南方科技大学
SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY

Robot Modeling & Control ME331

Section 10： Kinematics IX

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Outline

- Review
  - The Tool Velocity
  - The Analytical Jacobian
  - Singularities
- Static Force/Torque Relationships
- Inverse Velocity and Acceleration
- Manipulability
- Numerical Solution to Inverse Kinematics

Velocity Kinematics

- **The Tool Velocity**

$$v_{\text{tool}} = v_6 + \omega_6 \times r$$

$$\begin{aligned} \omega_6^{\text{tool}} \times r^{\text{tool}} &= R^T \omega_6^6 \times (R^T d) \\ &= -R^T d \times R^T \omega_6^6 = -S(R^T d) R^T \omega_6^6 = -R^T S(d) R R^T \omega_6^6 = -R^T S(d) \omega_6^6 \end{aligned}$$

To express the free vector v_6 in coordinates relative to the tool frame, we apply the rotational transformation

$$v_6^{\text{tool}} = R^T v_6^6$$

Combining above Equations to obtain the linear velocity of the tool frame and the angular velocity of the tool frame, we have

$$v_{\text{tool}}^{\text{tool}} = R^T v_6^6 - R^T S(d) \omega_6^6$$
$$\omega_{\text{tool}}^{\text{tool}} = R^T \omega_6^6$$

which can be written as the matrix equation

$$\xi_{\text{tool}}^{\text{tool}} = \begin{bmatrix} R^T & -R^T S(d) \\ 0_{3\times 3} & R^T \end{bmatrix} \xi_6^6$$

Velocity Kinematics

- **The Tool Velocity**

$$\xi_{\mathrm{tool}}^{\mathrm{tool}} = \begin{bmatrix} R^T & -R^T S(d) \\ 0_{3\times3} & R^T \end{bmatrix} \xi_6^6$$

In many cases, it is useful to solve the inverse problem: compute the required end effector-velocity to produce a desired tool velocity. Since

$$\begin{bmatrix} R & S(d)R \\ 0_{3\times3} & R \end{bmatrix} = \begin{bmatrix} R^T & -R^T S(d) \\ 0_{3\times3} & R^T \end{bmatrix}^{-1}$$

We can obtain $\xi_6^6 = \begin{bmatrix} R & S(d)R \\ 0_{3\times3} & R \end{bmatrix} \xi_{\mathrm{tool}}^{\mathrm{tool}}$

This gives the general expression for transforming velocities between two rigidly attached moving frames

$$\xi_A^A = \begin{bmatrix} R_B^A & S(d_B^A)R_B^A \\ 0_{3\times3} & R_B^A \end{bmatrix} \xi_B^B$$

Velocity Kinematics

- **The Analytical Jacobian**

The Jacobian matrix derived above is sometimes called the **geometric Jacobian** to distinguish it from the **analytical Jacobian**, denoted $J_a(q)$, which is based on a minimal representation for the orientation of the end-effector frame. Let

$$X = \begin{bmatrix} d(q) \\ \alpha(q) \end{bmatrix}$$

denote the end-effector pose, where $d(q)$ is the usual vector from the origin of the base frame to the origin of the end-effector frame and α denotes a minimal representation for the orientation of the end-effector frame relative to the base frame.

For example, let $\alpha = (\varphi, \theta, \psi)$ be a vector of Euler angles.

Then we seek an expression of the form

$$\dot{X} = \begin{bmatrix} \dot{d} \\ \dot{\alpha} \end{bmatrix} = J_a(q)\dot{q}$$

to define the analytical Jacobian.

Velocity Kinematics

- **The Analytical Jacobian**

If $R = R_{z,\varphi} R_{y,\theta} R_{z,\psi}$ is the Euler angle transformation, then

$$\dot{R} = S(\omega)R$$

in which ω defining the angular velocity is given by

$$\omega = \begin{bmatrix} c_\psi s_\theta \dot{\phi} - s_\psi \dot{\theta} \\ s_\psi s_\theta \dot{\phi} + c_\psi \dot{\theta} \\ \dot{\psi} + c_\theta \dot{\phi} \end{bmatrix} = \begin{bmatrix} c_\psi s_\theta & -s_\psi & 0 \\ s_\psi s_\theta & c_\psi & 0 \\ c_\theta & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = B(\alpha)\dot{\alpha}$$

The components of ω are called nutation (章动), spin (旋转), and precession (进动), respectively. Combining the above relationship with the previous definition of the Jacobian

$$\begin{bmatrix} v \\ \omega \end{bmatrix} = \begin{bmatrix} \dot{d} \\ \omega \end{bmatrix} = J(q)\dot{q}$$

yields

$$J(q)\dot{q} = \begin{bmatrix} v \\ \omega \end{bmatrix} = \begin{bmatrix} \dot{d} \\ B(\alpha)\dot{\alpha} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & B(\alpha) \end{bmatrix} \begin{bmatrix} \dot{d} \\ \dot{\alpha} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & B(\alpha) \end{bmatrix} J_a(q)\dot{q}$$

Velocity Kinematics

- **Singularities：Decoupling of Singularities**

The manipulator configurations themselves are geometric quantities, independent of the frames used to describe them.

Recognizing this fact allows us to decouple the determination of singular configurations, for those manipulators with spherical wrists, into two simpler problems:

1) The first is to determine so-called **arm singularities**, that is, singularities resulting from motion of the arm, which consists of the first three or more links.

2) The second is to determine the **wrist singularities** resulting from motion of the spherical wrist.

Consider the case that n = 6, that is, the manipulator consists of a 3-DOF arm with a 3-DOF spherical wrist. In this case the Jacobian is a 6 × 6 matrix and a configuration *q* is singular if and only if

$$\det J(q) \quad = \quad 0$$

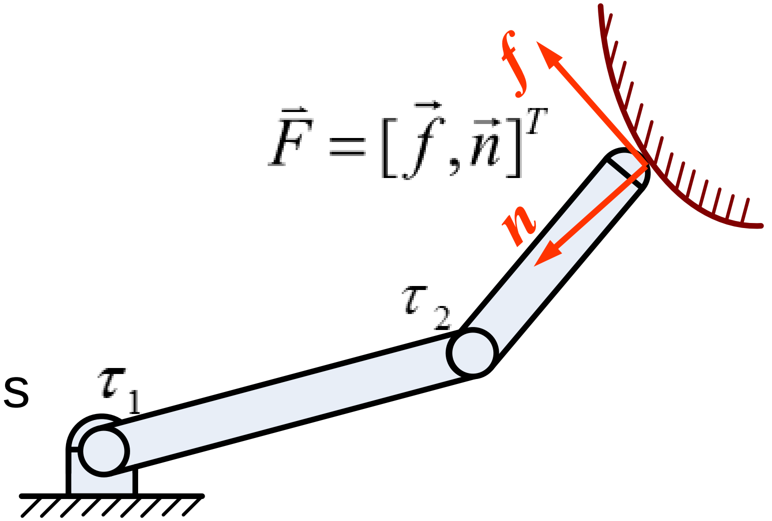
Velocity Kinematics

- **Static Force/Torque Relationships**

Interaction of the manipulator with the environment produces forces and moments at the end effector or tool. These, in turn, produce torques at the joints of the robot.

quantitative relationship?

$$(\tau_1, \tau_2) \Longleftrightarrow (\vec{f}, \vec{n})$$



This relationship is important for

- Development of path planning methods
- Derivation of the dynamic equations
- Design of force control algorithms

Let $F = (F_x, F_y, F_z, n_x, n_y, n_z)$ represent the vector of forces and moments at the end effector. Let τ denote the corresponding vector of joint torques. Then F and τ are related by

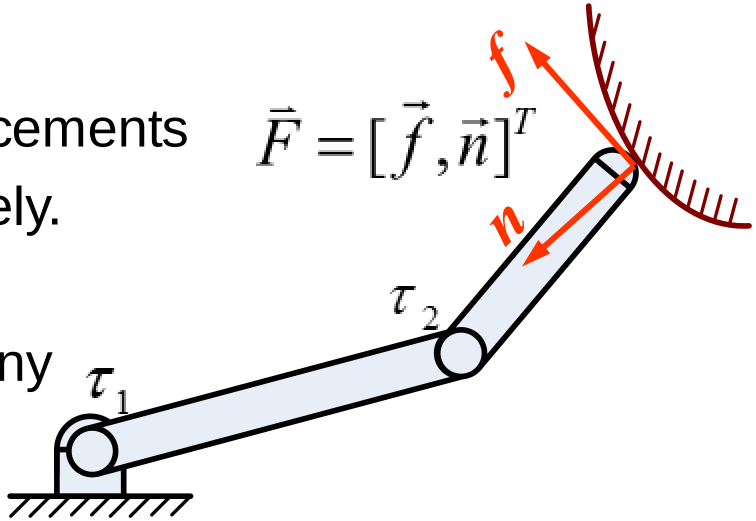
$$\tau = J^T(q)F$$

Velocity Kinematics

- **Static Force/Torque Relationships**

$$\tau = J^T(q)F$$

An easy way to derive this relationship is through the so-called **principle of virtual work （虚功原理）**.



Let δX and δq represent infinitesimal displacements in the task space and joint space, respectively. These displacements are called **virtual displacements** if they are consistent with any constraints imposed on the system.

$$\delta X = J(q)\delta q$$

The virtual work δw of the system is

$$\begin{aligned} \delta w &= F^T \delta X - \tau^T \delta q \\ &= F^T J(q)\delta q - \tau^T \delta q \\ &= (F^T J(q) - \tau^T)\delta q \end{aligned}$$

