南方科技大学
SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY

Robot Modeling & Control ME331

Section 7： Kinematics VI

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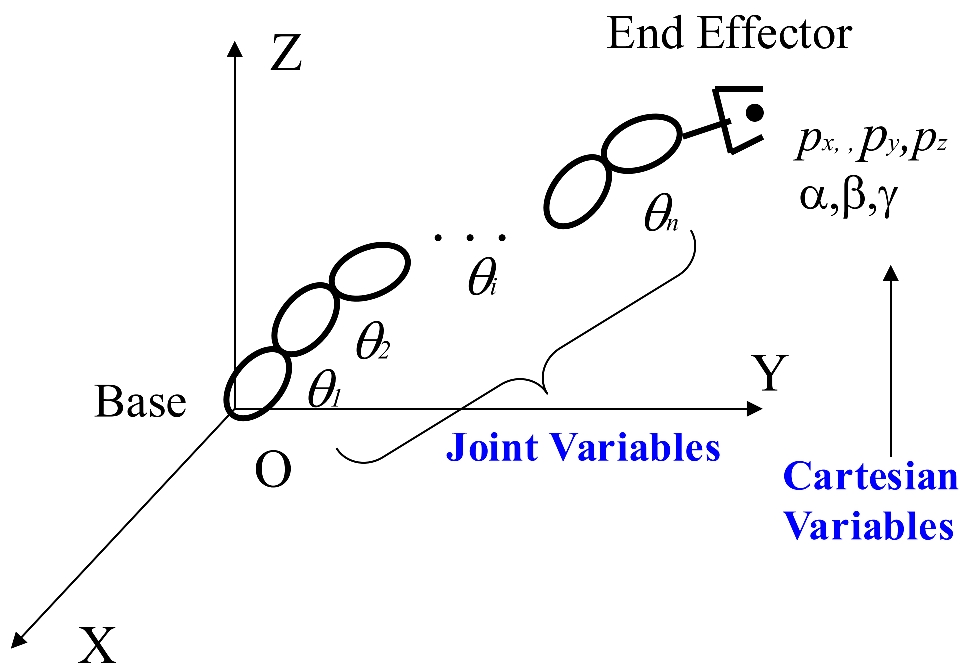
Outline

- Review
  - Inverse Kinematics
    - Kinematic Decoupling
    - Inverse Position: A Geometric Approach
- Inverse Orientation
- Jacobian
    - Definition
    - Calculation
- Angular Velocity
- Skew Symmetric Matrix

Inverse Kinematics

Given the position and orientation of the end-effector, find the joint variables that achieve such configuration.

$$\boldsymbol{\theta} = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \cdot \\ \cdot \\ \cdot \\ \theta_n \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} p_x \\ p_y \\ p_z \\ \alpha \\ \beta \\ \gamma \end{bmatrix}$$



(Joint) $\theta \longleftarrow \mathbf{x}$ **(Cartesian)**

Inverse Kinematics

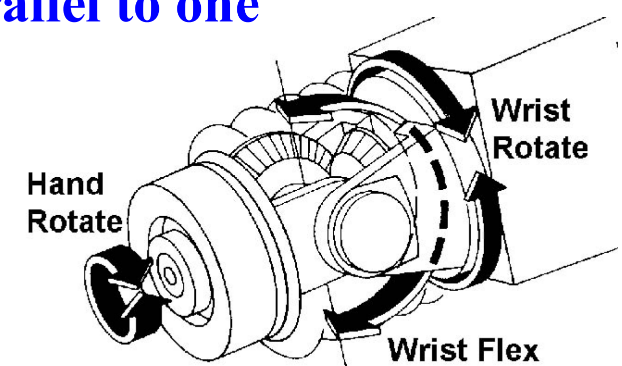
Inverse Kinematics

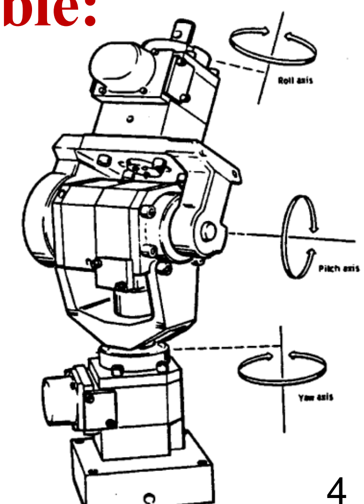
- **Transformation Matrix**

$$\begin{bmatrix} n_x & s_x & a_x & p_x \\ n_y & s_y & a_y & p_y \\ n_z & s_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = T_0^1 T_1^2 T_2^3 T_3^4 T_4^5 T_5^6 = T(\theta) \Rightarrow \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \\ \theta_5 \\ \theta_6 \end{bmatrix}$$

Special cases make the closed-form arm solution possible:

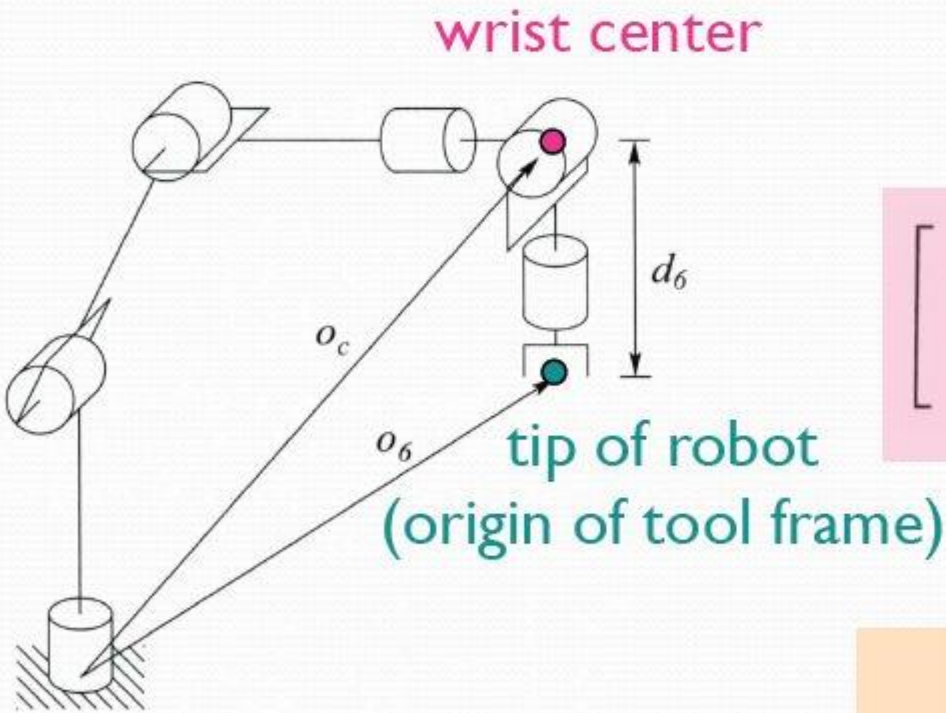
1. **Three adjacent joint axes intersecting**
2. **Three adjacent joint axes parallel to one**





