Robot Modeling & Control ME331

Section 3： Kinematics II

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Outline

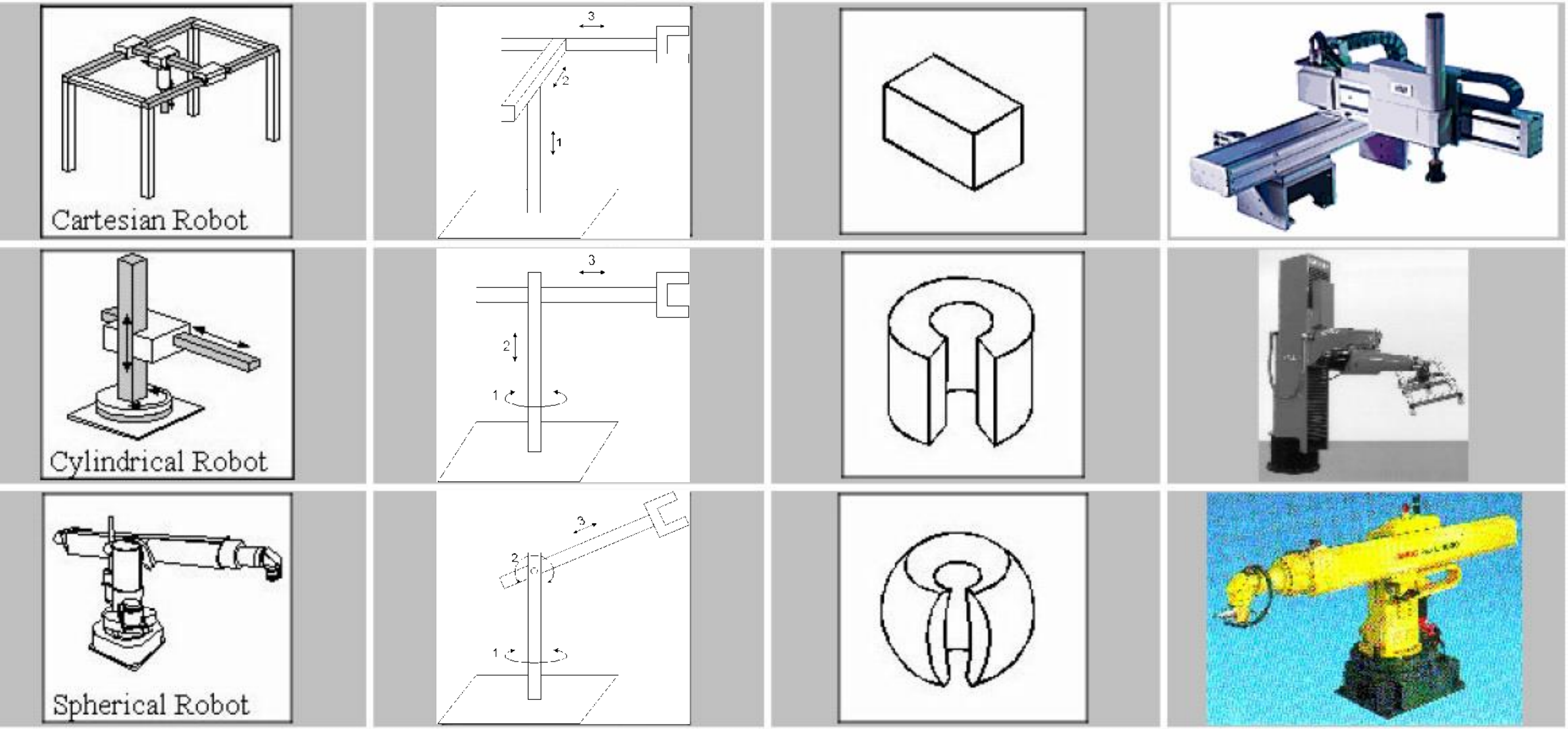
- Review
  - Robot Geometry
  - Robot Specification
    - Number of Axes, DOF, Working space, Payload
    - Precision, Repeatability
  - Rotation Matrix
    - Basic Rotation Matrix
    - Composite Rotation Matrix
- Kinematics
  - Homogeneous Matrix
    - Composite Homogeneous Transformation Matrix
    - Geometric Interpretation
  - Orientation Representation
    - Euler Angle I & II
    - Yaw-Pitch-Roll (YPR)

Review

- **Manipulator（Robot arms, Industrial robot）**
  – A chain of rigid bodies (links) connected by joints (revolute or prismatic)
- **Manipulator Specification**
  – DOF, Redundant Robot
  – Workspace, Payload
  – Accuracy — How accurately a specified point can be reached
  – Repeatability — How accurately the same position can be reached if the motion is repeated many times