If the manipulator is in equilibrium

$$\delta w = 0$$

$$\tau^T = F^T J(q)$$

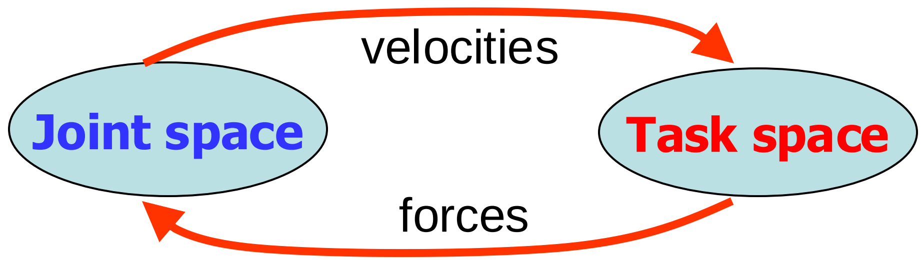
$$\tau = J^T(q)F$$

Velocity Kinematics

- **Static Force/Torque Relationships**

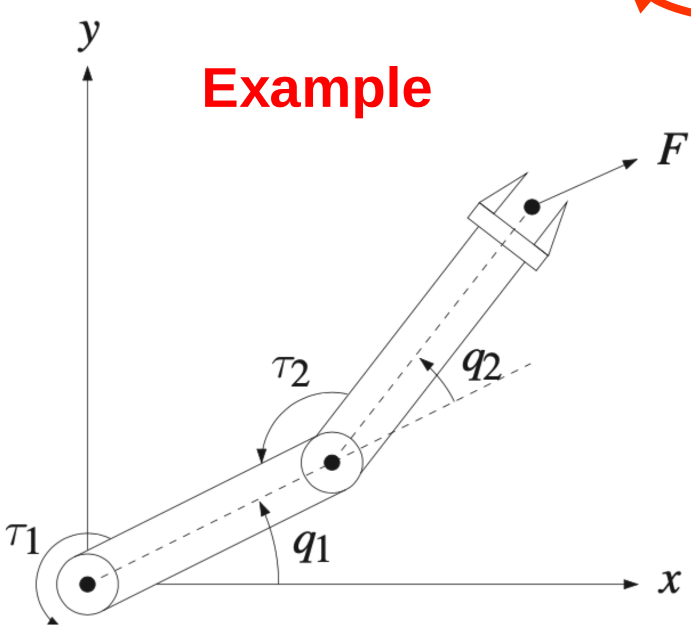
The end-effector forces are related to the joint torques by the transpose of the manipulator Jacobian.

$$\xi = J(q)\dot{q}$$



$$\tau = J^T(q)F$$

Example



A force F applied at the end of link two. The resulting joint torques $\tau = (\tau_1, \tau_2)$ are given as

$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} -a_1 s_1 - a_2 s_{12} & a_1 c_1 + a_2 c_{12} & 0 & 0 & 0 & 1 \\ -a_2 s_{12} & a_2 c_{12} & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} F_x \\ F_y \\ F_z \\ n_x \\ n_y \\ n_z \end{bmatrix}$$

Velocity Kinematics

- **Inverse Velocity**

The inverse velocity problem is the problem of finding the joint velocities $\dot{q}$ that produce the desired end-effector velocity.

$$\xi = J_{6\times n}\dot{q}$$

- For the case when $n = 6$ and J is nonsingular

$$\dot{q} = J^{-1}\xi$$

- For the case when $n < 6$ there will be a solution if and only if ξ lies in the range space of the Jacobian.

  This can be determined by the following simple rank test. A vector ξ belongs to the range of J if and only if

$$\operatorname{rank} J(q) = \operatorname{rank} [J(q) \mid \xi]$$

  This is a standard result from linear algebra, and several algorithms exist, such as Gaussian elimination, for solving such systems of linear equations.

Velocity Kinematics

• Inverse Velocity

- For the case when $n > 6$, the manipulators are redundant. We can solve for $\dot{q}$ using the right pseudoinverse $J^+ = J^T(JJ^T)^{-1}$.

$$\dot{q} = J^+\xi + (I - J^+J)b$$

Homework_1

in which $b \in \mathbb{R}^n$ is an arbitrary vector.

- ✓ In general, for $n > 6$, $(I - J^+J) \neq 0$, and all vectors of the form $(I - J^+J)b$ lie in the null space of J.
- ✓ This means that, if $\dot{q}'$ is a joint velocity vector such that $\dot{q}' = (I - J^+J)b$, then when the joints move with velocity $\dot{q}'$, the end effector will remain fixed since $J(q)\dot{q}' = 0$.
- ✓ Thus, if $\dot{q}$ is a solution, then so is $\dot{q} + \dot{q}'$ with $\dot{q}' = (I - J^+J)b$, for any value of b.
- ✓ If the goal is to minimize the resulting joint velocities, we choose $b = 0$.

Homework_1

Velocity Kinematics

- **Inverse Acceleration**

We can apply a similar approach when the analytical Jacobian is used in place of the manipulator Jacobian. Recall that the joint velocities and the end-effector velocities are related by the analytical Jacobian as

$$\dot{X} = J_a(q)\dot{q}$$

Differentiating the above equation yields an expression for the acceleration

$$\ddot{X} = \left(\frac{d}{dt}J_a(q)\right)\dot{q} + J_a(q)\ddot{q}$$

Thus, given a vector $\ddot{X}$ of end-effector accelerations, the instantaneous joint acceleration vector $\ddot{q}$ is given as a solution of

$$J_a(q)\ddot{q} = \ddot{X} - \left(\frac{d}{dt}J_a(q)\right)\dot{q}$$

For 6-DOF manipulators the inverse acceleration equations can therefore be written as

$\ddot{q} = J_a(q)^{-1}\left[\ddot{X} - \left(\frac{d}{dt}J_a(q)\right)\dot{q}\right]$, provided $\det J_a(q) \neq 0$.

Velocity Kinematics

- **Manipulability**

$$\xi = J\dot{q}$$

- We can think of J as scaling the input $\dot{q}$ to produce the output ξ. It is often useful to characterize quantitatively the effects of this scaling.
- In systems with a single input and a single output, this kind of characterization is given in terms of the so-called impulse response of a system, which essentially characterizes how the system responds to a unit input.
- In this multidimensional case, the analogous concept is to characterize the output in terms of an input that has unit norm.

Consider the set of all robot joint velocities $\dot{q}$ such that

$$\| \dot{q} \|^2 = \dot{q}_1^2 + \dot{q}_2^2 + \cdots + \dot{q}_n^2 \leq 1$$

If we use the minimum norm solution $\dot{q} = J^+\xi = J^{\mathrm{T}}(JJ^{\mathrm{T}})^{-1}\xi$, we obtain

$$\| \dot{q} \|^2 = \dot{q}^T\dot{q} = (J^+\xi)^T J^+\xi = \xi^T(JJ^T)^{-1}\xi$$ ***Homework_2***

This equation gives us a quantitative characterization of the scaling effected by the Jacobian.

Velocity Kinematics

- **Manipulability**

$$\| \dot{q} \|^2 = \xi^T (JJ^T)^{-1} \xi \leq 1$$

- If the manipulator Jacobian is full rank, that is, if rank $J = m$, then the above equation defines an m-dimensional ellipsoid that is known as the **manipulability ellipsoid**.
- If the input (joint velocity) vector has unit norm, then the output (end-effector velocity) will lie within the ellipsoid.

We can more easily see that the above equation defines an ellipsoid by replacing the Jacobian by its **Singular Value Decomposition (SVD)**.

For square matrices, we can use tools such as the determinant, eigenvalues, and eigenvectors to analyze their properties. However, for nonsquare matrices these tools simply do not apply. Their generalizations are captured by the **SVD**.

Velocity Kinematics

- **Manipulability:** Singular Value Decomposition (SVD)

For $J \in \mathbb{R}^{m\times n}$, we have $JJ^T \in \mathbb{R}^{m\times m}$. It is easily seen that JJ^T is symmetric and positive semi-definite. Therefore, JJ^T has real and nonnegative eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq 0$.

The **singular values (奇异值)** for the matrix J are given by the square roots of the eigenvalues of JJ^T,

$$\sigma_i = \sqrt{\lambda_i},\ i = 1, \cdots, m$$

The singular value decomposition (SVD) of the matrix J is then given by

$$J = U\Sigma V^T$$

in which $U \in \mathbb{R}^{m\times m}$, $V \in \mathbb{R}^{n\times n}$ are orthogonal matrices, and Σ is given by

$$\Sigma = \left(\begin{array}{cccc|c} \sigma_1 & & & & \\ & \sigma_2 & & & \\ & & \ddots & & 0 \\ & & & \sigma_m & \end{array}\right)_{\mathrm{m}\times\mathrm{n}}$$

where $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_m \geq 0$.

Velocity Kinematics

- **Manipulability:** Manipulability Ellipsoid

$$\| \dot{q} \|^2 = \xi^T (JJ^T)^{-1} \xi \leq 1$$

By replacing the Jacobian with its SVD $J = U\Sigma V^{\mathrm{T}}$,we can obtain

$$\xi^{\mathrm{T}} (JJ^{\mathrm{T}})^{-1} \xi = \left(U^{\mathrm{T}} \xi\right)^{\mathrm{T}} \Sigma_m^{-2} (U^{\mathrm{T}} \xi)$$

In which

$$\boldsymbol{\Sigma_m^{-2}} = \begin{bmatrix} \sigma_1^{-2} & & & \\ & \sigma_2^{-2} & & \\ & & \ddots & \\ & & & \sigma_m^{-2} \end{bmatrix}$$

If we make the substitution $\omega = U^{\mathrm{T}} \xi$, then

$$\omega^{\mathrm{T}} \Sigma_m^{-2} \omega = \sum \frac{\omega_i^2}{\sigma_i^2} \leq 1$$

This is the equation for an axis-aligned ellipse in a new coordinate system that is obtained by rotation according to the orthogonal matrix U^{T}.

Velocity Kinematics

- **Manipulability:** Manipulability Measure

$$\sum \frac{\omega_i^2}{\sigma_i^2} \leq 1$$

The volume of the ellipsoid is given by

$$\text{volume} = K\sigma_1\sigma_2 \cdots \sigma_m$$

in which K is a constant that depends only on the dimension of the ellipsoid.

The **manipulability measure** (可操作性度量) is given by

$$\mu = \sigma_1\sigma_2 \cdots \sigma_m$$

Now, consider the special case when the robot is not redundant, that is, $J \in \mathbb{R}^{m\times m}$. Recall that the determinant of a product is equal to the product of the determinants, and that a matrix and its transpose have the same determinant. Thus, we have

$$\det JJ^T = \lambda_1^2\lambda_2^2 \cdots \lambda_m^2$$

In which $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m$ are the eigenvalues of J. This leads to

$$\mu = \sqrt{\det JJ^T} = |\lambda_1\lambda_2 \cdots \lambda_m| = |\det J|$$

Velocity Kinematics

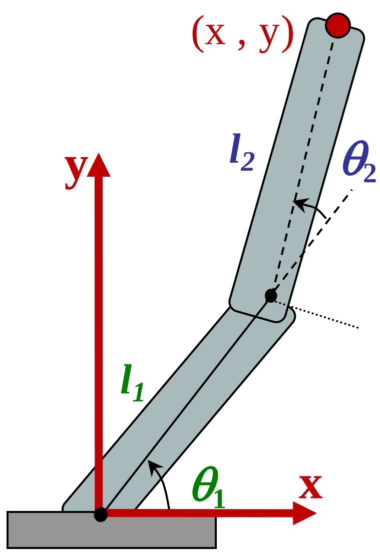
- **Manipulability:** Manipulability Measure

$$\mu = \sqrt{\det JJ^T} = |\lambda_1 \lambda_2 \cdots \lambda_m| = |\det J|$$

The manipulability μ has the following properties.

