Robot Modeling & Control ME331

Section 13： Dynamics III

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Outline

- Review
  - D'Alembert's Principle
  - Kinetic and Potential Energy
  - Equations of Motion
- Some Common Configurations
  - Elbow Manipulator
  - Elbow Manipulator with Remotely Driven Link
  - Five-Bar Linkage
- Analysis of inertial couplings and gravity term
- Properties of Dynamic Equations

The Euler–Lagrange Equations

D'Alembert's Principle

$$\sum_{i=1}^{k} f_i^T \delta r_i - \sum_{i=1}^{k} \dot{p}_i^T \delta r_i = 0 \quad \Rightarrow \quad \sum_{j=1}^{n} \left\{ \frac{d}{dt}\frac{\partial K}{\partial \dot{q}_j} - \frac{\partial K}{\partial q_j} - \psi_j \right\} \delta q_j = 0$$

$$\frac{d}{dt}\frac{\partial K}{\partial \dot{q}_j} - \frac{\partial K}{\partial q_j} = \psi_j, \qquad j = 1, \cdots n$$

where the generalized force ψ_j is the sum of an externally applied generalized force τ_j and another one due to a potential field force as

$$\psi_j = -\frac{\partial P}{\partial q_j} + \tau_j$$

Then we can obtain the **Euler–Lagrange Equation** form

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_j} - \frac{\partial \mathcal{L}}{\partial q_j} = \tau_j$$

where $\mathcal{L} = K - P$ is the Lagrangian.

Kinetic and Potential Energy

The Inertial Tensor

$$K = \frac{1}{2} m v^T v + \frac{1}{2} \omega^T \mathcal{J} \omega$$

where the above linear and angular velocity vectors, v and ω, respectively, are expressed in the inertial frame.

If we denote as $\mathcal{J}$ the inertia tensor expressed in the inertial frame. If we denote as I the inertia tensor expressed in the body-attached frame, then the two matrices are related via a similarity transformation according to

$$\mathcal{J} = R I R^T$$

where R is the orientation transformation from the body-attached frame and the inertial frame.

Let the mass density of the object be represented as a function of position, $\rho(x, y, z)$. Then I the inertia tensor in the body attached frame is computed as

$$I = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$$

Kinetic and Potential Energy

Kinetic Energy for an n-Link Robot

For Jacobian matrices J_{v_i} and J_{ω_i}, of dimension $3 \times n$, we have

$$v_i = J_{v_i}(q)\dot{q}$$

$$\omega_i = J_{\omega_i}(q)\dot{q}$$

$$\mathcal{J}_i = R_i I_i R_i^T$$

Then the overall kinetic energy of the manipulator equals

$$K = \frac{1}{2}\sum_{i=1}^{n}\{m_i {v_i}^T v_i + {\omega_i}^T \mathcal{J}_i \omega_i\}$$

$$= \frac{1}{2}\dot{q}^T\left[\sum_{i=1}^{n}\{m_i J_{v_i}(q)^T J_{v_i}(q) + J_{\omega_i}(q)^T R_i(q) I_i R_i(q)^T J_{\omega_i}(q)\}\right]\dot{q}$$

$$= \frac{1}{2}\dot{q}^T D(q)\dot{q}$$

where $D(q)$ is a configuration dependent matrix called the **inertia matrix**.

Remark: The inertia matrix is symmetric and positive.

Equations of Motion

We specialize the Euler–Lagrange equations to the case when two conditions hold.

1) The kinetic energy is a quadratic function of the vector $\dot{q}$ of the form

$$K = \frac{1}{2}\dot{q}^T D(q)\dot{q} = \frac{1}{2}\sum_{i,j} d_{ij}(q)\,\dot{q}_i\dot{q}_j$$

where $d_{ij}(q)$ are the entries of the $n \times n$ inertia matrix $D(q)$, which is symmetric and positive definite for each $q \in \mathbb{R}^n$.

2) The potential energy $P = P(q)$ is independent of $\dot{q}$.

The Euler–Lagrange equations for such a system can be derived as follows. The Lagrangian can be written as

$$L = K - P = \frac{1}{2}\sum_{i,j} d_{ij}(q)\,\dot{q}_i\dot{q}_j - P(q)$$

Thus, for each $k = 1, \cdots, n$, the Euler–Lagrange equations can be written

$$\sum_{j} d_{kj}(q)\,\ddot{q}_j + \sum_{i,j}\frac{\partial d_{kj}}{\partial q_i}\dot{q}_i\dot{q}_j - \frac{1}{2}\sum_{i,j}\frac{\partial d_{ij}}{\partial q_k}\dot{q}_i\dot{q}_j + \frac{\partial P}{\partial q_k} = \tau_k$$

Equations of Motion

$$\sum_j d_{kj}(q)\,\ddot{q}_j + \sum_{i,j}\left\{\frac{\partial d_{kj}}{\partial q_i} - \frac{1}{2}\frac{\partial d_{ij}}{\partial q_k}\right\}\dot{q}_i\dot{q}_j + \frac{\partial P}{\partial q_k} = \tau_k$$

$$\sum_{i,j}\left\{\frac{\partial d_{kj}}{\partial q_i} - \frac{1}{2}\frac{\partial d_{ij}}{\partial q_k}\right\}\dot{q}_i\dot{q}_j = \sum_{i,j} c_{ijk}\dot{q}_i\dot{q}_j$$

where $c_{ijk} := \frac{1}{2}\left\{\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k}\right\}$ are known as **Christoffel symbols**.

For a fixed k, we have $c_{ijk} = c_{jik}$, which reduces the effort involved in computing these symbols by a factor of about one half.

Finally, if we define

$$g_k = \frac{\partial P}{\partial q_k}$$

then we can write the Euler–Lagrange equations as

$$\sum_{j=1}^{n} d_{kj}(q)\ddot{q}_j + \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ijk}(q)\dot{q}_i\dot{q}_j + g_k(q) = \tau_k, \qquad k = 1,\cdots,n$$

Equations of Motion

$$\sum_{j=1}^{n} d_{kj}(q)\ddot{q}_j + \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ijk}(q)\dot{q}_i\dot{q}_j + g_k(q) = \tau_k, \qquad k = 1, \cdots, n$$

The first type involves the second derivative of the generalized coordinates.

