南方科技大学
SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY

Robot Modeling & Control ME331

Section 9： Kinematics VIII

Chenglong Fu（付成龙）
Dept. of MEE， SUSTech

Outline

- Review
  - Addition of Angular Velocities
  - Linear Velocity
  - Derivation of the Jacobian
- The Tool Velocity
- The Analytical Jacobian
- Singularities

Velocity Kinematics

Addition of Angular Velocities

$$\omega_{0,2}^{0} = \omega_{0,1}^{0} + R_1^0 \omega_{1,2}^{1}$$

The above reasoning can be extended to any number of coordinate systems.

In particular, suppose that we are given

$$R_n^0 = R_1^0 R_2^1 \cdots R_n^{n-1}$$

Extending the above reasoning we obtain

$$\dot{R}_n^0 = S\left(\omega_{0,n}^{0}\right) R_n^0$$

In which

$$\omega_{0,n}^{0} = \omega_{0,1}^{0} + R_1^0 \omega_{1,2}^{1} + R_2^0 \omega_{2,3}^{2} + R_3^0 \omega_{3,4}^{3} + \ldots + R_{n-1}^0 \omega_{n-1,n}^{n-1}$$

$$= \omega_{0,1}^{0} + \omega_{1,2}^{0} + \omega_{2,3}^{0} + \omega_{3,4}^{0} + \ldots + \omega_{n-1,n}^{0}$$

Velocity Kinematics

Linear Velocity of a Point Attached to a Moving Frame

Suppose the point p is rigidly attached to the frame $o_1x_1y_1z_1$, and the motion of the frame $o_1x_1y_1z_1$ relative to $o_0x_0y_0z_0$ is

$$H_1^0(t) = \begin{bmatrix} R_1^0(t) & o_1^0(t) \\ 0 & 1 \end{bmatrix}$$

For simplicity we omit the argument t and the subscripts and superscripts on R_1^0 and o_1^0, and write

$$p^0 = Rp^1 + o$$

Differentiating the above expression using the product rule gives

$$\begin{aligned} \dot{p}^0 &= \dot{R}p^1 + \dot{o} \\ &= S(\omega)Rp^1 + \dot{o} \\ &= \omega \times r + v \end{aligned}$$

where $r = Rp^1$ is the vector from o_1 to p expressed in the orientation of the frame $o_0x_0y_0z_0$, and v is the rate at which the origin o_1 is moving.

Velocity Kinematics

Derivation of the Jacobian

Consider an *n*-link manipulator with joint variables $q_1, \cdots, q_n$.

$$T_n^0(q) = \begin{bmatrix} R_n^0(q) & o_n^0(q) \\ 0 & 1 \end{bmatrix}$$

The objective of this section is to relate the linear and angular velocity of the end effector to the vector of joint velocities $\dot{q}(t)$.

Let $S(\omega_n^0) = \dot{R}_n^0 (R_n^0)^T$ define the angular velocity vector ω_n^0 of the end effector, and let $v_n^0 = \dot{o}_n^0$ denote the linear velocity of the end effector.

We seek expressions of the form

$$\begin{bmatrix} v_n^0 \\ \omega_n^0 \end{bmatrix} = \begin{bmatrix} J_v \\ J_w \end{bmatrix} \dot{q}$$

$$\xi = J\dot{q}$$

where ξ is sometimes called a body velocity. Note that this velocity vector is not the derivative of a position variable.

Velocity Kinematics

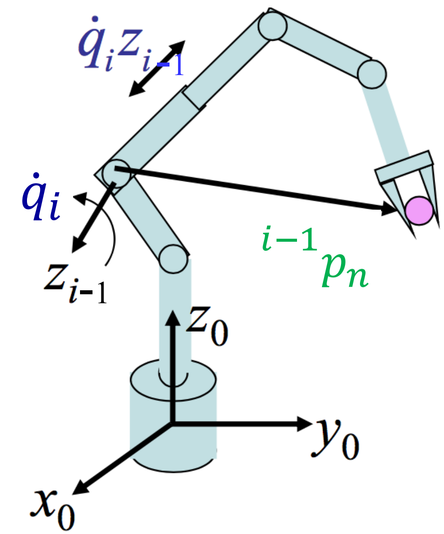
Combining the Linear and Angular Velocity Jacobians

As we have seen in the preceding section, the upper half of the Jacobian J_v is given as

$$J_v = [J_{v_1} \cdots J_{v_n}]$$

in which the i^{th} column J_{v_i} is

$$J_{v_i} = \begin{cases} z_{i-1} \times (o_n - o_{i-1}) & \text{for revolute joint } i \\ z_{i-1} & \text{for prismatic joint } i \end{cases}$$



The lower half of the Jacobian is given as

$$J_\omega = [J_{\omega_1} \cdots J_{\omega_n}]$$

in which the i^{th} column J_{ω_i} is

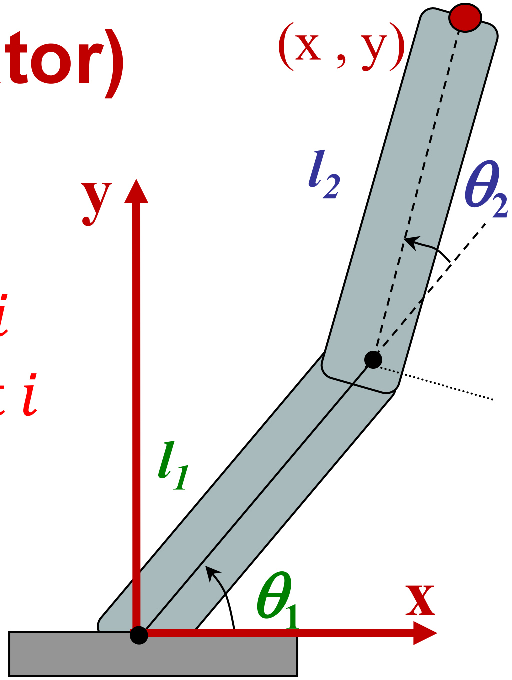
$$J_{\omega_i} = \begin{cases} z_{i-1} & \text{for revolute joint } i \\ 0 & \text{for prismatic joint } i \end{cases}$$

The only quantities needed to compute the Jacobian are the unit vectors z_i and the coordinates of the origins $o_1, \ldots, o_n$.