Inverse Kinematics

Kinematic Decoupling



$$o = o_c^0 + d_6 R \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} = \begin{bmatrix} o_x - d_6 r_{13} \\ o_y - d_6 r_{23} \\ o_z - d_6 r_{33} \end{bmatrix}$$

position

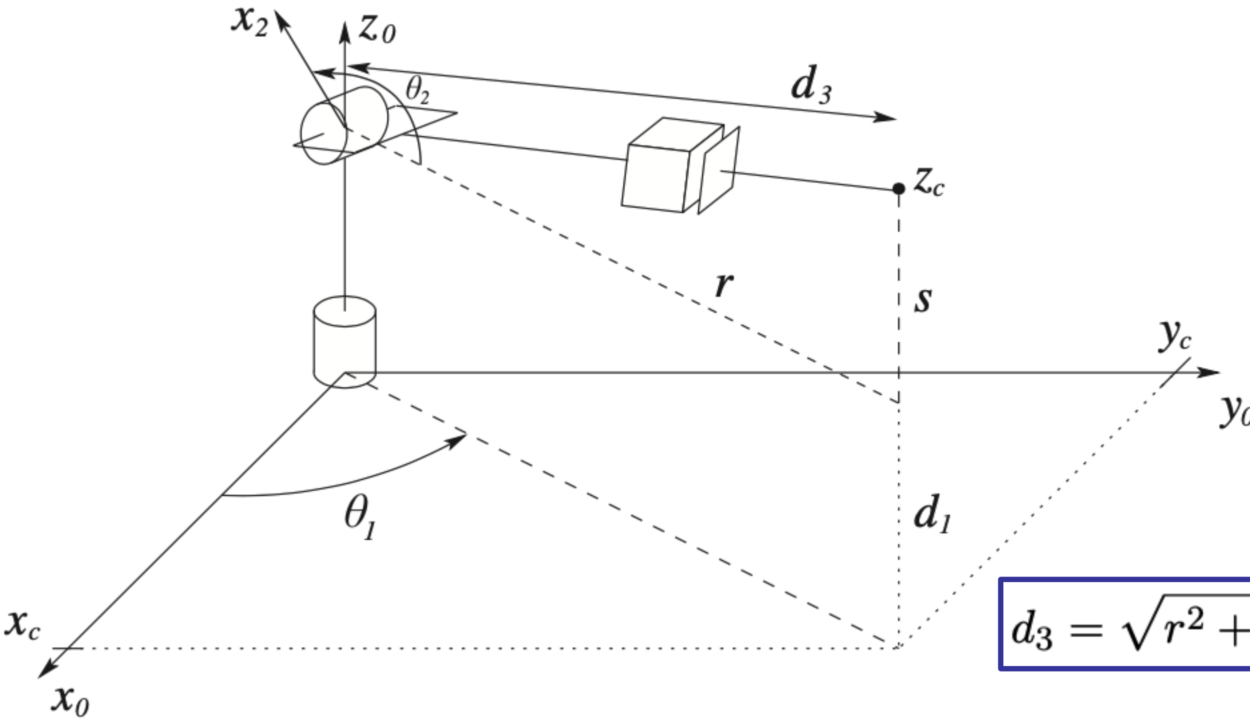
$$R = R_3^0 R_6^3$$

$$R_6^3 = (R_3^0)^{-1} R = (R_3^0)^T R$$

orientation

Inverse Position

A Geometric Approach



$$\boxed{\theta_1 \quad = \quad \mathrm{Atan2}(x_c, y_c)}$$

$$\boxed{\theta_1 \quad = \quad \pi + \mathrm{Atan2}(x_c, y_c)}$$

$$\boxed{\theta_2 = \mathrm{Atan2}(r, s) + \frac{\pi}{2}}$$

$$\boxed{d_3 = \sqrt{r^2 + s^2} \quad = \quad \sqrt{x_c^2 + y_c^2 + (z_c - d_1)^2}}$$

- The general idea of the geometric approach is to solve for joint variable θ_i by projecting the manipulator onto the $x_{i-1} - y_{i-1}$ plane and solving a simple trigonometry problem.
- For example, to solve for θ_1, we project the arm onto the $x_0 - y_0$ plane and use trigonometry to find θ_1.