The second type involves quadratic terms in the first derivatives of $\dot{q}$, where the coefficients may depend on q. These latter terms are further classified into those involving a product of the type $\dot{q}_i^2$ and those involving a product of the type $\dot{q}_i\dot{q}_j$ where $i \neq j$.

- ✓ Terms of the type $\dot{q}_i^2$ are called centrifugal (离心力).
- ✓ Terms of the type $\dot{q}_i\dot{q}_j$ are called Coriolis terms (科氏力).

The third type of terms are those involving only q but not its derivatives. This third type arises from differentiating the potential energy.

It is common to write the above equation in matrix form as

$$\boldsymbol{D}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{\tau}$$

Equations of Motion

$$\sum_{j=1}^{n} d_{kj}(q)\ddot{q}_j + \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ijk}(q)\dot{q}_i\dot{q}_j + g_k(q) = \tau_k, \qquad k = 1, \cdots, n$$

$$\boldsymbol{D}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{\tau}$$

where the $(k, j)^{th}$ element of the matrix $\boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})$ is defined as

$$c_{kj} = \sum_{i=1}^{n} c_{ijk}(q)\dot{q}_i = \sum_{i=1}^{n} \frac{1}{2}\left\{\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k}\right\}\dot{q}_i$$

and the gravity vector $\boldsymbol{g}(\boldsymbol{q})$ is given by

$$\boldsymbol{g}(\boldsymbol{q}) = [g_1(q), \cdots, g_n(q)]^T$$

In summary, the development of equations of motion is very general and applies to any mechanical system whose kinetic energy is a quadratic function of the vector $\dot{q}$ and whose potential energy is independent of $\dot{q}$.

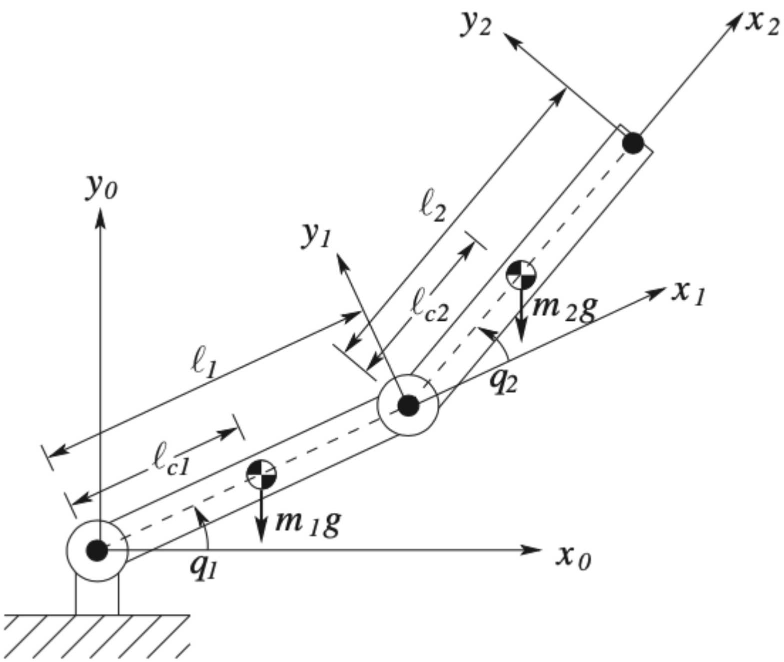
Equations of Motion

- **Derive dynamic equations - Lagrangian method**
  - **Coordinate system, generalized coordinates, generalized forces**
  - **Calculate the position and velocity of CoM for each link**
  - **Calculate the total kinetic energy of robot** $K(q, \dot{q})$
  - **Calculate the total potential energy of robot** $P(q)$
  - **Lagrange function** $L = K - P$
  - **Substituting into dynamic equation**

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = \tau_i \quad \Rightarrow \quad \boldsymbol{D}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{\tau}$$

Some Common Configurations

Planar Elbow Manipulator



q_i : joint angle (generalized coordinate);
m_i: the mass of link i;
l_i : the length of link i;
l_{ci}: the distance from the previous joint to the center of mass of link i;
I_i : the moment of inertia of link i about an axis coming out of the page, passing through the center of mass of link i.

We will use the Denavit–Hartenberg joint variables as generalized coordinates.

First, we consider the velocity terms.

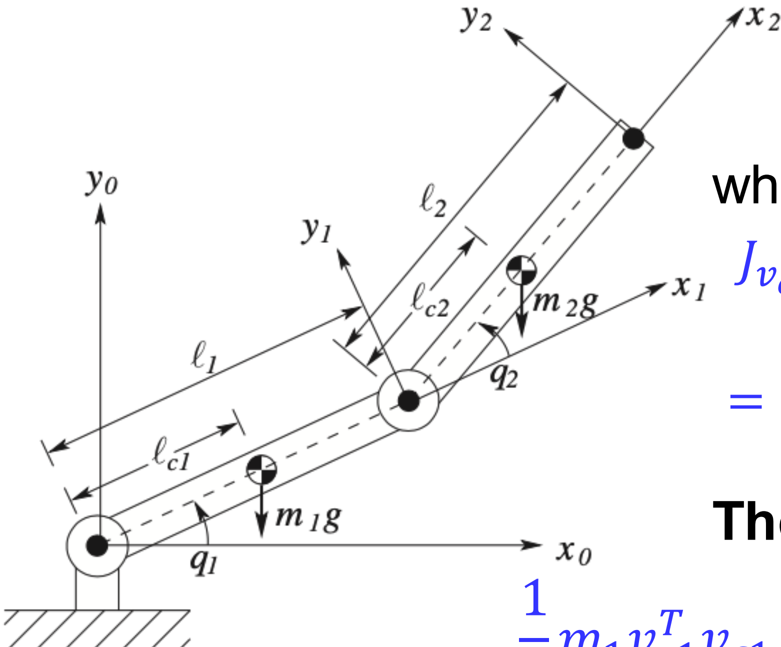
$$v_{c1} = J_{v_{c1}} \dot{q}$$

where

$$J_{v_{c1}} = \begin{bmatrix} -l_{c1} \sin q_1 & 0 \\ l_{c1} \cos q_1 & 0 \\ 0 & 0 \end{bmatrix}$$

