南方科技大学
SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY

Robot Modeling & Control ME331

Section 6： Kinematics V

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Outline

- **Review**
  - D-H Representation
  - Forward Kinematics Equations
  - Yaw-Pitch-Roll
    - atan2(x,y)
- **Inverse Kinematics**
  - General Problem
  - Kinematic Decoupling
  - Inverse Position: A Geometric Approach
  - Example: Stanford Arm

Review

- **Steps to derive kinematics model:**
  - Assign D-H coordinates frames
  - Find link parameters
  - Transformation matrices of adjacent joints
  - Calculate kinematics matrix
  - When necessary, Euler angle representation

Review

• D-H方法总结

连杆i（关节$i+1$）的坐标系$O_iX_iY_iZ_i$			
原点O_i	X_i坐标轴	Y_i坐标轴	Z_i坐标轴
位于Z_i轴线上，且与Z_i轴和Z_{i-1}轴的公垂线的交点	沿Z_i轴和Z_{i-1}轴的公垂线，指向离开Z_{i-1}轴的方向	根据轴X_i Z_i按右手直角坐标系法则确定	沿着$i+1$关节的运动轴线

连杆的参数			
名称	含义	正负	性质
转角θ_i	X_i轴和X_{i-1}轴两轴线之间夹角	右手法则z_{i-1}	关节转动时为变量
距离d_n	**X_i轴和X_{i-1}轴两轴线公垂线长度**	沿z_{i-1}正向+	关节移动时为变量
长度a_n	Z_i轴和Z_{i-1}轴两轴线的公垂线长度	与X_i正向一致	尺寸参数,常量
扭角α_i	Z_i轴和Z_{i-1}轴线之间的扭角	右手法则X_i	尺寸参数,常量

Review

- D-H transformation matrix for adjacent coordinate frames, i and i-1.
  - The position and orientation of the ***i***-th frame coordinate can be expressed in the ***(i-1)***-th frame by the following 4 successive elementary transformations:

Reference coordinate

Target coordinate

$$T_i^{i-1} = T(z_{i-1}, d_i) \quad T(z_{i-1}, \theta_i) \quad T(x_i, a_i) \quad T(x_i, \alpha_i)$$

$$= \begin{bmatrix} C\theta_i & -C\alpha_i S\theta_i & S\alpha_i S\theta_i & a_i C\theta_i \\ S\theta_i & C\alpha_i C\theta_i & -S\alpha_i C\theta_i & a_i S\theta_i \\ 0 & S\alpha_i & C\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Review

- Kinematics Equations
  - chain product of successive coordinate transformation matrices of T_i^{i-1}
  - T_n^0 specifies the location of the n-th coordinate frame w.r.t. the base coordinate system

$$T_n^0 = T_1^0 \, T_2^1 \cdots T_n^{n-1}$$

$$= \begin{bmatrix} \boxed{R_n^0} & \boxed{P_n^0} \\ \mathbf{0} & 1 \end{bmatrix} = \begin{bmatrix} n & s & a & P_n^0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Orientation matrix

Position vector

Review

- Forward Kinematics

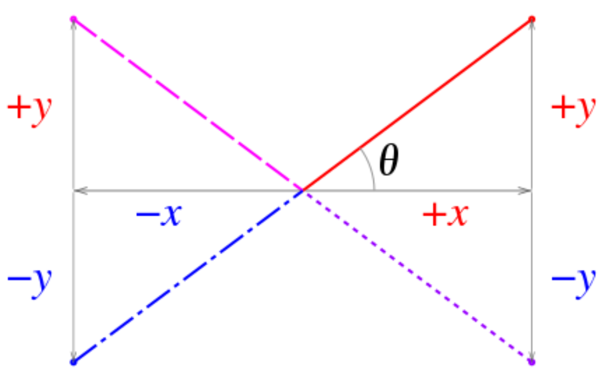
$$\begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \\ \theta_5 \\ \theta_6 \end{bmatrix} \Rightarrow \begin{bmatrix} p_x \\ p_y \\ p_z \\ \phi \\ \theta \\ \varphi \end{bmatrix}$$

- Kinematics Transformation Matrix

$$T = \begin{bmatrix} n_x & s_x & a_x & p_x \\ n_y & s_y & a_y & p_y \\ n_z & s_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Why use Euler angle representation?

What is $a\tan 2(y,x)$?



$$\theta = a\tan 2(y,x) = \begin{cases} 0^\circ \le \theta \le 90^\circ & for \quad +x \quad and \quad +y \\ 90^\circ \le \theta \le 180^\circ & for \quad -x \quad and \quad +y \\ -180^\circ \le \theta \le -90^\circ & for \quad -x \quad and \quad -y \\ -90^\circ \le \theta \le 0^\circ & for \quad +x \quad and \quad -y \end{cases}$$

Review

- Yaw-Pitch-Roll Representation

$$T = R_{z,\phi} R_{y,\theta} R_{x,\psi} \quad \Rightarrow \quad R_{z,\phi}^{-1} T = R_{y,\theta} R_{x,\psi}$$

$$\begin{bmatrix} C\phi & S\phi & 0 & 0 \\ -S\phi & C\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} n_x & s_x & a_x & 0 \\ n_y & s_y & a_y & 0 \\ n_z & s_z & a_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} C\theta & 0 & S\theta & 0 \\ 0 & 1 & 0 & 0 \\ -S\theta & 0 & C\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & C\psi & -S\psi & 0 \\ 0 & S\psi & C\psi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} C\phi \cdot n_x + S\phi \cdot n_y & XX & XX & 0 \\ -S\phi \cdot n_x + C\phi \cdot n_y & \boxed{-S\phi \cdot s_x + C\phi \cdot s_y} & \boxed{-S\phi \cdot a_x + C\phi \cdot a_y} & 0 \\ n_z & XX & XX & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} C\theta & S\theta S\psi & S\theta C\psi & 0 \\ 0 & \boxed{C\psi} & \boxed{-S\psi} & 0 \\ -S\theta & C\theta S\psi & C\theta C\psi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(Equation A)

Review

- Compare LHS and RHS of Equation A, we have:

$$-\sin\phi\cdot n_x+\cos\phi\cdot n_y=0 \quad\Rightarrow\quad \phi=a\tan 2(n_y,n_x)$$

$$\begin{cases}\cos\phi\cdot n_x+\sin\phi\cdot n_y=\cos\theta\\ n_z=-\sin\theta\end{cases} \quad\Rightarrow\quad \theta=a\tan 2(-n_z,\cos\phi\cdot n_x+\sin\phi\cdot n_y)$$

$$\begin{cases}-\sin\phi\cdot s_x+\cos\phi\cdot s_y=\cos\psi\\ -\sin\phi\cdot a_x+\cos\phi\cdot a_y=-\sin\psi\end{cases}$$

$$\Downarrow$$

$$\psi=a\tan 2(\sin\phi\cdot a_x-\cos\phi\cdot a_y,-\sin\phi\cdot s_x+\cos\phi\cdot s_y)$$