- z_i are given by the first three elements in the third column of T_i^0.
- o_i is given by the first three elements of the fourth column of T_i^0.

Velocity Kinematics

- **Example (2-DoF Planar Manipulator)**
  - **Given l_1, l_2, *find*: *Jacobian***
    （Use vector product method）



$$J_{v_i} = \begin{cases} z_{i-1} \times (o_n - o_{i-1}) & \text{for revolute joint } i \\ z_{i-1} & \text{for prismatic joint } i \end{cases}$$

$$z_0 = [0 \quad 0 \quad 1]^{\mathrm{T}} \qquad z_1 = [0 \quad 0 \quad 1]^{\mathrm{T}}$$

$$o_2 - o_1 = [l_2 \cos(\theta_1 + \theta_2) \quad l_2 \sin(\theta_1 + \theta_2) \quad 0]^{\mathrm{T}}$$

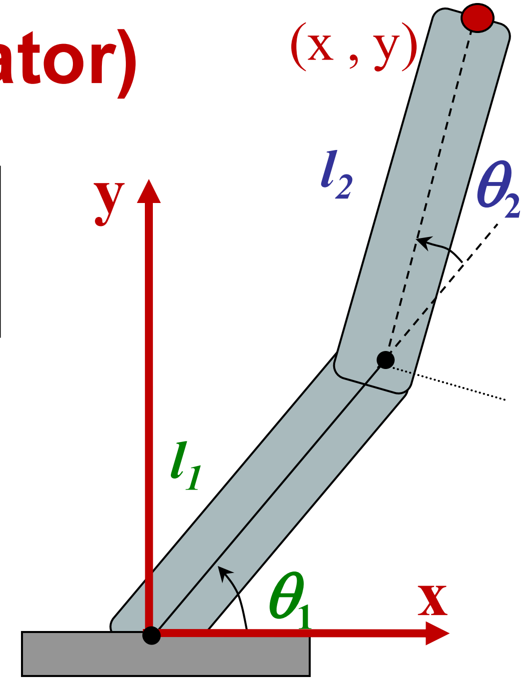
$$o_2 - o_0 = [l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2) \quad l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2) \quad 0]^{\mathrm{T}}$$

$$J_{v_1} = z_0 \times (o_2 - o_0) = \begin{bmatrix} -l_1 \sin\theta_1 - l_2 \sin(\theta_1 + \theta_2) \\ l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2) \\ 0 \end{bmatrix}$$

$$J_{v_2} = z_1 \times (o_2 - o_1) = \begin{bmatrix} -l_2 \sin(\theta_1 + \theta_2) \\ l_2 \cos(\theta_1 + \theta_2) \\ 0 \end{bmatrix} \qquad \boldsymbol{J} = [J_{v_1} \quad J_{v_2}]$$

Velocity Kinematics

- **Example (2-DoF Planar Manipulator)**



$$J_{v_1} = z_0 \times (o_2 - o_0) = \begin{bmatrix} -l_1 \sin\theta_1 - l_2 \sin(\theta_1 + \theta_2) \\ l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2) \\ 0 \end{bmatrix}$$

$$J_{v_2} = z_1 \times (o_2 - o_1) = \begin{bmatrix} -l_2 \sin(\theta_1 + \theta_2) \\ l_2 \cos(\theta_1 + \theta_2) \\ 0 \end{bmatrix}$$

$$\boldsymbol{J_v} = [J_{v_1} \quad J_{v_1}]$$

$$J_{\omega_1} = z_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad J_{\omega_2} = z_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

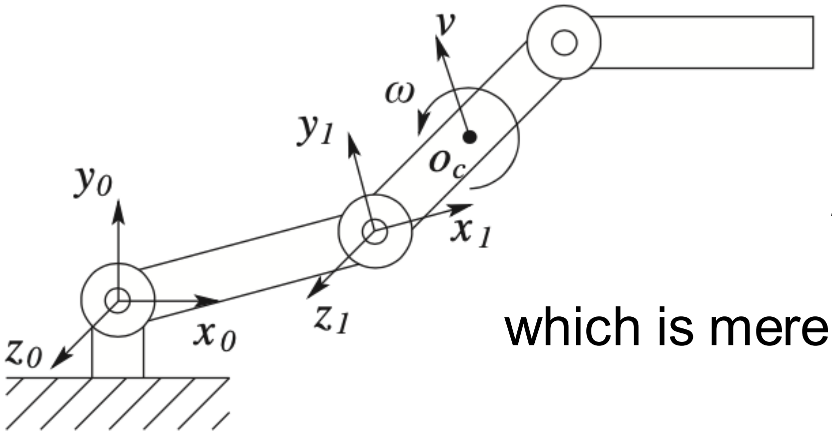
$$\boldsymbol{J_\omega} = [J_{\omega_1} \quad J_{\omega_2}]$$

$$\boldsymbol{J} = \begin{bmatrix} \boldsymbol{J_v} \\ \boldsymbol{J_\omega} \end{bmatrix} = \begin{bmatrix} -l_1 s_1 - l_2 s_{12} & -l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

Velocity Kinematics

- **Example: Jacobian for an Arbitrary Point on a Link**

Consider the three-link planar manipulator. Suppose we wish to compute the linear velocity v and the angular velocity ω of the center of link 2.