Some Common Configurations

Planar Elbow Manipulator



Similarly,

$$v_{c2} = J_{v_{c2}} \dot{q}$$

where

$$J_{v_{c2}} = \begin{bmatrix} -l_1 \sin q_1 - l_{c2} \sin(q_1 + q_2) & -l_{c2} \sin(q_1 + q_2) \\ l_1 \cos q_1 + l_{c2} \cos(q_1 + q_2) & l_{c2} \cos(q_1 + q_2) \\ 0 & 0 \end{bmatrix}$$

The translational part of the kinetic energy is

$$\frac{1}{2} m_1 v_{c1}^T v_{c1} + \frac{1}{2} m_2 v_{c2}^T v_{c2} = \frac{1}{2} \dot{q}^T \{ m_1 J_{v_{c1}}^T J_{v_{c1}} + m_2 J_{v_{c2}}^T J_{v_{c2}} \} \dot{q}$$

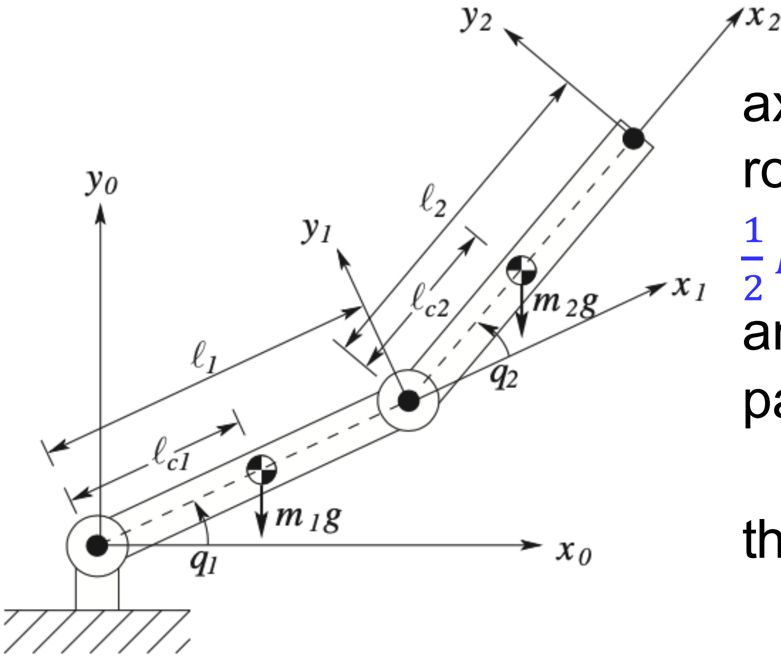
Next, we consider the angular velocity terms. Because of the particularly simple nature of this manipulator, many of the potential difficulties do not arise. First, it is clear that

$$\omega_1 = \dot{q}_1 k, \qquad \omega_2 = (\dot{q}_1 + \dot{q}_2) k$$

when expressed in the base inertial frame.

Some Common Configurations

Planar Elbow Manipulator



Moreover, since ω_i is aligned with the z-axes of each joint coordinate frame, the rotational kinetic energy reduces simply to $\frac{1}{2}I_i\omega_i^2$, where I_i is the moment of inertia about an axis through the center of mass of link i parallel to the z_i-axis.

Hence, **the rotational kinetic energy** of the overall system is

$$\frac{1}{2}\dot{q}^T\left\{I_1\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}+I_2\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\right\}\dot{q}$$

To form the inertia matrix $D(q)$, we merely have to add the translational kinetic energy and the rotational kinetic energy respectively. Thus

$$D(q)=m_1 J_{v_{c1}}^T J_{v_{c1}}+m_2 J_{v_{c2}}^T J_{v_{c2}}+\begin{bmatrix}I_1+I_2 & I_2\\ I_2 & I_2\end{bmatrix}$$

Carrying out the above multiplications and using the standard trigonometric identities leads to

Some Common Configurations

Planar Elbow Manipulator

$$
\begin{aligned}
d_{11} &= m_1\ell_{c1}^2 + m_2(\ell_1^2 + \ell_{c2}^2 + 2\ell_1\ell_{c2}\cos q_2) + I_1 + I_2 \\
d_{12} &= d_{21} = m_2(\ell_{c2}^2 + \ell_1\ell_{c2}\cos q_2) + I_2 \\
d_{22} &= m_2\ell_{c2}^2 + I_2
\end{aligned}
$$

Now, we can compute the Christoffel symbols. This gives

$$
\begin{aligned}
c_{111} &= \frac{1}{2}\frac{\partial d_{11}}{\partial q_1} = 0 \\
c_{121} &= c_{211} = \frac{1}{2}\frac{\partial d_{11}}{\partial q_2} = -m_2\ell_1\ell_{c2}\sin q_2 = h \\
c_{221} &= \frac{\partial d_{12}}{\partial q_2} - \frac{1}{2}\frac{\partial d_{22}}{\partial q_1} = h \\
c_{112} &= \frac{\partial d_{21}}{\partial q_1} - \frac{1}{2}\frac{\partial d_{11}}{\partial q_2} = -h \\
c_{122} &= c_{212} = \frac{1}{2}\frac{\partial d_{22}}{\partial q_1} = 0 \\
c_{222} &= \frac{1}{2}\frac{\partial d_{22}}{\partial q_2} = 0
\end{aligned}
$$