- In general, $\mu = 0$ holds if and only if rank(J) < m, that is, when J is not full rank.
- Suppose that there is some error in the measured velocity $\Delta\xi$. We can bound the corresponding error in the computed joint velocity $\Delta\dot{q}$ by

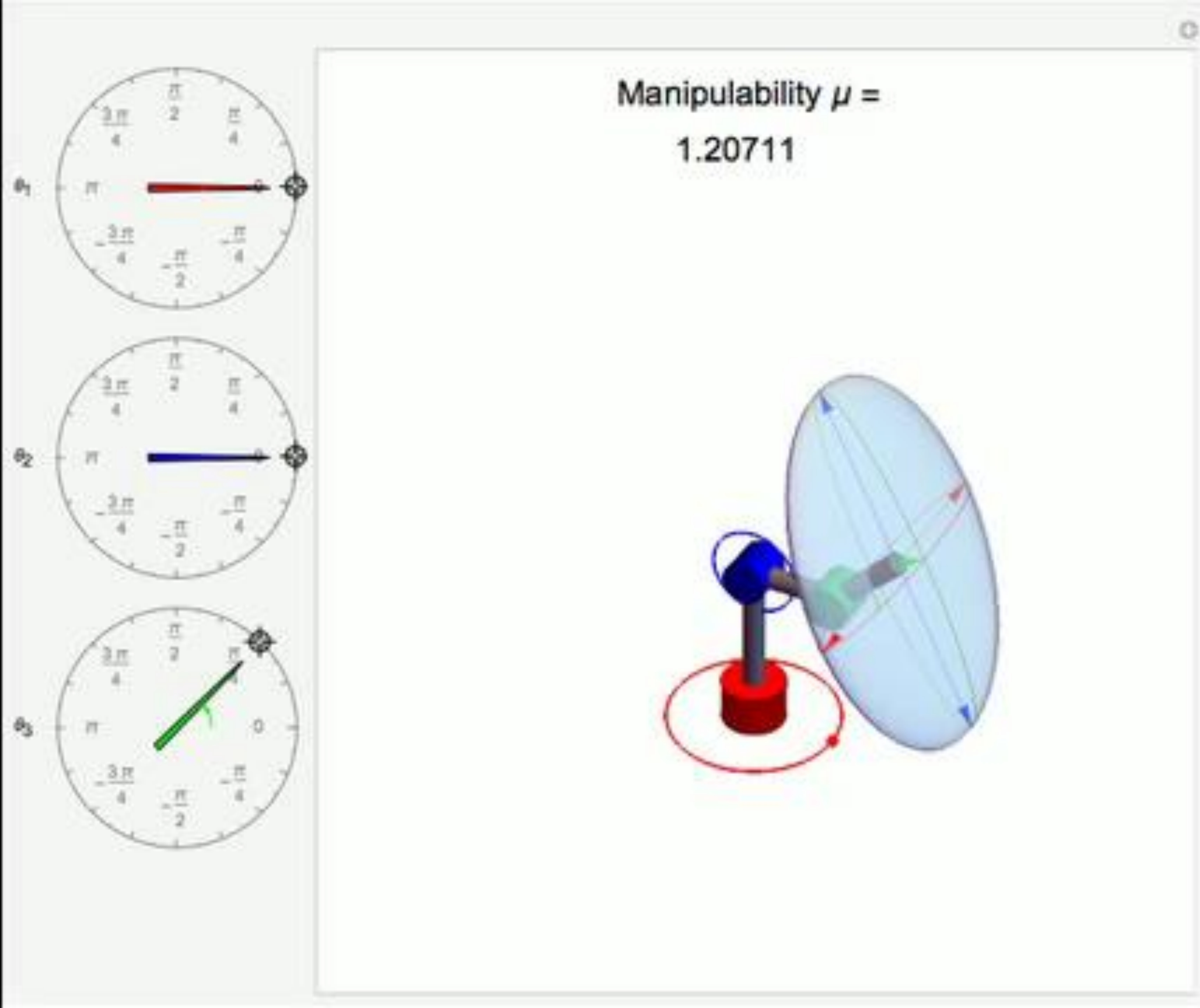
$$(\sigma_1)^{-1} \le \frac{\|\Delta\dot{q}\|}{\|\Delta\xi\|} \le (\sigma_m)^{-1}$$



Example (Two-link Planar Arm). Consider the two-link planar arm and the task of positioning in the plane. The Jacobian is given by

$$J = \begin{bmatrix} -a_1 s_1 - a_2 s_{12} & -a_2 s_{12} \\ a_1 c_1 + a_2 c_{12} & a_2 c_{12} \end{bmatrix}$$

$$\mu = |\det(J)| = l_1 l_2 \, |\sin\theta_2|$$

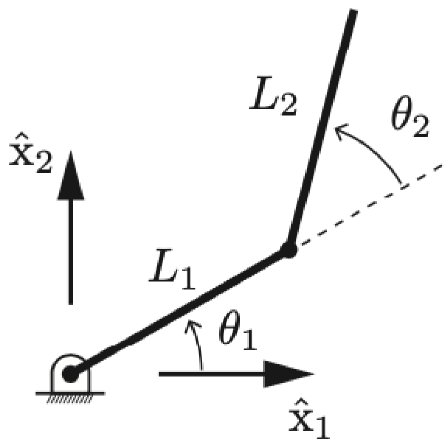


Manipulability μ =
1.20711
θ1
θ2
θ3

Velocity Kinematics

- **Manipulability:** Manipulability Measure

Example: Two-link Planar Arm. (manipulability ellipsoid)



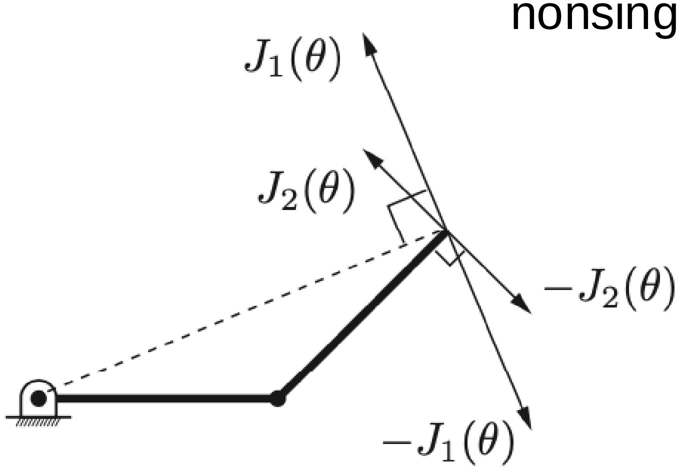
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -L_1 \sin\theta_1 - L_2 \sin(\theta_1 + \theta_2) & -L_2 \sin(\theta_1 + \theta_2) \\ L_1 \cos\theta_1 + L_2 \cos(\theta_1 + \theta_2) & L_2 \cos(\theta_1 + \theta_2) \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

Writing the two columns of $J(\theta)$ as $J_1(\theta)$ and $J_2(\theta)$, and the tip velocity $\dot{x}$ as v_{tip}, the above equation becomes

$$v_{\text{tip}} = J_1(\theta)\dot{\theta}_1 + J_2(\theta)\dot{\theta}_2$$

Now let's substitute $L_1 = L_2 = 1$ and consider the robot at the nonsingular postures: $\theta = (0, \pi/4)$. The Jacobians $J(\theta)$ is

$$J\left(\begin{bmatrix} 0 \\ \pi/4 \end{bmatrix}\right) = \begin{bmatrix} -0.71 & -0.71 \\ 1.71 & 0.71 \end{bmatrix}$$



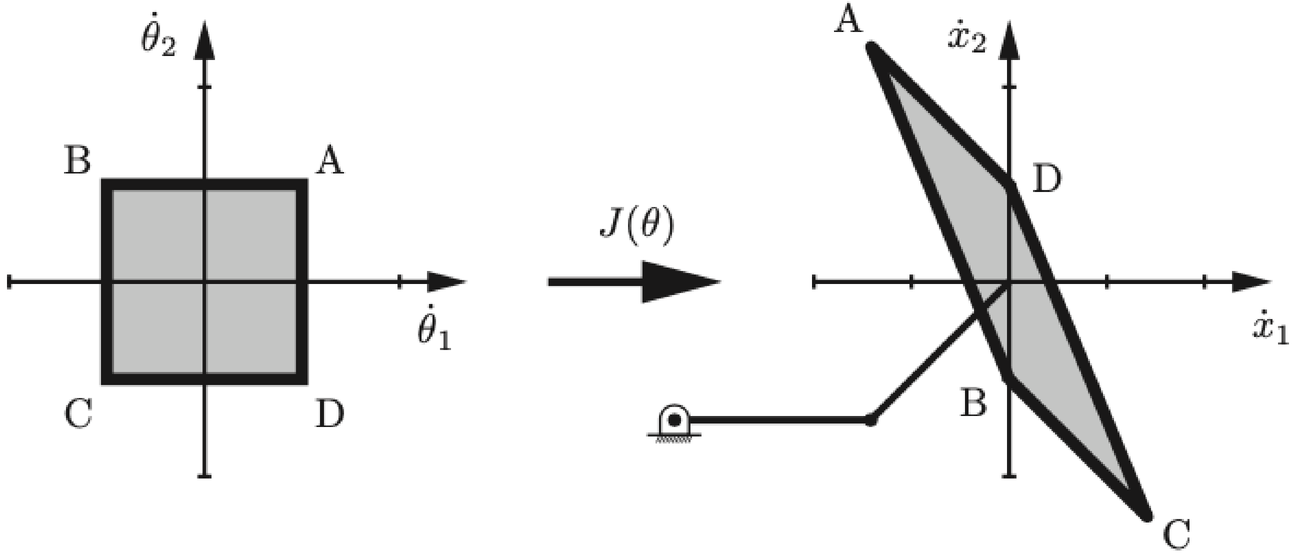
Columns 1 and 2 of the Jacobian correspond to the endpoint velocity when $\dot{\theta}_1 = 1$ (and $\dot{\theta}_2 = 0$) and when $\dot{\theta}_2 = 1$ (and $\dot{\theta}_1 = 0$), respectively.

Velocity Kinematics

- **Manipulability:** Manipulability Measure

Example: Two-link Planar Arm. (manipulability ellipsoid)

The Jacobian can be used to map bounds on the rotational speed of the joints to bounds on v_{tip}.



