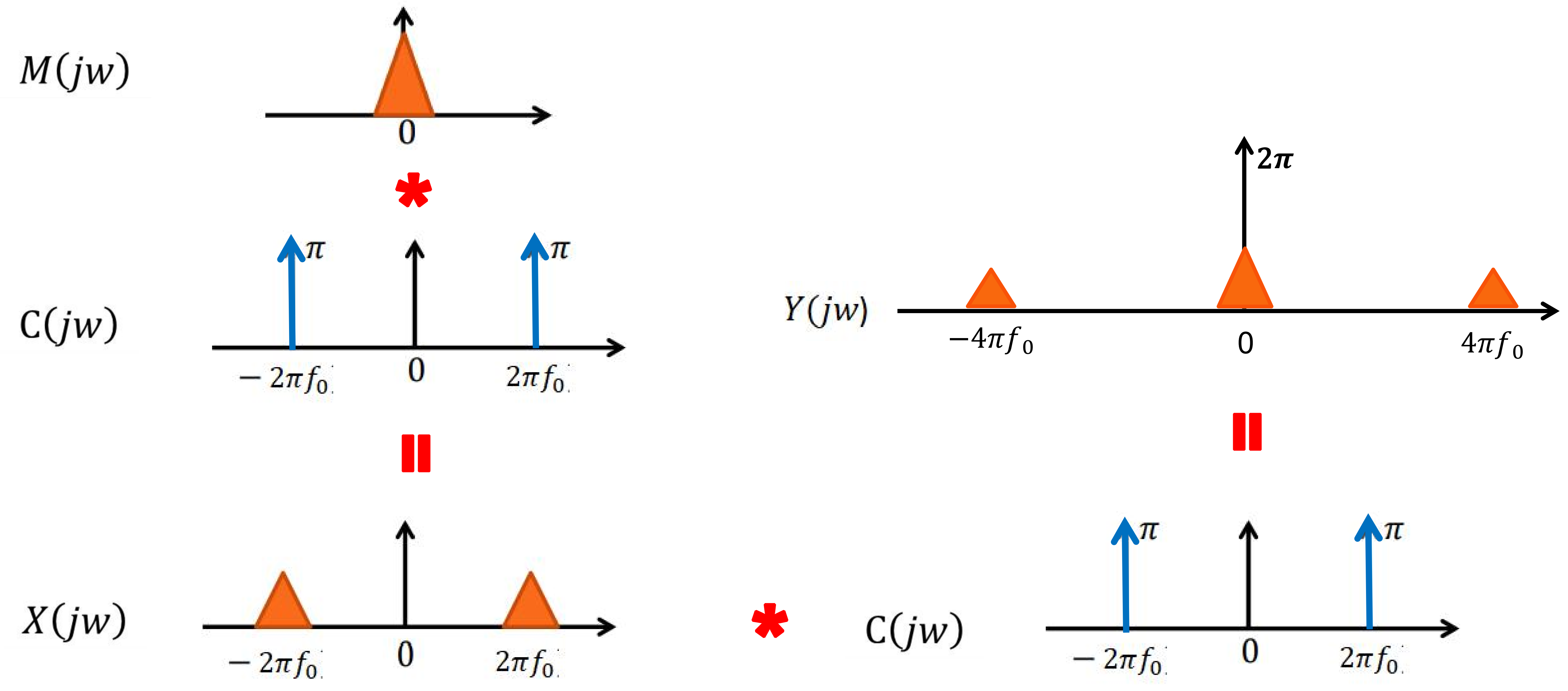
Deodulation——Spectrum shifting again

$$y(t) = m(t)cos(2\pi f_0 t)cos(2\pi f_0 t)$$

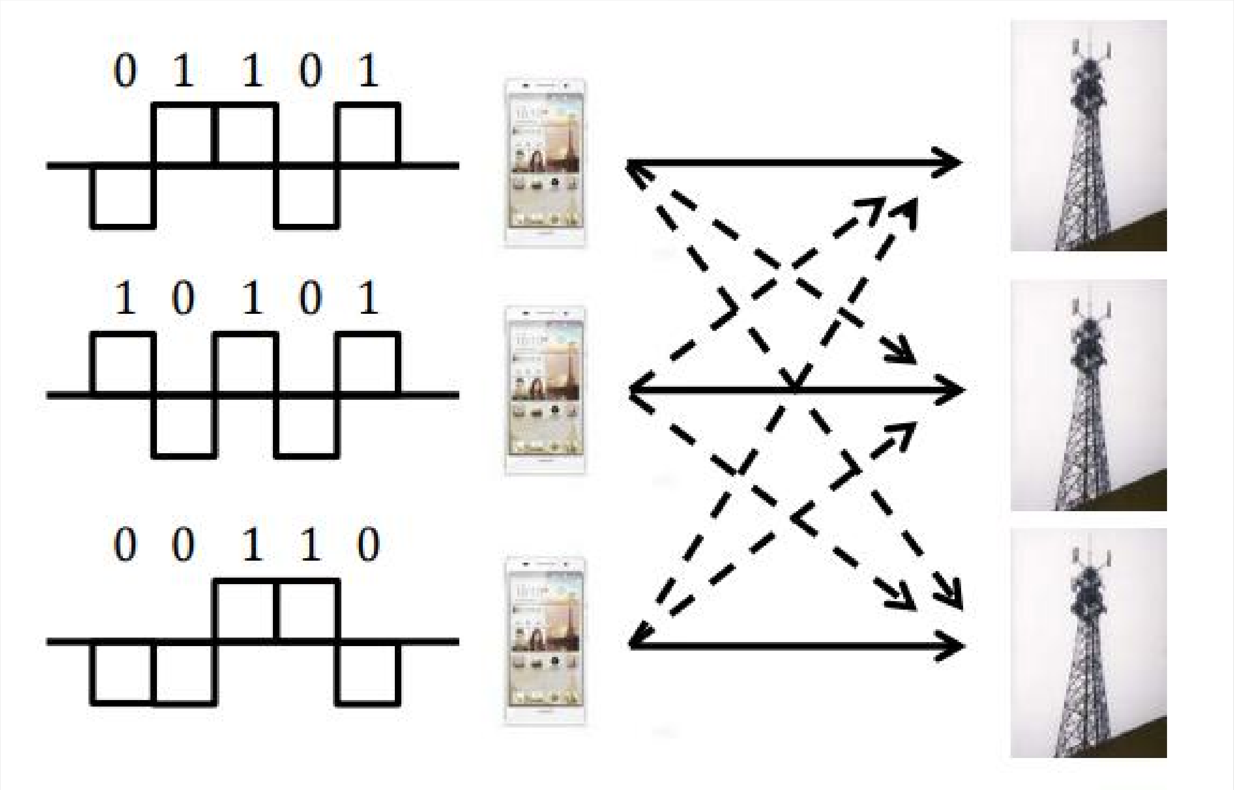


$$Y(jw) = \frac{1}{4}M(j(w - 4\pi f_0 t) + \frac{1}{4}M(j(w + 4\pi f_0 t) + \frac{1}{2}M(jw)$$

Multiply the amplitude modulation wave with the original carrier signal again, the spectrum will be "shifted" again.the spectrum of the original signal can be found in low frequency. The difference from the original spectrum