Some Common Configurations

Planar Elbow Manipulator

For each link, the potential energy is its mass multiplied by the gravitational acceleration and the height of its center of mass. Thus

$$
\begin{aligned}
P_1 &= m_1 g \ell_{c1} \sin q_1 \\
P_2 &= m_2 g(\ell_1 \sin q_1 + \ell_{c2} \sin(q_1 + q_2))
\end{aligned}
$$

and so the total potential energy is

$$
P = P_1 + P_2 = (m_1 \ell_{c1} + m_2 \ell_1) g \sin q_1 + m_2 \ell_{c2} g \sin(q_1 + q_2)
$$

Therefore, the functions g_k become

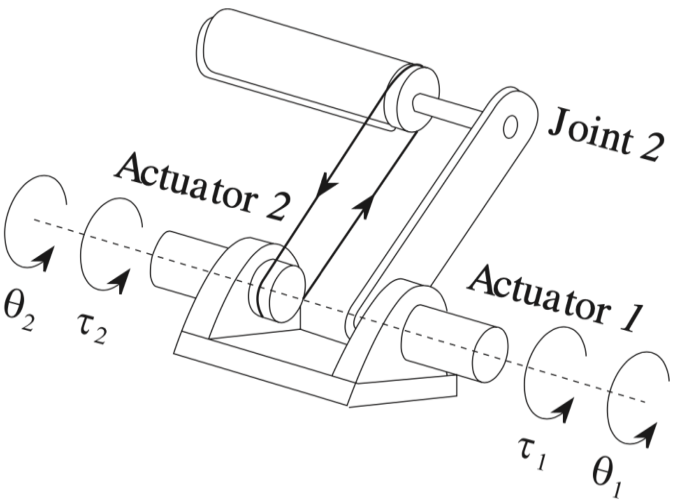
$$
\begin{aligned}
g_1 &= \frac{\partial P}{\partial q_1} = (m_1 \ell_{c1} + m_2 \ell_1) g \cos q_1 + m_2 \ell_{c2} g \cos(q_1 + q_2) \\
g_2 &= \frac{\partial P}{\partial q_2} = m_2 \ell_{c2} g \cos(q_1 + q_2)
\end{aligned}
$$

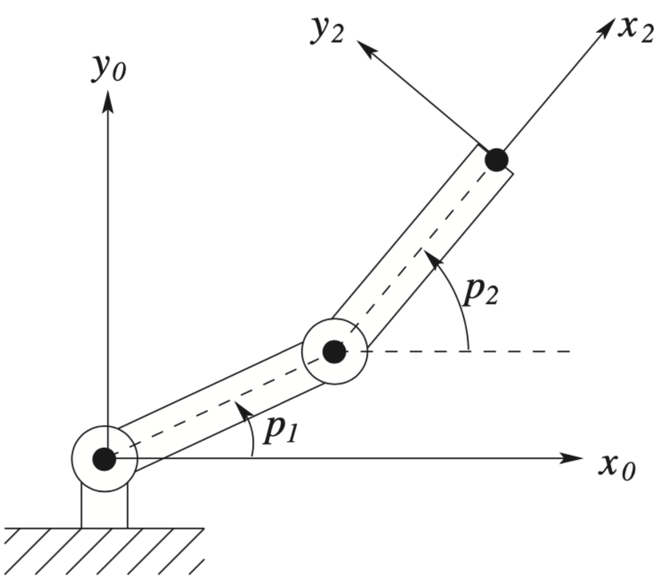
Finally, we can write down the dynamical equations of the system as

$$
\begin{aligned}
d_{11}\ddot{q}_1 + d_{12}\ddot{q}_2 + c_{121}\dot{q}_1\dot{q}_2 + c_{211}\dot{q}_2\dot{q}_1 + c_{221}\dot{q}_2^2 + g_1 &= \tau_1 \\
d_{21}\ddot{q}_1 + d_{22}\ddot{q}_2 + c_{112}\dot{q}_1^2 + g_2 &= \tau_2
\end{aligned}
$$

Some Common Configurations

Planar Elbow Manipulator with Remotely Driven Link





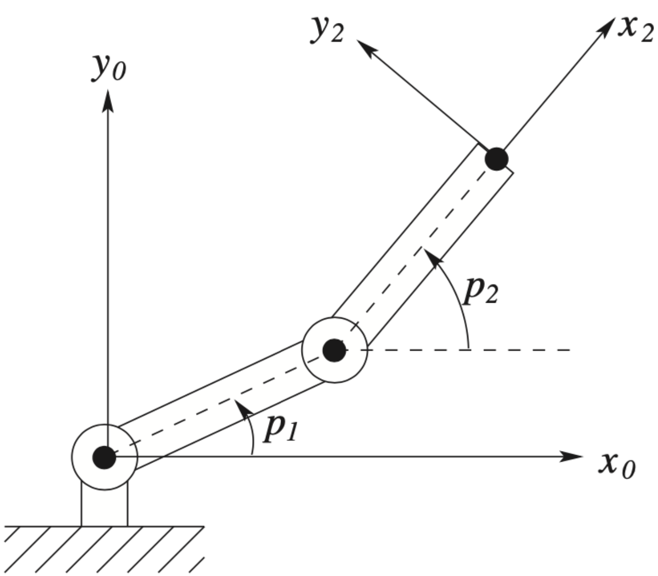
We illustrate the use of Lagrangian equations in a situation where the generalized coordinates are not the joint variables.

Consider the planar elbow manipulator that both joints are driven by motors mounted at the base. The first joint is turned directly by one of the motors, while the other is turned via a gearing mechanism or a timing belt.

In this case, one should choose the generalized coordinates as shown in the left figure, because the angle p_2 is determined by driving motor number 2, and is not affected by the angle p_1. We will derive the dynamical equations for this configuration and show that some simplifications will result.

Some Common Configurations

Planar Elbow Manipulator with Remotely Driven Link



Since p_1 and p_2 are not the joint angles used earlier, we cannot use the velocity Jacobians (vector product method) in order to find the kinetic energy of each link.