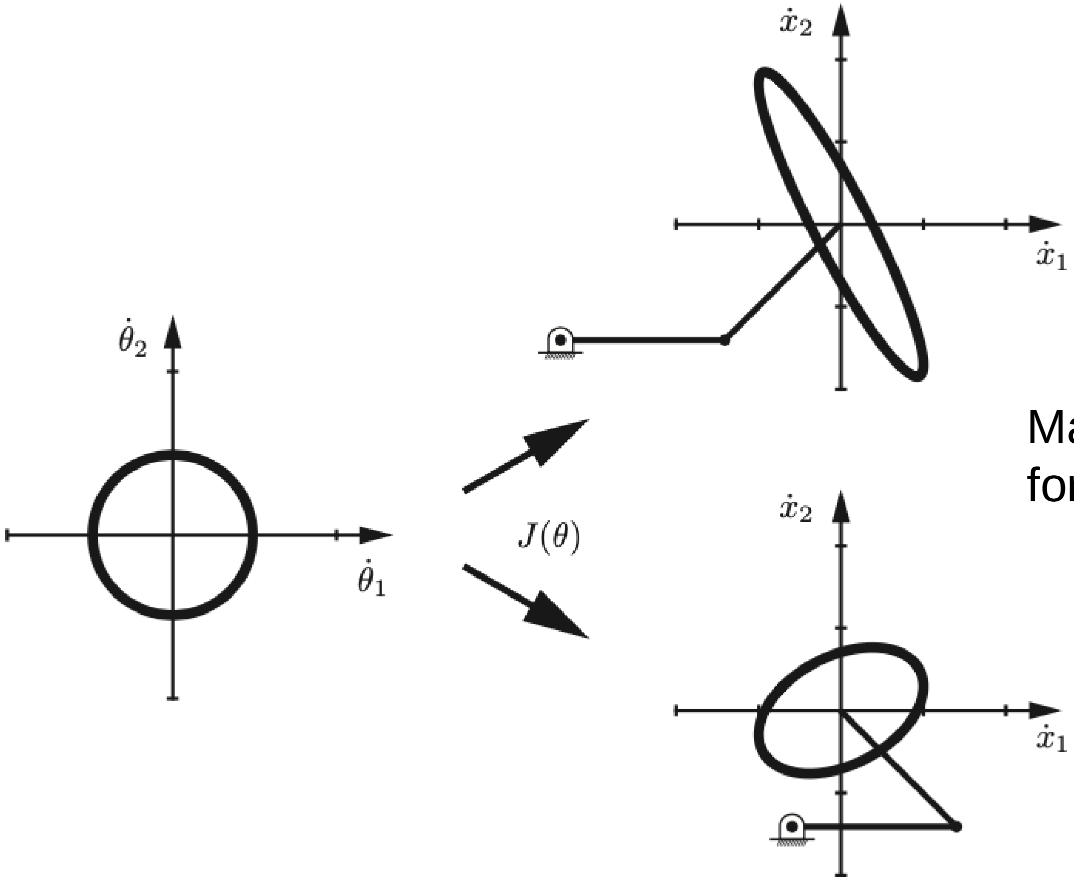
- Mapping the set of possible joint velocities, represented as a square in the $\dot{\theta}_1$–$\dot{\theta}_2$ space, through the Jacobian to find the parallelogram of possible end-effector velocities.
- The extreme points A, B, C, and D in the joint velocity space map to the extreme points A, B, C, and D in the end-effector velocity space.

Velocity Kinematics

- **Manipulability:** Manipulability Measure

Example: Two-link Planar Arm. (manipulability ellipsoid)

Rather than mapping a polygon of joint velocities through the Jacobian, we could instead map a unit circle of joint velocities in the $\dot{\theta}_1$–$\dot{\theta}_2$ plane.



Manipulability ellipsoids for two different postures

Velocity Kinematics

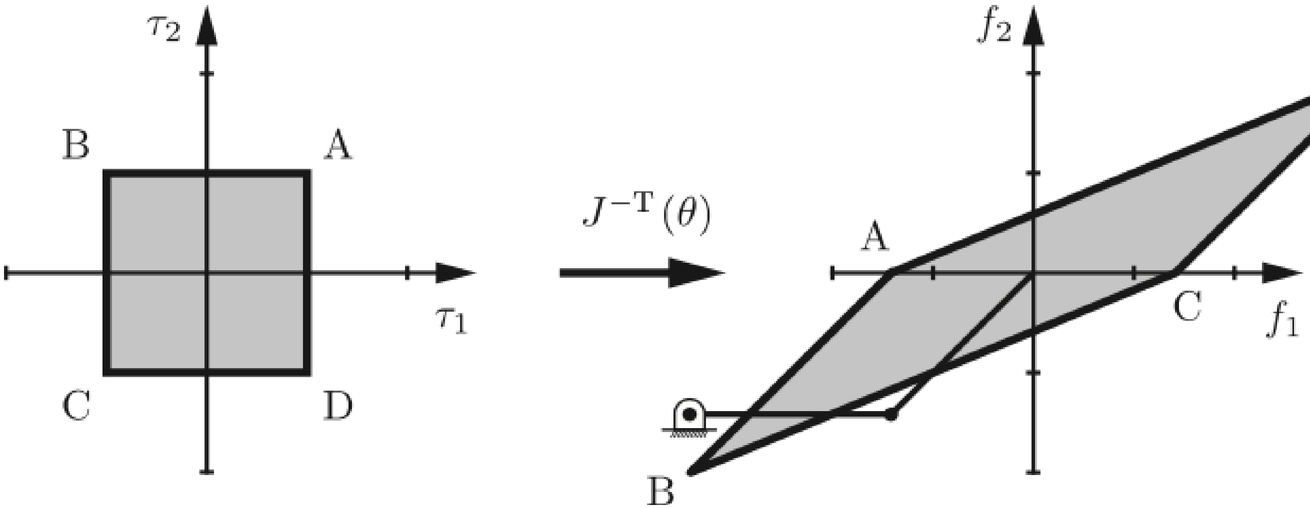
- **Manipulability:** Manipulability Measure

Example: Two-link Planar Arm. (force ellipsoid)

The Jacobian also plays a central role in static analysis. Suppose that an external force is applied to the robot tip. What are the joint torques required to resist this external force?

$$\tau = J^T(\theta) f_{\text{tip}}$$

$$f_{\text{tip}} = J^{-\text{T}}(\theta)\tau$$

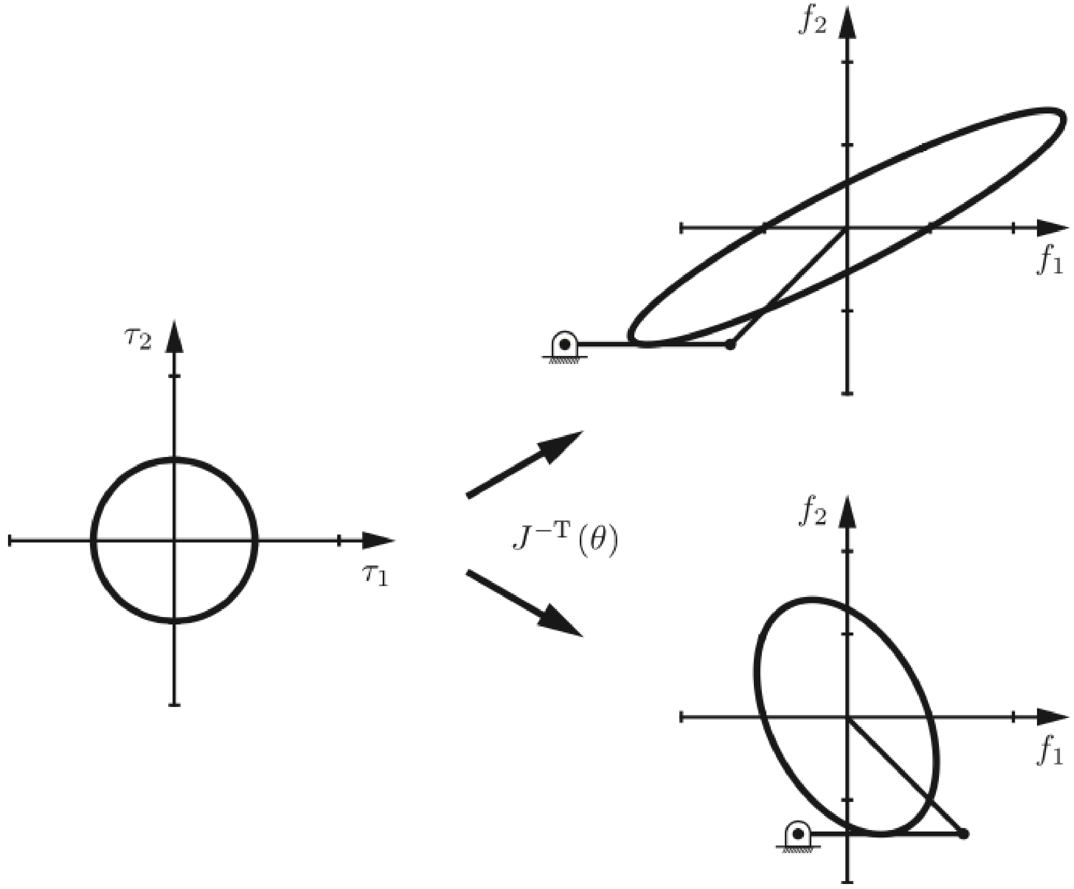


Mapping joint torque bounds to tip force bounds

Velocity Kinematics

- **Manipulability:** Manipulability Measure

Example: Two-link Planar Arm. (force ellipsoid)

