南方科技大学
SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY

Robot Modeling & Control ME331

Section 8： Kinematics VII

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Outline

- Review
  - Jacobian
  - Skew Symmetric Matrix
  - Time Derivative of Rotation Matrices
- Angular Velocity: The General Case
- Addition of Angular Velocities
- Linear Velocity
- Derivation of the Jacobian

Review--Jacobian Matrix

$$\begin{bmatrix} x \\ y \\ z \\ \phi \\ \theta \\ \varphi \end{bmatrix} = h(\begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{bmatrix})_{6\times 1} = \begin{bmatrix} h_1(q_1, q_2, \cdots, q_6) \\ h_2(q_1, q_2, \cdots, q_6) \\ h_3(q_1, q_2, \cdots, q_6) \\ h_4(q_1, q_2, \cdots, q_6) \\ h_5(q_1, q_2, \cdots, q_6) \\ h_6(q_1, q_2, \cdots, q_6) \end{bmatrix}_{6\times 1}$$

$$Y_{6\times 1} = h(q_{n\times 1})$$

$$\dot{Y}_{6\times 1} = \frac{d}{dt} h(q_{n\times 1}) = \frac{dh(q)}{dq} \frac{dq}{dt} = \frac{dh(q)}{dq} \dot{q}$$

$$\dot{Y}_{6\times 1} = J_{6\times n} \dot{q}_{n\times 1}$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \boxed{\left[\frac{dh(q)}{dq} \right]_{6\times n}} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix}_{n\times 1}$$

$$\boxed{J = \frac{dh(q)}{dq}}$$

Jacobian is a function of q.
It is not a constant!

Jacobian Matrix

- **Physical Interpretation**

$$\dot{Y} = J\dot{q} = \begin{bmatrix} J_{11} & J_{12} & \cdots & J_{16} \\ J_{21} & J_{22} & \cdots & J_{26} \\ \vdots & \vdots & \vdots & \vdots \\ J_{61} & J_{62} & \cdots & J_{66} \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \\ \dot{q}_5 \\ \dot{q}_6 \end{bmatrix}$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} J_{11}\dot{q}_1 + J_{12}\dot{q}_2 + \cdots + J_{16}\dot{q}_6 \\ J_{21}\dot{q}_1 + J_{22}\dot{q}_2 + \cdots + J_{26}\dot{q}_6 \\ J_{31}\dot{q}_1 + J_{32}\dot{q}_2 + \cdots + J_{36}\dot{q}_6 \\ J_{41}\dot{q}_1 + J_{42}\dot{q}_2 + \cdots + J_{46}\dot{q}_6 \\ J_{51}\dot{q}_1 + J_{52}\dot{q}_2 + \cdots + J_{56}\dot{q}_6 \\ J_{61}\dot{q}_1 + J_{62}\dot{q}_2 + \cdots + J_{66}\dot{q}_6 \end{bmatrix}$$

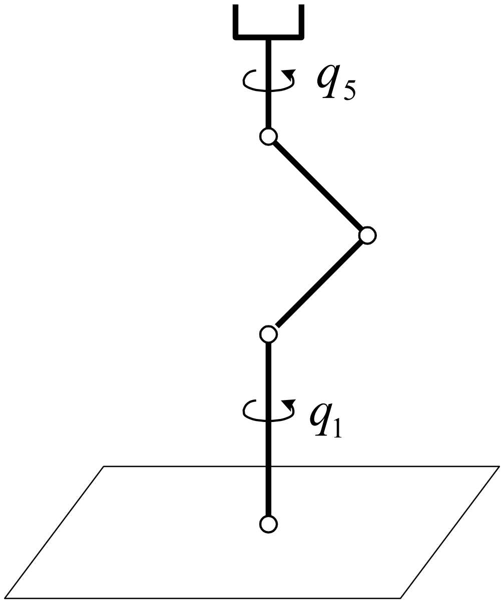
How each individual joint space velocity contribute to task space velocity.