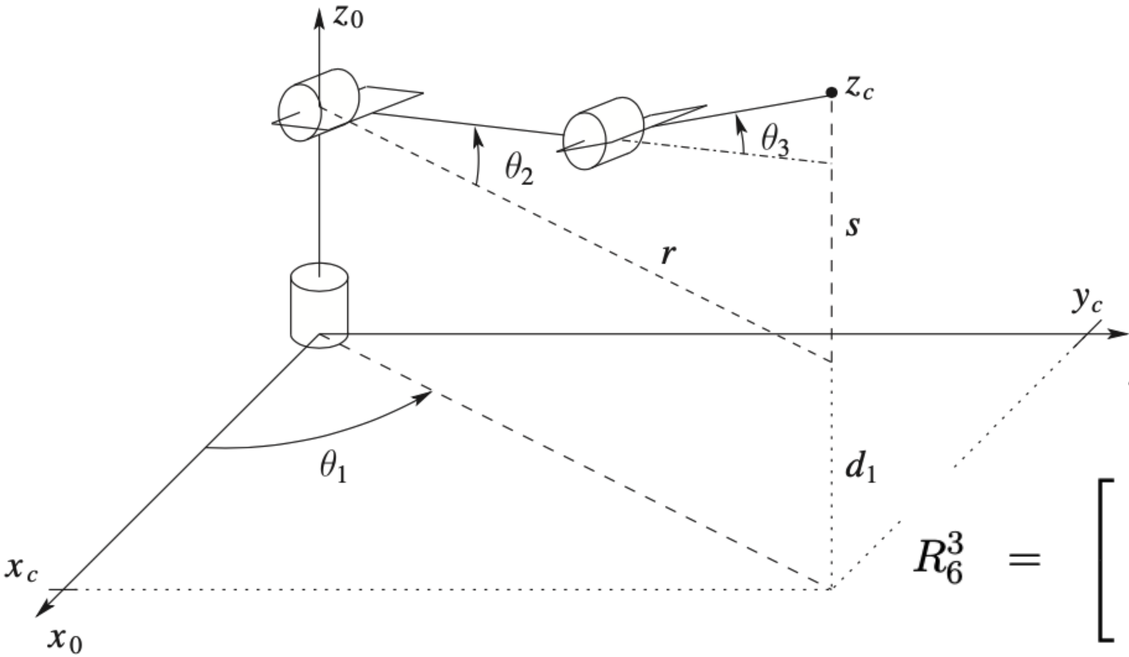
Inverse Orientation

- The inverse orientation problem is to find the values of the final three joint variables corresponding to a given orientation with respect to the frame $o_3x_3y_3z_3$.
- For a spherical wrist, this can be interpreted as the problem of finding a set of Euler angles corresponding to a given rotation matrix R.
- The rotation matrix obtained for the spherical wrist has the same form as the rotation matrix for the Euler transformation. Therefore, we can use the previous method to solve for the three joint angles of the spherical wrist.

$$\theta_4 = \varphi,\ \theta_5 = \theta,\ \theta_6 = \psi$$

Inverse Orientation

Example: Articulated Manipulator with Spherical Wrist



Link	a_i	α_i	d_i	θ_i
1	0	90	d_1	θ_1
2	a_2	0	0	θ_2
3	a_3	0	0	θ_3

$$R_3^0 = \begin{bmatrix} c_1c_{23} & -c_1s_{23} & s_1 \\ s_1c_{23} & -s_1s_{23} & -c_1 \\ s_{23} & c_{23} & 0 \end{bmatrix}$$

$$R_6^3 = \begin{bmatrix} c_4c_5c_6 - s_4s_6 & -c_4c_5s_6 - s_4c_6 & c_4s_5 \\ s_4c_5c_6 + c_4s_6 & -s_4c_5s_6 + c_4c_6 & s_4s_5 \\ -s_5c_6 & s_5s_6 & c_5 \end{bmatrix}$$

The equation to be solved for the final three variables is therefore $\boxed{R_6^3 = (R_3^0)^T R}$

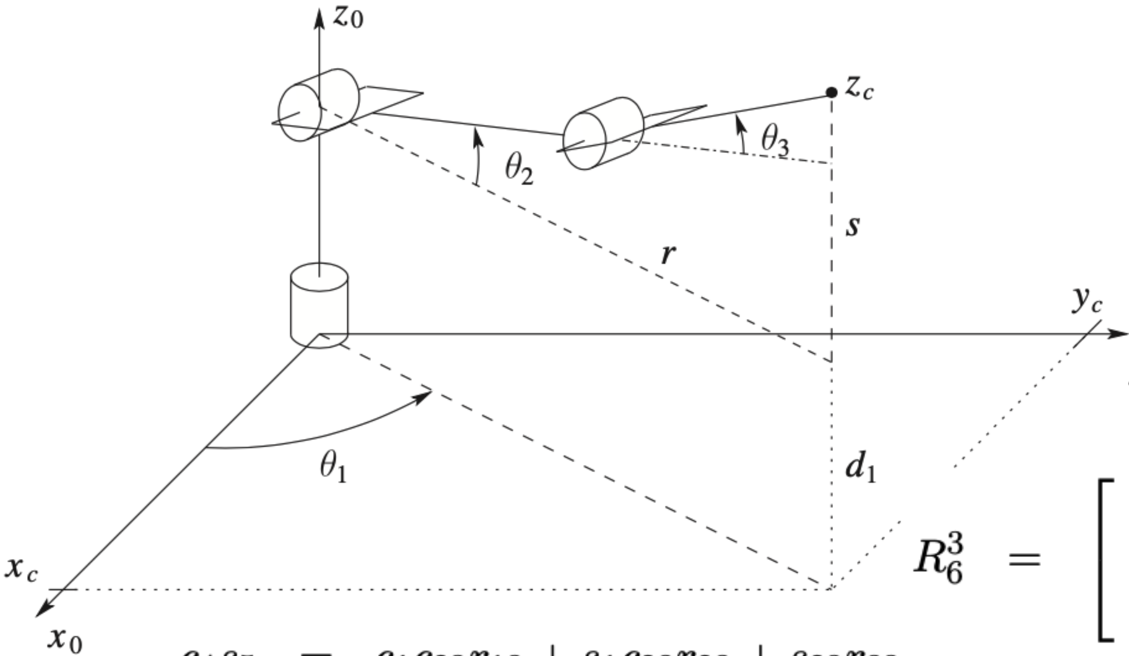
The three equations given by the third column in the above matrix equation are

$$\begin{aligned} c_4s_5 &= c_1c_{23}r_{13} + s_1c_{23}r_{23} + s_{23}r_{33} \\ s_4s_5 &= -c_1s_{23}r_{13} - s_1s_{23}r_{23} + c_{23}r_{33} \\ c_5 &= s_1r_{13} - c_1r_{23} \end{aligned}$$

$$\begin{aligned} \theta_5 &= \text{Atan2}\left(s_1r_{13} - c_1r_{23}, \pm\sqrt{1 - (s_1r_{13} - c_1r_{23})^2}\right) \\ \theta_4 &= \text{Atan2}(c_1c_{23}r_{13} + s_1c_{23}r_{23} + s_{23}r_{33}, \\ &\qquad -c_1s_{23}r_{13} - s_1s_{23}r_{23} + c_{23}r_{33}) \\ \theta_6 &= \text{Atan2}(-s_1r_{11} + c_1r_{21}, s_1r_{12} - c_1r_{22}) \end{aligned}$$