Inverse Kinematics

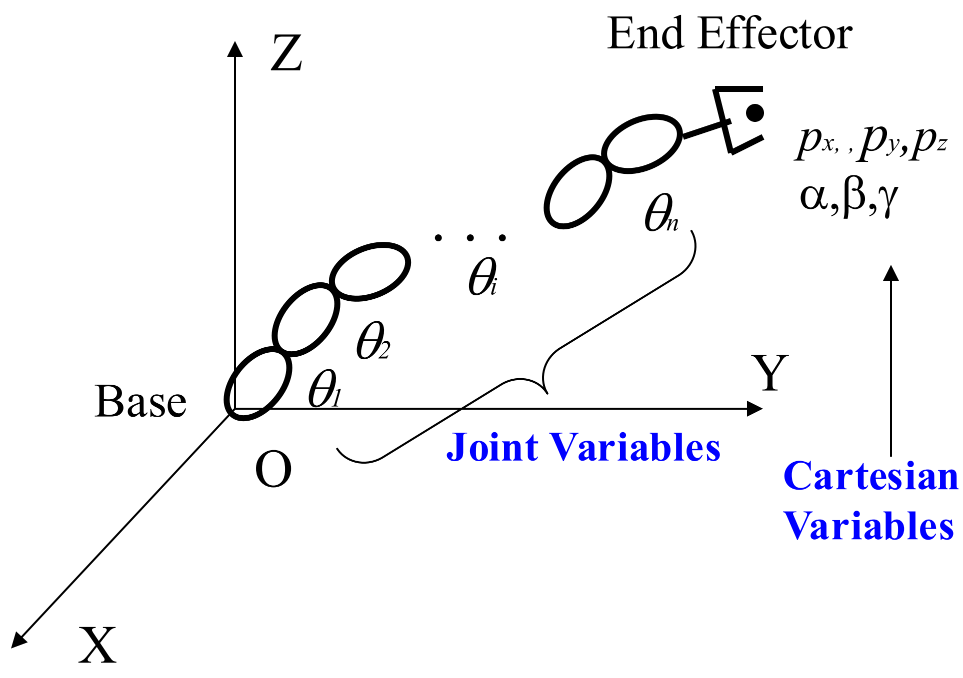
Outline: Inverse Kinematics (IK)

- Problem formulation
- Existence
- Multiple Solutions
- Algebraic Approach
- Geometric Approach
- Example

Inverse Kinematics

Given the position and orientation of the end-effector, find the joint variables that achieve such configuration.

$$\boldsymbol{\theta} = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \cdot \\ \cdot \\ \cdot \\ \theta_n \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} p_x \\ p_y \\ p_z \\ \alpha \\ \beta \\ \gamma \end{bmatrix}$$



$$\text{(Joint)} \quad \theta \longleftarrow \mathbf{x} \quad \text{(Cartesian)}$$

Inverse Kinematics

Inverse Kinematics

The general IK problem can be stated as follows:

Given a 4×4 homogeneous transformation

$$H = \begin{bmatrix} R & o \\ 0 & 1 \end{bmatrix} \in SE(3) = \mathbf{R}^3 \times SO(3)$$

with $R \in SO(3)$, find (one or all) solutions of the equation

$$T_n^0(q_1, \cdots, q_n) = H \qquad (*)$$

where

$$T_n^0(q_1, \cdots, q_n) = T_1^0(q_1) \cdots T_n^{n-1}(q_n)$$

Here, H represents the desired position and orientation of the end-effector, and our task is to find the values for the joint variables $q_1, \cdots, q_n$, so that $T_n^0(q_1, \cdots, q_n) = H.$ $(*)$

Inverse Kinematics

- Equation (✱) results in twelve nonlinear equations in n unknown variables, which can be written as

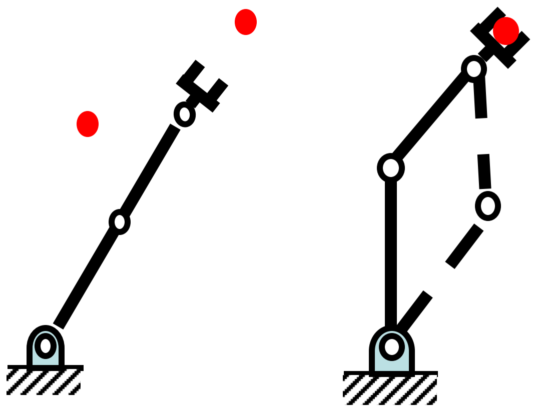
$$T_{ij}(q_1, \ldots, q_n) = h_{ij}\ , i = 1, 2, 3, j = 1, \ldots, 4,$$

where T_{ij} , h_{ij} refer to the twelve nontrivial entries of T_n^0 and H, respectively. (Since the bottom row of both T_n^0 and H are (0,0,0,1), four of the sixteen equations represented by (✱) are trivial.)

- Whereas the Forward Kinematics problem always has a unique solution that can be obtained simply by evaluating the forward equations, the IK problem may or may not have a solution.
- Even if a solution exists, it may or may not be unique.

Inverse Kinematics

- **Given the numerical value of T_n^0, find $\theta_1, \cdots \theta_n$.**
- **For 6 DOF arm (12 equations and 6 unknown)**
- **Among 9 corresponding to rotation, only 3 are independent**
- **3 from orientation, 3 from position: 6 equation, 6 known.**
- **Nonlinear equations: difficult to solve**
  - Existence of solution
  - Multiple solution
  - Method of solution



- **Existence:**
  - The existence relates to the manipulator's workspace
  - **Workspace**: set of all points that manipulator can reach
  - For a solution to exist the point should be in manipulator's work space

Inverse Kinematics

- **Transformation Matrix**

$$\begin{bmatrix} n_x & s_x & a_x & p_x \\ n_y & s_y & a_y & p_y \\ n_z & s_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = T_1^0 T_2^1 T_3^2 T_4^3 T_5^4 T_6^5 = T(\theta) \Rightarrow \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \\ \theta_5 \\ \theta_6 \end{bmatrix}$$

Special cases make the closed-form arm solution possible:

1. **Three adjacent joint axes intersecting (PUMA， Stanford)**
2. **Three adjacent joint axes parallel to one another (ASEA， MINIMOVER)**

Example

- **For the Stanford manipulator, which is an example of a spherical (RRP) manipulator with a spherical wrist, suppose that the desired position and orientation of the final frame are given by**

$$H = \begin{bmatrix} 0 & 1 & 0 & -0.154 \\ 0 & 0 & 1 & 0.763 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example

To find the corresponding joint variables θ_1, θ_2, d_3, θ_4, θ_5, and θ_6 we must solve the following simultaneous set of nonlinear trigonometric equations:

$$
\begin{aligned}
c_1[c_2(c_4c_5c_6 - s_4s_6) - s_2s_5c_6] - s_1(s_4c_5c_6 + c_4s_6) &= 0 \\
s_1[c_2(c_4c_5c_6 - s_4s_6) - s_2s_5c_6] + c_1(s_4c_5c_6 + c_4s_6) &= 0 \\
-s_2(c_4c_5c_6 - s_4s_6) - c_2s_5c_6 &= 1 \\
c_1[-c_2(c_4c_5s_6 + s_4c_6) + s_2s_5s_6] - s_1(-s_4c_5s_6 + c_4c_6) &= 1 \\
s_1[-c_2(c_4c_5s_6 + s_4c_6) + s_2s_5s_6] + c_1(-s_4c_5s_6 + c_4c_6) &= 0 \\
s_2(c_4c_5s_6 + s_4c_6) + c_2s_5s_6 &= 0 \\
c_1(c_2c_4s_5 + s_2c_5) - s_1s_4s_5 &= 0 \\
s_1(c_2c_4s_5 + s_2c_5) + c_1s_4s_5 &= 1 \\
-s_2c_4s_5 + c_2c_5 &= 0 \\
c_1s_2d_3 - s_1d_2 + d_6(c_1c_2c_4s_5 + c_1c_5s_2 - s_1s_4s_5) &= -0.154 \\
s_1s_2d_3 + c_1d_2 + d_6(c_1s_4s_5 + c_2c_4s_1s_5 + c_5s_1s_2) &= 0.763 \\
c_2d_3 + d_6(c_2c_5 - c_4s_2s_5) &= 0
\end{aligned}
$$

$$
H = \begin{bmatrix} 0 & 1 & 0 & -0.154 \\ 0 & 0 & 1 & 0.763 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Inverse Kinematics