Jacobian Matrix

- **Inverse Jacobian**

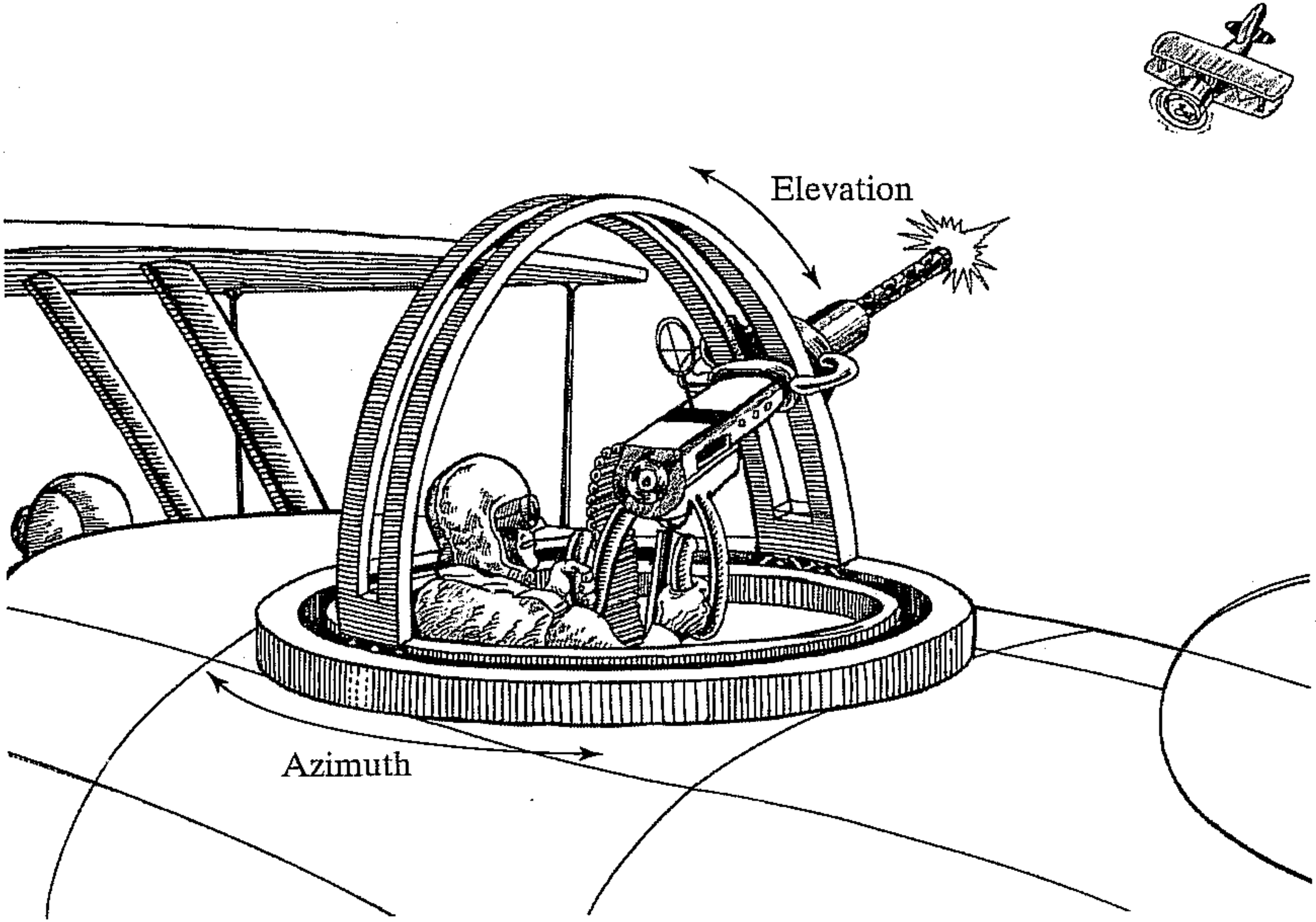
$$\dot{Y} = J\dot{q} = \begin{bmatrix} J_{11} & J_{12} & \cdots & J_{16} \\ J_{21} & J_{22} & \cdots & J_{26} \\ \vdots & \vdots & \vdots & \vdots \\ J_{61} & J_{62} & \cdots & J_{66} \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \\ \dot{q}_5 \\ \dot{q}_6 \end{bmatrix}$$

$$\dot{q} = J^{-1}\dot{Y}$$



- **Singularity**
  - rank(J) < min{6,n}, Jacobian Matrix is less than full rank
  - Jacobian is non-invertable
  - **Boundary Singularities**: occur when the tool tip is on the surface of the work envelop.
  - **Interior Singularities**: occur inside the work envelope when two or more of the axes of the robot form a straight line, i.e., collinear

Jacobian Matrix

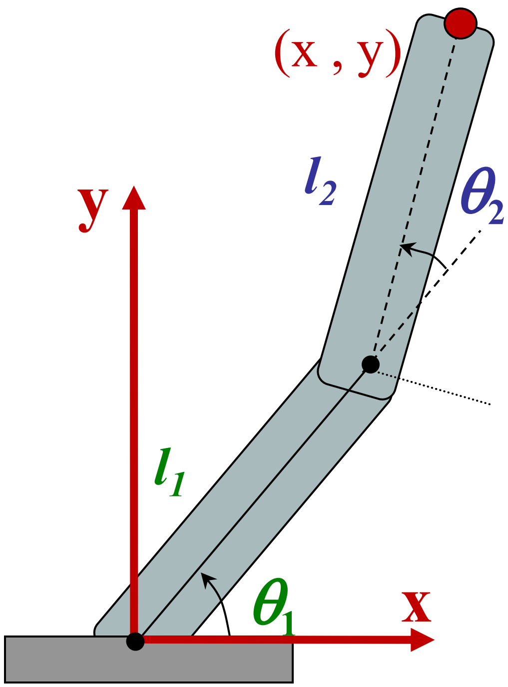


Example

- **Find the singularity configuration of the 2-DOF planar robot arm**

$$\dot{Y} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = J \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

$$J = \begin{bmatrix} -l_1 \sin\theta_1 - l_2 \sin(\theta_1 + \theta_2) & -l_2 \sin(\theta_1 + \theta_2) \\ l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2) & l_2 \cos(\theta_1 + \theta_2) \end{bmatrix}$$