Inverse Orientation

Example: Articulated Manipulator with Spherical Wrist



Link	a_i	α_i	d_i	θ_i
1	0	90	d_1	θ_1
2	a_2	0	0	θ_2
3	a_3	0	0	θ_3

$$R_3^0 = \begin{bmatrix} c_1c_{23} & -c_1s_{23} & s_1 \\ s_1c_{23} & -s_1s_{23} & -c_1 \\ s_{23} & c_{23} & 0 \end{bmatrix}$$

$$R_6^3 = \begin{bmatrix} c_4c_5c_6 - s_4s_6 & -c_4c_5s_6 - s_4c_6 & c_4s_5 \\ s_4c_5c_6 + c_4s_6 & -s_4c_5s_6 + c_4c_6 & s_4s_5 \\ -s_5c_6 & s_5s_6 & c_5 \end{bmatrix}$$

$$\begin{aligned} c_4s_5 &= c_1c_{23}r_{13} + s_1c_{23}r_{23} + s_{23}r_{33} \\ s_4s_5 &= -c_1s_{23}r_{13} - s_1s_{23}r_{23} + c_{23}r_{33} \\ c_5 &= s_1r_{13} - c_1r_{23} \end{aligned}$$

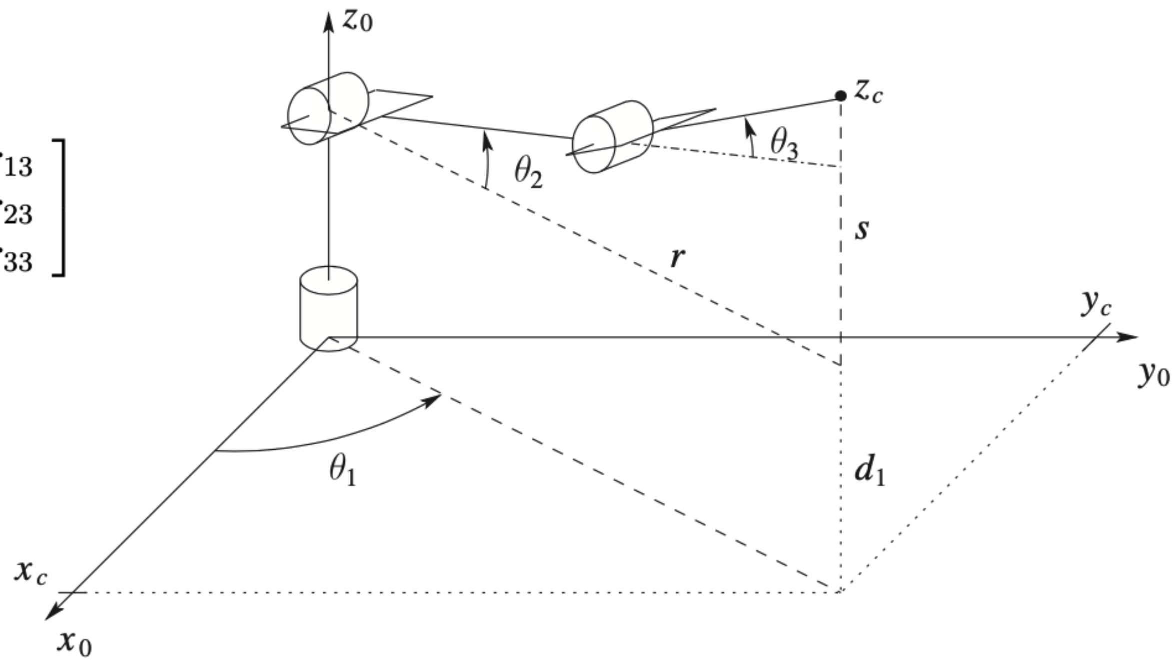
$$\begin{aligned} \theta_4 = \ & \text{Atan2}(c_1c_{23}r_{13} + s_1c_{23}r_{23} + s_{23}r_{33}, \\ & -c_1s_{23}r_{13} - s_1s_{23}r_{23} + c_{23}r_{33}) \end{aligned}$$

- If $s_5 = 0$, then joint axes z_3 and z_5 are collinear.
- This is a singular configuration and only the sum θ_4+θ_6 can be determined.
- One solution is to choose θ_4 arbitrarily and then determine θ_6.

Inverse Orientation

Example: Complete Solution

To summarize the geometric approach for solving the inverse kinematics equations, we give here one solution to the inverse kinematics of the six DOF elbow manipulator.



Given

$$o = \begin{bmatrix} o_x \\ o_y \\ o_z \end{bmatrix}, \quad R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

then with

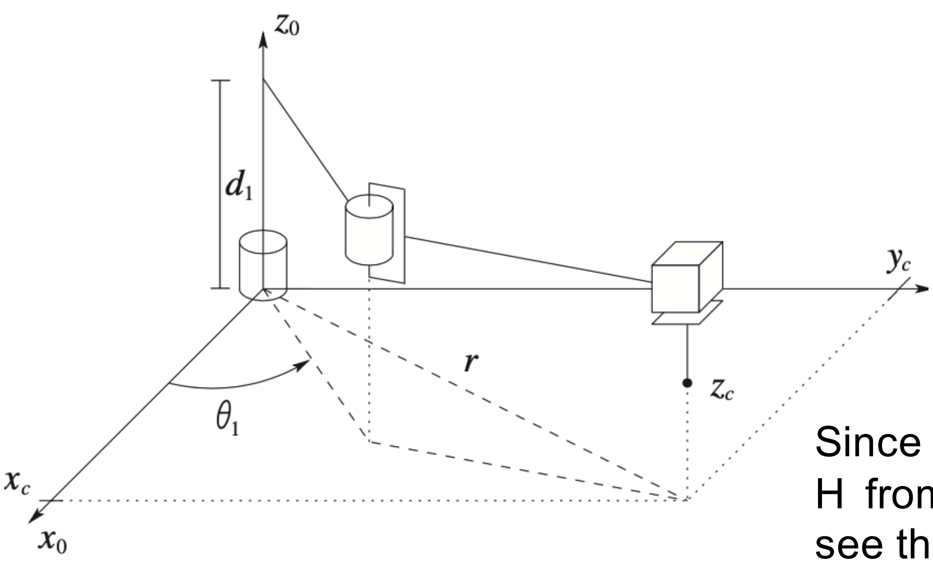
$$\begin{aligned} x_c &= o_x - d_6 r_{13} \\ y_c &= o_y - d_6 r_{23} \\ z_c &= o_z - d_6 r_{33} \end{aligned}$$