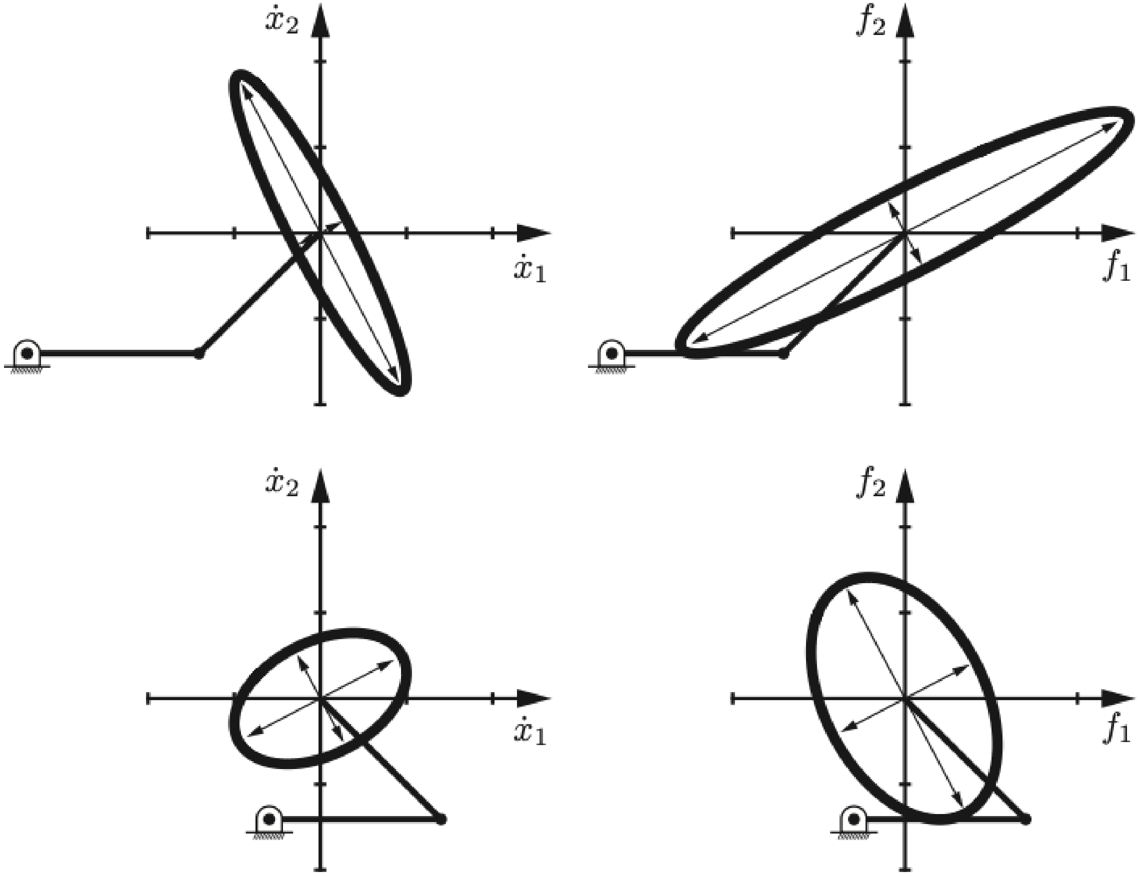


Force ellipsoids for two different postures

Velocity Kinematics

- **Manipulability:** Manipulability Measure

manipulability ellipsoid v.s. force ellipsoid



If it is easy to generate a tip velocity in a given direction then it is difficult to generate a force in that same direction, and vice versa.

Velocity Kinematics

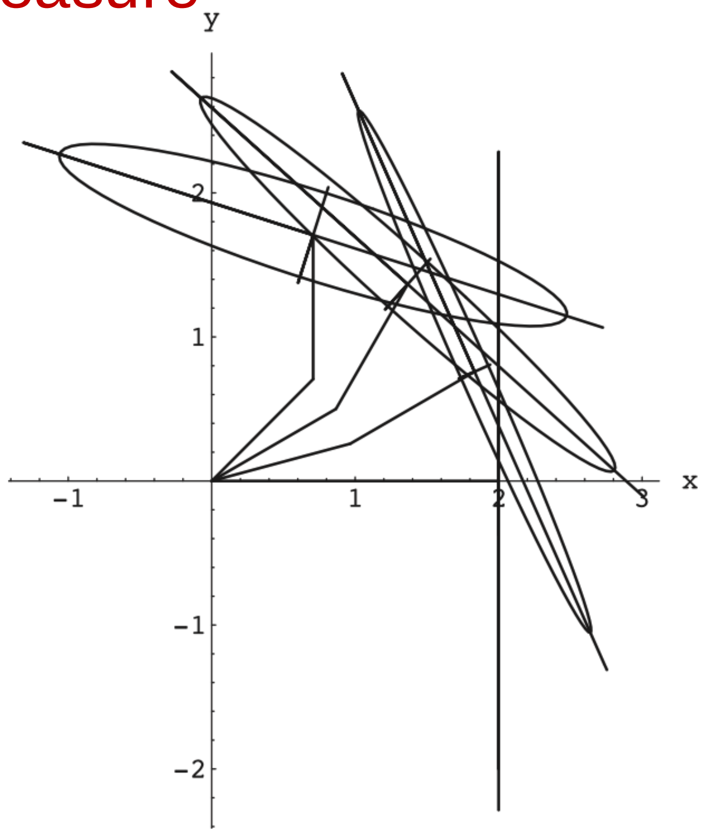
- **Manipulability:** Manipulability Measure

$$\mu = |\det(J)| = l_1 l_2 \,|\sin\theta_2|$$

We can use manipulability to determine the optimal configurations in which to perform certain tasks. It is desirable to perform a task in the configuration for which the end effector has the maximum manipulability.

For the two-link arm the maximum manipulability is obtained for $\theta_2 = \pm\pi/2$.

Manipulability can also be used to aid in the design of manipulators. For example, suppose that we wish to design a two-link planar arm whose total link length $l_1 + l_2$ is fixed.



Manipulability ellipsoids

We need only find l_1 and l_2 to maximize the product $l_1 l_2$. This is achieved when $l_1 = l_2$. Thus, to maximize manipulability, the link lengths should be chosen to be equal.

Velocity Kinematics

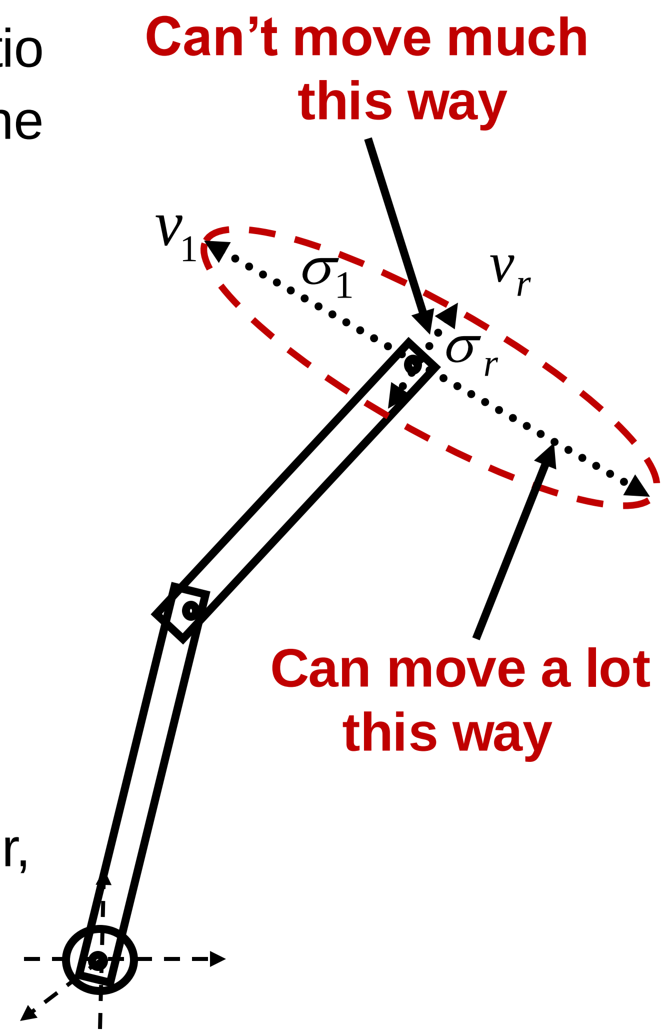
- **Manipulability:** Condition Number & Dexterity Index

The **condition number** is defined as the ratio between the largest singular value and the minimum singular value of the Jacobian matrix J :

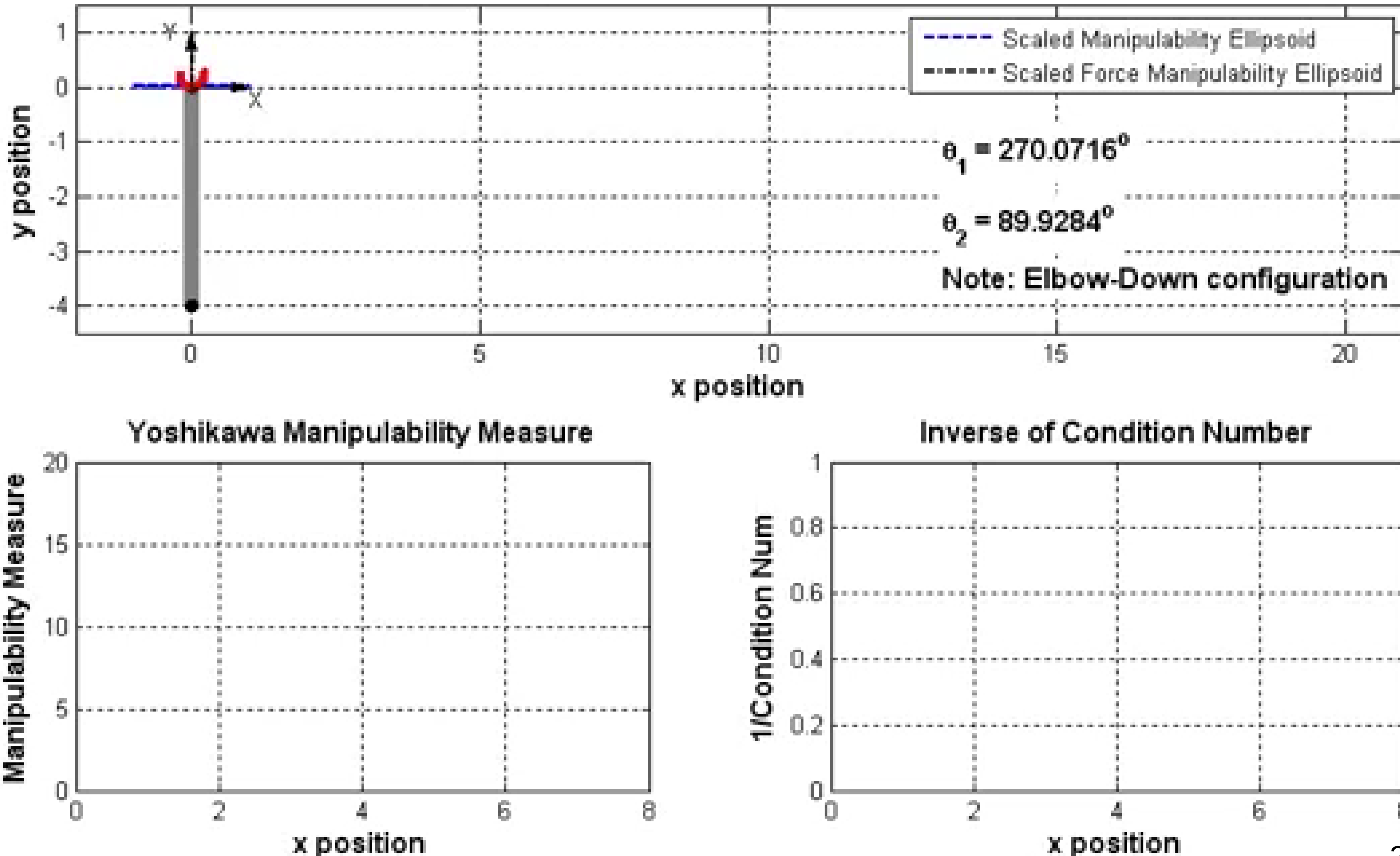
$$\kappa \equiv \frac{\sigma_{\max}}{\sigma_{\min}}$$