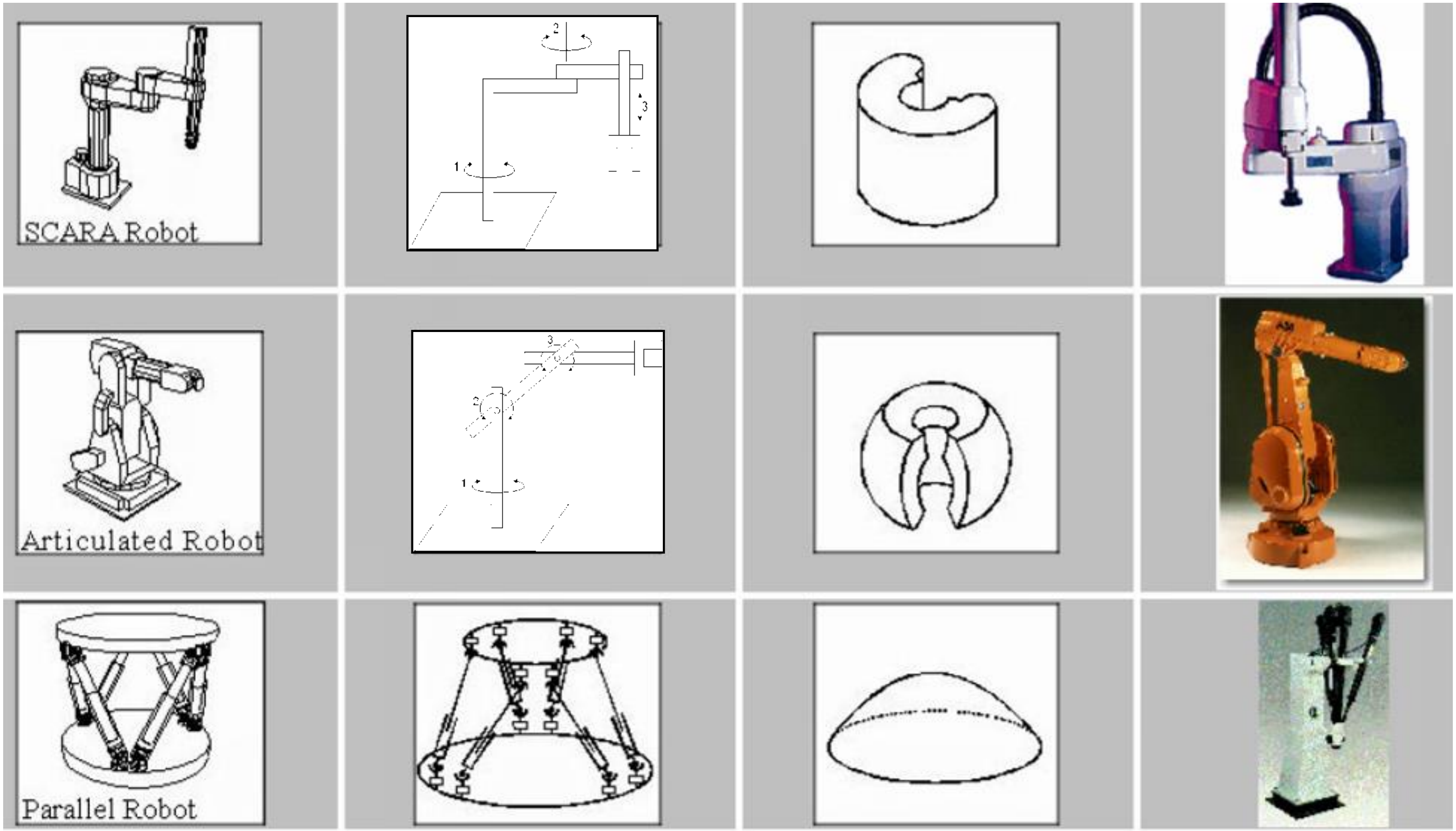
Review

- **Manipulators：**



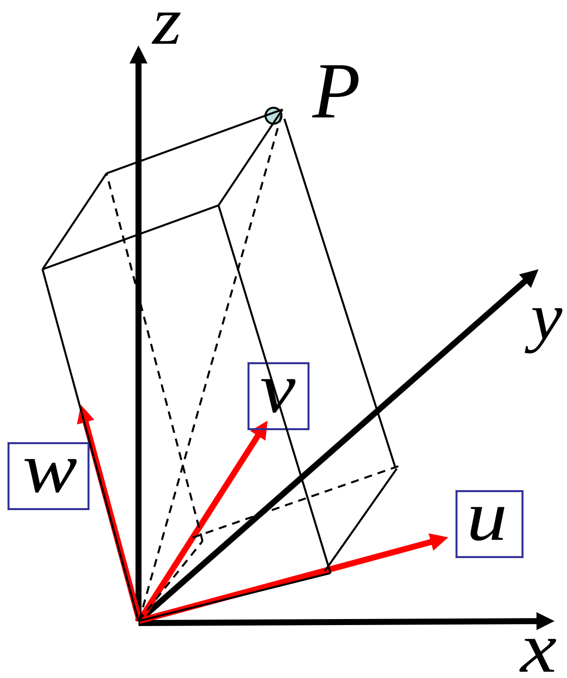
Review

- **Manipulators**:



Review

- **Basic Rotation**
  - p_x, p_y, and p_z represented the projections of P onto OX, OY, OZ axes, respectively.
  - Since $P = p_u \mathrm{i_u} + p_v \mathrm{j_v} + p_w \mathrm{k_w}$
  - Then



$$p_x = \mathrm{i_x} \cdot P = \mathrm{i_x} \cdot \mathrm{i_u} p_u + \mathrm{i_x} \cdot \mathrm{j_v} p_v + \mathrm{i_x} \cdot \mathrm{k_w} p_w$$

$$p_y = \mathrm{j_y} \cdot P = \mathrm{j_y} \cdot \mathrm{i_u} p_u + \mathrm{j_y} \cdot \mathrm{j_v} p_v + \mathrm{j_y} \cdot \mathrm{k_w} p_w$$

$$p_z = \mathrm{k_z} \cdot P = \mathrm{k_z} \cdot \mathrm{i_u} p_u + \mathrm{k_z} \cdot \mathrm{j_v} p_v + \mathrm{k_z} \cdot \mathrm{k_w} p_w$$

Review

- **Basic Rotation Matrices**

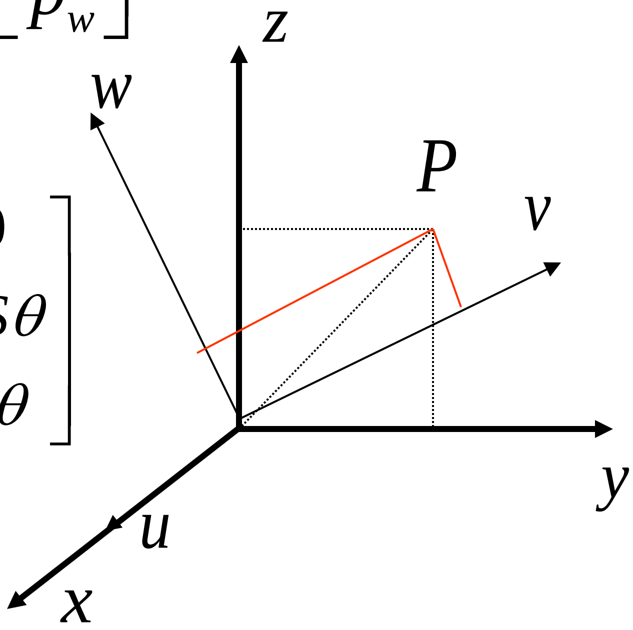
$$P_{xyz} = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = \begin{bmatrix} \mathrm{i_x \cdot i_u} & \mathrm{i_x \cdot j_v} & \mathrm{i_x \cdot k_w} \\ \mathrm{j_y \cdot i_u} & \mathrm{j_y \cdot j_v} & \mathrm{j_y \cdot k_w} \\ \mathrm{k_z \cdot i_u} & \mathrm{k_z \cdot j_v} & \mathrm{k_z \cdot k_w} \end{bmatrix} \begin{bmatrix} p_u \\ p_v \\ p_w \end{bmatrix} = RP_{uvw}$$

$$P_{xyz} = RP_{uvw}$$

$$P_{uvw} = QP_{xyz}$$

$$Q = R^{-1} = R^T$$

$$Rot(x,\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\theta & -S\theta \\ 0 & S\theta & C\theta \end{bmatrix}$$



Review

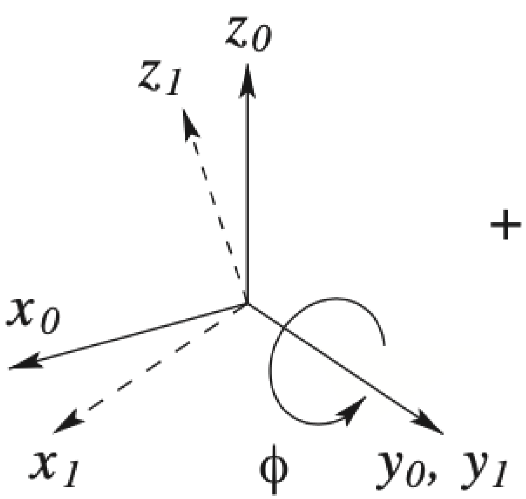
- **Basic Rotation Matrices**
  - **Rotation about *x*-axis with θ** ⇒ $$Rot(x,\theta)=\begin{bmatrix}1 & 0 & 0\\ 0 & C\theta & -S\theta\\ 0 & S\theta & C\theta\end{bmatrix}$$
  - **Rotation about *y*-axis with θ** ⇒ $$Rot(y,\theta)=\begin{bmatrix}C\theta & 0 & S\theta\\ 0 & 1 & 0\\ -S\theta & 0 & C\theta\end{bmatrix}$$
  - **Rotation about *z*-axis with θ** ⇒ $$Rot(z,\theta)=\begin{bmatrix}C\theta & -S\theta & 0\\ S\theta & C\theta & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$P_{xyz}=RP_{uvw}$$

Review: Composition of Rotations

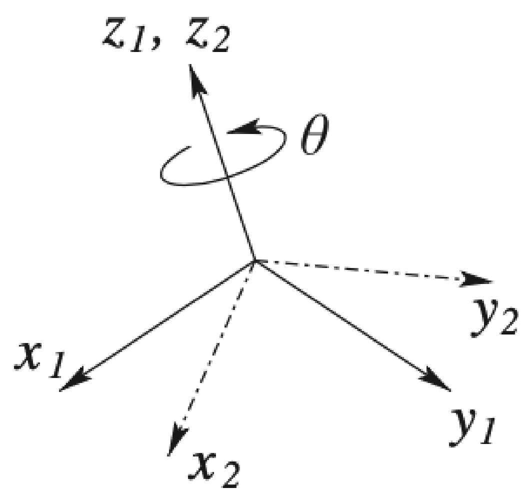
- Matrix multiplications do not commute
  - Rotation with Respect to the Current Frame:

    if rotating coordinate OUVW is rotating about its current principal axes, then ***post-multiply*** the previous (resultant) rotation matrix with an appropriate basic rotation matrix