$$\begin{aligned} \theta_1 &= \text{Atan2}(x_c, y_c) \\ \theta_2 &= \text{Atan2}\left(\sqrt{x_c^2 + y_c^2 - d^2}, z_c - d_1\right) \\ &\quad -\text{Atan2}(a_2 + a_3 c_3, a_3 s_3) \\ \theta_3 &= \text{Atan2}\left(D, \pm\sqrt{1 - D^2}\right), \\ &\quad \textit{with } D = \frac{x_c^2 + y_c^2 - d^2 + (z_c - d_1)^2 - a_2^2 - a_3^2}{2 a_2 a_3} \end{aligned}$$

$$\begin{aligned} \theta_4 &= \text{Atan2}(c_1 c_{23} r_{13} + s_1 c_{23} r_{23} + s_{23} r_{33}, \\ &\quad -c_1 s_{23} r_{13} - s_1 s_{23} r_{23} + c_{23} r_{33}) \\ \theta_5 &= \text{Atan2}\left(s_1 r_{13} - c_1 r_{23}, \pm\sqrt{1 - (s_1 r_{13} - c_1 r_{23})^2}\right) \\ \theta_6 &= \text{Atan2}(-s_1 r_{11} + c_1 r_{21}, s_1 r_{12} - c_1 r_{22}) \end{aligned}$$

Inverse Orientation

Example: SCARA Manipulator



The inverse kinematics solution is given as the set of solutions of the equation

$$T_4^0 = \begin{bmatrix} R & o \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} c_{12}c_4 + s_{12}s_4 & s_{12}c_4 - c_{12}s_4 & 0 & a_1c_1 + a_2c_{12} \\ s_{12}c_4 - c_{12}s_4 & -c_{12}c_4 - s_{12}s_4 & 0 & a_1s_1 + a_2s_{12} \\ 0 & 0 & -1 & -d_3 - d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Since the SCARA has only four DOF, not every possible H from SE(3) allows a solution. In fact we can easily see that there is no solution unless *R* is of the form

$$R = \begin{bmatrix} c_\alpha & s_\alpha & 0 \\ s_\alpha & -c_\alpha & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

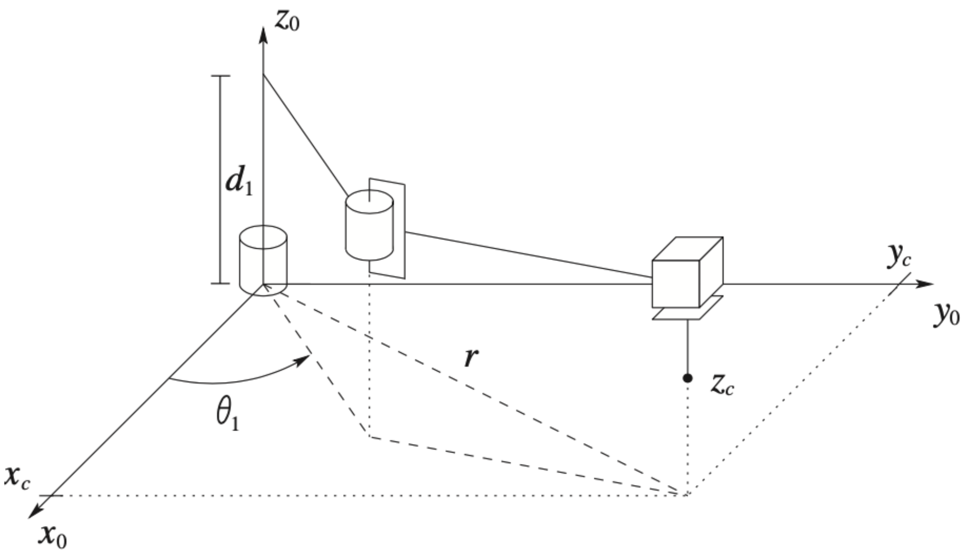
The sum $\theta_1 + \theta_2 - \theta_4$ is determined by $\boxed{\theta_1 + \theta_2 - \theta_4 = \alpha = \text{Atan2}(r_{11}, r_{12})}$

Projecting the manipulator configuration onto the $x_0 - y_0$ plane yields the geometry shown in the above figure. Using the law of cosines

$$c_2 = \frac{o_x^2 + o_y^2 - a_1^2 - a_2^2}{2a_1a_2} \Rightarrow \boxed{\theta_2 = \text{Atan2}\left(c_2, \pm\sqrt{1 - c_2^2}\right)}$$

Inverse Orientation

Example: SCARA Manipulator



$$\theta_2 \quad = \quad \text{Atan2}\left(c_2, \pm\sqrt{1-c_2^2}\right)$$

The sum $\theta_1 + \theta_2 - \theta_4$ is determined by

$$\theta_1 + \theta_2 - \theta_4 = \alpha = \text{Atan2}(r_{11}, r_{12})$$

The value for θ_1 is then obtained as $\theta_1 \quad = \quad \text{Atan2}(o_x, o_y) - \text{Atan2}(a_1 + a_2 c_2, a_2 s_2)$

We may now determine θ_4 from the sum equation as

$$\begin{aligned} \theta_4 &= \theta_1 + \theta_2 - \alpha \\ &= \theta_1 + \theta_2 - \text{Atan2}(r_{11}, r_{12}) \end{aligned}$$

Finally d_3 is given as $d_3 \quad = \quad d_1 - o_z - d_4$

Jacobian Matrix

Joint space		Task space	Joint velocity space		Task velocity space
$\begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \\ \theta_5 \\ \theta_6 \end{bmatrix}$	**Forward** $\Rightarrow$ **Kinematics** $\Leftarrow$ **Inverse**	$\begin{bmatrix} x \\ y \\ z \\ \phi \\ \theta \\ \varphi \end{bmatrix}$	$\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \\ \dot{\theta}_4 \\ \dot{\theta}_5 \\ \dot{\theta}_6 \end{bmatrix}$	**Jacobian Matrix** $\Leftrightarrow$	$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$

Joint Space

Task Space

Jacobian Matrix: Relationship between joint space velocity with task space velocity