Method of solution:

- All proposed manipulator solution strategies can be spit into two broad classes: closed-form solutions and numerical solutions.
- Numerical solutions generally are much slower than the corresponding closed-form solution; in fact, that, for most uses, we are not interested in the numerical approach to solution of kinematics.
- We will restrict our attention to closed-form solution methods.

Inverse Kinematics

Closed form solution method :

- “Closed form" means a solution method based on analytic expressions or on the solution of a polynomial of degree 4 or less, such that non-iterative calculations suffice to arrive at a solution.
- Within the class of closed-form solutions, we distinguish two methods of obtaining the solution: algebraic and geometric.
- Any geometric methods brought to bear are applied by means of algebraic expressions, so the two methods are similar. The methods differ perhaps in approach only.

Inverse Kinematics

Why closed-form solution methods?

1. In certain applications, such as tracking a welding seam whose location is provided by a vision system, the inverse kinematic equations must be solved at a rapid rate, say every 20 ms, and having closed form expressions rather than an iterative search is a practical necessity.
2. The kinematic equations in general have multiple solutions. Having closed form solutions allows one to develop rules for choosing a particular solution among several.

Inverse Kinematics

Kinematic Decoupling Approach

- A sufficient condition that a manipulator with six revolute joints have a closed-form solution is that three neighboring joint axes intersect at a point.
- For 6-DoF manipulators, with the last three joints intersecting at a point, it is possible to decouple the IK problem into two simpler problems:

  1) inverse position kinematics;

  2) inverse orientation kinematics.
- Using kinematic decoupling, we can consider the position and orientation problems independently.

Inverse Kinematics

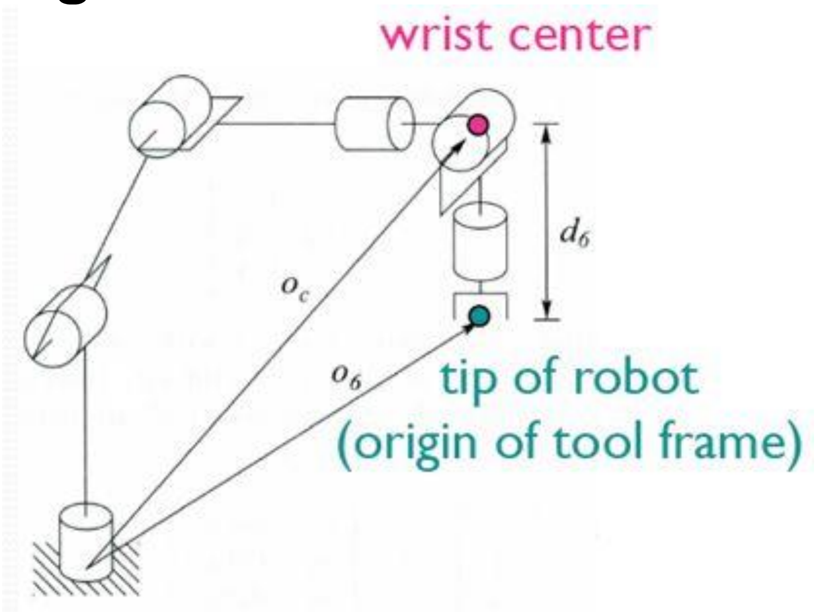
Spherical wrist

- The assumption of a spherical wrist means that the axes z_3, z_4, and z_5 intersect at o_c and hence the origins o_4 and o_5 assigned by the DH-convention will always be at the wrist center o_c.
- Therefore, the motion of the final three links about these axes will not change the position of o_c.
- Thus, the position of the wrist center is a function of only the first three joint variables.

Inverse Kinematics

Kinematic Decoupling

- In this way, the inverse kinematics problem may be separated into two simpler problems.
  - First, finding the position of the intersection of the wrist axes, called the wrist center.
  - Then finding the orientation of the wrist.



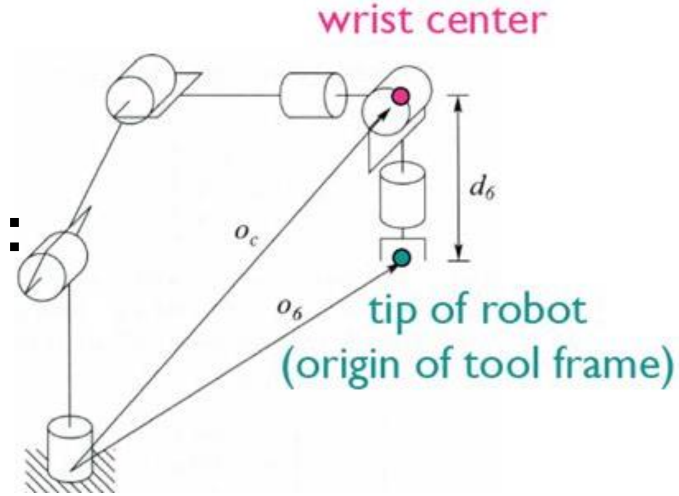
Inverse Kinematics

Kinematic Decoupling

- **Inverse kinematic equation**

$$T_6^0(q_1, \cdots, q_6) = H$$

can be represented as two equations:



$$R_6^0(q_1, \cdots, q_6) = R$$
$$o_6^0(q_1, \cdots, q_6) = o_6$$

- **By the spherical wrist, the origin of the tool frame (whose desired coordinates are given by o_6) is simply obtained by a translation of distance d_6 along z_5 from o_c.**

$$o_6 = o_c^0 + R\begin{bmatrix} 0 \\ 0 \\ d_6 \end{bmatrix} = o_c^0 + d_6 R\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Inverse Kinematics