In this case we have that

$$J(q) = \begin{bmatrix} z_0 \times (o_c - o_0) & z_1 \times (o_c - o_1) & 0 \\ z_0 & z_1 & 0 \end{bmatrix}$$

which is merely the usual Jacobian with o_c in place of o_n.

homework_1

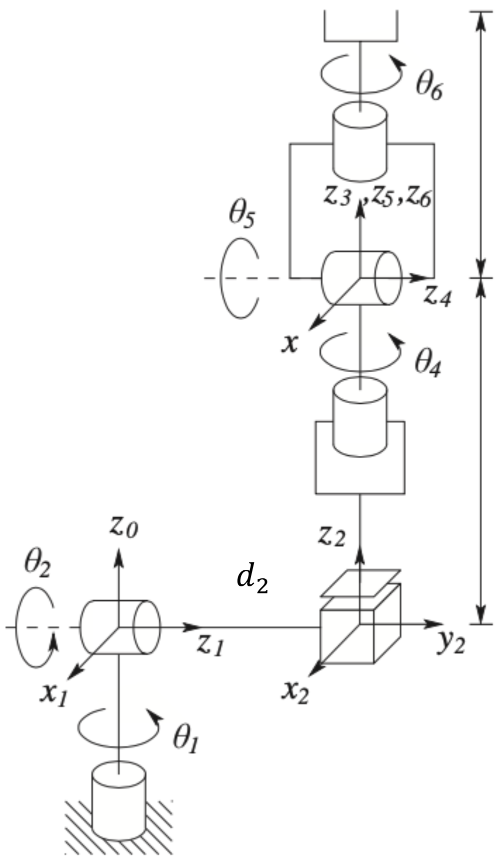
- The third column of the Jacobian is zero, since the velocity of the second link is unaffected by motion of the third link.
- We are treating only kinematic effects here. Reaction forces on link 2 due to the motion of link 3 will influence the motion of link 2.
- In this case the vector o_c must be computed as it is not given directly by the T matrices

Velocity Kinematics

- **Example: Stanford Manipulator**

Consider the Stanford Manipulator. Joint 3 is prismatic and that $o_3=o_4=o_5$. Denoting this common origin by o we see that the columns of the Jacobian have the form

homework_2



$$J_i = \begin{bmatrix} z_{i-1} \times (o_6 - o_{i-1}) \\ z_{i-1} \end{bmatrix} \quad i=1,2$$

$$o_0 = (0,0,0) = o_1$$

$$J_3 = \begin{bmatrix} z_2 \\ 0 \end{bmatrix}$$

$$J_i = \begin{bmatrix} z_{i-1} \times (o_6 - o_{i-1}) \\ z_{i-1} \end{bmatrix} \quad i=4,5,6$$

$$o_3 = \begin{bmatrix} c_1 s_2 d_3 - s_1 d_2 \\ s_1 s_2 d_3 + c_1 d_2 \\ c_2 d_3 \end{bmatrix}$$

$$o_6 = \begin{bmatrix} c_1 s_2 d_3 - s_1 d_2 + d_6(c_1 c_2 c_4 s_5 + c_1 c_5 s_2 - s_1 s_4 s_5) \\ s_1 s_2 d_3 + c_1 d_2 + d_6(c_1 s_4 s_5 + c_2 c_4 s_1 s_5 + c_5 s_1 s_2) \\ c_2 d_3 + d_6(c_2 c_5 - c_4 s_2 s_5) \end{bmatrix}$$

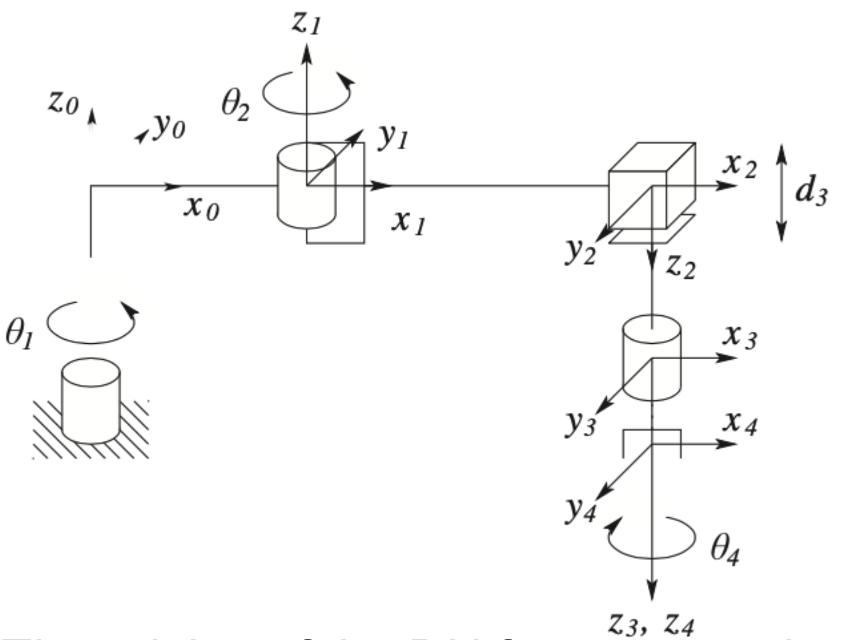
$$z_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad z_1 = \begin{bmatrix} -s_1 \\ c_1 \\ 0 \end{bmatrix} \quad z_2 = \begin{bmatrix} c_1 s_2 \\ s_1 s_2 \\ c_2 \end{bmatrix} \quad z_3 = \begin{bmatrix} c_1 s_2 \\ s_1 s_2 \\ c_2 \end{bmatrix}$$

$$z_4 = \begin{bmatrix} -c_1 c_2 s_4 - s_1 c_4 \\ -s_1 c_2 s_4 + c_1 c_4 \\ s_2 s_4 \end{bmatrix} \quad z_5 = \begin{bmatrix} c_1 c_2 c_4 s_5 - s_1 s_4 s_5 + c_1 s_2 c_5 \\ s_1 c_2 c_4 s_5 + c_1 s_4 s_5 + s_1 s_2 c_5 \\ -s_2 c_4 s_5 + c_2 c_5 \end{bmatrix}$$