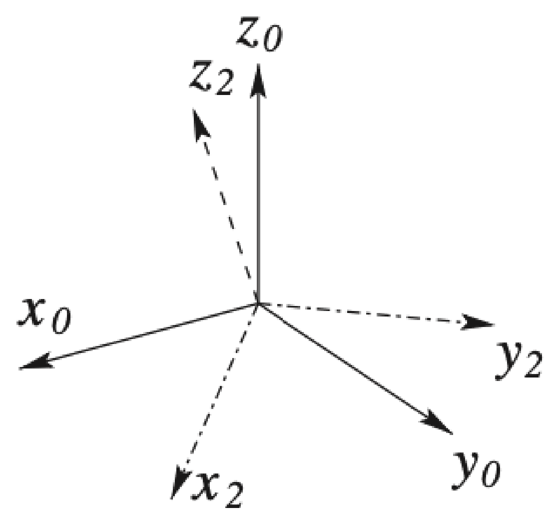
$\boldsymbol{R_{y,\phi}}$

+



$\boldsymbol{R_{z,\theta}}$

=

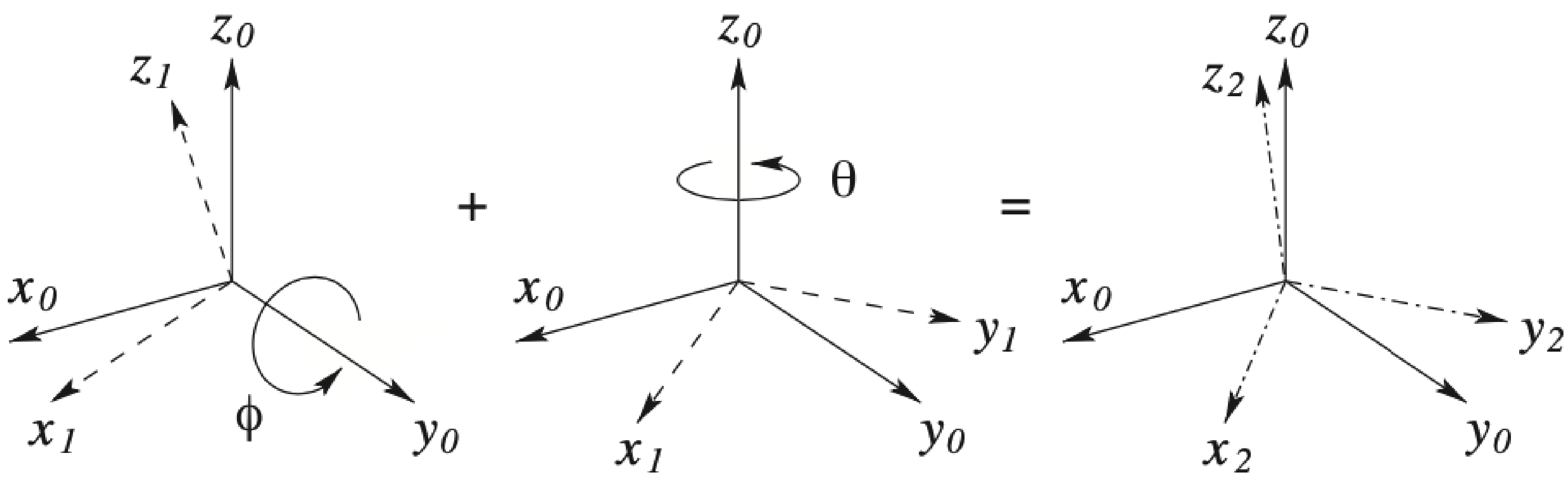


$\boldsymbol{R = R_{y,\phi}\, R_{z,\theta}}$

Review: Composition of Rotations

– Rotation with Respect to the Fixed Frame:

- if rotating coordinate O-U-V-W is rotating about principal axis of OXYZ frame, then ***Pre-multiply*** the previous (resultant) rotation matrix with an appropriate basic rotation matrix



$R_{y,\phi}$ $\qquad$ $R_{z,\theta}$ $\qquad$ $R = R_{z,\theta} R_{y,\phi}$

Example 4

- **Find the rotation matrix for the following operations:**

Rotation ϕ about OY axis

Rotation θ about OW axis

Rotation α about OU axis

Answer...

$$R = Rot(y,\phi) I_3 Rot(w,\theta) Rot(u,\alpha)$$

$$= \begin{bmatrix} C\phi & 0 & S\phi \\ 0 & 1 & 0 \\ -S\phi & 0 & C\phi \end{bmatrix} \begin{bmatrix} C\theta & -S\theta & 0 \\ S\theta & C\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\alpha & -S\alpha \\ 0 & S\alpha & C\alpha \end{bmatrix}$$

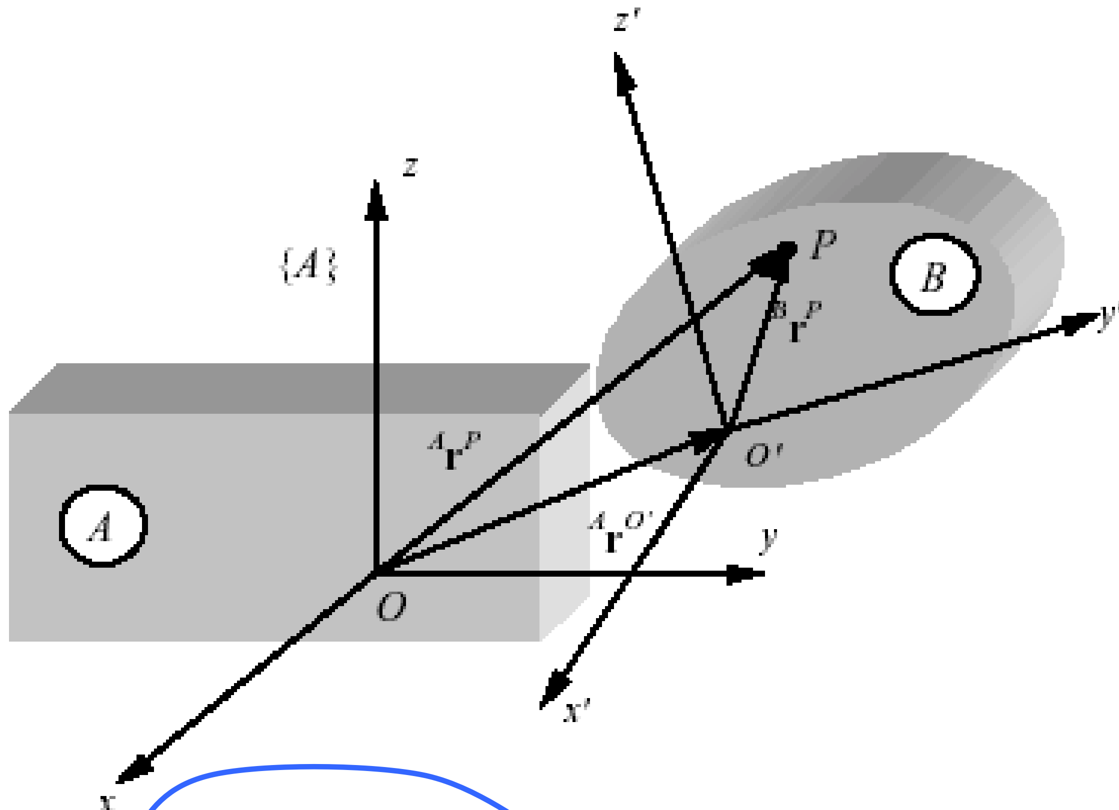
$$= \begin{bmatrix} C\phi C\theta & S\phi S\alpha - C\phi S\theta C\alpha & C\phi S\theta S\alpha + S\phi C\alpha \\ S\theta & C\theta C\alpha & -C\theta S\alpha \\ -S\phi C\theta & S\phi S\theta C\alpha + C\phi S\alpha & C\phi C\alpha - S\phi S\theta S\alpha \end{bmatrix}$$