Kinematic Decoupling

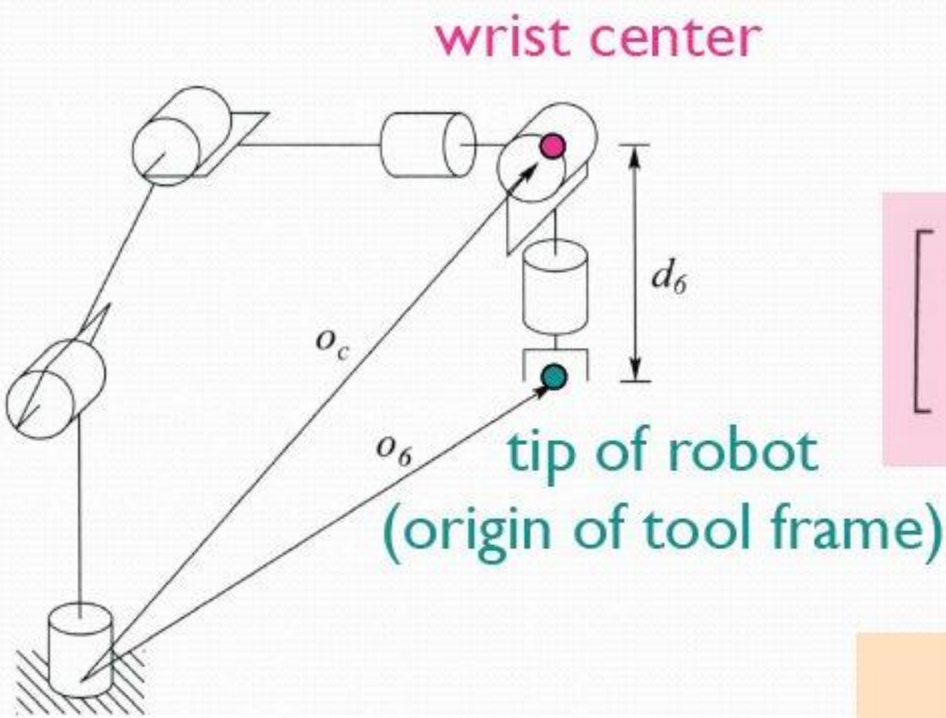
- In order to have the end-effector of the robot at the point with coordinates given by o_6 and with the orientation given by $R = (r_{ij})$, it is necessary and sufficient that the wrist center o_c have coordinates given by

$$o_c^0 = o_6 - d_6 R \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} o_{cx} \\ o_{cy} \\ o_{cz} \end{bmatrix} = \begin{bmatrix} o_{6x} - d_6 r_{13} \\ o_{6y} - d_6 r_{23} \\ o_{6z} - d_6 r_{33} \end{bmatrix}$$

- Using this equation, we can calculate the first three joint variables, and therefore, R_3^0.

Inverse Kinematics

Kinematic Decoupling



$$o = o_c^0 + d_6 R \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} = \begin{bmatrix} o_x - d_6 r_{13} \\ o_y - d_6 r_{23} \\ o_z - d_6 r_{33} \end{bmatrix}$$

position

$$R = R_3^0 R_6^3$$

$$R_6^3 = (R_3^0)^{-1} R = (R_3^0)^T R$$

orientation

Inverse Kinematics

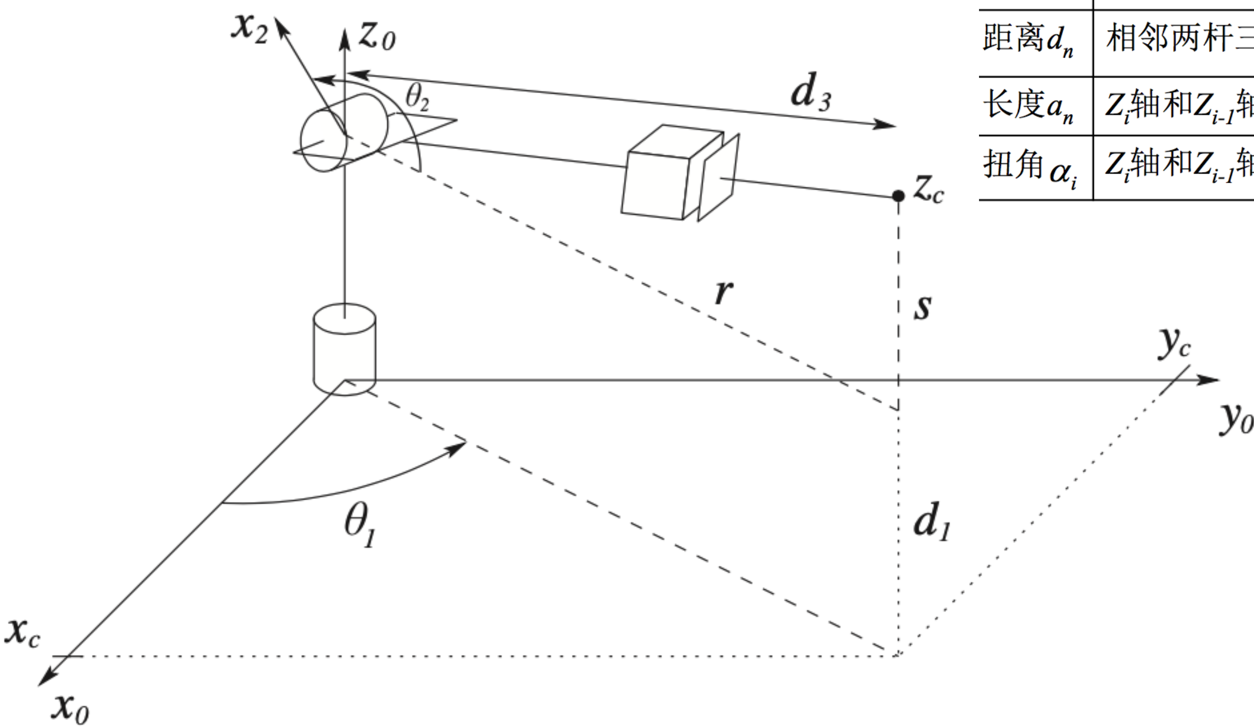
Inverse Position: A Geometric Approach

- **For the common kinematic arrangements that we consider, we can use a geometric approach to find the variables, q_1, q_2, q_3 corresponding to $\mathbf{o}^0_c$.**
- **The general idea of the geometric approach is to solve for joint variable q_i by projecting the manipulator onto the $x_{i-1} - y_{i-1}$ plane and solving a simple trigonometry problem.**
- **For example, to solve for θ_1, we project the arm onto the $x_0 - y_0$ plane and use trigonometry to find θ_1. We will illustrate this method with two important examples: the spherical (RRP) and the articulated (RRR) arms.**

Inverse Position: A Geometric Approach

Spherical Configuration

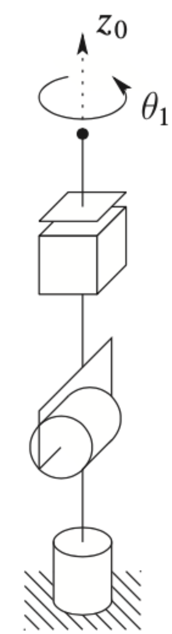
名称	含义	正负
转角θ_i	X_i轴和X_{i-1}轴两轴线之间夹角	右手法则z_{i-1}
距离d_n	相邻两杆三轴线两条公垂线的距离	沿z_{i-1}正向+
长度a_n	Z_i轴和Z_{i-1}轴两轴线的公垂线长度	与X_i正向一致
扭角α_i	Z_i轴和Z_{i-1}轴线之间的扭角	右手法则X_i



First three joints of a spherical manipulator.