Velocity Kinematics

- **Example: SCARA Manipulator**

This Jacobian is a 6 × 4 matrix. Since joints 1, 2, and 4 are revolute and joint 3 is prismatic, and since $o_4 - o_3$ is parallel to z_3 (and thus, $z_3 \times (o_4 - o_3) = 0$), the Jacobian is of the form



$$J = \begin{bmatrix} z_0 \times (o_4 - o_0) & z_1 \times (o_4 - o_1) & z_2 & 0 \\ z_0 & z_1 & 0 & z_3 \end{bmatrix}$$

$z_0 = z_1 = k$, and $z_2 = z_3 = -k$.

Therefore the Jacobian is

$$J = \begin{bmatrix} -a_1 s_1 - a_2 s_{12} & -a_2 s_{12} & 0 & 0 \\ a_1 c_1 + a_2 c_{12} & a_2 c_{12} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & -1 \end{bmatrix}$$

The origins of the DH frames are given by

$$o_1 = \begin{bmatrix} a_1 c_1 \\ a_1 s_1 \\ 0 \end{bmatrix} \qquad o_2 = \begin{bmatrix} a_1 c_1 + a_2 c_{12} \\ a_1 s_1 + a_2 s_{12} \\ 0 \end{bmatrix} \qquad o_4 = \begin{bmatrix} a_1 c_1 + a_2 c_{12} \\ a_1 s_2 + a_2 s_{12} \\ d_3 - d_4 \end{bmatrix}$$

Velocity Kinematics

• The Tool Velocity

Many tasks require that a tool be attached to the end effector. In such cases, it is necessary to relate the velocity of the tool frame to the velocity of the end-effector frame.

Suppose that the tool is rigidly attached to the end effector, and the fixed spatial relationship between the end effector and the tool frame is given by the constant homogeneous transformation matrix

$$T_{\text{tool}}^{6} = \begin{bmatrix} R_{\text{tool}}^{6} & d \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R & d \\ 0 & 1 \end{bmatrix}$$

We will assume that the end effector velocity is given and expressed in coordinates relative to the end-effector frame, that is, we are given ξ_6^6.

We will derive the velocity of the tool expressed in coordinates relative to the tool frame, that is, we will derive $\xi_{\text{Tool}}^{\text{Tool}}$.

Since the two frames are rigidly attached, the angular velocity of the tool frame is the same as the angular velocity of the end-effector frame.

$$R_{\text{tool}}^{0} = R_6^0 R \Rightarrow \dot{R}_{\text{tool}}^{0} = \dot{R}_6^0 R \Rightarrow S(\omega_{\text{tool}}^{0}) R_{\text{tool}}^{0} = S(\omega_6^0) R_6^0 R \Rightarrow S(\omega_{\text{tool}}^{0}) = S(\omega_6^0)$$

Velocity Kinematics

- **The Tool Velocity**

To obtain the tool angular velocity relative to the tool frame we apply a rotational transformation

$$\omega_{\text{tool}}^{\text{tool}} = \omega_6^{\text{tool}} = R_6^{\text{tool}} \omega_6^6 = {R_{\text{tool}}^6}^T \omega_6^6 = R^T \omega_6^6$$

If the end-effector frame is moving with body velocity $\xi = (v_6, \omega_6)$, then the linear velocity of the origin of the tool frame, which is rigidly attached to the end-effector frame, is given by

$$v_{\text{tool}} = v_6 + \omega_6 \times r$$

where r is the vector from the origin of the end-effector frame to the origin of the tool frame.

since d gives the coordinates of the origin of the tool frame with respect to the end-effector frame, and therefore we can express r in coordinates relative to the tool frame as $r^{\text{tool}} = R^T d$. Thus, we write $\omega_6 \times r$ in coordinates with respect to the tool frame as

$$\begin{aligned} \omega_6^{\text{tool}} \times r^{\text{tool}} &= R^T \omega_6^6 \times (R^T d) \\ &= -R^T d \times R^T \omega_6^6 = -S(R^T d) R^T \omega_6^6 = -R^T S(d) R R^T \omega_6^6 = -R^T S(d) \omega_6^6 \end{aligned}$$

Velocity Kinematics

- **The Tool Velocity**

$$v_{\text{tool}} = v_6 + \omega_6 \times r$$
$$\omega_6^{\text{tool}} \times r^{\text{tool}} = -R^T S(d) \omega_6^6$$

To express the free vector v_6 in coordinates relative to the tool frame, we apply the rotational transformation

$$v_6^{\text{tool}} = R^T v_6^6$$

Combining above Equations to obtain the linear velocity of the tool frame and the angular velocity of the tool frame, we have

$$v_{\text{tool}}^{\text{tool}} = R^T v_6^6 - R^T S(d) \omega_6^6$$
$$\omega_{\text{tool}}^{\text{tool}} = R^T \omega_6^6$$

which can be written as the matrix equation

$$\xi_{\text{tool}}^{\text{tool}} = \begin{bmatrix} R^T & -R^T S(d) \\ 0_{3\times 3} & R^T \end{bmatrix} \xi_6^6$$

Velocity Kinematics

- **The Tool Velocity**

$$\xi_{\text{tool}}^{\text{tool}} = \begin{bmatrix} R^T & -R^T S(d) \\ 0_{3\times 3} & R^T \end{bmatrix} \xi_6^6$$