Det(J)=0 ⇨ **Not full rank**

$$\theta_2 = 0, \quad \pi$$

Velocity Kinematics

Skew-Symmetric Matrices

If $a = (a_x, a_y, a_z)^T$ is a 3-vector, we define the skew symmetric $S(a)$ as：

$$S(a) = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}$$

Example： Define the the skew symmetric matrices with

$$\boldsymbol{i} = [1\ 0\ 0]^{\mathrm{T}},\ \boldsymbol{j} = [0\ 1\ 0]^{\mathrm{T}},\ \boldsymbol{k} = [0\ 0\ 1]^{\mathrm{T}}.$$

$$\boldsymbol{S}(\boldsymbol{i}) = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{-1} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \end{bmatrix} \quad \boldsymbol{S}(\boldsymbol{j}) = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{-1} & \mathbf{0} & \mathbf{0} \end{bmatrix} \quad \boldsymbol{S}(\boldsymbol{k}) = \begin{bmatrix} \mathbf{0} & \mathbf{-1} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

Velocity Kinematics

Properties of Skew-Symmetric Matrices

- Operator S is linear, that is $S(\alpha a + \beta b) = \alpha S(a) + \beta S(b)$
  for any vectors $a, b \in \mathbb{R}^3$ and scalars $\alpha, \beta \in R$
- For any vectors $a, p \in \mathbb{R}^3$, $S(a)p = a \times p$
- For $R \in SO(3)$ and $a \in \mathbb{R}^3$, $RS(a)R^T = S(Ra)$

The left-hand side of the equation represents a similarity transformation of the matrix $S(a)$. The equation says, therefore, that the matrix representation of $S(a)$ in a coordinate frame rotated by R is the same as the skew-symmetric matrix $S(Ra)$ corresponding to the vector a rotated by R.

- For $n \times n$ Skew-Symmetric Matrix S, and any vector $X \in \mathbb{R}^n$,

$$X^T S X = 0$$

Velocity Kinematics

The Derivative of a Rotation Matrix

Suppose $R = R(\theta) \in SO(3)$. Since R is orthogonal, it follows

$$R(\theta)R(\theta)^{\mathrm{T}} = I$$

Differentiating both sides of the equation with respect to θ using the product rule gives

$$\left[\frac{d}{d\theta}R\right]R(\theta)^{\mathrm{T}} + R(\theta)\left[\frac{d}{d\theta}R^{\mathrm{T}}\right] = 0 \quad (*)$$

Define $S := \left[\frac{d}{d\theta}R\right]R(\theta)^{\mathrm{T}}$, then $S^{\mathrm{T}} = R\,(\theta)\left[\frac{d}{d\theta}R^{\mathrm{T}}\right]$

Eqn (*) indicates S is skew-symmetric, then:

$$\left[\frac{d}{d\theta}R\right] = SR(\theta)$$

Computing the derivative of the rotation matrix R is equivalent to a matrix multiplication by a skew- symmetric matrix S.

Velocity Kinematics

The Derivative of a Rotation Matrix

Example: if $R = R_{x,\theta}$, find $\frac{\mathrm{d}}{\mathrm{d}\theta} R_{x,\theta}$

$$R_{x,\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$S = \left[\frac{d}{d\theta} R\right] R^{\mathrm{T}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\sin\theta & -\cos\theta \\ 0 & \cos\theta & -\sin\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = S(i)$$

Thus, $\frac{\mathrm{d}}{\mathrm{d}\theta} R_{x,\theta} = S(i)\ R_{x,\theta}$

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}\theta} R_{k,\theta} = S(k)\ R_{k,\theta}}$$

Velocity Kinematics

Angular Velocity: The General Case

Consider the general case of angular velocity about an arbitrary, possibly moving, axis. Suppose that a rotation matrix $R(t)$,

$$\dot{R}(t) = S\big(\omega(t)\big)R(t)$$

where $S\big(\omega(t)\big)$ is skew symmetric. $\omega(t)$ is the **angular velocity** of the rotating frame with respect to the fixed frame at time t.

To see that ω is the angular velocity vector, consider a point p rigidly attached to a moving frame. The coordinates of p relative to the fixed frame are given by $p^0 = R_1^0 p^1$. Differentiating this expression we obtain

$$\begin{aligned} \frac{d}{dt}p^0 &= \dot{R}_1^0 p^1 \\ &= S(\omega)R_1^0 p^1 \\ &= \omega \times R_1^0 p^1 \\ &= \omega \times p^0 \end{aligned}$$

Velocity Kinematics

Angular Velocity: The General Case

$$\dot{R}(t) = S\big(\omega(t)\big)R(t)$$

– The above equation shows the relationship between angular velocity and the derivative of a rotation matrix.

– In particular, if the instantaneous orientation of a frame $o_1x_1y_1z_1$ with respect to a frame $o_0x_0y_0z_0$ is given by R_1^0, then the angular velocity of frame $o_1x_1y_1z_1$ is directly related to the derivative of R_1^0 by the above equation.