Pre-multiply if rotate about the OXYZ axes
Post-multiply if rotate about the OUVW axes

Coordinate Transformations

• Position vector of *P* in {*B*} is transformed to position vector of *P* in {*A*}

• Description of {*B*} as seen from an observer in {*A*}



Rotation of {*B*} with respect to {*A*}

Translation of the origin of {*B*} with respect to origin of {*A*}

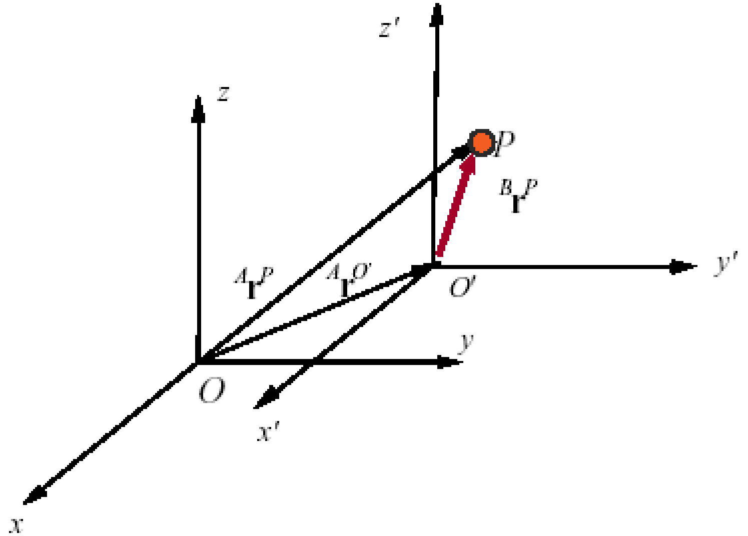
Coordinate Transformations

- **Two Special Cases**

$$^{A}r^{P} = {}^{A}R_{B}\,{}^{B}r^{P} + {}^{A}r^{o'}$$

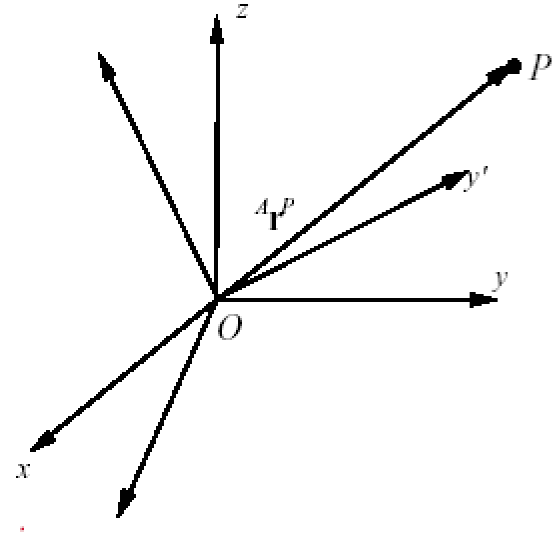
1. **Translation only**
   - **Axes of {*B*} and {*A*} are parallel**

$$^{A}R_{B} = I$$



2. **Rotation only**
   - **Origins of {*B*} and {*A*} are coincident**

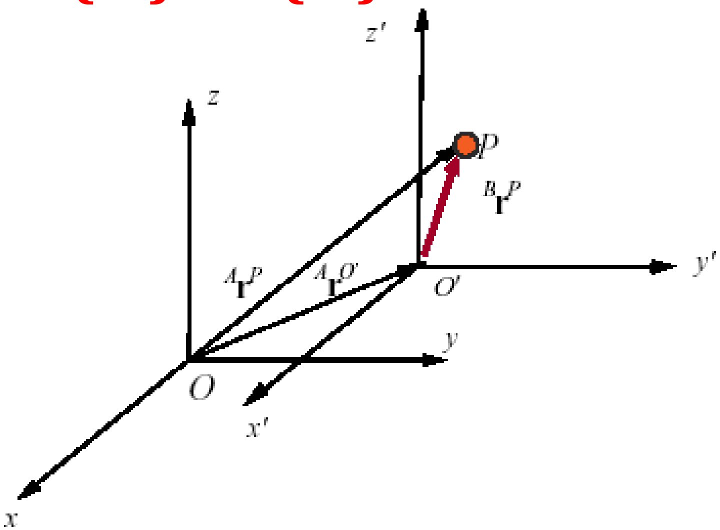
$$^{A}r^{o'} = 0$$



Homogeneous Representation

- **Coordinate transformation from {*B*} to {*A*}**

$$ {}^{A}r^{P} = {}^{A}R_{B}\,{}^{B}r^{P} + {}^{A}r^{o'} $$



$$ \begin{bmatrix} {}^{A}r^{P} \\ 1 \end{bmatrix} = \begin{bmatrix} {}^{A}R_{B} & {}^{A}r^{o'} \\ 0_{1\times 3} & 1 \end{bmatrix} \begin{bmatrix} {}^{B}r^{P} \\ 1 \end{bmatrix} $$

- **Homogeneous transformation matrix**

$$ {}^{A}T_{B} = \begin{bmatrix} {}^{A}R_{B} & {}^{A}r^{o'} \\ 0_{1\times 3} & 1 \end{bmatrix} = \begin{bmatrix} \boxed{R_{3\times 3}} & \boxed{P_{3\times 1}} \\ 0 & \boxed{1} \end{bmatrix} $$

Rotation matrix

Position vector

Scaling

Homogeneous Representation

- **Special cases**

1. Translation

$$^{A}T_{B} = \begin{bmatrix} I_{3\times 3} & {}^{A}r^{o'} \\ 0_{1\times 3} & 1 \end{bmatrix}$$