In many cases, it is useful to solve the inverse problem: compute the required end effector-velocity to produce a desired tool velocity. Since

$$\begin{bmatrix} R & S(d)R \\ 0_{3\times 3} & R \end{bmatrix} = \begin{bmatrix} R^T & -R^T S(d) \\ 0_{3\times 3} & R^T \end{bmatrix}^{-1}$$

We can obtain $\xi_6^6 = \begin{bmatrix} R & S(d)R \\ 0_{3\times 3} & R \end{bmatrix} \xi_{\text{tool}}^{\text{tool}}$

This gives the general expression for transforming velocities between two rigidly attached moving frames

$$\xi_A^A = \begin{bmatrix} R_B^A & S(d_B^A)R_B^A \\ 0_{3\times 3} & R_B^A \end{bmatrix} \xi_B^B$$

Velocity Kinematics

- **The Analytical Jacobian**

The Jacobian matrix derived above is sometimes called the **geometric Jacobian** to distinguish it from the **analytical Jacobian**, denoted $J_a(q)$, which is based on a minimal representation for the orientation of the end-effector frame. Let

$$X = \begin{bmatrix} d(q) \\ \alpha(q) \end{bmatrix}$$

denote the end-effector pose, where $d(q)$ is the usual vector from the origin of the base frame to the origin of the end-effector frame and α denotes a minimal representation for the orientation of the end-effector frame relative to the base frame.

For example, let $\alpha = (\varphi, \theta, \psi)$ be a vector of Euler angles.

Then we seek an expression of the form

$$\dot{X} = \begin{bmatrix} \dot{d} \\ \dot{\alpha} \end{bmatrix} = J_a(q)\dot{q}$$

to define the analytical Jacobian.

Velocity Kinematics

- **The Analytical Jacobian**

If $R = R_{z,\varphi} R_{y,\theta} R_{z,\psi}$ is the Euler angle transformation, then

$$\dot{R} = S(\omega)R$$

in which ω defining the angular velocity is given by

$$\omega = \begin{bmatrix} c_\psi s_\theta \dot{\phi} - s_\psi \dot{\theta} \\ s_\psi s_\theta \dot{\phi} + c_\psi \dot{\theta} \\ \dot{\psi} + c_\theta \dot{\phi} \end{bmatrix} = \begin{bmatrix} c_\psi s_\theta & -s_\psi & 0 \\ s_\psi s_\theta & c_\psi & 0 \\ c_\theta & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = B(\alpha)\dot{\alpha}$$

The components of ω are called nutation (章动), spin (旋转), and precession (进动), respectively. Combining the above relationship with the previous definition of the Jacobian

$$\begin{bmatrix} v \\ \omega \end{bmatrix} = \begin{bmatrix} \dot{d} \\ \omega \end{bmatrix} = J(q)\dot{q}$$

yields

$$J(q)\dot{q} = \begin{bmatrix} v \\ \omega \end{bmatrix} = \begin{bmatrix} \dot{d} \\ B(\alpha)\dot{\alpha} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & B(\alpha) \end{bmatrix} \begin{bmatrix} \dot{d} \\ \dot{\alpha} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & B(\alpha) \end{bmatrix} J_a(q)\dot{q}$$

Velocity Kinematics

- **The Analytical Jacobian**

$$J(q)\dot{q} = \begin{bmatrix} v \\ \omega \end{bmatrix} = \begin{bmatrix} \dot{d} \\ B(\alpha)\dot{\alpha} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & B(\alpha) \end{bmatrix} \begin{bmatrix} \dot{d} \\ \dot{\alpha} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & B(\alpha) \end{bmatrix} J_a(q)\dot{q}$$

Thus, the analytical Jacobian, $J_a(q)$, may be computed from the geometric Jacobian as

$$J_a(q) = \begin{bmatrix} I & 0 \\ 0 & B^{-1}(\alpha) \end{bmatrix} J(q)$$

provided det B(α) ≠ 0.

We will discuss the notion of Jacobian singularities, which are configurations where the Jacobian loses rank. Singularities of the matrix B(α) are called representational singularities (表象奇点).

It can easily be shown that B(α) is invertible provided $\sin(\theta) \neq 0$. This means that the singularities of the analytical Jacobian include the singularities of the geometric Jacobian, ***J***, as defined in the next section, together with the representational singularities.

Velocity Kinematics

- **Singularities (奇异点)**

The 6 × n Jacobian *J*(q) defines a mapping

$$\xi = J(q)\dot{q}$$

between the vector $\dot{q}$ of joint velocities and the vector ξ = (*v*,*ω*) of end-effector velocities. This implies that all possible end-effector velocities are linear combinations of the columns of the Jacobian matrix,

$$\xi = J_1\dot{q}_1 + J_2\dot{q}_2 + \cdots + J_n\dot{q}_n$$

For example, for the two-link planar arm, the Jacobian matrix has two columns. It is easy to see that the linear velocity of the end effector must lie in the *x y*-plane, since neither column has a nonzero entry for the third row.

The rank of a matrix is not necessarily constant. Indeed, the rank of the manipulator Jacobian matrix will depend on the configuration *q*. Configurations for which rank *J*(q) is less than its maximum value are called **singularities** or **singular configurations**.

Velocity Kinematics

- **Singularities**

Identifying manipulator singularities is important for several reasons.

- Singularities represent configurations from which certain directions of motion may be unattainable.
- At singularities, bounded end-effector velocities may correspond to unbounded joint velocities.
- At singularities, bounded joint torques may correspond to unbounded end-effector forces and torques. (We will see this in Force Control Chapter.)
- Singularities often correspond to points on the boundary of the manipulator workspace, that is, to points of maximum reach of the manipulator.

Velocity Kinematics

- **Singularities：Decoupling of Singularities**

The manipulator configurations themselves are geometric quantities, independent of the frames used to describe them.