– When there is a possibility of ambiguity, we will use the notation $\omega_{i,j}$ to denote the angular velocity that corresponds to the derivative of the rotation matrix R_j^i.

– Since ω is a free vector, we can express it with respect to any coordinate system of our choosing.

– We use a superscript to denote the reference frame. For example, $\omega_{1,2}^0$ would give the angular velocity that corresponds to the derivative of R_2^1, expressed in coordinates relative to frame $o_0x_0y_0z_0$.

Velocity Kinematics

Angular Velocity: The General Case

$$\dot{R}(t) = S\big(\omega(t)\big)R(t)$$

– In cases where the angular velocities specify rotation relative to the base frame, we will often simplify the subscript, for example, using ω_2 to represent the angular velocity that corresponds to the derivative of R_2^0.

Example: Suppose that $R(t) = R_{x,\theta(t)}$. Then $\dot{R}(t)$ is computed using the chain rule as

$$\dot{R} = \frac{dR}{dt}$$

$$= \frac{dR}{d\theta}\frac{d\theta}{dt}$$

$$= \dot{\theta}S(i)R(t)$$

$$= S\big(\omega(t)\big)R(t)$$

in which $\omega = i\dot{\theta}$ is the angular velocity. Note, i = (1, 0, 0).

Velocity Kinematics

Addition of Angular Velocities

We now derive the expressions for the composition of angular velocities of two moving frames $o_1x_1y_1z_1$ and $o_2x_2y_2z_2$ relative to the fixed frame $o_0x_0y_0z_0$.

We assume that the three frames share a common origin. Let the relative orientations of the frames $o_1x_1y_1z_1$ and $o_2x_2y_2z_2$ be given by the rotation matrices $R_1^0(t)$ and $R_2^1(t)$ (both time-varying).

In the derivations that follow, we will use the notation $\omega_{i,j}$ to denote the angular velocity vector that corresponds to the time derivative of the rotation matrix R_j^i . Since we can express this vector relative to the coordinate frame of our choosing, we again use a superscript to define the reference frame. Thus, $\omega_{i,j}^k$ would denote the angular velocity vector corresponding to the derivative of R_j^i, expressed relative to frame k.

Velocity Kinematics

Addition of Angular Velocities

$$R_2^0(t) = R_1^0(t)R_2^1(t)$$

Taking derivatives of both sides of the Eqn. with respect to time yields

$$\dot{R}_2^0 = \dot{R}_1^0 R_2^1 + R_1^0 \dot{R}_2^1$$

The term $\dot{R}_2^0$ on the left-hand side can be written $\dot{R}_2^0 = S(\omega_{0,2}^0)R_2^0$

where $\omega_{0,2}^0$ denotes the total angular velocity experienced by frame $o_2x_2y_2z_2$. This angular velocity results from the combined rotations expressed by R_1^0 and R_2^1.

The first term on the right-hand side is $\dot{R}_1^0 R_2^1 = S\left(\omega_{0,1}^0\right)R_1^0 R_2^1 = S\left(\omega_{0,1}^0\right)R_2^0$

where $\omega_{0,1}^0$ denotes the angular velocity of frame $o_1x_1y_1z_1$ that results from the changing R_1^0, and this angular velocity vector is expressed relative to the coordinate system $o_0x_0y_0z_0$.

Velocity Kinematics

Addition of Angular Velocities

$$\dot{R}_2^0 = \dot{R}_1^0 R_2^1 + R_1^0 \dot{R}_2^1$$

The term $\dot{R}_2^0$ on the left-hand side can be written $\dot{R}_2^0 = S(\omega_{0,2}^0) R_2^0$

The first term on the right-hand side is $\dot{R}_1^0 R_2^1 = S\left(\omega_{0,1}^0\right) R_1^0 R_2^1 = S\left(\omega_{0,1}^0\right) R_2^0$

The second term on the right-hand side is

$$\begin{aligned} R_1^0 \dot{R}_2^1 &= R_1^0 S\left(\omega_{1,2}^1\right) R_2^1 \\ &= R_1^0 S\left(\omega_{1,2}^1\right) {R_1^0}^T R_1^0 R_2^1 = S\left(R_1^0 \omega_{1,2}^1\right) R_1^0 R_2^1 \\ &= S\left(R_1^0 \omega_{1,2}^1\right) R_2^0 \end{aligned}$$

where $\omega_{1,2}^1$ denotes the angular velocity of frame $o_2x_2y_2z_2$ that corresponds to the changing R_2^1, expressed relative to the coordinate system $o_1x_1y_1z_1$.

Thus, the product $R_1^0 \omega_{1,2}^1$ expresses this angular velocity relative to the coordinate system $o_0x_0y_0z_0$. In other words, $R_1^0 \omega_{1,2}^1$ gives the coordinates of the free vector $\omega_{1,2}$ with respect to frame 0.