is that the amplitude is half of the original, which can be compensated by amplification.



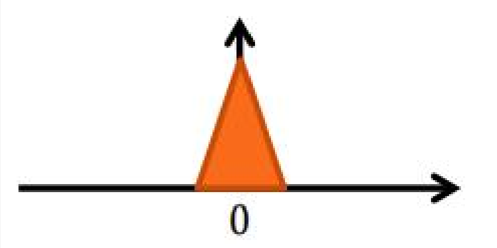
$M(jw)$
0
$C(jw)$
π
π
$-2\pi f_0$
0
$2\pi f_0$
$X(jw)$
$-2\pi f_0$
0
$2\pi f_0$
$Y(jw)$
2π
$-4\pi f_0$
0
$4\pi f_0$
$C(jw)$
π
π
$-2\pi f_0$
0
$2\pi f_0$



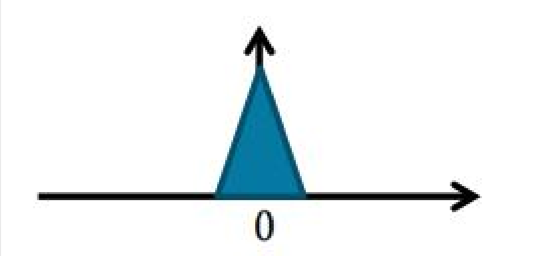
- Each phone wants to deliver information to its base station
- Therefore, there is cross-talk in the wireless channel
- How can we solve this issue?