Recognizing this fact allows us to decouple the determination of singular configurations, for those manipulators with spherical wrists, into two simpler problems:

1) The first is to determine so-called **arm singularities**, that is, singularities resulting from motion of the arm, which consists of the first three or more links.

2) The second is to determine the **wrist singularities** resulting from motion of the spherical wrist.

Consider the case that n = 6, that is, the manipulator consists of a 3-DOF arm with a 3-DOF spherical wrist. In this case the Jacobian is a 6 × 6 matrix and a configuration *q* is singular if and only if

$$\det J(q) \quad = \quad 0$$

Velocity Kinematics

- **Singularities：Decoupling of Singularities**

$$\det J(q) = 0$$

If we now partition the Jacobian J into 3 × 3 blocks as

$$J = [J_P \mid J_O] = \left[\begin{array}{c|c} J_{11} & J_{12} \\ \hline J_{21} & J_{22} \end{array} \right]$$

then, since the final three joints are always revolute

$$J_O = \begin{bmatrix} z_3 \times (o_6 - o_3) & z_4 \times (o_6 - o_4) & z_5 \times (o_6 - o_5) \\ z_3 & z_4 & z_5 \end{bmatrix}$$

Since the wrist axes intersect at a common point o, if we choose the coordinate frames so that $o_3 = o_4 = o_5 = o_6 = o$, then J_O becomes

$$J_O = \begin{bmatrix} 0 & 0 & 0 \\ z_3 & z_4 & z_5 \end{bmatrix}$$

In this case the Jacobian matrix has the block triangular form

$$J = \begin{bmatrix} J_{11} & 0 \\ J_{21} & J_{22} \end{bmatrix}$$

Velocity Kinematics

- **Singularities：Decoupling of Singularities**

In this case the Jacobian matrix has the block triangular form

$$J = \begin{bmatrix} J_{11} & 0 \\ J_{21} & J_{22} \end{bmatrix}$$

with determinant

$$\det J = \det J_{11} \det J_{22}$$

where J_{11} and J_{22} are each 3×3 matrices. J_{11} has i^{th} column $z_{i-1}\times(o-o_{i-1})$ if joint i is revolute, and z_{i-1} if joint i is prismatic, while

$$J_{22} = [z_3 \; z_4 \; z_5]$$

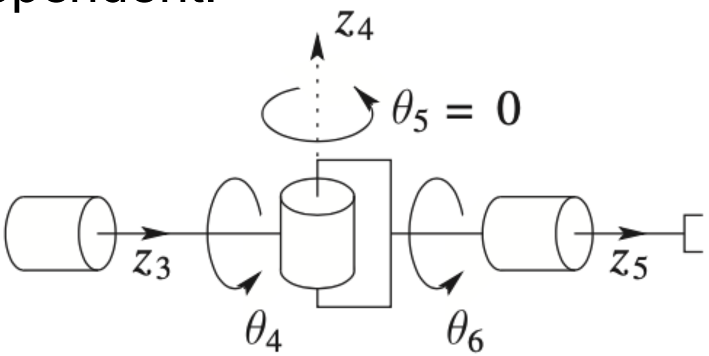
Therefore the set of singular configurations of the manipulator is the union of the set of arm configurations satisfying det J_{11} = 0 and the set of wrist configurations satisfying det J_{22} = 0.

Velocity Kinematics

- **Singularities：Wrist Singularities**

$$J_{22} = [z_3 \; z_4 \; z_5]$$

A spherical wrist is in a singular configuration whenever the vectors z_3, z_4, and z_5 are linearly dependent.



This happens when the joint axes z_3 and z_5 are collinear, that is, when $\theta_5 = 0$ or π. These are the only singularities of the spherical wrist. They are unavoidable without imposing mechanical limits on the wrist design to restrict its motion in such a way that z_3 and z_5 are prevented from lining up.

When any two revolute-joint axes are collinear a singularity results, since an equal and opposite rotation about the axes results in no net motion of the end effector.

Velocity Kinematics

- **Singularities：Arm Singularities**

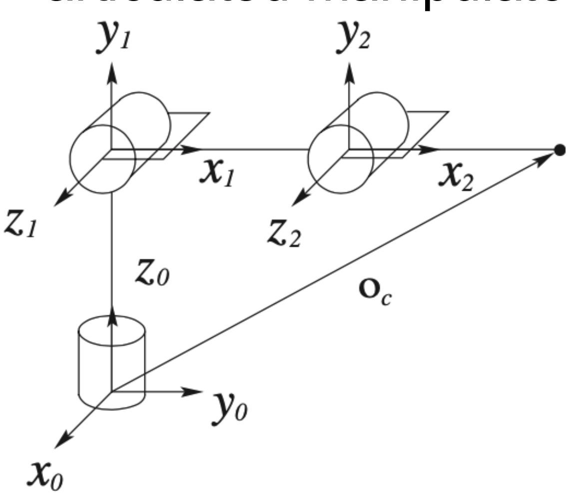
To investigate arm singularities we need only to compute det J_{11} which is done using

$$J_{v_i} = \begin{cases} z_{i-1} \times (o_n - o_{i-1}) & \text{for revolute joint } i \\ z_{i-1} & \text{for prismatic joint } i \end{cases}$$

but with the wrist center o in place of o_n.

We will determine the singularities for three common arms, the elbow manipulator, the spherical manipulator and the SCARA manipulator.