Velocity Kinematics

Addition of Angular Velocities

$$\dot{R}_2^0 = \dot{R}_1^0 R_2^1 + R_1^0 \dot{R}_2^1$$

The term $\dot{R}_2^0$ on the left-hand side can be written $\dot{R}_2^0 = S(\omega_{0,2}^0) R_2^0$

The first term on the right-hand side is $\dot{R}_1^0 R_2^1 = S\left(\omega_{0,1}^0\right) R_1^0 R_2^1 = S\left(\omega_{0,1}^0\right) R_2^0$

The second term on the right-hand side is $R_1^0 \dot{R}_2^1 = S\left(R_1^0 \omega_{1,2}^1\right) R_2^0$

Now, combining the above expressions we have shown that

$$S\left(\omega_{0,2}^0\right) R_2^0 = \left\{S\left(\omega_{0,1}^0\right) + S\left(R_1^0 \omega_{1,2}^1\right)\right\} R_2^0$$

Since $S(a) + S(b) = S(a + b)$, we see that

$$\omega_{0,2}^0 = \omega_{0,1}^0 + R_1^0 \omega_{1,2}^1$$

In other words, the angular velocities can be added once they are expressed relative to the same coordinate frame, in this case $o_0x_0y_0z_0$.

The above reasoning can be extended to any number of coordinate systems.

Velocity Kinematics

Addition of Angular Velocities

$$\omega_{0,2}^{0} = \omega_{0,1}^{0} + R_1^0 \omega_{1,2}^{1}$$

The above reasoning can be extended to any number of coordinate systems.

In particular, suppose that we are given

$$R_n^0 = R_1^0 R_2^1 \cdots R_n^{n-1}$$

Extending the above reasoning we obtain

$$\dot{R}_n^0 = S\left(\omega_{0,n}^{0}\right) R_n^0$$

In which

$$\omega_{0,n}^{0} = \omega_{0,1}^{0} + R_1^0 \omega_{1,2}^{1} + R_2^0 \omega_{2,3}^{2} + R_3^0 \omega_{3,4}^{3} + \ldots + R_{n-1}^0 \omega_{n-1,n}^{n-1}$$

$$= \omega_{0,1}^{0} + \omega_{1,2}^{0} + \omega_{2,3}^{0} + \omega_{3,4}^{0} + \ldots + \omega_{n-1,n}^{0}$$

Velocity Kinematics

Linear Velocity of a Point Attached to a Moving Frame

Suppose the point p is rigidly attached to the frame $o_1x_1y_1z_1$, and that $o_1x_1y_1z_1$ is rotating relative to the frame $o_0x_0y_0z_0$. Then the coordinates of p with respect to the frame $o_0x_0y_0z_0$ are given by

$$p^0 = R_1^0 p^1$$

The velocity $\dot{p}^0$ is then given by the product rule for differentiation as

$$\begin{aligned}\dot{p}^0 &= \dot{R}_1^0(t)p^1 + R_1^0(t)\dot{p}^1 \\ &= S(\omega^0)R_1^0(t)p^1 \\ &= S(\omega^0)p^0 \\ &= \omega^0 \times p^0\end{aligned}$$

Now, suppose that the motion of the frame $o_1x_1y_1z_1$ relative to $o_0x_0y_0z_0$ is more general. Suppose that the homogeneous transformation relating the two frames is time-dependent, so that

$$H_1^0(t) = \begin{bmatrix} R_1^0(t) & o_1^0(t) \\ 0 & 1 \end{bmatrix}$$

Velocity Kinematics

Linear Velocity of a Point Attached to a Moving Frame

$$H_1^0(t) = \begin{bmatrix} R_1^0(t) & o_1^0(t) \\ 0 & 1 \end{bmatrix}$$

For simplicity we omit the argument t and the subscripts and superscripts on R_1^0 and o_1^0, and write

$$p^0 = Rp^1 + o$$

Differentiating the above expression using the product rule gives

$$\begin{aligned} \dot{p}^0 &= \dot{R}p^1 + \dot{o} \\ &= S(\omega)Rp^1 + \dot{o} \\ &= \omega \times r + v \end{aligned}$$

where $r = Rp^1$ is the vector from o_1 to p expressed in the orientation of the frame $o_0x_0y_0z_0$, and v is the rate at which the origin o_1 is moving.

If the point p is moving relative to the frame $o_1x_1y_1z_1$, then we must add to the term v the term $R(t)\dot{p}^1$, which is the rate of change of the coordinates p^1 expressed in the frame $o_0x_0y_0z_0$.

Velocity Kinematics

Derivation of the Jacobian

Consider an *n*-link manipulator with joint variables $q_1, \cdots, q_n$.

$$T_n^0(q) = \begin{bmatrix} R_n^0(q) & o_n^0(q) \\ 0 & 1 \end{bmatrix}$$

The objective of this section is to relate the linear and angular velocity of the end effector to the vector of joint velocities $\dot{q}(t)$.

Let $S(\omega_n^0) = \dot{R}_n^0 (R_n^0)^T$ define the angular velocity vector ω_n^0 of the end effector, and let $v_n^0 = \dot{o}_n^0$ denote the linear velocity of the end effector.