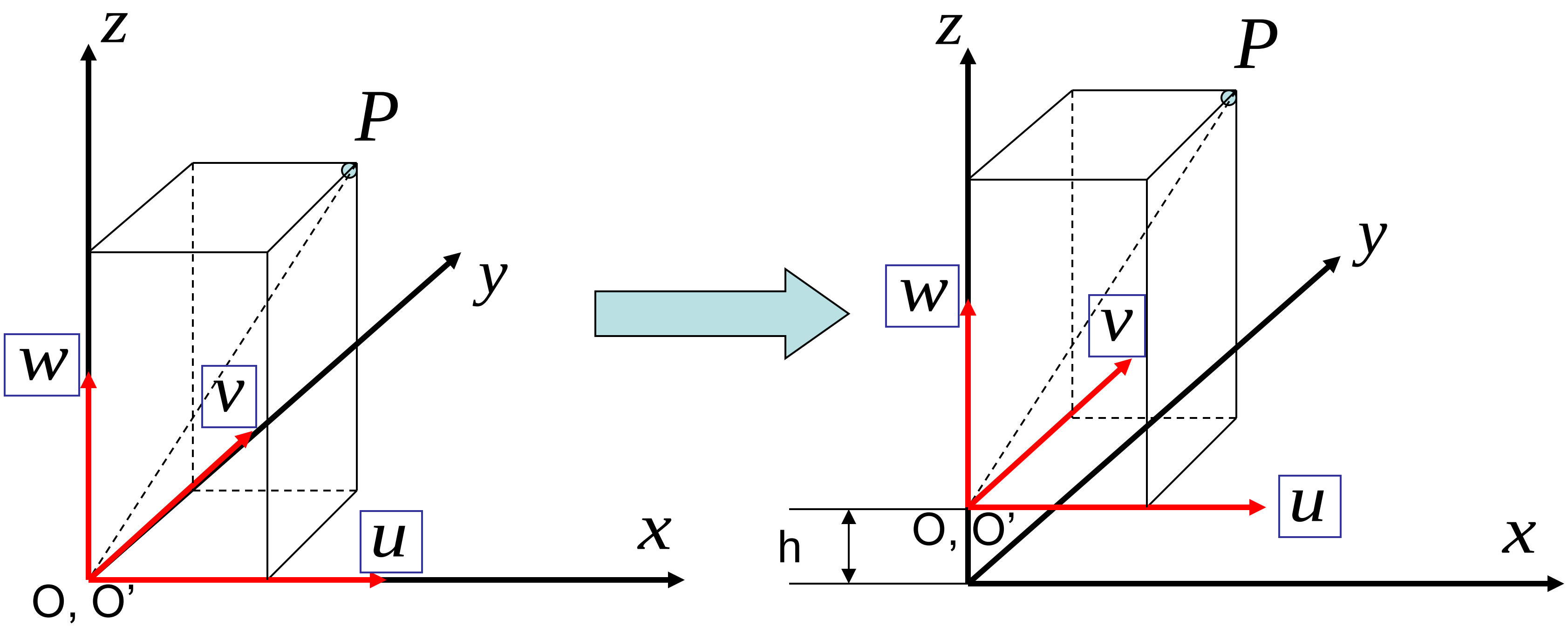
2. Rotation

$$^{A}T_{B} = \begin{bmatrix} {}^{A}R_{B} & 0_{3\times 1} \\ 0_{1\times 3} & 1 \end{bmatrix}$$

Example 5

- **Translation along Z-axis with h:**

$$Trans(z,h) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & h \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & h \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_u \\ p_v \\ p_w \\ 1 \end{bmatrix} = \begin{bmatrix} p_u \\ p_v \\ p_w + h \\ 1 \end{bmatrix}$$

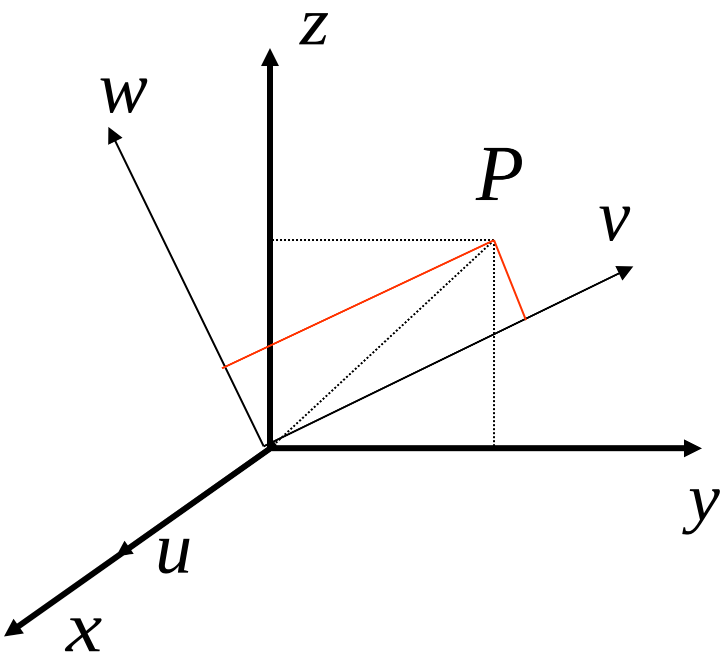


Example 6

- **Rotation about the X-axis by**

$$Rot(x,\theta)=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & C\theta & -S\theta & 0\\ 0 & S\theta & C\theta & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$\begin{bmatrix}x\\ y\\ z\\ 1\end{bmatrix}=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & C\theta & -S\theta & 0\\ 0 & S\theta & C\theta & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}p_u\\ p_v\\ p_w\\ 1\end{bmatrix}$$



Homogeneous Transformation

- **Composite Homogeneous Transformation Matrix**
- **Rules:**
  - Transformation (rotation/translation) w.r.t (X,Y,Z) (OLD FRAME), using pre-multiplication
  - Transformation (rotation/translation) w.r.t (U,V,W) (NEW FRAME), using post-multiplication

Example 7

- **Find the homogeneous transformation matrix (T) for the following operations:**

Rotation α about OX axis

Translation of a along OX axis

Translation of d along OZ axis

Rotation of θ about OZ axis

$$Answer: T = T_{z,\theta} T_{z,d} T_{x,a} T_{x,\alpha}$$

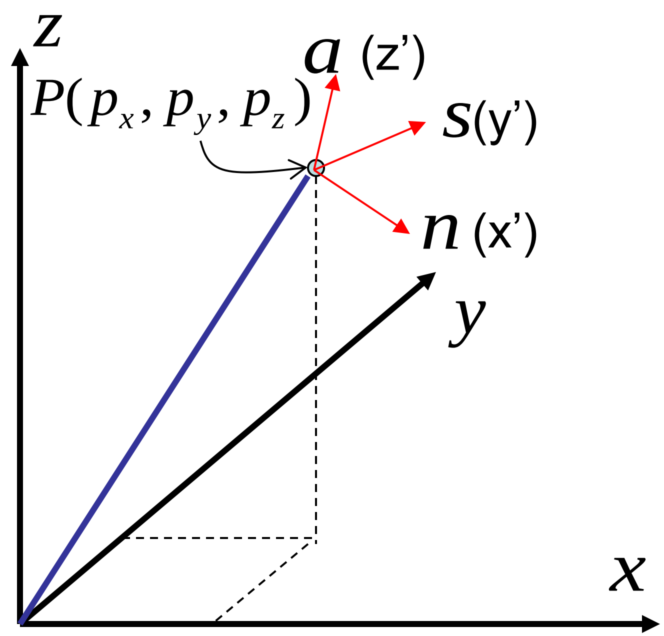
$$= \begin{bmatrix} C\theta & -S\theta & 0 & 0 \\ S\theta & C\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & C\alpha & -S\alpha & 0 \\ 0 & S\alpha & C\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Homogeneous Representation

- **A frame in space (Geometric Interpretation)**

$$F = \begin{bmatrix} R_{3\times3} & P_{3\times1} \\ 0 & 1 \end{bmatrix}$$

$$F = \begin{bmatrix} \boxed{\begin{matrix} n_x \\ n_y \\ n_z \end{matrix}} & \begin{matrix} s_x \\ s_y \\ s_z \end{matrix} & \begin{matrix} a_x \\ a_y \\ a_z \end{matrix} & \begin{matrix} p_x \\ p_y \\ p_z \end{matrix} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Principal axis ***n*** w.r.t. the reference coordinate system