Jacobian Matrix

Forward kinematics

$$\begin{bmatrix} x \\ y \\ z \\ \phi \\ \theta \\ \varphi \end{bmatrix} = h(\begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{bmatrix})_{6\times 1} = \begin{bmatrix} h_1(q_1, q_2, \cdots, q_6) \\ h_2(q_1, q_2, \cdots, q_6) \\ h_3(q_1, q_2, \cdots, q_6) \\ h_4(q_1, q_2, \cdots, q_6) \\ h_5(q_1, q_2, \cdots, q_6) \\ h_6(q_1, q_2, \cdots, q_6) \end{bmatrix}_{6\times 1} \Rightarrow Y_{6\times 1} = h(q_{n\times 1})$$

$$\Downarrow$$

$$\dot{Y}_{6\times 1} = \frac{d}{dt} h(q_{n\times 1}) = \frac{dh(q)}{dq}\frac{dq}{dt} = \frac{dh(q)}{dq}\dot{q}$$

$$\Downarrow$$

$$\dot{Y}_{6\times 1} = J_{6\times n}\dot{q}_{n\times 1}$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \boxed{\left[\frac{dh(q)}{dq}\right]_{6\times n}} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix}_{n\times 1} \Leftarrow$$

$$\boxed{J = \frac{dh(q)}{dq}}$$

Jacobian Matrix

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \left[\frac{dh(q)}{dq} \right]_{6\times n} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix}_{n\times 1}$$

Jacobian is a function of *q*, it is not a constant!

$$J = \left(\frac{dh(q)}{dq} \right)_{6\times n} = \begin{bmatrix} \frac{\partial h_1}{\partial q_1} & \frac{\partial h_1}{\partial q_2} & \cdots & \frac{\partial h_1}{\partial q_n} \\ \frac{\partial h_2}{\partial q_1} & \frac{\partial h_2}{\partial q_2} & \cdots & \frac{\partial h_2}{\partial q_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial h_6}{\partial q_1} & \frac{\partial h_6}{\partial q_2} & \cdots & \frac{\partial h_6}{\partial q_n} \end{bmatrix}_{6\times n}$$

Jacobian Matrix

Forward Kinematics

$$T_0^6 = \left[\begin{array}{ccc|c} n & s & a & p \\ \hline 0 & 0 & 0 & 1 \end{array}\right]_{4\times 4}$$

$$p = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} h_1(q) \\ h_2(q) \\ h_3(q) \end{bmatrix}$$

$$\{n, s, a\} \rightarrow \begin{bmatrix} \phi(q) \\ \theta(q) \\ \psi(q) \end{bmatrix} = \begin{bmatrix} h_4(q) \\ h_5(q) \\ h_6(q) \end{bmatrix}$$

$$Y_{6\times 1} = h(q) = \begin{bmatrix} h_1(q) \\ h_2(q) \\ \vdots \\ h_6(q) \end{bmatrix}$$

$$\dot{Y}_{6\times 1} = J_{6\times n} \dot{q}_{n\times 1}$$

$$\dot{Y} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} V \\ \Omega \end{bmatrix}$$

Linear velocity

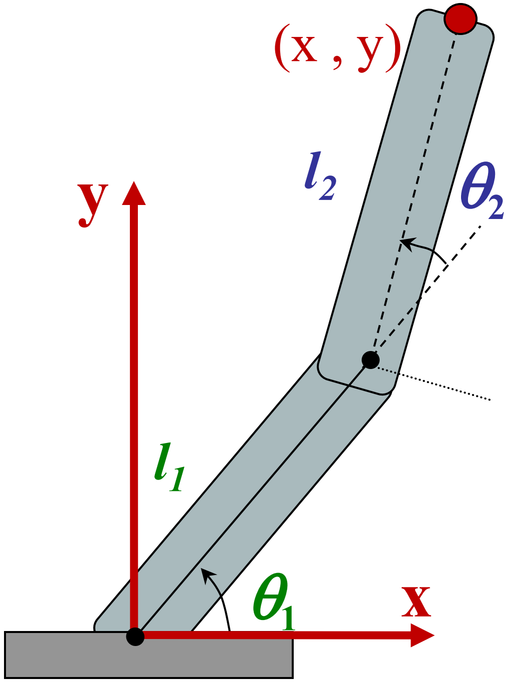
$$V = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix}$$

Angular velocity

$$\Omega = \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix}$$

Example

- **2-DOF planar robot arm**
  - **Given *l_1, l_2*, *Find: Jacobian***



$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2) \\ l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2) \end{bmatrix} = \begin{bmatrix} h_1(\theta_1, \theta_2) \\ h_2(\theta_1, \theta_2) \end{bmatrix}$$

$$\dot{Y} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = J \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

$$J = \begin{bmatrix} \dfrac{\partial h_1}{\partial \theta_1} & \dfrac{\partial h_1}{\partial \theta_2} \\ \dfrac{\partial h_2}{\partial \theta_1} & \dfrac{\partial h_2}{\partial \theta_2} \end{bmatrix} = \begin{bmatrix} -l_1 \sin\theta_1 - l_2 \sin(\theta_1 + \theta_2) & -l_2 \sin(\theta_1 + \theta_2) \\ l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2) & l_2 \cos(\theta_1 + \theta_2) \end{bmatrix}$$

Jacobian Matrix

- **Physical Interpretation**

$$\dot{Y} = J\dot{q} = \begin{bmatrix} J_{11} & J_{12} & \cdots & J_{16} \\ J_{21} & J_{22} & \cdots & J_{26} \\ \vdots & \vdots & \vdots & \vdots \\ J_{61} & J_{62} & \cdots & J_{66} \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \\ \dot{q}_5 \\ \dot{q}_6 \end{bmatrix}$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} J_{11}\dot{q}_1 + J_{12}\dot{q}_2 + \cdots + J_{16}\dot{q}_6 \\ J_{21}\dot{q}_1 + J_{22}\dot{q}_2 + \cdots + J_{26}\dot{q}_6 \\ J_{31}\dot{q}_1 + J_{32}\dot{q}_2 + \cdots + J_{36}\dot{q}_6 \\ J_{41}\dot{q}_1 + J_{42}\dot{q}_2 + \cdots + J_{46}\dot{q}_6 \\ J_{51}\dot{q}_1 + J_{52}\dot{q}_2 + \cdots + J_{56}\dot{q}_6 \\ J_{61}\dot{q}_1 + J_{62}\dot{q}_2 + \cdots + J_{66}\dot{q}_6 \end{bmatrix}$$

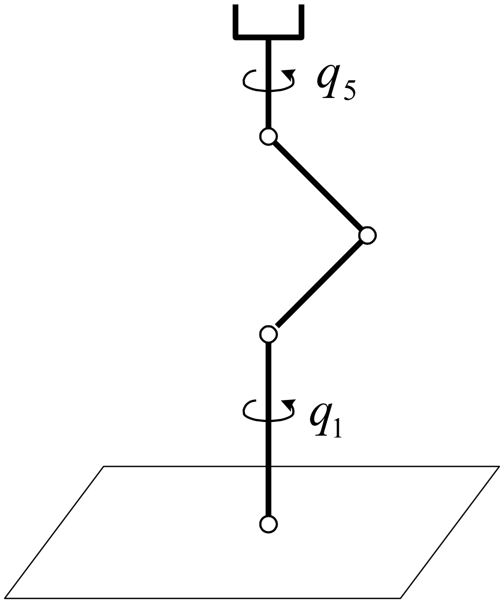
How each individual joint space velocity contribute to task space velocity.