$$\boxed{\theta_1 = \text{Atan2}(x_c, y_c)}$$

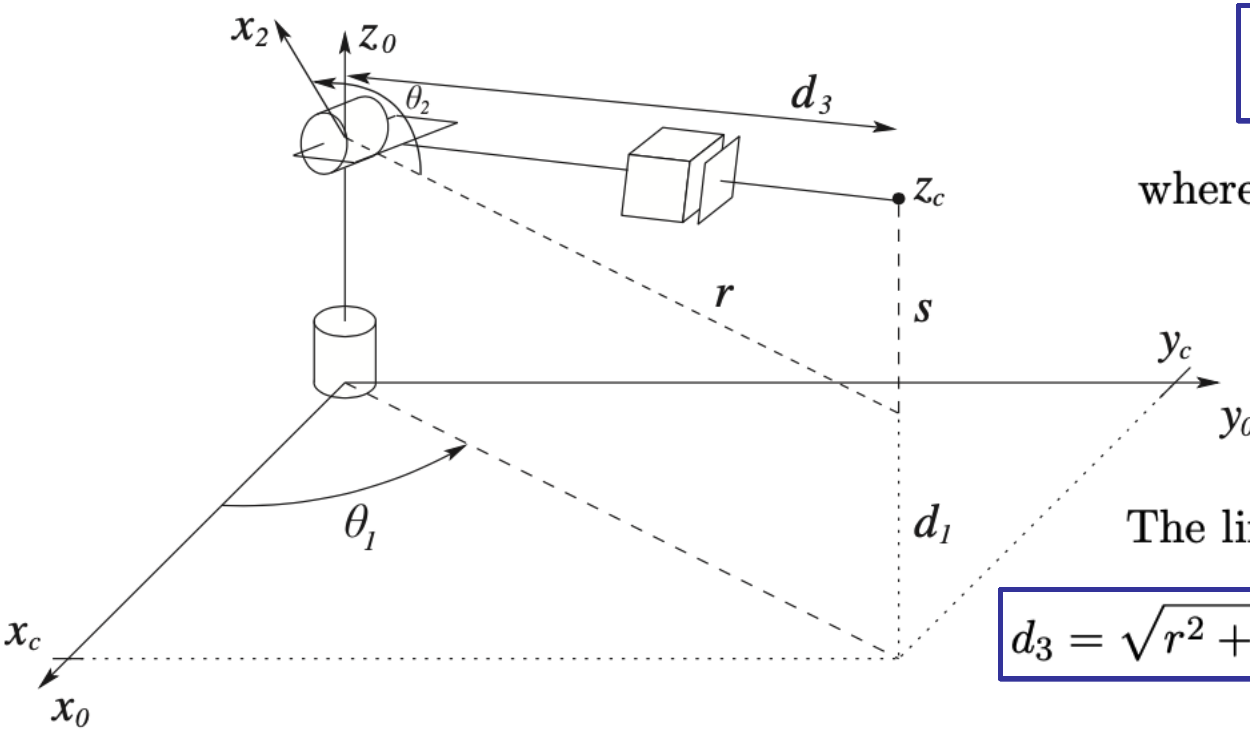
$$\boxed{\theta_1 = \pi + \text{Atan2}(x_c, y_c)}$$



Singular configuration for a spherical manipulator in which the wrist center lies on the z_0 axis.

Inverse Position: A Geometric Approach

Spherical Configuration



First three joints of a spherical manipulator.

$$\boxed{\theta_1 = \mathrm{Atan2}(x_c, y_c)}$$

$$\boxed{\theta_1 = \pi + \mathrm{Atan2}(x_c, y_c)}$$

$$\boxed{\theta_2 = \mathrm{Atan2}(r, s) + \frac{\pi}{2}}$$

where $r^2 = x_c^2 + y_c^2$ and $s = z_c - d_1$.

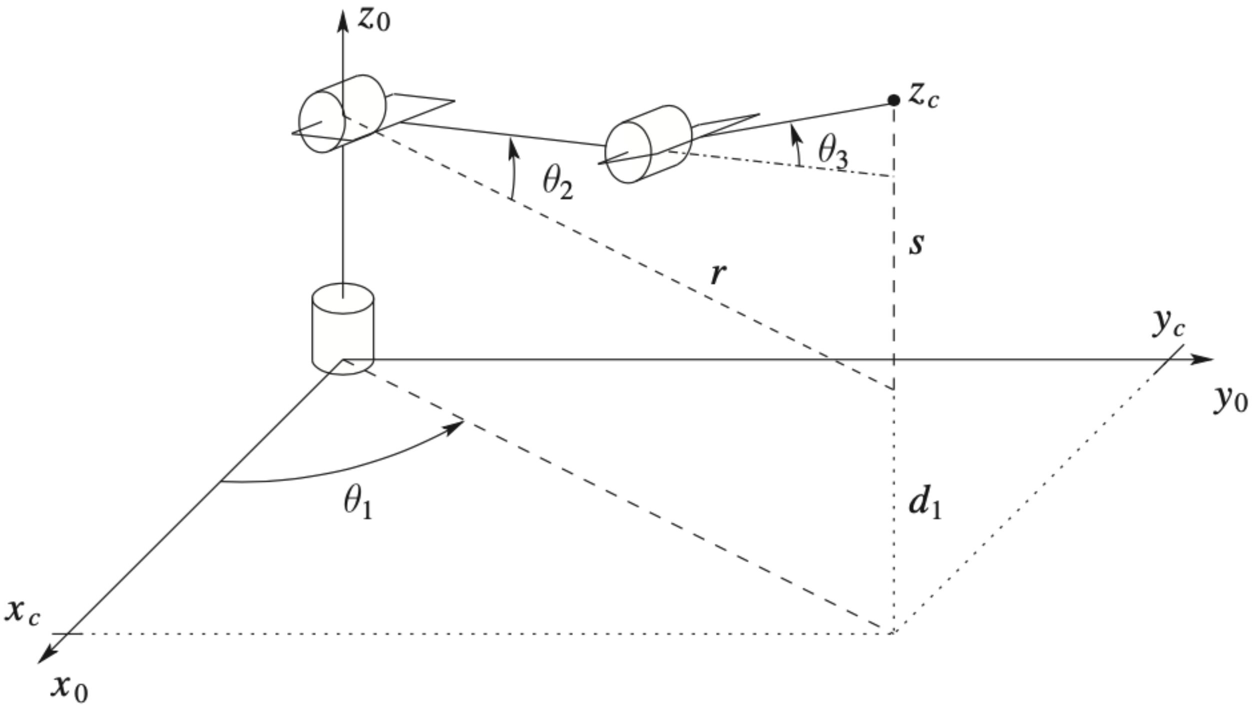
The linear distance d_3 is found as

$$\boxed{d_3 = \sqrt{r^2 + s^2} = \sqrt{x_c^2 + y_c^2 + (z_c - d_1)^2}}$$

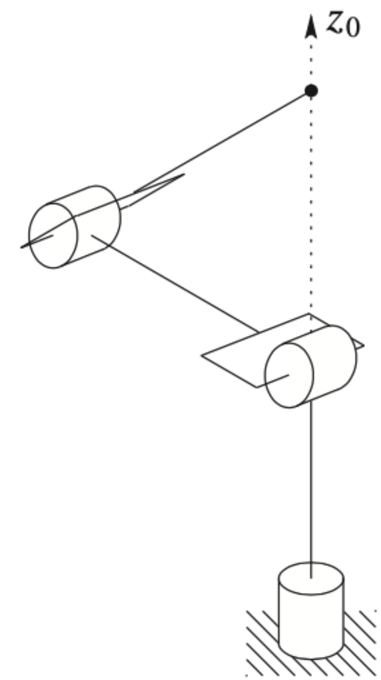
The negative square root solution for d_3 is disregarded and thus in this case we obtain two solutions to the inverse position kinematics as long as the wrist center does not intersect z_0.