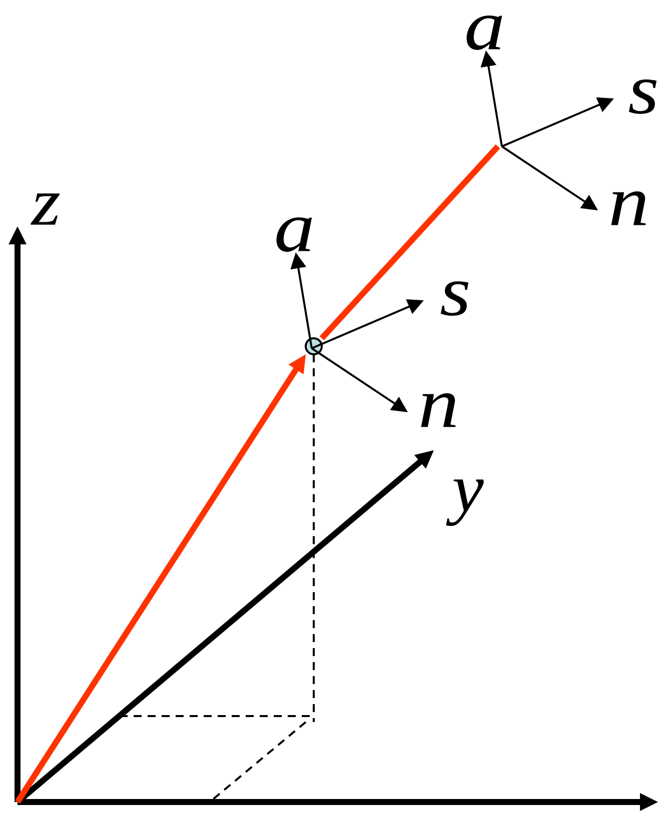
$$P_{xyz}= \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = \begin{bmatrix} \mathrm{i_x \cdot i_u} & \mathrm{i_x \cdot j_v} & \mathrm{i_x \cdot k_w} \\ \mathrm{j_y \cdot i_u} & \mathrm{j_y \cdot j_v} & \mathrm{j_y \cdot k_w} \\ \mathrm{k_z \cdot i_u} & \mathrm{k_z \cdot j_v} & \mathrm{k_z \cdot k_w} \end{bmatrix} \begin{bmatrix} p_u \\ p_v \\ p_w \end{bmatrix} = RP_{uvw}$$

Homogeneous Transformation

- **Translation**

$$F_{new} = \begin{bmatrix} 1 & 0 & 0 & d_x \\ 0 & 1 & 0 & d_y \\ 0 & 0 & 1 & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} n_x & s_x & a_x & p_x \\ n_y & s_y & a_y & p_y \\ n_z & s_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

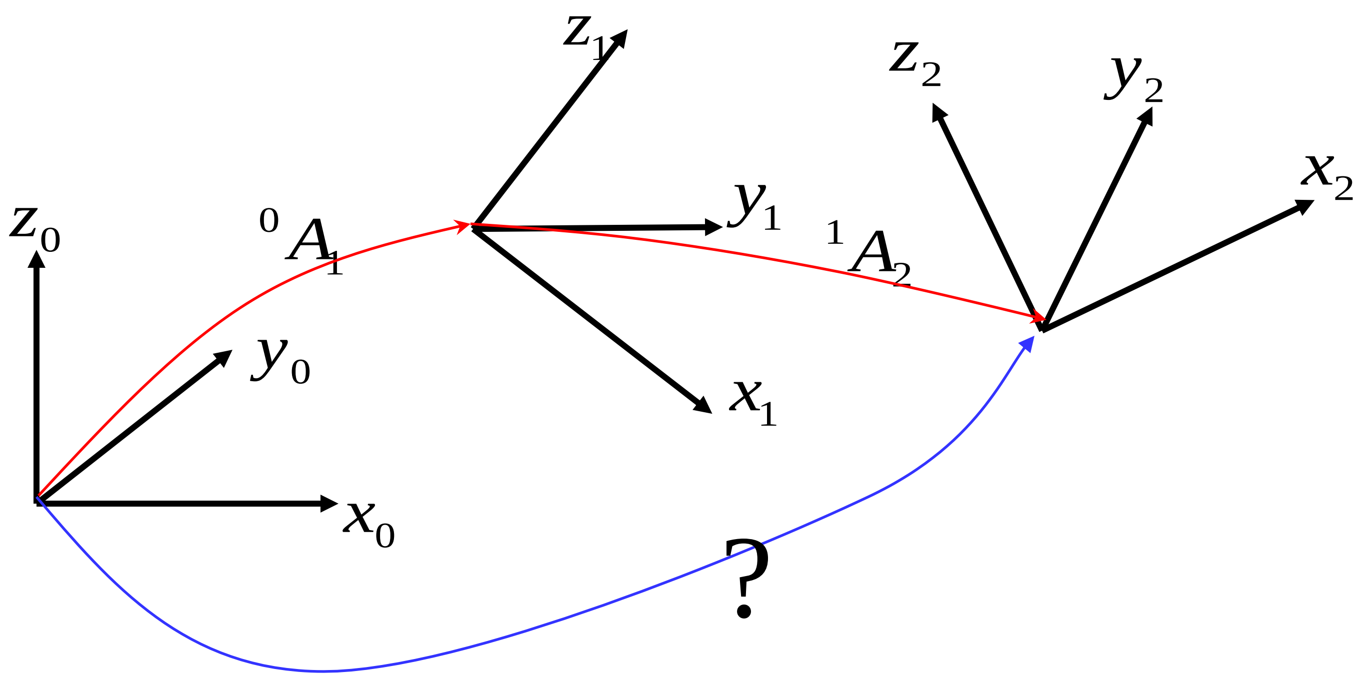
$$= \begin{bmatrix} n_x & s_x & a_x & p_x + d_x \\ n_y & s_y & a_y & p_y + d_y \\ n_z & s_z & a_z & p_z + d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



$$F_{new} = Trans(d_x, d_y, d_z) \times F_{old}$$

Homogeneous Transformation

Composite Homogeneous Transformation Matrix

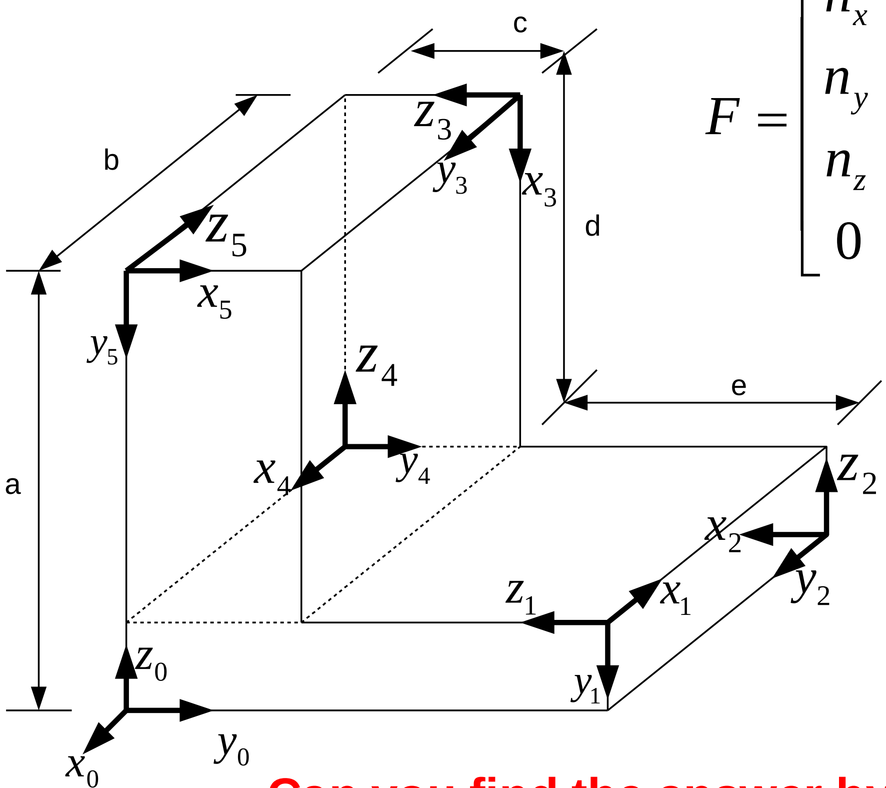


${}^{i-1}A_i$ **Transformation matrix for adjacent coordinate frames**

${}^0A_2 = {}^0A_1\,{}^1A_2$ **Chain product of successive coordinate transformation matrices**

Example 8

- For the figure shown below, find the 4x4 homogeneous transformation matrices ${}^{i-1}A_i$ and ${}^{0}A_i$ for i=1, 2, 3, 4, 5



$$F = \begin{bmatrix} n_x & s_x & a_x & p_x \\ n_y & s_y & a_y & p_y \\ n_z & s_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^{0}A_1 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & e+c \\ 0 & -1 & 0 & a-d \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^{1}A_2 = \begin{bmatrix} 0 & -1 & 0 & b \\ 0 & 0 & -1 & a-d \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^{0}A_2 = \begin{bmatrix} 0 & 1 & 0 & -b \\ -1 & 0 & 0 & e+c \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Can you find the answer by observation based on the geometric interpretation of homogeneous transformation matrix?