Instead, we have to carry out the analysis directly. It is easy to see that

$$v_{c1} = \begin{bmatrix} -\ell_{c1}\sin p_1 & 0 \\ \ell_{c1}\cos p_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \end{bmatrix}$$

$$v_{c2} = \begin{bmatrix} -\ell_1\sin p_1 & -\ell_{c2}\sin p_2 \\ \ell_1\cos p_1 & \ell_{c2}\cos p_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \end{bmatrix}$$

$$\omega_1 = \dot{p}_1 k, \qquad \omega_2 = \dot{p}_2 k$$

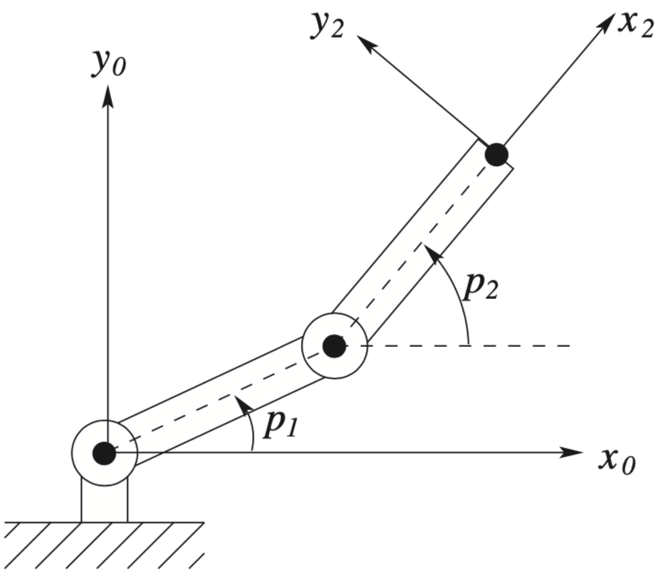
Hence, the kinetic energy of the manipulator equals $K = \frac{1}{2}\dot{p}^T D(p)\dot{p}$

where

$$D(p) = \begin{bmatrix} m_1\ell_{c1}^2 + m_2\ell_1^2 + I_1 & m_2\ell_1\ell_{c2}\cos(p_2 - p_1) \\ m_2\ell_1\ell_{c2}\cos(p_2 - p_1) & m_2\ell_{c2}^2 + I_2 \end{bmatrix}$$

Some Common Configurations

Planar Elbow Manipulator with Remotely Driven Link



Computing the Christoffel symbols gives

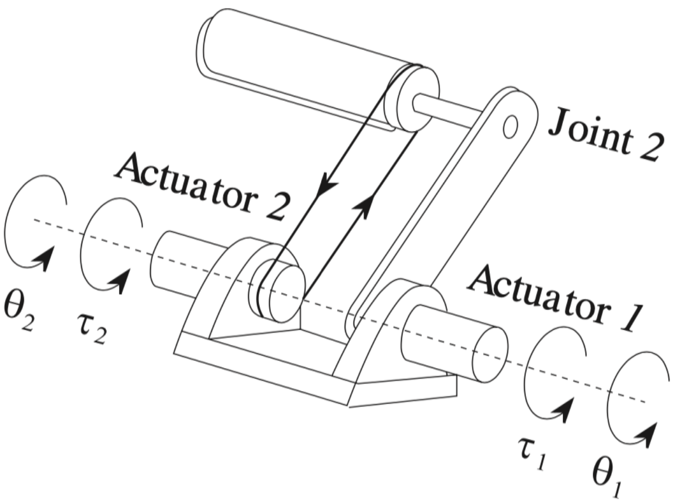
$$
\begin{aligned}
c_{111} &= \frac{1}{2}\frac{\partial d_{11}}{\partial p_1} = 0 \\
c_{121} &= c_{211} = \frac{1}{2}\frac{\partial d_{11}}{\partial p_2} = 0 \\
c_{221} &= \frac{\partial d_{12}}{\partial p_2} - \frac{1}{2}\frac{\partial d_{22}}{\partial p_1} = -m_2\ell_1\ell_{c2}\sin(p_2 - p_1) \\
c_{112} &= \frac{\partial d_{21}}{\partial p_1} - \frac{1}{2}\frac{\partial d_{11}}{\partial p_2} = m_2\ell_1\ell_{c2}\sin(p_2 - p_1) \\
c_{212} &= = c_{122} = \frac{1}{2}\frac{\partial d_{22}}{\partial p_1} = 0 \\
c_{222} &= \frac{1}{2}\frac{\partial d_{22}}{\partial p_2} = 0
\end{aligned}
$$

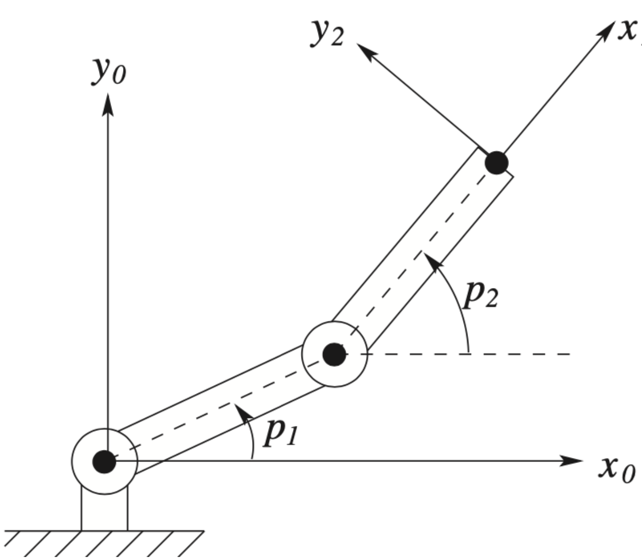
Next, the potential energy of the manipulator, in terms of p_1 and p_2, equals

$$
P = m_1 g \ell_{c1} \sin p_1 + m_2 g(\ell_1 \sin p_1 + \ell_{c2} \sin p_2)
$$

Some Common Configurations

Planar Elbow Manipulator with Remotely Driven Link





The potential energy of the manipulator equals

$$P = m_1 g \ell_{c1} \sin p_1 + m_2 g(\ell_1 \sin p_1 + \ell_{c2} \sin p_2)$$

Hence, the gravitational generalized forces are

$$\begin{aligned} g_1 &= (m_1 \ell_{c1} + m_2 \ell_1) g \cos p_1 \\ g_2 &= m_2 \ell_{c2} g \cos p_2 \end{aligned}$$