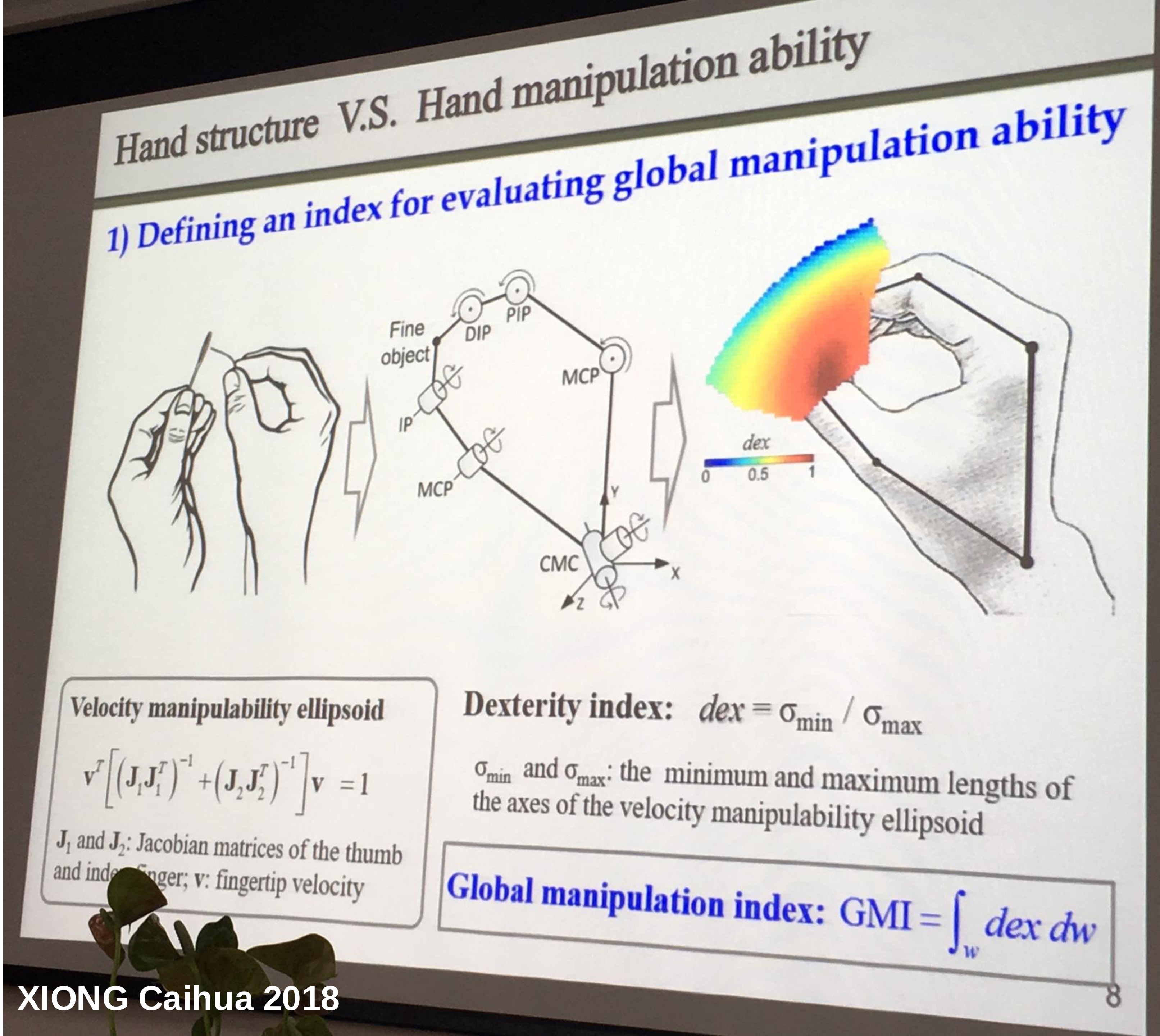
- $1 \leq \kappa \leq \infty$
- When κ is low (i.e., close to 1) then the manipulability ellipsoid is nearly spherical or isotropic, meaning that it is equally easy to move in any direction. This situation is generally desirable.
- As the robot approaches a singularity, however, κ goes to infinity.

The **dexterity index** is defined as $dex \equiv \frac{\sigma_{\min}}{\sigma_{\max}}$.



Velocity Kinematics





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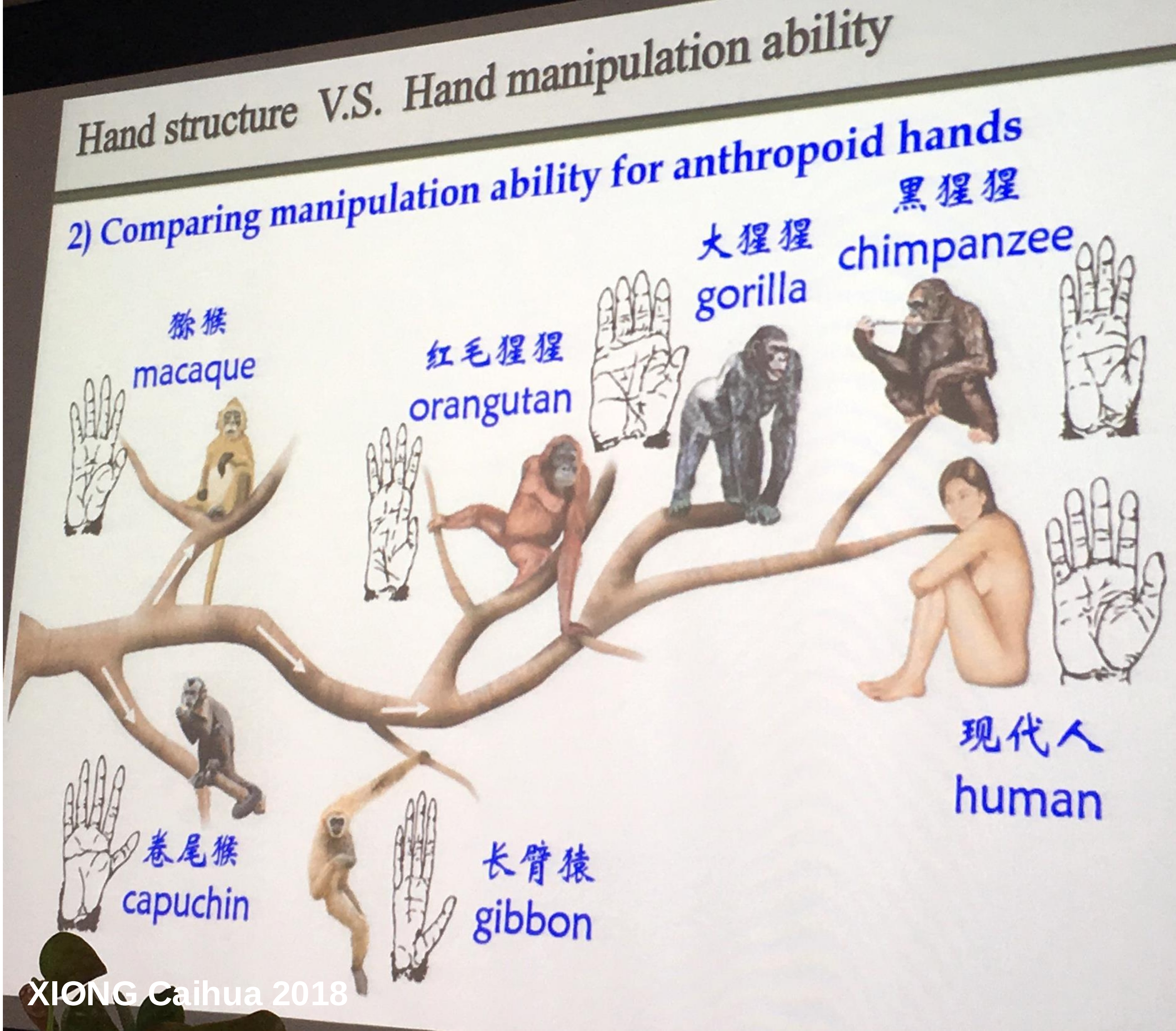

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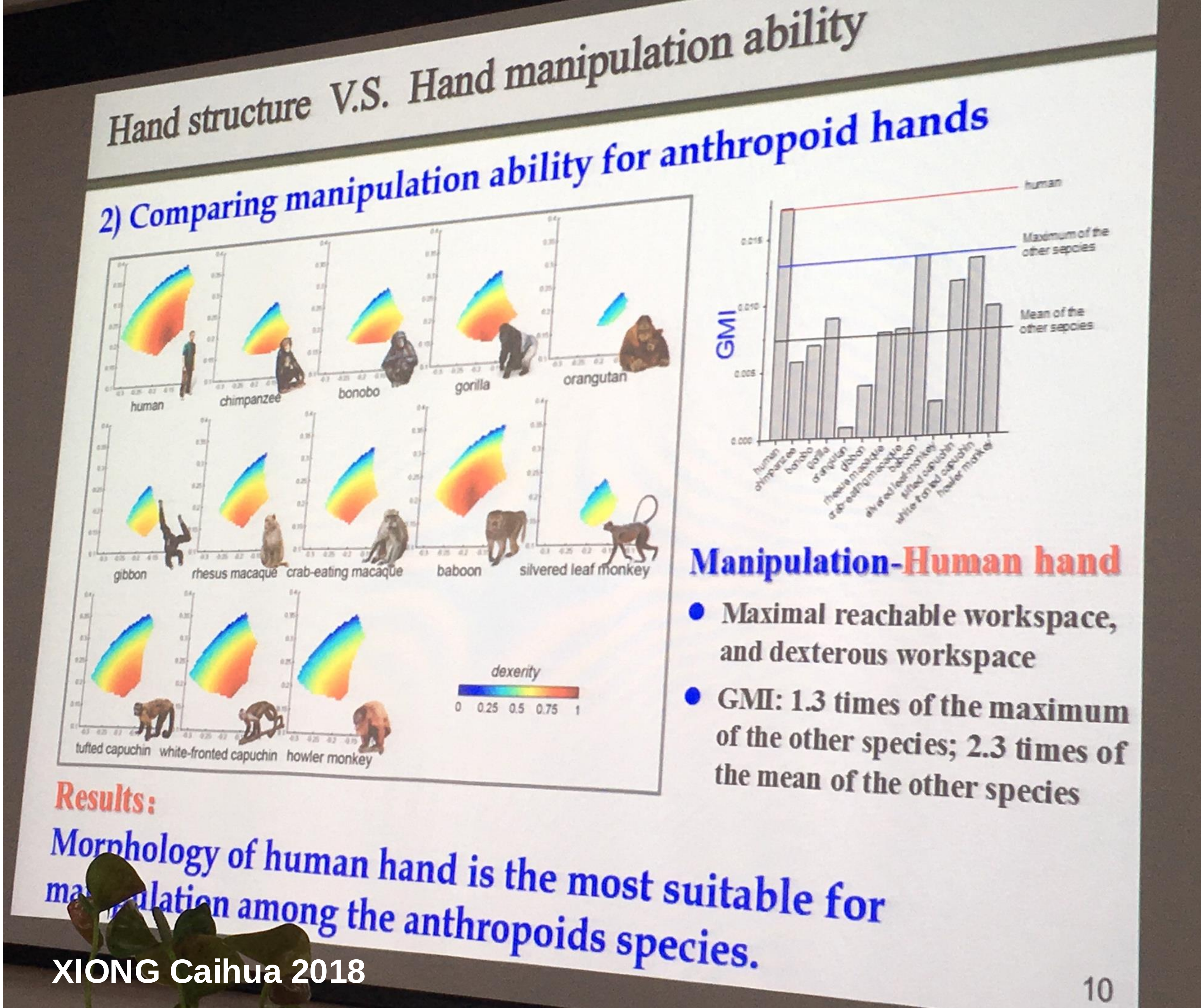