Orientation Representation

$$F = \begin{bmatrix} R_{3\times 3} & P_{3\times 1} \\ 0 & 1 \end{bmatrix}$$

- **Rotation matrix representation needs 9 elements to completely describe the orientation of a rotating rigid body.**
- **Any simple way?**

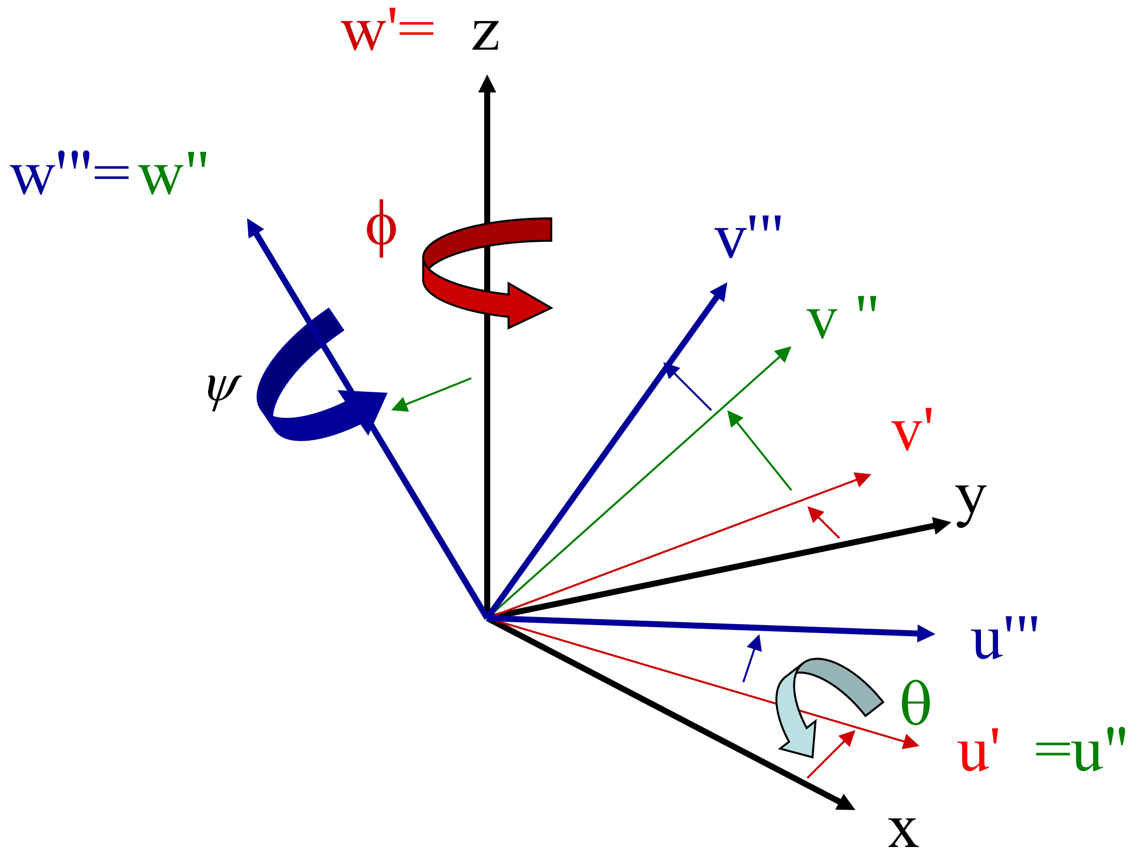
Euler Angles Representation

Orientation Representation

- Euler Angles Representation (ϕ, θ, ψ)
  - Many different types
  - Description of Euler angle representations

	Euler Angle I	Euler Angle II	**Yaw-Pitch-Roll**
Sequence	ϕ about OZ axis	ϕ about OZ axis	ψ about OX axis
of	θ about OU axis	θ about OV axis	θ about OY axis
Rotations	ψ about OW axis	ψ about OW axis	ϕ about OZ axis

Euler Angle I, Animated



欧拉角 I
ϕ about OZ axis
θ about OU axis
ψ about OW axis

Orientation Representation

- **Euler Angle I**

$$R_{z\phi} = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}, R_{u'\theta} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix},$$

$$R_{w''\varphi} = \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

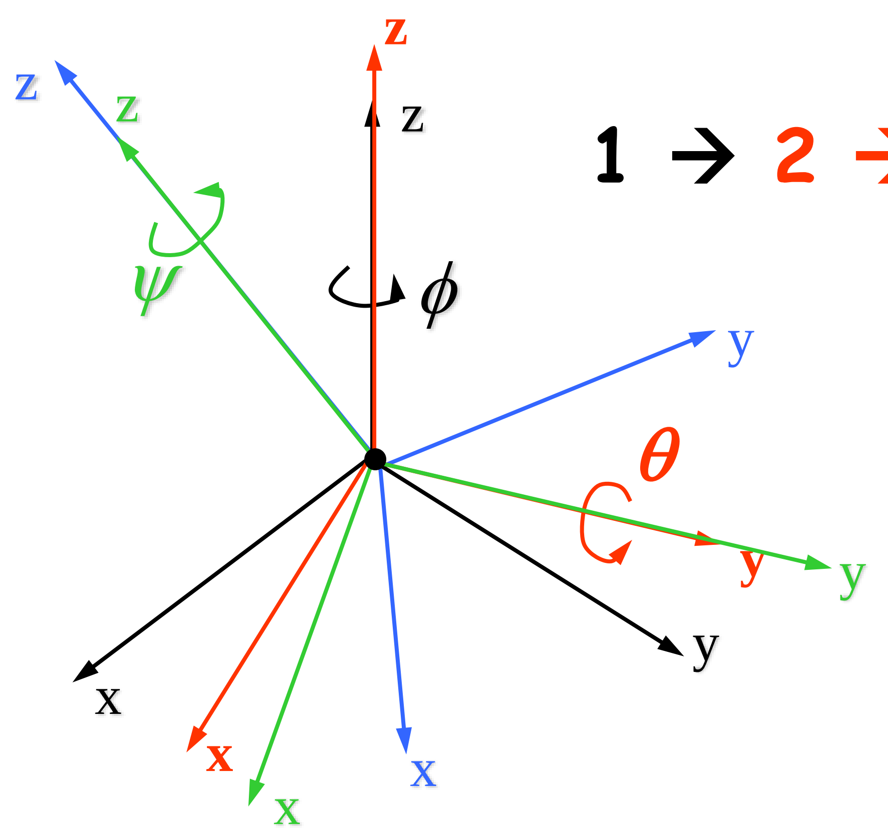
Euler Angle I

Resultant Eulerian Rotation Matrix:

$$R = R_{z\phi} R_{u'\theta} R_{w''\varphi}$$

$$\begin{pmatrix} \cos\phi\cos\psi - \sin\phi\sin\psi\cos\theta & -\cos\phi\sin\psi - \sin\phi\cos\psi\cos\theta & \sin\psi\sin\theta \\ \sin\phi\cos\psi + \cos\phi\sin\psi\cos\theta & -\sin\phi\sin\psi + \cos\phi\cos\psi\cos\theta & -\cos\phi\sin\theta \\ \sin\psi\sin\theta & \cos\psi\sin\theta & \cos\theta \end{pmatrix}$$