Example 1 (Elbow Manipulator Singularities). Consider the three-link articulated manipulator with coordinate frames attached.



$$J_{11} = \begin{bmatrix} -a_2s_1c_2 - a_3s_1c_{23} & -a_2s_2c_1 - a_3s_{23}c_1 & -a_3c_1s_{23} \\ a_2c_1c_2 + a_3c_1c_{23} & -a_2s_1s_2 - a_3s_1s_{23} & -a_3s_1s_{23} \\ 0 & a_2c_2 + a_3c_{23} & a_3c_{23} \end{bmatrix}$$

$$\det J_{11} = -a_2a_3s_3(a_2c_2 + a_3c_{23})$$

$$s_3 = 0, \quad \textit{that is, } \theta_3 = 0 \textit{ or } \pi$$

$$a_2c_2 + a_3c_{23} = 0$$

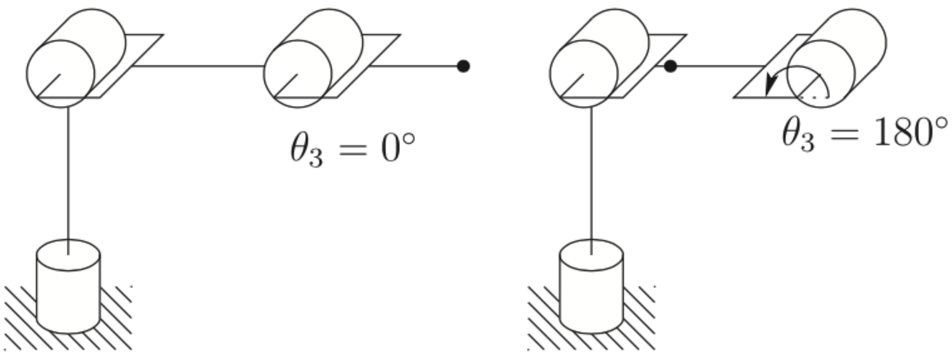
Homework_3

Velocity Kinematics

- **Singularities：Arm Singularities**

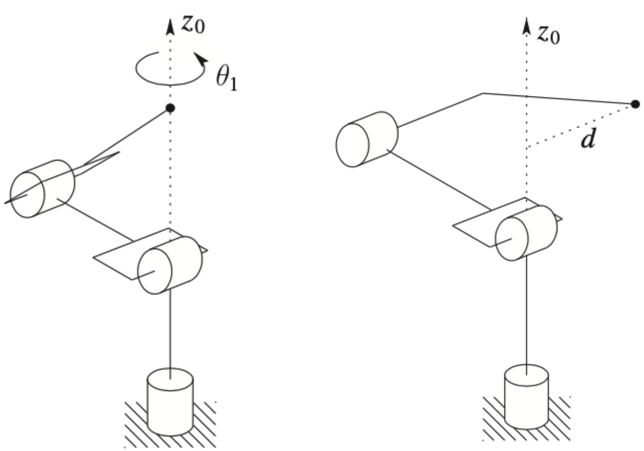
Example 1 (Elbow Manipulator Singularities). Consider the three-link articulated manipulator with coordinate frames attached.

$$s_3 = 0, \quad \textit{that is}, \ \theta_3 = 0 \ \textit{or} \ \pi$$



The situation arises when the elbow is fully extended or fully retracted as shown.

$$a_2c_2 + a_3c_{23} = 0$$



This configuration occurs when the wrist center intersects the axis of the base rotation, z_0.

There are infinitely many singular configurations and infinitely many solutions to the inverse position kinematics when the wrist center is along this axis.

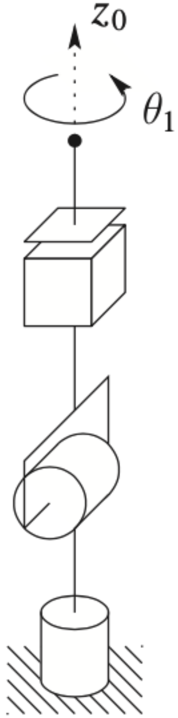
For an elbow manipulator with an offset, the wrist center cannot intersect z_0.

Velocity Kinematics

- **Singularities：Arm Singularities**

Example 2 (Spherical Manipulator Singularities).

Consider the spherical arm. This manipulator is in a singular configuration when the wrist center intersects z_0 as shown since, as before, any rotation about the base leaves this point fixed.



Homework_4

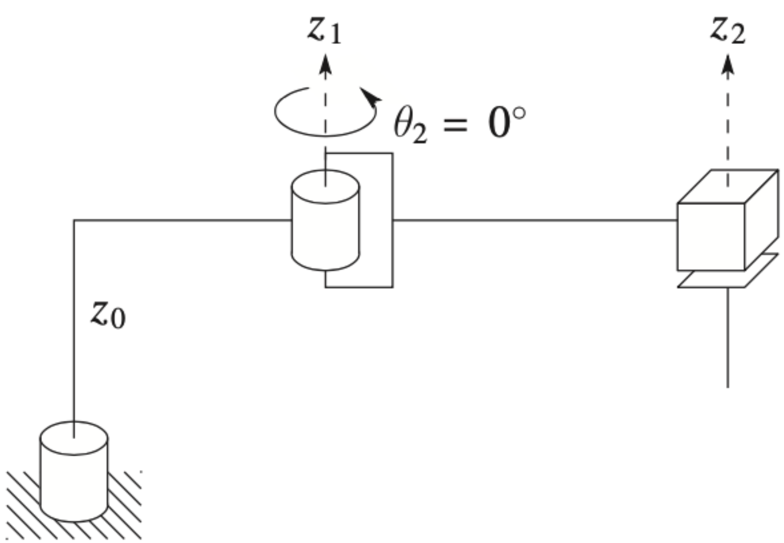
Singularity of spherical manipulator with no offsets.

Velocity Kinematics

- **Singularities：Arm Singularities**

Example 3 (SCARA Manipulator Singularities).

We can see that the only singularity of the SCARA arm is when the elbow is geometrically fully extended or fully retracted.