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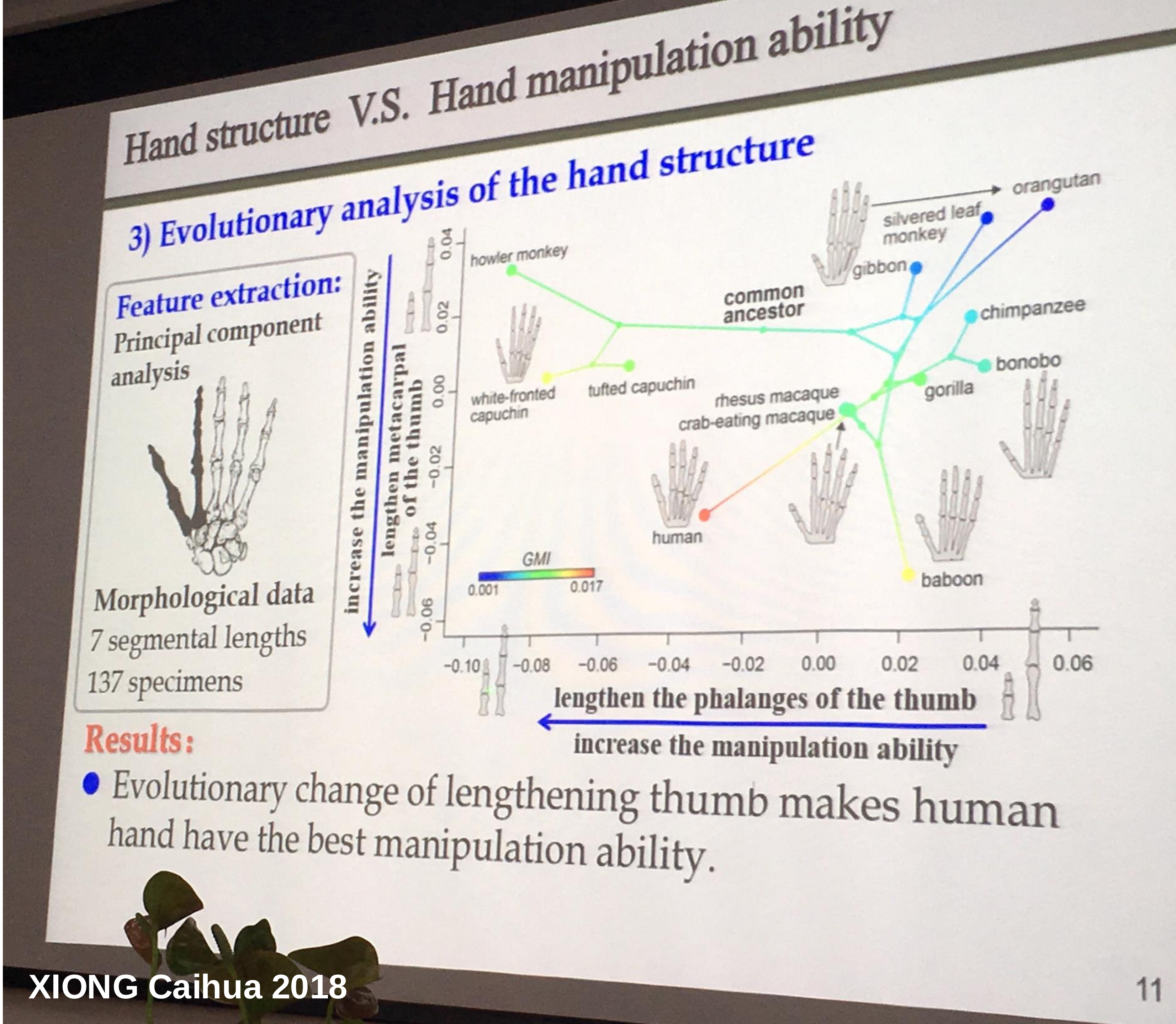

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Velocity Kinematics

- **Numerical Solution to Inverse Kinematics**

Let $x^d \in \mathbb{R}^m$ be a vector of Cartesian coordinates. For example, x^d could represent the wrist center point ($m = 3$) or the end-effector position and orientation ($m = 6$). The forward kinematics for an n-link manipulator, in this case, is a function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$. If we set

$$G(q) = x^d - f(q)$$

then a solution to the inverse kinematics is a configuration q^d satisfying $G(q^d) = x^d - f(q^d) = 0$.

Below we will give details of the most common algorithms to iteratively solve for q^d given x^d.

1) The first based on the **Jacobian inverse**, which is similar to the Newton–Raphson method for root finding.
2) The second is based on the **Jacobian transpose** and is derived as a gradient search algorithm.

Velocity Kinematics

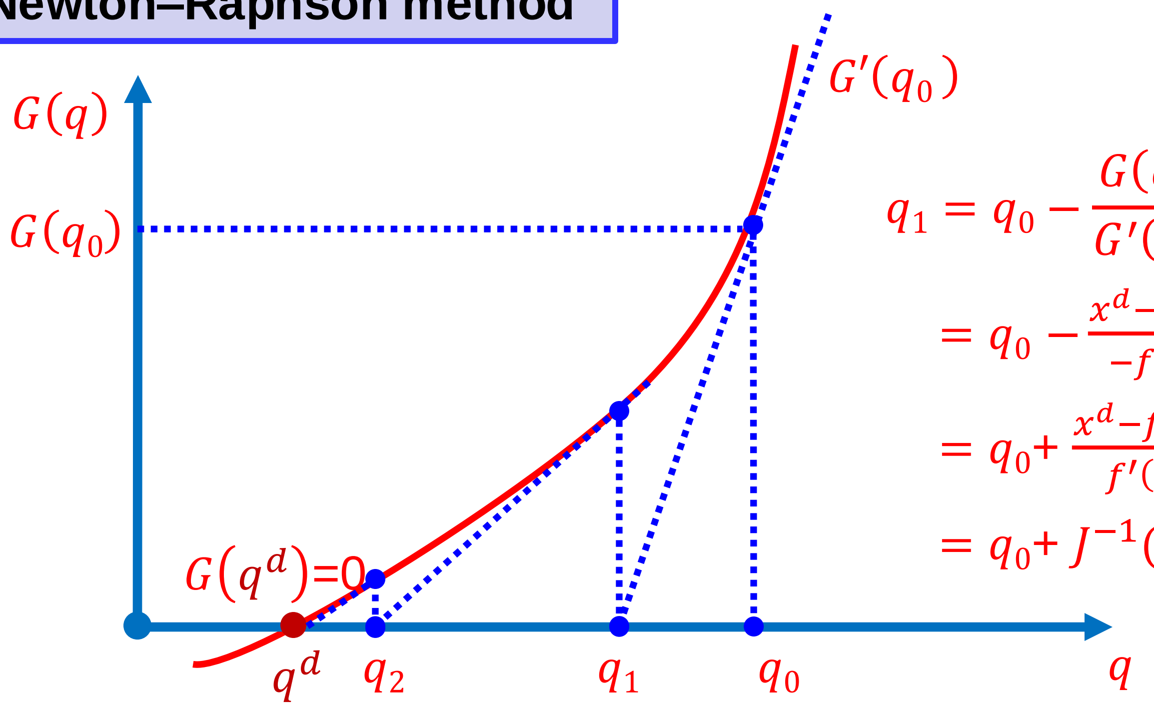
- **Numerical Solution to Inverse Kinematics**

1) Jacobian Inverse Method

The forward kinematics is a function $f : \mathbb{R}^n \to \mathbb{R}^m$.

If we set $G(q) = x^d - f(q)$, then a solution to the inverse kinematics is a configuration q^d satisfying $G\left(q^d\right) = x^d - f\left(q^d\right) = 0$.