Finally, the equations of motion are

$$\begin{aligned} d_{11}\ddot{p}_1 + d_{12}\ddot{p}_2 + c_{221}\dot{p}_2^2 + g_1 &= \tau_1 \\ d_{21}\ddot{p}_1 + d_{22}\ddot{p}_2 + c_{112}\dot{p}_1^2 + g_2 &= \tau_2 \end{aligned}$$

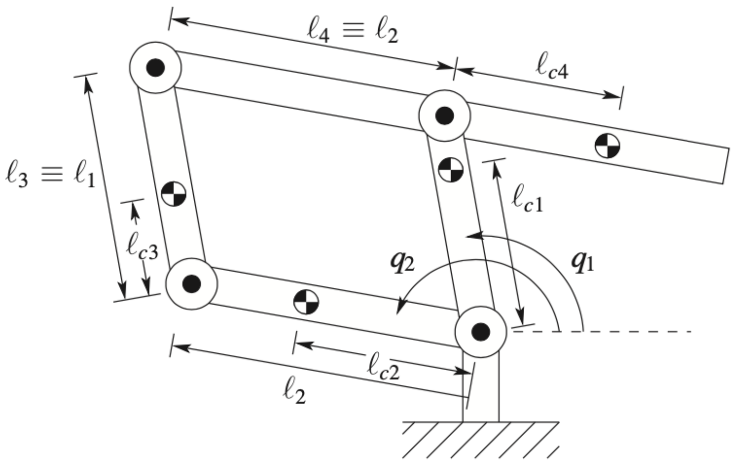
We see that by driving the second joint remotely from the base we have eliminated the Coriolis forces, but we still have the centrifugal forces coupling the two joints.

Some Common Configurations

Planar Elbow Manipulator with Remotely Driven Link

Some Common Configurations

Five-Bar Linkage

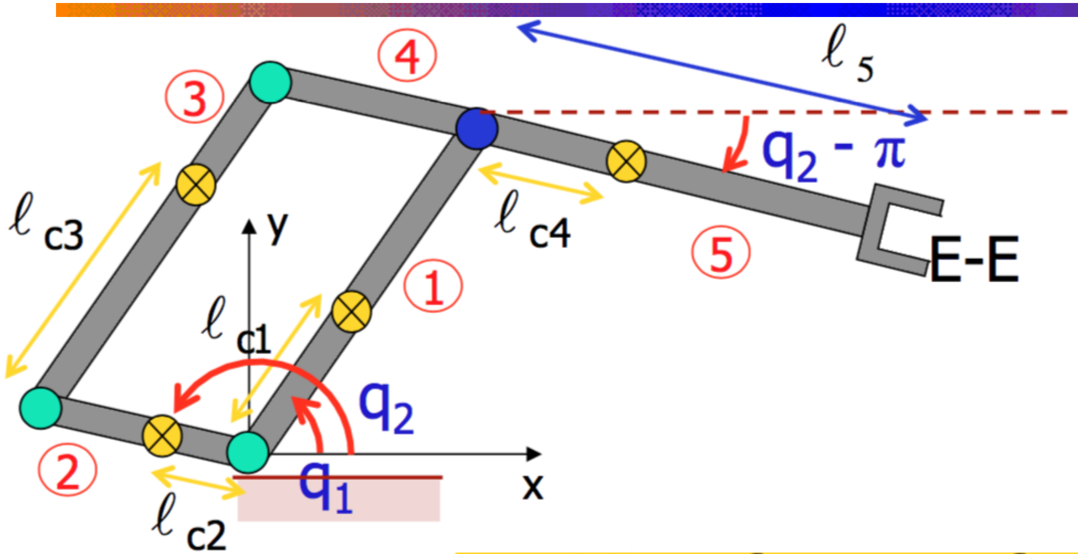


The mechanism in the left figure is called a five-bar linkage. Clearly, there are only four bars in the left figure, but in the theory of mechanisms it is a convention to count the ground as an additional linkage.

It is assumed that the lengths of links 1 and 3 are the same, and that the two lengths marked l_2 are the same.

- In this way the closed path is in fact a parallelogram, which greatly simplifies the computations.
- Notice that the quantities l_{c1} and l_{c3} need not be equal.
- There are in fact only two degrees of freedom, identified as q_1 and q_2.
- Thus, in contrast to the earlier mechanisms we studied, this one is a closed kinematic chain (though of a particularly simple kind).
- As a result, we cannot use the earlier results on Jacobian matrices (vector product method).

Robots with parallelogram structure



$\otimes$ center of mass:
arbitrary ℓ_{ci}

parallelogram:
$\ell_1 = \ell_3$
$\ell_2 = \ell_4$

position of center of masses

$$p_{c1} = \begin{pmatrix} \ell_{c1}c_1 \\ \ell_{c1}s_1 \end{pmatrix} \quad p_{c2} = \begin{pmatrix} \ell_{c2}c_2 \\ \ell_{c2}s_2 \end{pmatrix} \quad p_{c3} = \begin{pmatrix} \ell_2 c_2 \\ \ell_2 s_2 \end{pmatrix} + \begin{pmatrix} \ell_{c3}c_1 \\ \ell_{c3}s_1 \end{pmatrix} \quad p_{c4} = \begin{pmatrix} \ell_1 c_1 \\ \ell_1 s_1 \end{pmatrix} - \begin{pmatrix} \ell_{c4}c_2 \\ \ell_{c4}s_2 \end{pmatrix}$$