SCARA manipulator singularity.

$$J_{11} = \begin{bmatrix} \alpha_1 & \alpha_3 & 0 \\ \alpha_2 & \alpha_4 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

where

$$\begin{aligned} \alpha_1 &= -a_1 s_1 - a_2 s_{12} \\ \alpha_2 &= a_1 c_1 + a_2 c_{12} \\ \alpha_3 &= -a_1 s_{12} \\ \alpha_4 &= a_1 c_{12} \end{aligned}$$

The rank of J_{11} will be less than three when $\alpha_1\alpha_4 - \alpha_2\alpha_3 = 0$.

It is easy to compute this quantity and show that it is equivalent to

$$s_2 = 0, \quad \textit{which implies} \quad \theta_2 = 0, \pi$$

Summary

Velocity Kinematics

- The Tool Velocity
- The Analytical Jacobian
- Singularities

Homework 9

Homework 9 is posted at http://bb.sustech.edu.cn

Due date: March 31, 2025

Next class: March 31, 2025（Tuesday）

作业要求（Requirements）：

1. **文件格式为以自己姓名学号作业序号命名的pdf文件；**

   （File name: YourSID_ YourName_09.pdf）

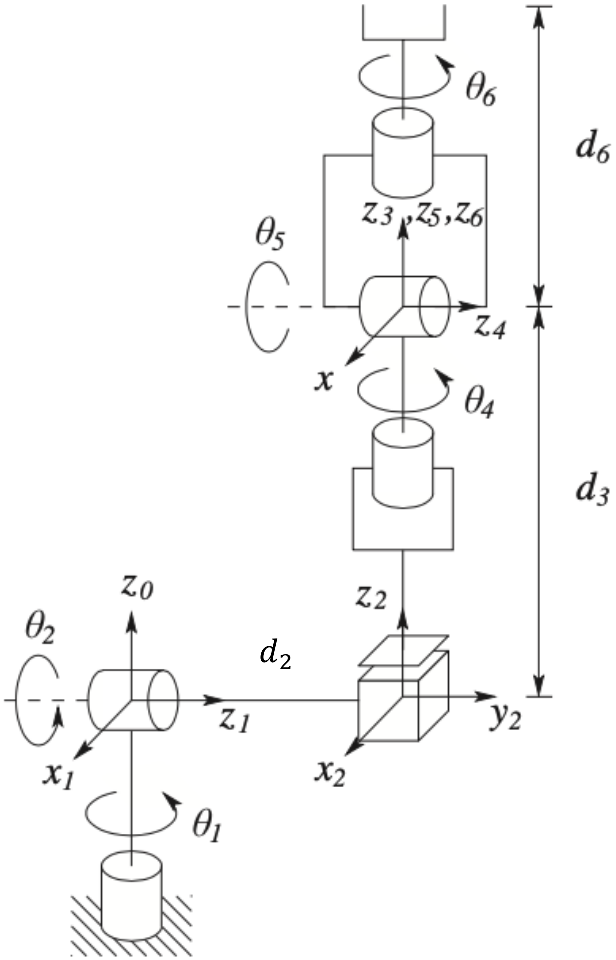
2. **作业里也写上自己的姓名和学号。**

   （Write your name and SID in the homework）

Homework 9

1. *For the three-link planar manipulator of PPT Slide 9, compute the vector o^c and derive the manipulator Jacobian matrix.*

2. *Complete the derivation of the Jacobian for the Stanford Manipulator in PPT Slide 10.*



Homework 9

3. Compute the Jacobian J_{11} for the 3-link elbow manipulator in PPT Slide 25 and show that it agrees with the result in PPT slide 25. Show that the determinant of this matrix agrees with the result in PPT slide 25.

4. Compute the Jacobian J_{11} for the three-link spherical manipulator in PPT Slide 27, and find the singular configuration according to Jacobian matrix

机器人全身运动通用方法

General Methods for Robot Whole-Body Motion

主讲人：王东林 研究员

时间：2025. 3. 24(周一) 15:00-16:00

地点：工学院北楼514会议室

王东林 研究员
西湖大学
人工智能系副主任

Bio:

王东林，特聘研究员，国家科技创新2030重大项目首席科学家，西湖大学人工智能系副主任，西湖大学机器智能实验室主任，西湖机器人科技（杭州）有限公司创始人。西安交通大学电信学院工学学士（2003）和硕士学位（2006），加拿大卡尔加里大学电子与计算机工程系博士学位（2010）。2017年加盟西湖大学，主要研究方向为强化学习和机器人具身智能，构建了机器人强化具身智能方法体系，研发了足式机器人产品。近五年主持承担了科技部重大项目、国家自然科学基金委等多项国家重要项目。发表NeurIPS、ICML、ICLR、RSS、Nature子刊等人工智能和机器人顶会顶刊论文百余篇，实现了发明专利成果转化。研制的多款足式机器人智能产品，获得CCTV2、CCTV4、CCTV13、人民日报和央视新闻等主流媒体的广泛报道。

Abstract:

The core challenges in achieving universal autonomous intelligence for robots in open physical worlds lie in environmental unstructuredness, dynamic task variability, and autonomous perception-cognition integration. To address these challenges, this report proposes an innovative embodied intelligence framework centered on three pillars: task generality, interaction adaptability and environmental consistency. Starting from the core elements of embodied intelligence—data, computing power, algorithms, and sim-to-real technology—it elaborates on industry advancements and the latest laboratory achievements, including general-purpose embodied large models for robots, reinforcement learning-based decision-making models, algorithms mitigating simulation-to-reality gaps, and hardware deployments in quadrupedal and humanoid robots. These algorithms and technologies have significantly enhanced robotic capabilities in autonomous decision-making, robust operational performance, dynamic environmental adaptation, and industry-specific applications, while driving the deep integration of embodied large models with reinforcement learning and the co-evolution of software and hardware systems.

机械与能源工程系
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邀请人: 付成龙 教授