Newton–Raphson method



$$
\begin{aligned}
q_1 &= q_0 - \frac{G(q_0)}{G'(q_0)} \\
&= q_0 - \frac{x^d - f(q_0)}{-f'(q_0)} \\
&= q_0 + \frac{x^d - f(q_0)}{f'(q_0)} \\
&= q_0 + J^{-1}(q_0)\left(x^d - f(q_0)\right)
\end{aligned}
$$

Velocity Kinematics

- **Numerical Solution to Inverse Kinematics**

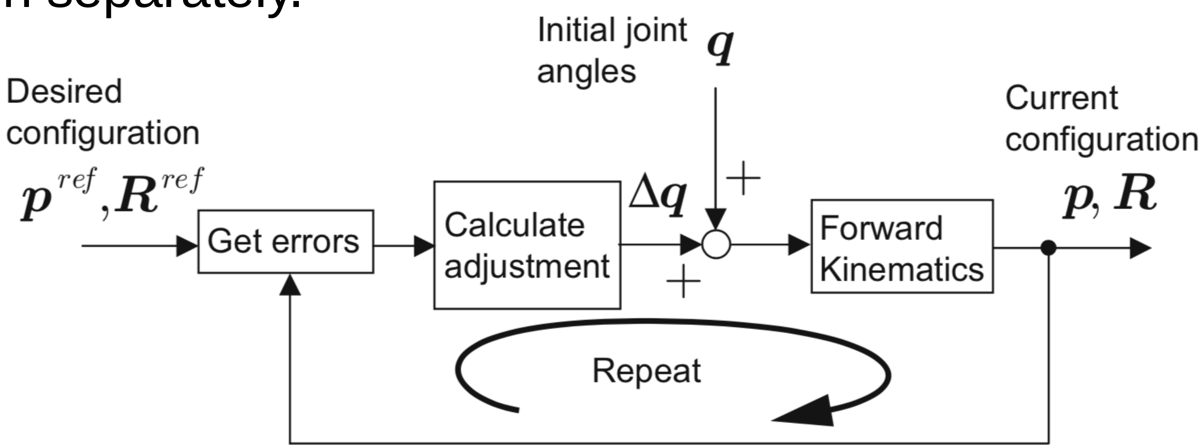
1) Jacobian Inverse Method

$$q_1 = q_0 + J^{-1}(q_0)(x^d - f(q_0))$$

To find a solution for q^d, we begin with an initial guess, q_0, and form a sequence of successive estimates, $q_0, q_1, q_2, \ldots$, as

$$q_k = q_{k-1} + \alpha_k J^{-1}(q_{k-1})\left(x^d - f(q_{k-1})\right), \qquad k = 1,2,\cdots$$

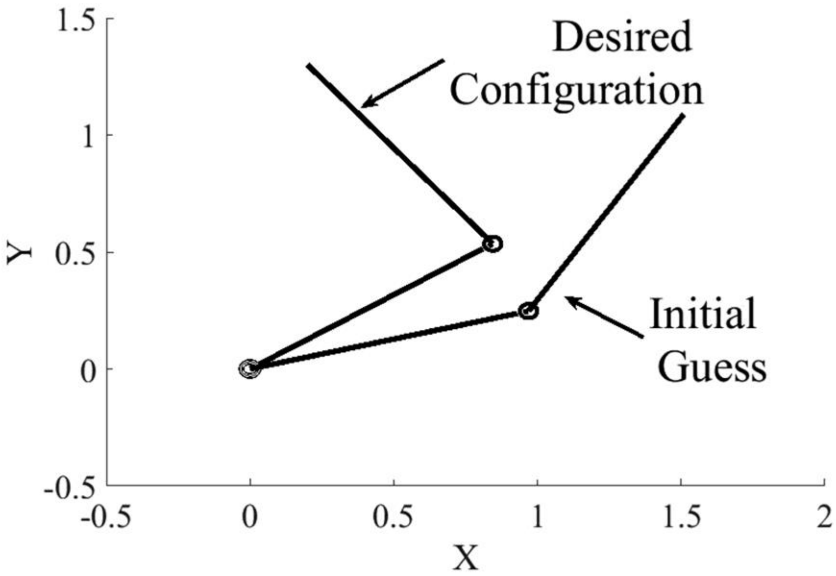
Note that a step size, $\alpha_k > 0$, which can be adjusted to aid convergence. The step size α_k may be chosen as a constant or as a function of k, a scalar or as a diagonal matrix, the last in order to scale each component of the configuration separately.



Velocity Kinematics

- **Numerical Solution to Inverse Kinematics**

1) Jacobian Inverse Method: Example



Iteration	θ_1	θ_2
1	-0.33284	2.6711
2	0.80552	2.1025
3	0.46906	1.9316
4	0.53554	1.7697
5	0.55729	1.7227
6	0.56308	1.7104
7	0.56455	1.7073
8	0.56492	1.7065
9	0.56501	1.7063
10	0.56503	1.7062

Desired end-effector coordinates are $x^d = (0.2, 1.3)$. The joint variables corresponding to x^d are $\theta_1 = 0.5650$, $\theta_2 = 1.7062$. The initial guess is $\theta_1 = 0.25$, $\theta_2 = 0.75$. The step size α was chosen as 0.75.

The algorithm converges to within 10^{-4} of the exact solution after 10 iterations with the given parameters.

Velocity Kinematics

- **Numerical Solution to Inverse Kinematics**

1) Jacobian Inverse Method

If the Jacobian is not square or not invertible, then one may use the pseudoinverse J^+ in place of J^{-1}. For $m \leq n$, we defined the right pseudoinverse as $J^+ = J^T(JJ^T)^{-1}$.

In this case, we can define the update rule for q_k as

$$q_k = q_{k-1} + \alpha_k J^+(q_{k-1})\left(x^d - f(q_{k-1})\right), \qquad k = 1,2,\cdots$$

Remark:

1) Only local convergence can be expected.
2) Since there are generally multiple solutions to the inverse kinematics, the particular configuration that results from running the algorithm is dependent on the initial guess.

Velocity Kinematics

- **Numerical Solution to Inverse Kinematics**

2) Jacobian Transpose Method

To find a solution q^d satisfying $f(q^d) = x^d$, we define an optimization problem

$$\min_q F(q) = \min_q \frac{1}{2}(f(q) - x^d)^T(f(q) - x^d)$$

The gradient of the above cost function $F(q)$ is given by

$$\nabla F(q) = J^T(q)(f(q) - x^d)$$

A gradient descent algorithm to minimize $F(q)$ is then

$$q_k = q_{k-1} + \alpha_k \nabla F(q_{k-1}) = q_{k-1} + \alpha_k J^T(q_{k-1})\left(x^d - f(q_{k-1})\right)$$

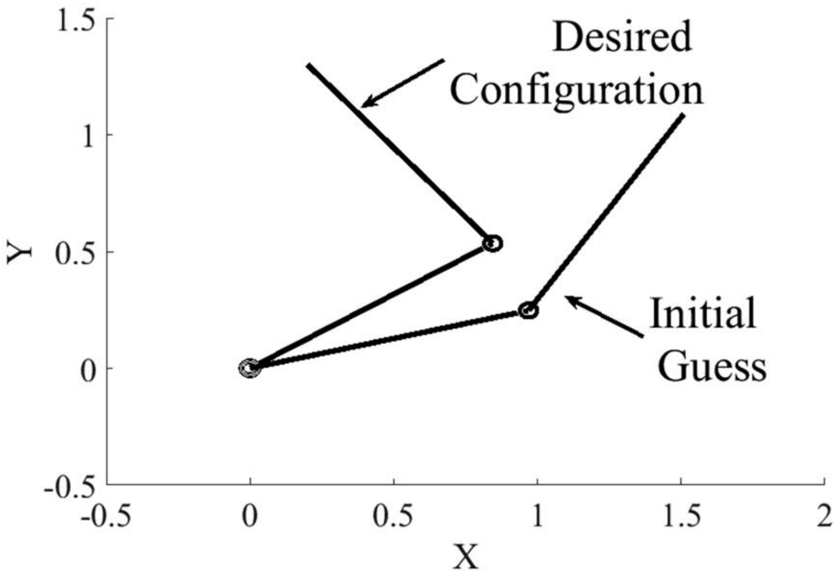
where, again, $\alpha_k > 0$ is the step size.

Remark: Jacobian transpose is easier to compute than the Jacobian inverse and does not suffer from configuration singularities. However, the convergence, in terms of number of iterations, may be slower with this method.

Velocity Kinematics

- **Numerical Solution to Inverse Kinematics**

2) Jacobian Transpose Method: Example



Iteration	θ_1	θ_2
1	1.8362	1.3412
2	0.4667	1.1025
3	1.1215	1.6233
4	0.45264	1.415
5	0.83519	1.7273
26	0.56522	1.7063
27	0.56492	1.7061
28	0.56514	1.7063
29	0.56498	1.7062
30	0.5650	1.7062

Desired end-effector coordinates are $x^d = (0.2, 1.3)$. The joint variables corresponding to x^d are $\theta_1 = 0.5650$, $\theta_2 = 1.7062$. The initial guess is $\theta_1 = 0.25$, $\theta_2 = 0.75$. The step size α was chosen as 0.75.

The algorithm converges to within 10^{-4} of the exact solution after 30 iterations with the given parameters.

Summary

- Static Force/Torque Relationships
- Inverse Velocity and Acceleration
- Manipulability
  - Manipulability Ellipsoid
  - Condition Number & Dexterity Index
- Numerical Solution to Inverse Kinematics
  - Jacobian Inverse Method
  - Jacobian Transpose Method

Homework 10

1. Suppose that $\dot{q}$ ***is a solution to*** $\xi = J\dot{q}$ ***for m < n.*** **(PPT Slide 12)**

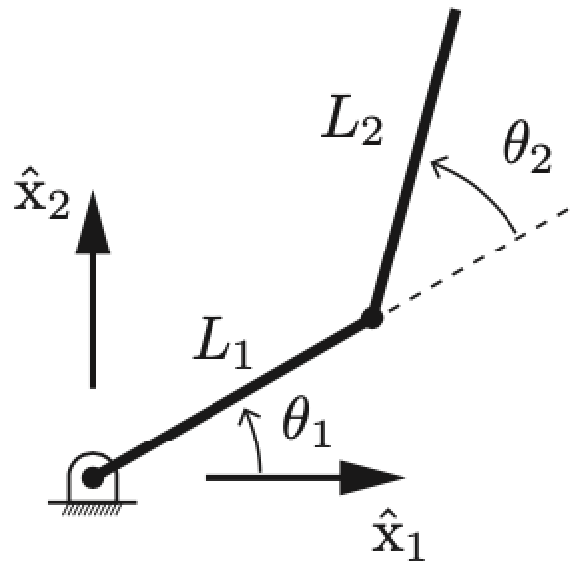
1) Show that $\dot{q} + (I - J^{+}J)b$ ***is also a solution to*** $\xi = J\dot{q}$ ***for any*** $b \in \mathbb{R}^n$***.***

2) Show that $b = 0$ ***gives the solution that minimizes the resulting joint velocities.***

2. Verify Equation $\| \dot{q} \|^2 = \dot{q}^T \dot{q} = (J^{+}\xi)^T J^{+}\xi = \xi^T (JJ^T)^{-1}\xi$ **(PPT Slide 14)**

3. Consider the two-link planar arm of the following figure and the task of positioning in the plane. $L_1 = L_2 = 1$. ***Plot manipulability ellipsoids for three different configurations:***

- 【$\theta_1 = \pi/6$, $\theta_2 = \pi/6$】
- 【$\theta_1 = \pi/4$, $\theta_2 = \pi/4$】
- 【$\theta_1 = \pi/2$, $\theta_2 = \pi/2$】



Homework 10

Homework 10 is posted at http://sakai.sustech.edu.cn

Due date: April 7, 2025

Next class: March 2, 2025

Dynamics I

作业要求（Requirements）：

1. **文件格式为以自己姓名学号作业序号命名的pdf文件；**

   （File name: YourSID_ YourName_10.pdf）

2. **作业里也写上自己的姓名和学号。**

   （Write your name and SID in the homework）