Jacobian Matrix

- **Inverse Jacobian**

$$\dot{Y} = J\dot{q} = \begin{bmatrix} J_{11} & J_{12} & \cdots & J_{16} \\ J_{21} & J_{22} & \cdots & J_{26} \\ \vdots & \vdots & \vdots & \vdots \\ J_{61} & J_{62} & \cdots & J_{66} \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \\ \dot{q}_5 \\ \dot{q}_6 \end{bmatrix}$$

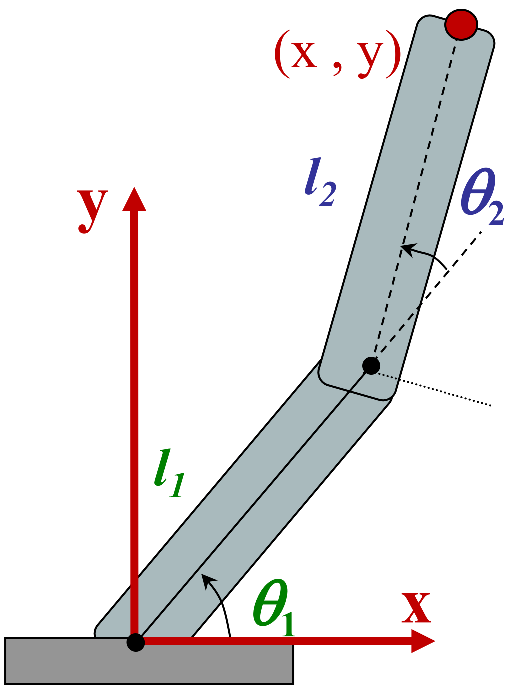
$$\dot{q} = J^{-1}\dot{Y}$$



- **Singularity**
  - **rank(J) < min{6,n}, Jacobian Matrix is less than full rank**
  - **Jacobian is non-invertable**
  - **Boundary Singularities: occur when the tool tip is on the surface of the work envelop.**
  - **Interior Singularities: occur inside the work envelope when two or more of the axes of the robot form a straight line, i.e., collinear**

Exercise

- **Find the singularity configuration of the 2-DOF planar robot arm**



Jacobian Matrix

Pseudoinverse (伪逆)

- Let A be an m×n matrix, and let A^+ be the pseudoinverse of A. If A is of full rank, then A^+ can be computed as:

$$A^+ = \begin{cases} A^T[AA^T]^{-1} & m \le n \\ A^{-1} & m = n \\ [A^T A]^{-1} A^T & m \ge n \end{cases}$$

- **Example:**

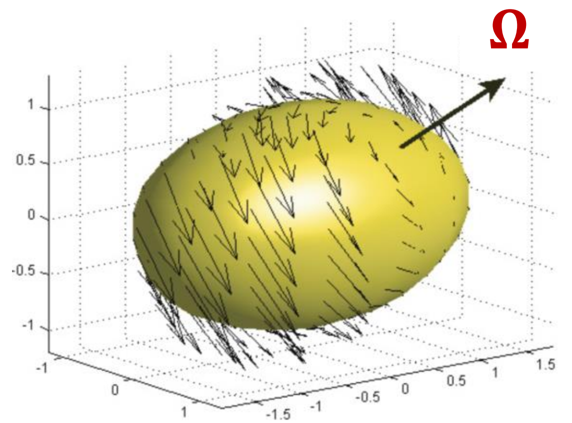
$$\begin{bmatrix} 1 & 0 & 2 \\ 1 & -1 & 0 \end{bmatrix} x = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

$$x = A^+ b = 1/9[-5,13,16]^T$$

$$A^+ = A^T[AA^T]^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 5 & 1 \\ 1 & 2 \end{bmatrix}^{-1} = \frac{1}{9} \begin{bmatrix} 1 & 4 \\ 1 & -5 \\ 4 & -2 \end{bmatrix}$$

Velocity Kinematics

Angular Velocity: The Fixed Axis Case



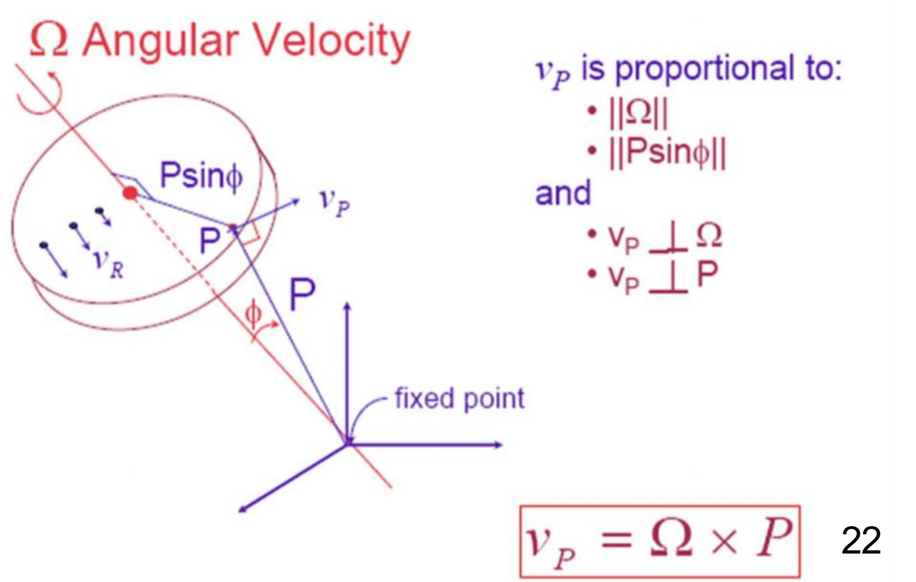
– When a rigid body moves in a pure rotation about a fixed axis, every point of the body moves in a circle.