We seek expressions of the form

$$\begin{bmatrix} v_n^0 \\ \omega_n^0 \end{bmatrix} = \begin{bmatrix} J_v \\ J_w \end{bmatrix} \dot{q}$$

$$\xi = J\dot{q}$$

where ξ is sometimes called a body velocity. Note that this velocity vector is not the derivative of a position variable.

Velocity Kinematics

Derivation of the Jacobian：Angular Velocity

$$\omega_{0,n}^{0} = \omega_{0,1}^{0} + R_1^0 \omega_{1,2}^{1} + R_2^0 \omega_{2,3}^{2} + R_3^0 \omega_{3,4}^{3} + \ldots + R_{n-1}^0 \omega_{n-1,n}^{n-1}$$

$$= \omega_{0,1}^{0} + \omega_{1,2}^{0} + \omega_{2,3}^{0} + \omega_{3,4}^{0} + \ldots + \omega_{n-1,n}^{0}$$

We can determine the angular velocity of the end effector relative to the base by expressing the angular velocity contributed by each joint in the orientation of the base frame and then summing these.

If the i^{th} joint is revolute, then the i^{th} joint variable q_i equals θ_i and the axis of rotation is z_{i-1}.

Slightly abusing notation, let ω_i^{i-1} represent the angular velocity of link i that is imparted by the rotation of joint i, expressed relative to frame $o_{i-1}x_{i-1}y_{i-1}z_{i-1}$. This angular velocity is expressed in the frame $i-1$ by

$$\omega_i^{i-1} = \dot{q}_i z_{i-1}^{i-1} = \dot{q}_i k$$

where k is the unit coordinate vector $(0, 0,1)$.

How about prismatic joint?

Velocity Kinematics

Derivation of the Jacobian：Angular Velocity

If the i^{th} joint is prismatic, then the motion of frame i relative to frame $i-1$ is a translation and

$$\omega_i^{i-1} = 0$$

Thus, if joint i is prismatic, the angular velocity of the end effector does not depend on q_i, which now equals d_i.

Therefore, the overall angular velocity of the end effector, ω_n^0, in the base frame is determined as

$$\omega_n^0 = \rho_1 \dot{q}_1 k + \rho_2 \dot{q}_2 R_1^0 k + \cdots + \rho_n \dot{q}_n R_{n-1}^0 k = \sum_{i=1}^{n} \rho_i \dot{q}_i z_{i-1}^0$$

in which ρ_i is equal to 1 if joint i is revolute and 0 if joint i is prismatic, since $z_{i-1}^0 = R_{i-1}^0 k$, $z_0^0 = k = [0,0,1]^T$.

Recall $\omega_n^0 = J_w \dot{q}$, then the lower half of the Jacobian J_w is thus given as

$$J_w = [\rho_1 z_0 \;\; \rho_2 z_1 \;\; \cdots \;\; \rho_n z_{n-1}]$$

We have omitted the superscripts for the unit vectors along the z-axes, since these are all referenced to the world frame.

Velocity Kinematics

Derivation of the Jacobian：Velocity

The linear velocity of the end effector is just $\dot{o}_n^0$. By the chain rule for differentiation

$$\dot{o}_n^0 = \sum_{i=1}^{n} \frac{\partial o_n^0}{\partial q_i} \dot{q}_i$$

Thus, we see that the i^{th} column of J_v, which we denote as J_{v_i} is given by

$$J_{v_i} = \frac{\partial o_n^0}{\partial q_i}$$

Furthermore, this expression is just the linear velocity of the end effector that would result if $\dot{q}_i$ were equal to one and the other $\dot{q}_j$ were zero.

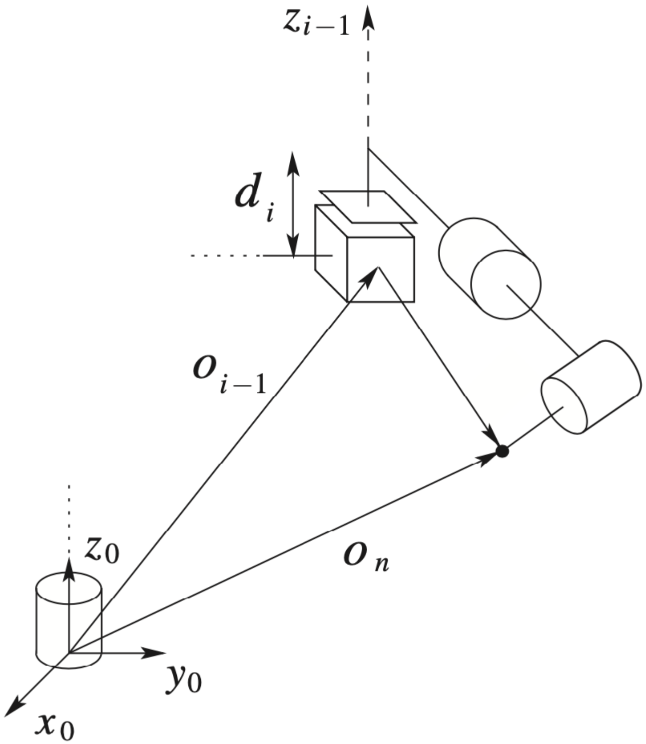
The i^{th} column of the Jacobian can be generated by holding all joints but the i^{th} joint fixed and actuating the i^{th} at unit velocity.