Inverse Position: A Geometric Approach

Articulated Configuration



First three joints of an elbow manipulator.



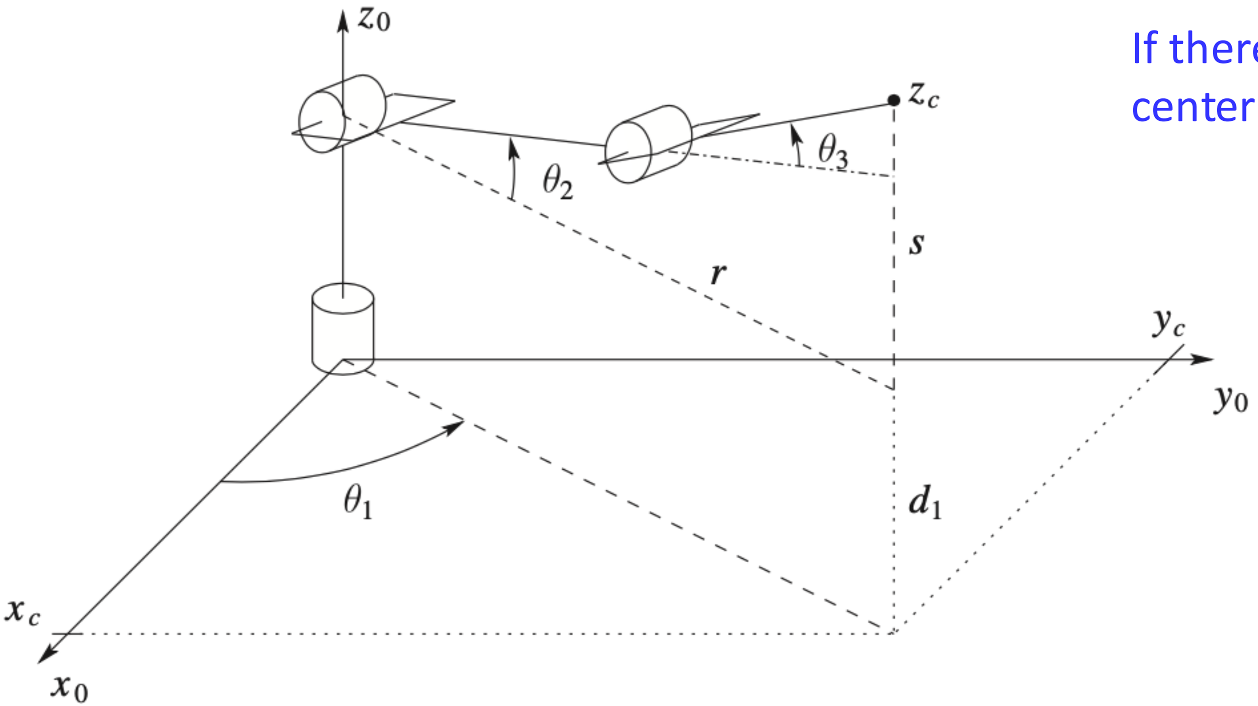
Singular configuration for an elbow manipulator in which the wrist center lies on the z_0 axis.

$$\boxed{\theta_1 = \text{Atan2}(x_c, y_c)}$$

$$\boxed{\theta_1 = \pi + \text{Atan2}(x_c, y_c)}$$

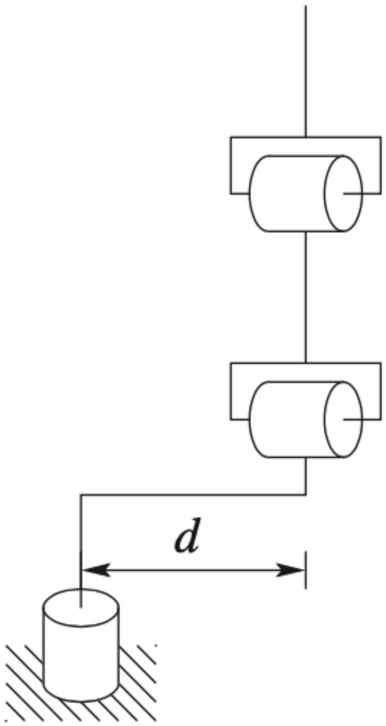
Inverse Position: A Geometric Approach

Articulated Configuration



First three joints of an elbow manipulator.

If there is an offset *d* then the wrist center cannot intersect z_0.



Elbow manipulator with shoulder offset.

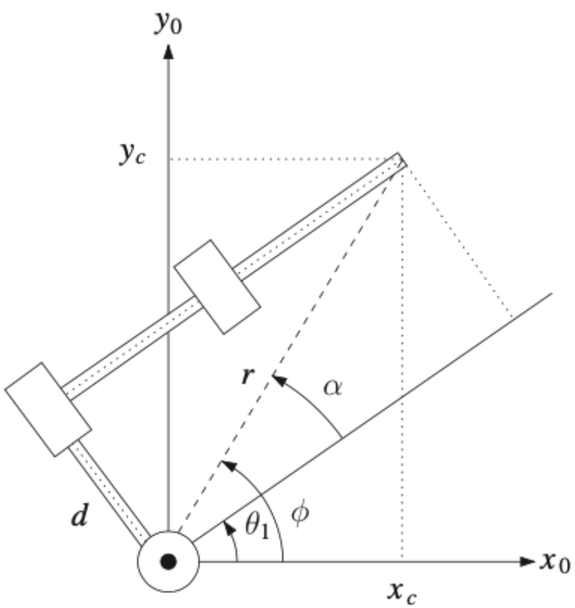
$$\boxed{\theta_1 = \text{Atan2}(x_c, y_c)}$$

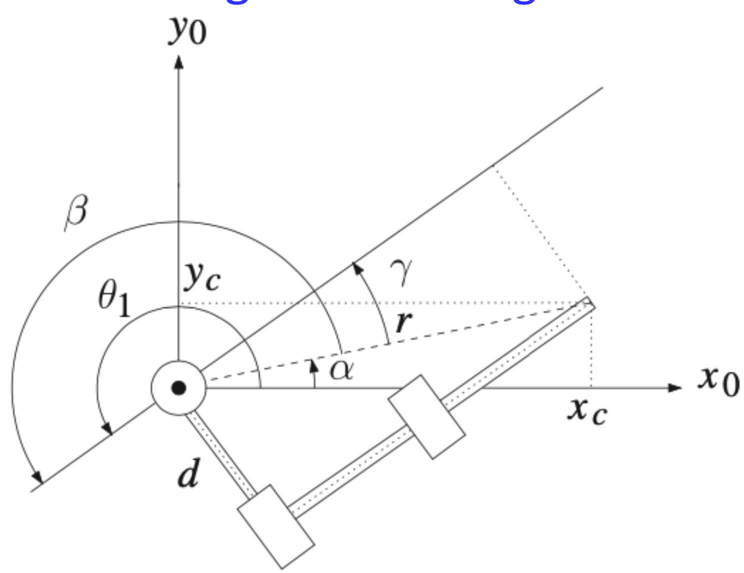
$$\boxed{\theta_1 = \pi + \text{Atan2}(x_c, y_c)}$$