– As the body rotates, a perpendicular from any point of the body to the axis sweeps out an angle $\boldsymbol{\theta}$. If $\boldsymbol{k}$ is a unit vector in the direction of the axis of rotation, then the angular velocity is given by:

$$\boldsymbol{\Omega} = \dot{\boldsymbol{\theta}}\boldsymbol{k}$$

– Given the angular velocity of the body, the linear velocity of any point on the body is given by the cross product $\boldsymbol{v_P}$:

$$\boldsymbol{v_P} = \boldsymbol{\Omega} \times \boldsymbol{P}$$



Velocity Kinematics

Skew-Symmetric Matrices

A 3×3 matrix $\boldsymbol{S}$ is **skew symmetric（反对称）**, if and only if

$$\boldsymbol{S}^{\mathrm{T}} + \boldsymbol{S} = \mathbf{0}$$

If $\boldsymbol{S} \in so(3)$ has components $s_{ij}\ (i,j = 1,2,3)$, then the above equation is equivalent to the nine equations：

$$s_{ij} + s_{ji} = 0 \quad i,j = 1,2,3$$

Therefore，

– the diagonal terms of $\boldsymbol{S}$ are zero: $s_{ii} = 0$;

– the off-diagonal terms $s_{ij}\ (i \neq j)$ satisfy $s_{ij} = -s_{ji}$.

$$S = \begin{bmatrix} 0 & -s_3 & s_2 \\ s_3 & 0 & -s_1 \\ -s_2 & s_1 & 0 \end{bmatrix}$$

Velocity Kinematics

Skew-Symmetric Matrices

If $a = (a_x, a_y, a_z)^T$ is a 3-vector, we define the skew symmetric $S(a)$ as：

$$S(a) = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}$$

Example： Define the the skew symmetric matrices with

$$\boldsymbol{i} = [1\ 0\ 0]^{\mathrm{T}},\ \boldsymbol{j} = [0\ 1\ 0]^{\mathrm{T}},\ \boldsymbol{k} = [0\ 0\ 1]^{\mathrm{T}}.$$

$$\boldsymbol{S}(\boldsymbol{i}) = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \end{bmatrix} \quad \boldsymbol{S}(\boldsymbol{j}) = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -\mathbf{1} & \mathbf{0} & \mathbf{0} \end{bmatrix} \quad \boldsymbol{S}(\boldsymbol{k}) = \begin{bmatrix} \mathbf{0} & -\mathbf{1} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

Velocity Kinematics

Properties of Skew-Symmetric Matrices

- Operator S is linear, that is $S(\alpha a + \beta b) = \alpha S(a) + \beta S(b)$ for any vectors $a, b \in \mathbb{R}^3$ and scalars $\alpha, \beta \in R$
- For any vectors $a, p \in \mathbb{R}^3$, $S(a)p = a \times p$
- For $R \in SO(3)$ and $a \in \mathbb{R}^3$, $RS(a)R^T = S(Ra)$

To show this, we use the fact that $R(a \times b) = Ra \times Rb$.

It says that if we first rotate the vectors a and b using the rotation transformation R and then form the cross product of the rotated vectors Ra and Rb, the result is the same as that obtained by first forming the cross product $a \times b$ and then rotating to obtain $R(a \times b)$.

$$\begin{aligned} RS(a)R^T b &= R(a \times R^T b) \\ &= (Ra) \times (RR^T b) \\ &= (Ra) \times b \\ &= S(Ra)b \end{aligned}$$

Velocity Kinematics

Properties of Skew-Symmetric Matrices

- For $R \in SO(3)$ and $a \in \mathbb{R}^3$, $RS(a)R^T = S(Ra)$

  The left-hand side of the equation represents a similarity transformation of the matrix $S(a)$. The equation says, therefore, that the matrix representation of $S(a)$ in a coordinate frame rotated by R is the same as the skew-symmetric matrix $S(Ra)$ corresponding to the vector a rotated by R.

- For $n \times n$ Skew-Symmetric Matrix S, and any vector $X \in \mathbb{R}^n$,

$$X^T SX = \ ?$$

$$X^T SX = 0$$

Velocity Kinematics

The Derivative of a Rotation Matrix

Suppose $R = R(\theta) \in SO(3)$. Since R is orthogonal， it follows

$$R(\theta)R(\theta)^{\mathrm{T}} = I$$

Differentiating both sides of the equation with respect to θ using the product rule gives

$$\left[\frac{d}{d\theta}R\right]R(\theta)^{\mathrm{T}} + R(\theta)\left[\frac{d}{d\theta}R^{\mathrm{T}}\right] = 0 \quad (*)$$

Define $S := \left[\frac{d}{d\theta}R\right]R(\theta)^{\mathrm{T}}$, then $S^{\mathrm{T}} = R\,(\theta)\left[\frac{d}{d\theta}R^{\mathrm{T}}\right]$

Eqn （*） indicates S is skew-symmetric，then：

$$\left[\frac{d}{d\theta}R\right] = SR(\theta)$$

Computing the derivative of the rotation matrix R is equivalent to a matrix multiplication by a skew-symmetric matrix S.

Velocity Kinematics

The Derivative of a Rotation Matrix

Example： if $R = R_{x,\theta}$， find $\frac{\mathrm{d}}{\mathrm{d}\theta} R_{x,\theta}$

$$R_{x,\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$S = \left[\frac{d}{d\theta} R\right] R^{\mathrm{T}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\sin\theta & -\cos\theta \\ 0 & \cos\theta & -\sin\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = S(i)$$

Thus， $\frac{\mathrm{d}}{\mathrm{d}\theta} R_{x,\theta} = S(i)\ R_{x,\theta}$

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}\theta} R_{k,\theta} = S(k)\ R_{k,\theta}}$$

Summary

Inverse Kinematics

- Inverse Orientation

Jacobian Matrix

- Definition
- Calculation（derivative method）

Angular Velocity: The Fixed Axis Case

Skew Symmetric Matrix

Homework 7

Homework 7 is posted at http://bb.sustech.edu.cn

Due date: March 24, 2025

Next class: March 19, 2024（Wednesday）

作业要求（Requirements）：

1. **文件格式为以自己姓名学号作业序号命名的pdf文件；**

   （File name: YourSID_ YourName_07.pdf）

2. **作业里也写上自己的姓名和学号。**

   （Write your name and SID in the homework）