This observation leads to a simple and intuitive derivation of the linear velocity Jacobian.

We now consider the two cases of prismatic and revolute joints separately.

Velocity Kinematics

Derivation of the Jacobian： Velocity (Case1: Prismatic Joints)



Left Figure illustrates the case when all joints are fixed except a single prismatic joint.

Since joint i is prismatic, it imparts a pure translation to the end effector.

The direction of translation is parallel to the axis z_{i-1} and the magnitude of the translation is $\dot{d}_i$, where d_i is the DH joint variable.

Thus, in the orientation of the base frame we have

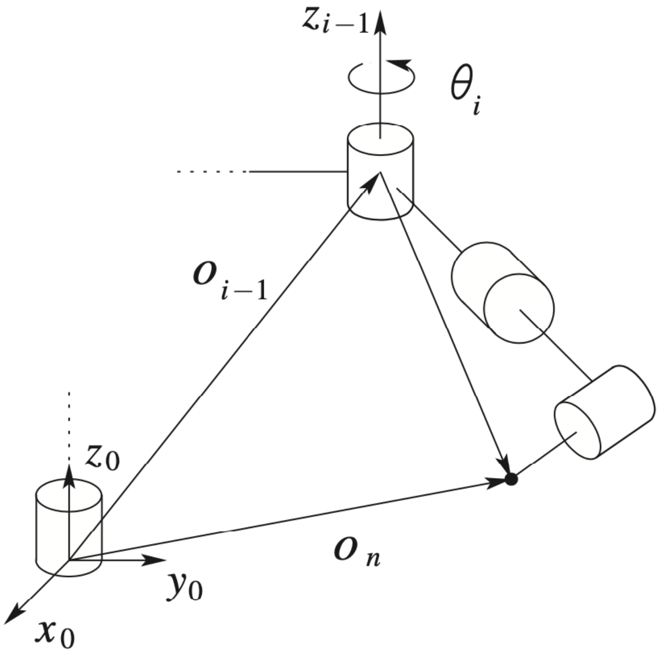
$$\dot{o}_n^0 = \dot{d}_i R_{i-1}^0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \dot{d}_i z_{i-1}^0$$

in which d_i is the joint variable for prismatic joint i. Thus, for the case of prismatic joints, after dropping the superscripts we have

$$J_{v_i} = z_{i-1}$$

Velocity Kinematics

Derivation of the Jacobian： Velocity (Case2: Revolute Joints)



Left Figure illustrates the case when all joints are fixed except a single revolute joint.

Since joint i is revolute, we have $q_i = \theta_i$.

Assuming that all joints are fixed except joint i, we see that the linear velocity of the end effector is simply of the form $\omega \times r$, where

$$\omega = \dot{\theta}_i z_{i-1}$$

$$r = o_n - o_{i-1}$$

Thus, putting these together and expressing the coordinates relative to the base frame, for a revolute joint we obtain

$$J_{v_i} = z_{i-1} \times (o_n - o_{i-1})$$

in which we have omitted the zero superscripts following our convention.

Velocity Kinematics

Combining the Linear and Angular Velocity Jacobians

As we have seen in the preceding section, the upper half of the Jacobian J_v is given as

$$J_v = [J_{v_1} \cdots J_{v_n}]$$

in which the i^{th} column J_{v_i} is

$$J_{v_i} = \begin{cases} z_{i-1} \times (o_n - o_{i-1}) & \text{for revolute joint } i \\ z_{i-1} & \text{for prismatic joint } i \end{cases}$$

The lower half of the Jacobian is given as

$$J_\omega = [J_{\omega_1} \cdots J_{\omega_n}]$$

in which the i^{th} column J_{ω_i} is

$$J_{\omega_i} = \begin{cases} z_{i-1} & \text{for revolute joint } i \\ 0 & \text{for prismatic joint } i \end{cases}$$

The only quantities needed to compute the Jacobian are the unit vectors z_i and the coordinates of the origins $o_1, \ldots, o_n$.

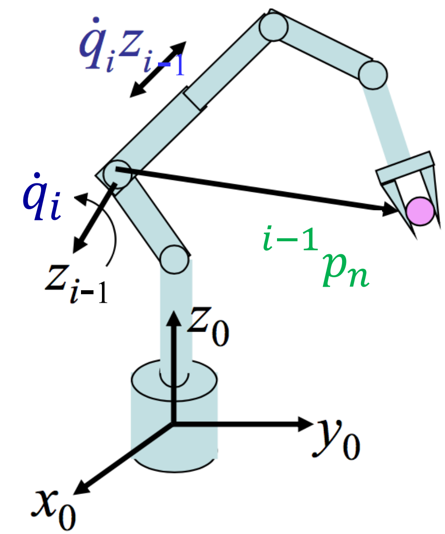
- z_i are given by the first three elements in the third column of T_i^0.
- o_i is given by the first three elements of the fourth column of T_i^0.