Inverse Position: A Geometric Approach

Articulated Configuration

These correspond to the so-called left arm and right arm configurations.





$$\boxed{\theta_1 = \phi - \alpha}$$

in which

$$\begin{aligned}
\phi &= \operatorname{Atan2}(x_c, y_c) \\
\alpha &= \operatorname{Atan2}\left(\sqrt{r^2 - d^2}, d\right) \\
&= \operatorname{Atan2}\left(\sqrt{x_c^2 + y_c^2 - d^2}, d\right)
\end{aligned}$$

$$\boxed{\theta_1 = \operatorname{Atan2}(x_c, y_c) + \operatorname{Atan2}\left(-\sqrt{r^2 - d^2}, -d\right)}$$

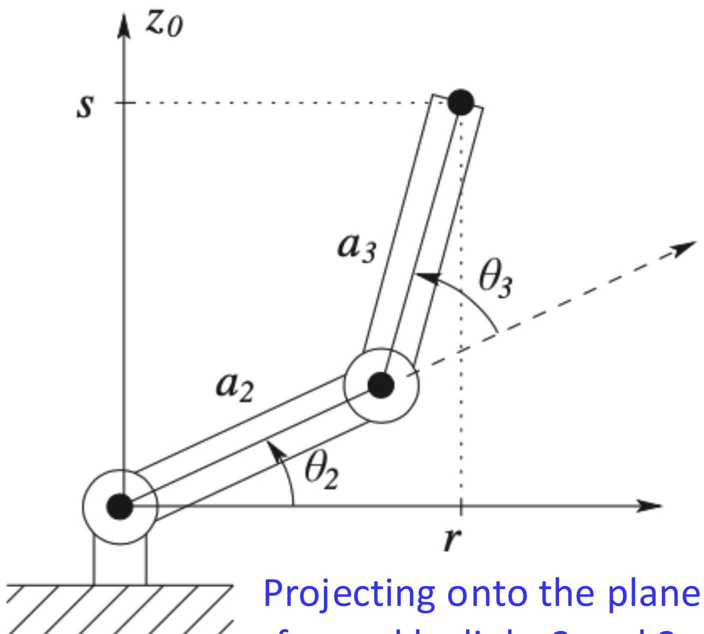
To see this, note that

$$\begin{aligned}
\theta_1 &= \alpha + \beta \\
\alpha &= \operatorname{Atan2}(x_c, y_c) \\
\beta &= \gamma + \pi \\
\gamma &= \operatorname{Atan2}\left(\sqrt{r^2 - d^2}, d\right)
\end{aligned}$$

Inverse Position: A Geometric Approach

Articulated Configuration

To find the angles θ_2, θ_3 for the elbow manipulator given θ_1, we consider the plane formed by the 2nd and 3rd links.



Projecting onto the plane formed by links 2 and 3.

$$\cos\theta_3 = \frac{r^2 + s^2 - a_2^2 - a_3^2}{2a_2a_3}$$

since $r^2 = x_c^2 + y_c^2 - d^2$ and $s = z_c - d_1$

$$\cos\theta_3 = \frac{x_c^2 + y_c^2 - d^2 + (z_c - d_1)^2 - a_2^2 - a_3^2}{2a_2a_3} := D$$

Hence, $\boxed{\theta_3 = \text{Atan2}\left(D, \pm\sqrt{1 - D^2}\right)}$

The two solutions for θ_3 correspond to the elbow down position and elbow up position, respectively.

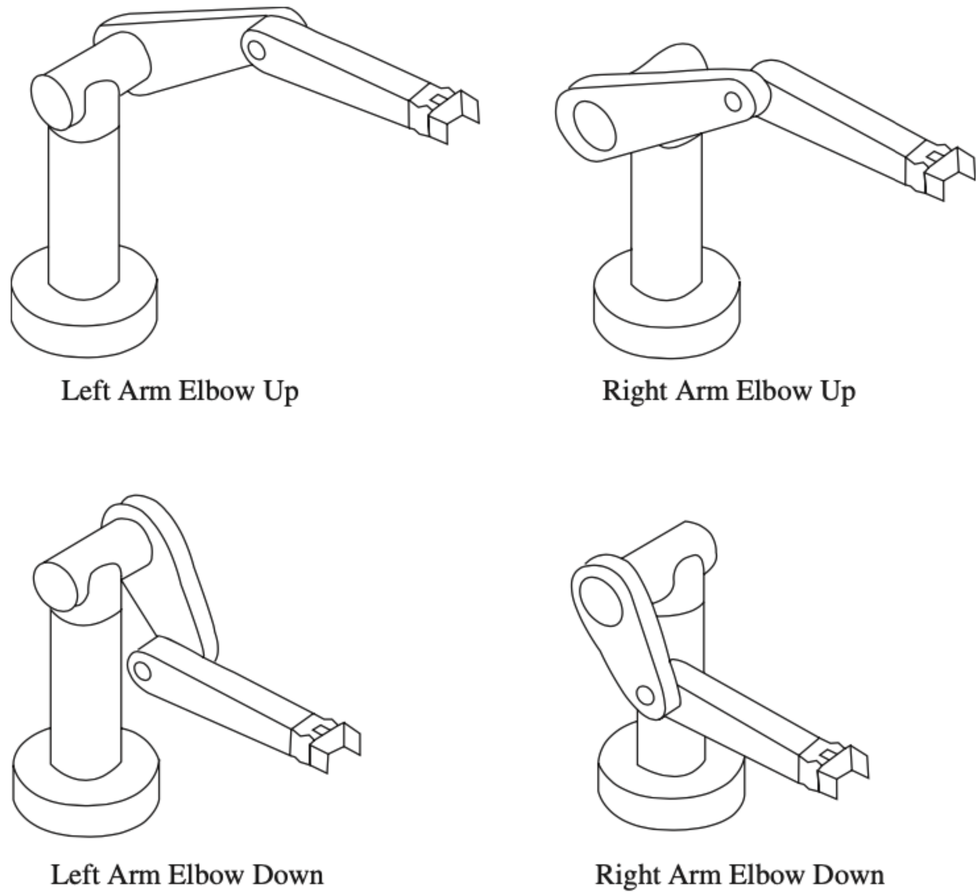
Similarly θ_2 is given as

$$\begin{aligned}\theta_2 &= \text{Atan2}(r, s) - \text{Atan2}(a_2 + a_3c_3, a_3s_3)\\ &= \text{Atan2}\left(\sqrt{x_c^2 + y_c^2 - d^2}, z_c - d_1\right) - \text{Atan2}(a_2 + a_3c_3, a_3s_3)\end{aligned}$$