The spectrum of the original signals $m_1(t)$ and $m_2(t)$

$M1(jw)$

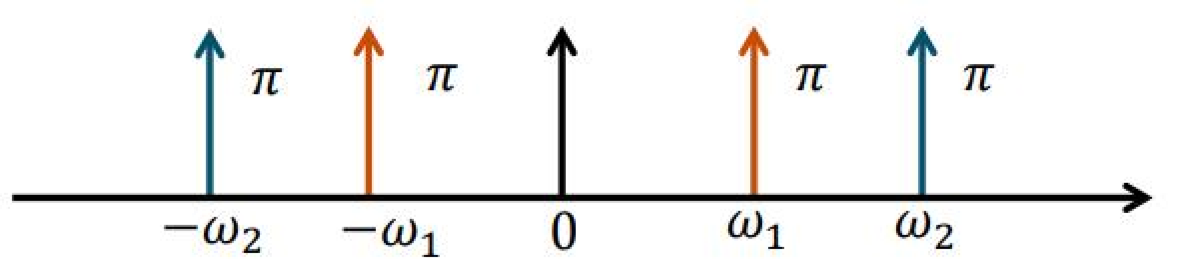


$M2(jw)$



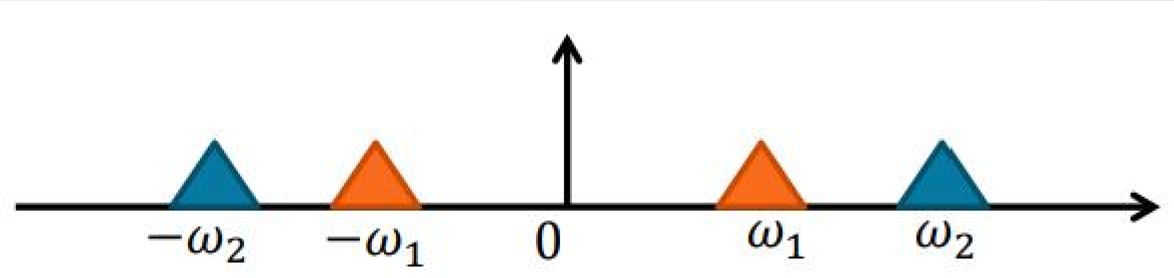
The spectrum of the carrier signals $\cos(\omega_1 t)$ and $\cos(\omega_2 t)$

$C1(jw) + C2(jw)$



The spectrum of AM signal $y(t) = m_1(t)cos(\omega_1 t) + m_2(t)cos(\omega_2 t)$

$Y(jw)$



Signals are distinguished in frequency domain

How to demodulate more signals amplititude modulation?

4.6 Amplitude Modulation and the Continuous-Time Fourier Transform

Intermediate Problems

(e). Determine analytically the Fourier transform of each of the signals

$$m(t)\cos(2\pi f_1 t)\cos(2\pi f_1 t),$$

$$m(t)\cos(2\pi f_1 t)\sin(2\pi f_1 t),$$

and

$$m(t)\sin(2\pi f_1 t)\sin(2\pi f_1 t)$$

$$m(t)\cos(2\pi f_1 t)\cos(2\pi f_2 t),$$

in terms of $M(j\omega)$, the Fourier transform of $m(t)$.

the secret of demodulation

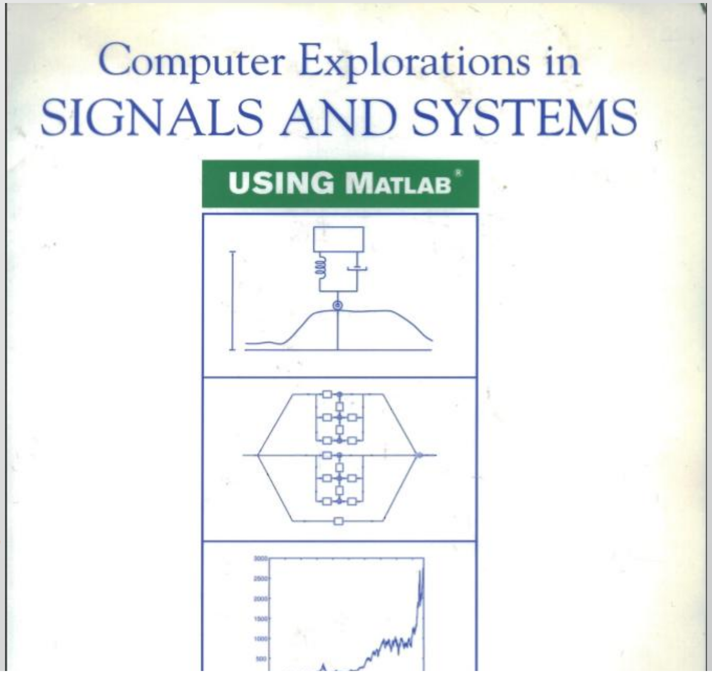
Lab4 Assignments

- Assignments：4.2、 **4.6**

When loading the file, you should have noticed that you have been transformed into Agent 008, the code-breaking sleuth. The last words of the aging Agent 007 were

"The future of technology lies in . . ."

at which point Agent 007 produced a floppy disk and keeled over. The floppy disk contained the MATLAB file **ctftmod.mat**. Your job is to decipher the message encoded in x(t) and complete Agent 007's prediction.



A	· -	H	····	O	- - -	V	··· -
B	- ···	I	··	P	· - - ·	W	· - -
C	- · - ·	J	· - - -	Q	- - · -	X	- ·· -
D	- ··	K	- · -	R	· - ·	Y	- · - -
E	·	L	· - ··	S	···	Z	- - ··
F	·· - ·	M	- -	T	-		
G	- - ·	N	- ·	U	·· -		

Tips :

1

4.6 , download file ctftmod.mat

2

4.6, Correction

$$x(t) = m_1(t)\cos(2\pi f_1 t) + m_2(t)\,\mathbf{sin}(2\pi f_2 t) + m_3(t)\sin(2\pi f_1 t)$$

3

4.6 (e) , add a foemula

$$m(t)\sin(2\pi f_1 t)\sin(2\pi f_1 t)$$