linear/angular velocities

$$v_{c1} = \begin{pmatrix} -\ell_{c1}s_1 \\ \ell_{c1}c_1 \end{pmatrix}\dot{q}_1 \quad v_{c2} = \begin{pmatrix} -\ell_{c2}s_2 \\ \ell_{c2}c_2 \end{pmatrix}\dot{q}_2 \quad v_{c3} = \begin{pmatrix} -\ell_{c3}s_1 \\ \ell_{c3}c_1 \end{pmatrix}\dot{q}_1 + \begin{pmatrix} -\ell_2 s_2 \\ \ell_2 s_2 \end{pmatrix}\dot{q}_2$$

$$v_{c4} = \begin{pmatrix} -\ell_1 s_1 \\ \ell_1 c_1 \end{pmatrix}\dot{q}_1 - \begin{pmatrix} -\ell_{c4}s_2 \\ \ell_{c4}c_2 \end{pmatrix}\dot{q}_2 \qquad \omega_1 = \omega_3 = \dot{q}_1 \qquad \omega_2 = \omega_4 = \dot{q}_2$$

Robots with parallelogram structure

T_i

$$T_1 = \frac{1}{2} m_1 \ell_{c1}^2 \dot{q}_1^2 + \frac{1}{2} I_{c1,zz} \dot{q}_1^2 \qquad T_2 = \frac{1}{2} m_2 \ell_{c2}^2 \dot{q}_2^2 + \frac{1}{2} I_{c2,zz} \dot{q}_2^2$$

$$T_3 = \frac{1}{2} I_{c3,zz} \dot{q}_1^2 + \frac{1}{2} m_3 \left(\ell_2^2 \dot{q}_2^2 + \ell_{c3}^2 \dot{q}_1^2 + 2 \ell_2 \ell_{c3} c_{2-1} \dot{q}_1 \dot{q}_2 \right)$$

$$T_4 = \frac{1}{2} I_{c4,zz} \dot{q}_2^2 + \frac{1}{2} m_4 \left(\ell_1^2 \dot{q}_1^2 + \ell_{c4}^2 \dot{q}_2^2 - 2 \ell_1 \ell_{c4} c_{2-1} \dot{q}_1 \dot{q}_2 \right)$$

$$T = \sum_{i=1}^{4} T_i$$

$$\mathbf{D(q)} = \begin{pmatrix} I_{c1,zz} + m_1 \ell_{c1}^2 + I_{c3,zz} + m_3 \ell_{c3}^2 + m_4 \ell_1^2 & \text{symm} \\ \left(m_3 \ell_2 \ell_{c3} - m_4 \ell_1 \ell_{c4} \right) c_{2-1} & I_{c2,zz} + m_2 \ell_{c2}^2 + I_{c4,zz} + m_4 \ell_{c4}^2 + m_3 \ell_2^2 \end{pmatrix}$$

structural condition in mechanical design $\boxed{m_3 \ell_2 \ell_{c3} = m_4 \ell_1 \ell_{c4}}$ (*)

$\Downarrow$

diagonal and constant $\Rightarrow$ centrifugal and Coriolis terms $\equiv 0$

mechanically DECOUPLED and LINEAR dynamic model (up to the gravity term g(q))

big advantage for the design of a motion control law!

Robots with parallelogram structure

from the y-components of vectors p_{ci}

U_i

$$U_1 = m_1 g_0 \ell_{c1} s_1 \qquad U_2 = m_2 g_0 \ell_{c2} s_2$$

$$U_3 = m_3 g_0 \left(\ell_2 s_2 + \ell_{c3} s_1\right) \qquad U_4 = m_4 g_0 \left(\ell_1 s_1 - \ell_{c4} s_2\right)$$

$$U = \sum_{i=1}^{4} U_i$$

$$g(q) = \left(\frac{\partial U}{\partial q}\right)^T = \begin{pmatrix} g_0\left(m_1 \ell_{c1} + m_3 \ell_{c3} + m_4 \ell_1\right) c_1 \\ g_0\left(m_2 \ell_{c2} + m_3 \ell_2 - m_4 \ell_{c4}\right) c_2 \end{pmatrix} = \begin{pmatrix} g_1(q_1) \\ g_2(q_2) \end{pmatrix}$$

gravity components are always "decoupled"

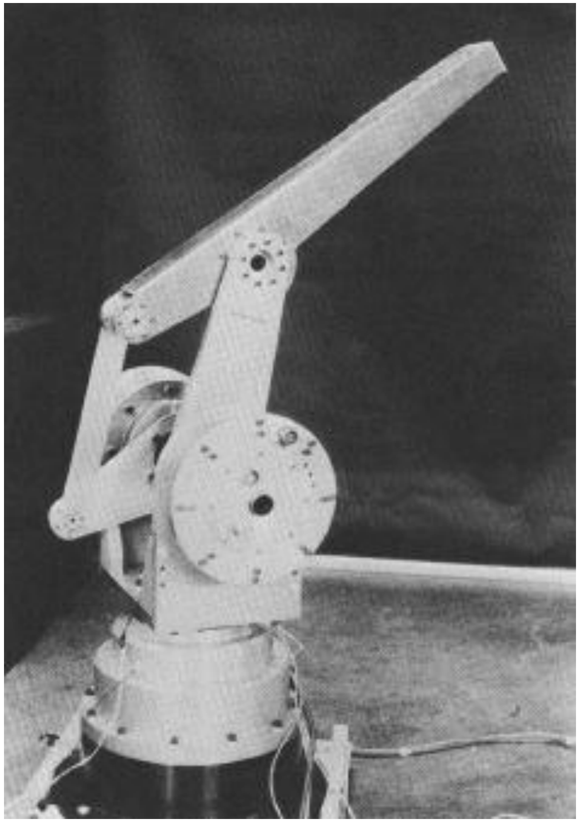
in addition, when (*) holds $\Rightarrow$