Euler Angle II, Animated



1 → 2 → 3 → 4

欧拉角 II
ϕ about OZ axis
θ about OV axis
ψ about OW axis

φ ：/fai/
θ ：/theta/
Ψ ：/psaɪ/

Orientation Representation

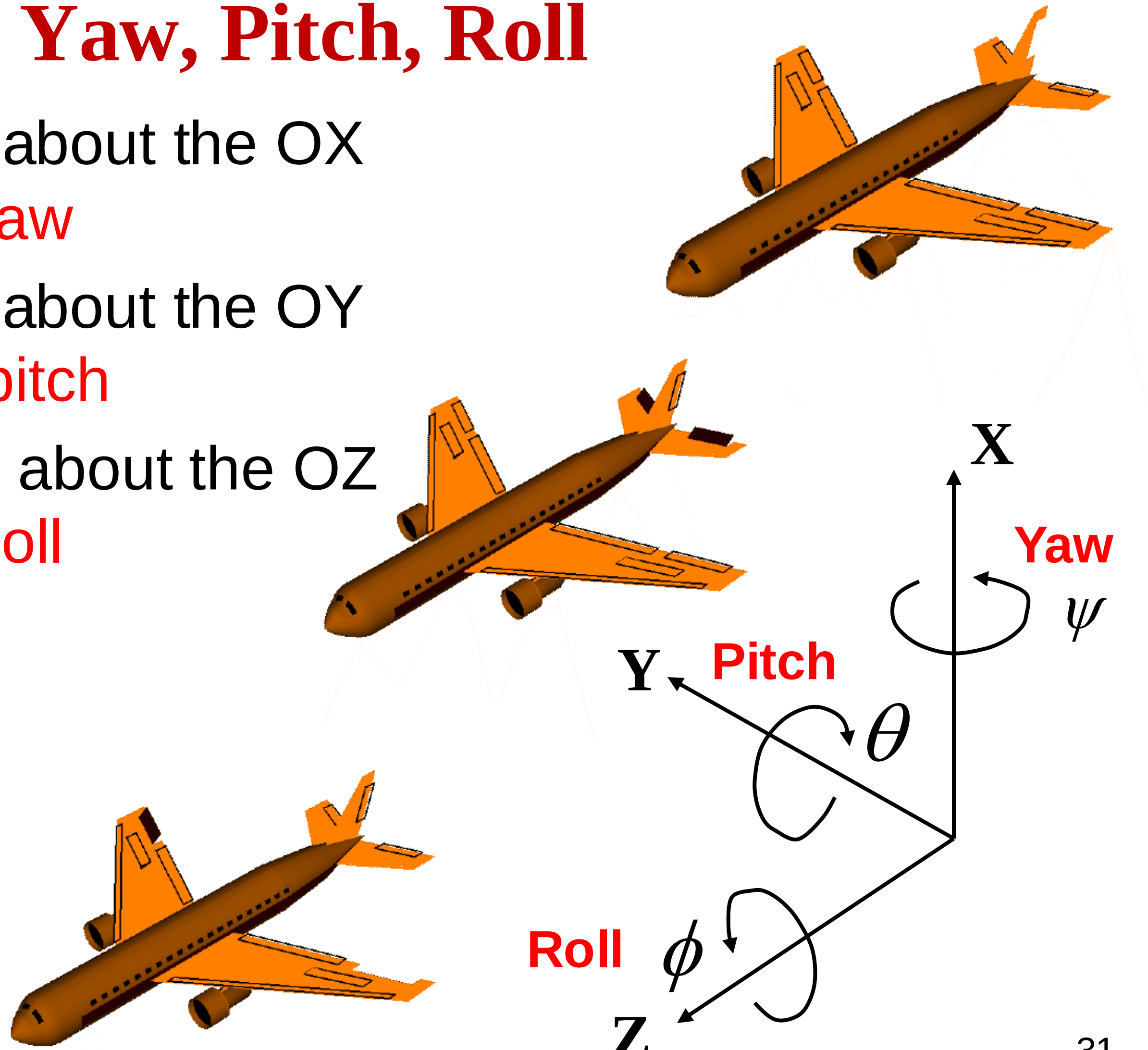
- **Matrix with Euler Angle II**

$$\begin{pmatrix} -\sin\phi\sin\varphi + \cos\phi\cos\varphi\cos\theta & -\sin\phi\cos\varphi - \sin\phi\cos\varphi\cos\theta & \cos\phi\sin\theta \\ \cos\phi\sin\varphi + \sin\phi\cos\varphi\cos\theta & \cos\phi\cos\varphi - \sin\phi\cos\varphi\cos\theta & \sin\varphi\sin\theta \\ -\cos\varphi\sin\theta & \sin\varphi\sin\theta & \cos\theta \end{pmatrix}$$

How to obtain this matrix ?

Orientation Representation

- **Description of Yaw, Pitch, Roll**
  - A rotation of ψ about the OX axis ($R_{x,\psi}$) -- yaw
  - A rotation of θ about the OY axis ($R_{y,\theta}$) -- pitch
  - A rotation of ϕ about the OZ axis ($R_{z,\phi}$) -- roll

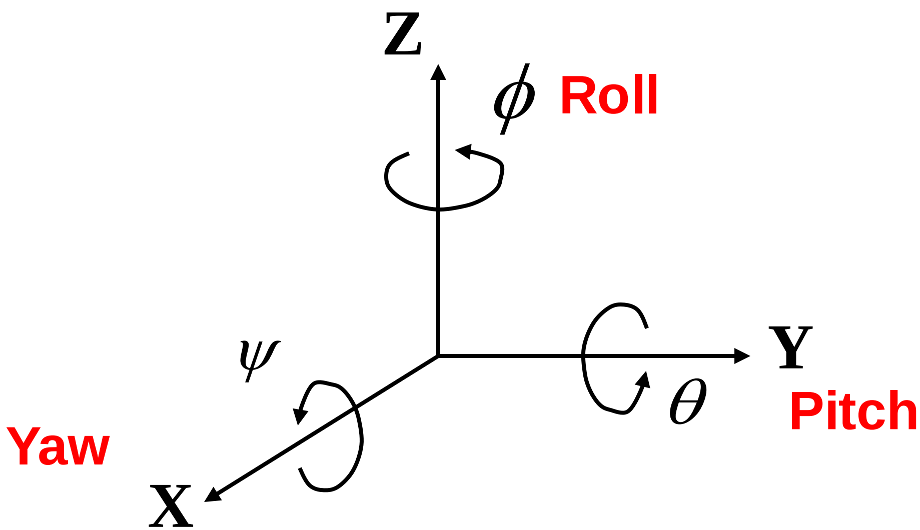


Exercise 1

- **How to obtain the rotation matrix for YPR?**

$$T = R_{z,\phi} R_{y,\theta} R_{x,\psi}$$

$$= \begin{bmatrix} C\phi & -S\phi & 0 & 0 \\ S\phi & C\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C\theta & 0 & S\theta & 0 \\ 0 & 1 & 0 & 0 \\ -S\theta & 0 & C\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & C\psi & -S\psi & 0 \\ 0 & S\psi & C\psi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Yaw-Pitch-Roll
ψ about OX axis
θ about OY axis
ϕ about OZ axis

Exercise 2

- **Geometric Interpretation?**

$$T = \begin{bmatrix} R_{3\times 3} & P_{3\times 1} \\ 0 & 1 \end{bmatrix}$$

- **Inverse Homogeneous Matrix?**

Summary

Homogeneous Matrix

- Composite Homogeneous Transformation
- Geometric Interpretation

Orientation Representation

- Euler Angle I & II
- Yaw-Pitch-Roll (YPR)

Homework 3

Homework 3 will be posted at http://sakai.sustech.edu.cn

Due date: March 3, 2025

Next class: March 3, 2025

Next class: Denavit-Hartenberg (D-H) Representation