Velocity Kinematics

CN

雅可比矩阵的计算公式(矢量积法)

$$J = \begin{bmatrix} J_v \\ J_w \end{bmatrix}$$



1) 雅可比矩阵的上半部分 $J_v = (J_{v_1} \ldots J_{v_n})$

其中，矩阵的第i列 J_{v_i}

$$J_{v_i} = \begin{cases} z_{i-1} \times (o_n - o_{i-1}), \text{转动关节} \\ z_{i-1} \qquad\qquad\qquad\quad\;, \text{平动关节} \end{cases}$$

2) 雅可比矩阵的下半部分 $J_\omega = (J_{\omega_1} \ldots J_{\omega_n})$

其中，矩阵的第i列 J_{ω_i}

$$J_{\omega_i} = \begin{cases} z_{i-1}, \text{转动关节} \\ 0 \quad\;\;, \text{平动关节} \end{cases}$$

Velocity Kinematics

CN

雅可比矩阵的计算公式(矢量积法)

$$J_{v_i} = \begin{cases} z_{i-1} \times (o_n - o_{i-1}) & \text{对于转动关节}i \\ z_{i-1} & \text{对于平动关节}i \end{cases}$$

$$J_{\omega_i} = \begin{cases} z_{i-1} & \text{对于转动关节}i \\ 0 & \text{对于平动关节}i \end{cases}$$

1) 计算雅可比矩阵仅需知道单位向量 z_i 以及原点 $o_{1,\ldots,}o_n$ 的坐标;

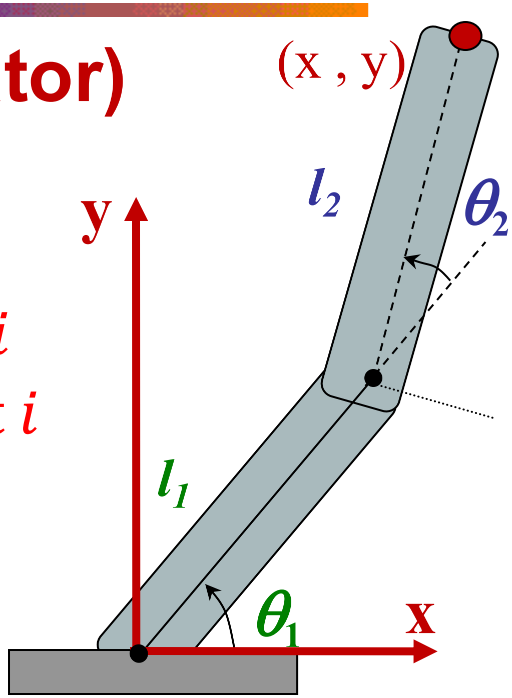
2) z_i 相对于基座坐标系的坐标，可由T_i^0 第3列中的3个元素给出;

3) o_i由T_i^0第4列中的3个元素给出;

4）该算法不仅适用于末端执行器的速度，也适用于计算机械臂上任何一点的速度，这对于计算质心速度推导动力学方程非常重要。

Velocity Kinematics

- **Example (2-DoF Planar Manipulator)**
  - **Given *l_1, l_2, find*: *Jacobian***
    （Use vector product method）



$$J_{v_i} = \begin{cases} z_{i-1} \times (o_n - o_{i-1}) & \text{for revolute joint } i \\ z_{i-1} & \text{for prismatic joint } i \end{cases}$$

$$z_0 = [0 \quad 0 \quad 1]^{\mathsf{T}} \qquad z_1 = [0 \quad 0 \quad 1]^{\mathsf{T}}$$

$$o_2 - o_1 = [l_2 \cos(\theta_1 + \theta_2) \quad l_2 \sin(\theta_1 + \theta_2) \quad 0]^{\mathsf{T}}$$

$$o_2 - o_0 = [l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2) \quad l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2) \quad 0]^{\mathsf{T}}$$

$$J_{v_1} = z_0 \times (o_2 - o_0) = \begin{bmatrix} -l_1 \sin\theta_1 - l_2 \sin(\theta_1 + \theta_2) \\ l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2) \\ 0 \end{bmatrix}$$

$$J_{v_2} = z_1 \times (o_2 - o_1) = \begin{bmatrix} -l_2 \sin(\theta_1 + \theta_2) \\ l_2 \cos(\theta_1 + \theta_2) \\ 0 \end{bmatrix} \qquad \boldsymbol{J} = [J_{v_1} \quad J_{v_1}]$$

Summary

Velocity Kinematics

Angular Velocity: The General Case

Addition of Angular Velocities

Linear Velocities

Derivation of the Jacobian

- Vector Product Method

Homework 8

Homework 8 is posted at http://bb.sustech.edu.cn

Due date: March 26, 2025

Next class: March 24, 2024（Monday）

Velocity Kinematics 3

作业要求（Requirements）：

1. **文件格式为以自己学号姓名作业序号命名的pdf文件；**

   （File name: YourSID_ YourName_08.pdf）

2. **作业里也写上自己的姓名和学号。**

   （Write your name and SID in the homework）