Inverse Position: A Geometric Approach

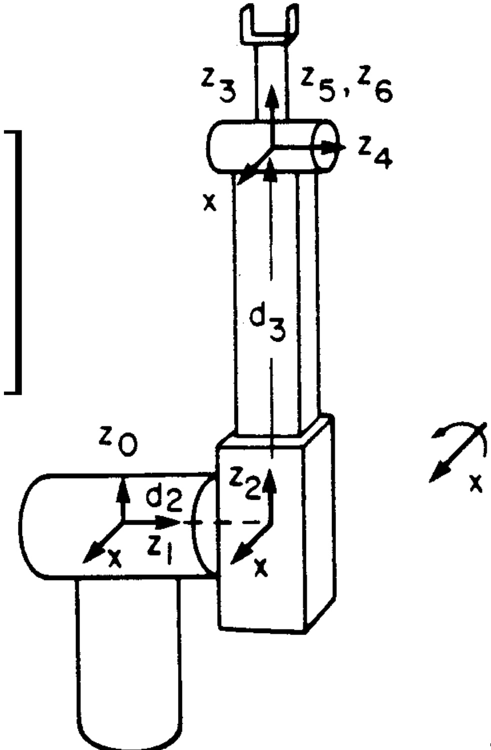
Articulated Configuration

There are four solutions to the inverse position kinematics as shown in the Figure. . These correspond to the situations left arm–elbow up, left arm–elbow down, right arm–elbow up and right arm–elbow down.



Example

- **Solving the inverse kinematics of Stanford arm**



$$T_6^0 = T_1^0 T_2^1 T_3^2 T_4^3 T_5^4 T_6^5 = \begin{bmatrix} n_x & s_x & a_x & p_x \\ n_y & s_y & a_y & p_y \\ n_z & s_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$(T_1^0)^{-1} T_6^0 = T_2^1 T_3^2 T_4^3 T_5^4 T_6^5 = T_6^1$$

$$T_6^1 = \begin{bmatrix} X & X & X & C\theta_1 p_x + S\theta_1 p_y \\ X & X & X & -p_z \\ X & X & X & -S\theta_1 p_x + C\theta_1 p_y \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} X & X & X & S\theta_2 \cdot d_3 \\ X & X & X & -C\theta_2 \cdot d_3 \\ X & X & X & 0.1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example

- **Solving the inverse kinematics of Stanford arm**

$$-\sin\theta_1 \cdot p_x + \cos\theta_1 \cdot p_y = 0.1 \qquad \textbf{Equation (1)}$$

$$\cos\theta_1 \cdot p_x + \sin\theta_1 \cdot p_y = \sin\theta_2 \cdot d_3 \qquad \textbf{Equation (2)}$$

$$-p_z = -\cos\theta_2 \cdot d_3 \qquad \textbf{Equation (3)}$$

In Equ. (1), let

$$p_x = r \cdot \cos\alpha, \qquad p_y = r \cdot \sin\alpha, \qquad r = \sqrt{p_x^2 + p_y^2}, \qquad \alpha = \mathrm{atan2}\left(\frac{p_y}{p_x}\right)$$

$$\sin\alpha \cdot \cos\theta_1 - \sin\theta_1 \cdot \cos\alpha = 0.1/r \Rightarrow \left.\begin{aligned} \sin(\alpha - \theta_1) &= 0.1/r \\ \cos(\alpha - \theta_1) &= \pm\sqrt{1 - (0.1/r)^2} \end{aligned}\right\}$$

$$\theta_1 = \mathrm{atan2}\left(\frac{p_y}{p_x}\right) - \mathrm{atan2}\left(\frac{0.1}{\pm\sqrt{r^2 - 0.1^2}}\right)$$

$$\theta_2 = \mathrm{atan2}\left(\frac{\cos\theta_1 p_x + \sin\theta_1 p_y}{p_z}\right)$$

$$d_3 = \frac{p_z}{\cos\theta_2}$$

Example

- **Solving the inverse kinematics of Stanford arm**

$$(T_4^3)^{-1}(T_3^2)^{-1}(T_2^1)^{-1}(T_1^0)^{-1}T_6^0 = T_5^4 T_6^5 = \begin{bmatrix} X & X & S\theta_5 & 0 \\ X & X & -C\theta_5 & 0 \\ X & X & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

From term (3,3)

$$-S\theta_4[C\theta_2(C\theta_1 a_x + S\theta_1 a_y) - S\theta_2 a_z] + C\theta_4(-S\theta_1 a_x + C\theta_1 a_y) = 0$$

$$\theta_4 = \text{atan2}\left(\frac{-S\theta_1 a_x + C\theta_1 a_y}{C\theta_2(C\theta_1 a_x + S\theta_1 a_y) - S\theta_2 a_z}\right)$$

$$\theta_5 = \text{atan2}\left(\frac{S\theta_5}{C\theta_5}\right)$$

From term (1,3), (2,3)

$$\begin{cases} S\theta_5 = C\theta_4(C\theta_2(C\theta_1 a_x + S\theta_1 a_y) - S\theta_2 a_z] + S\theta_4(-S\theta_1 a_x + C\theta_1 a_y) \\ C\theta_5 = S\theta_2(C\theta_1 a_x + S\theta_1 a_y) + C\theta_2 a_z) \end{cases}$$

Example

- **Solving the inverse kinematics of Stanford arm**

$$(T_5^4)^{-1}(T_4^3)^{-1}(T_3^2)^{-1}(T_2^1)^{-1}(T_1^0)^{-1}T_6^0 = T_6^5 = \begin{bmatrix} C\theta_6 & -S\theta_6 & 0 & 0 \\ S\theta_6 & C\theta_6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{cases} S\theta_6 = -C\theta_5\{C\theta_4[C\theta_2(C\theta_1 s_x + S\theta_1 s_y) - S\theta_2 s_z] + S\theta_4(-S\theta_1 s_x + C\theta_1 s_y)\} + S\theta_5[S\theta_2(C\theta_1 s_x + S\theta_1 s_y) + C\theta_2 s_z] \\ C\theta_6 = -S\theta_4[C\theta_2(C\theta_1 s_x + S\theta_1 s_y) - S\theta_2 s_z] + C\theta_4(-S\theta_1 s_x + C\theta_1 s_y) \end{cases}$$

$$\theta_6 = a\tan 2\left(\frac{S\theta_6}{C\theta_6}\right)$$

Inverse Kinematics: Solvability

- **Method of solution**: No general algorithm to solve nonlinear equations
- A manipulator is said to be solvable if the set of all joint variable associated to a given position and orientation can be determined.
  1. **Closed-form solutions**
  2. **Numerical solutions**
- Closed-form solutions are considered
- Algebraic and geometric solution
- All systems with revolute and prismatic joints having 6 DOF in a single series chain are solvable
- Only in special cases it could be solved analytically
- Robots with analytical solutions have many intersecting joints or many α_i are 0 or 90 degrees.
- Most manipulators designed to have closed-form solutions
- Closed-form solutions exist for decouple manipulator (three joints intersect.)

Summary

Inverse Kinematics

- General Inverse Kinematic Problem
- Kinematic Decoupling
- Inverse Position: A Geometric Approach
- Example: Stanford Arm (Algebraic Approach)

Homework 6

Homework 6：posted at http://bb.sustech.edu.cn

Due date: March 17, 2025

Next class: March 17, 2025（Monday）

作业要求（Requirements）：

1. **文件格式为以自己作业序号姓名学号命名的pdf文件；**
   （File name: YourSID_ YourName_06.pdf）
2. **作业里也写上自己的姓名和学号。**
   （Write your name and SID in the homework）