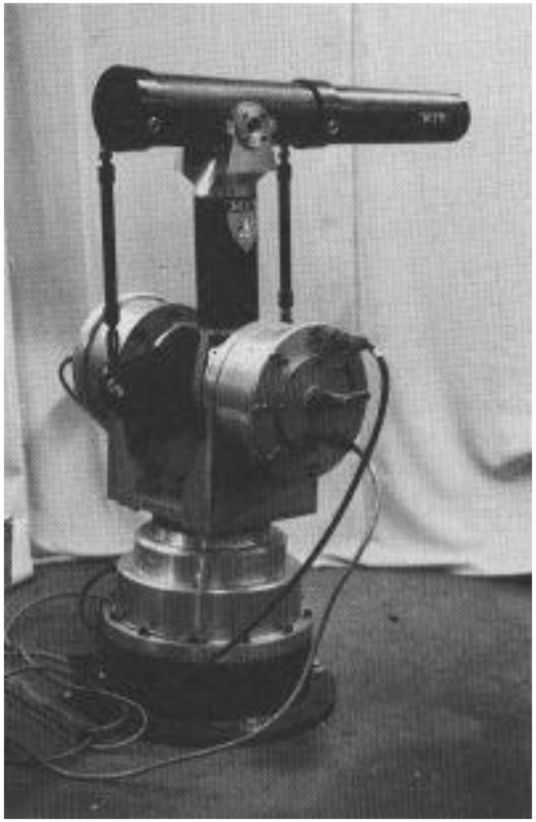
$$\boxed{\begin{aligned} b_{11}\ddot{q}_1 + g_1(q_1) &= u_1 \\ b_{22}\ddot{q}_2 + g_2(q_2) &= u_2 \end{aligned}}$$

u_i (non-conservative) torque producing work on q_i

further structural conditions in the mechanical design lead to $g(q) \equiv 0$!!

Robots with parallelogram structure

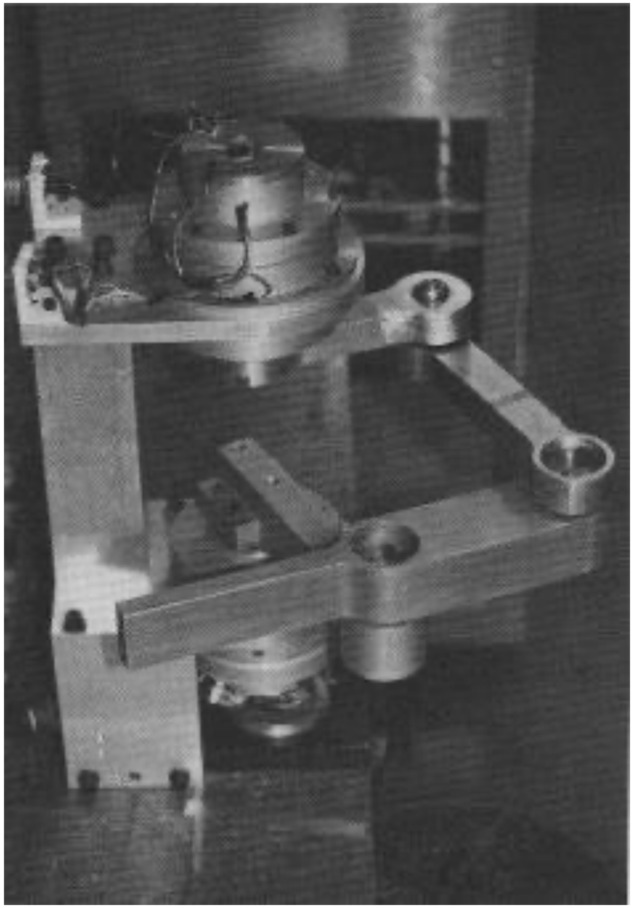




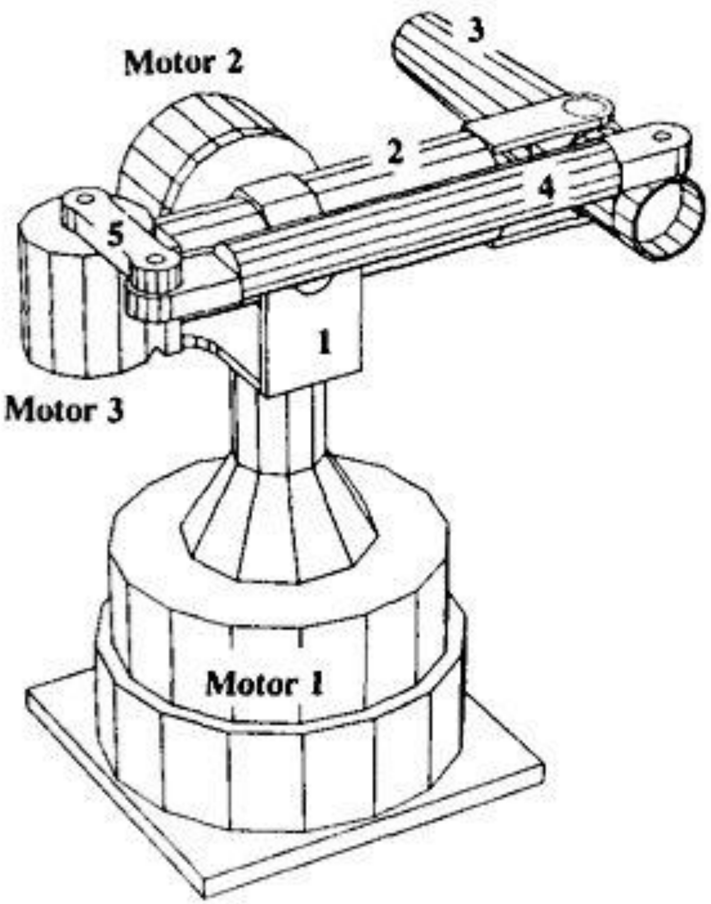
Comau Smart NJ130

MIT Direct Drive Mark II and Mark III

Robots with parallelogram structure

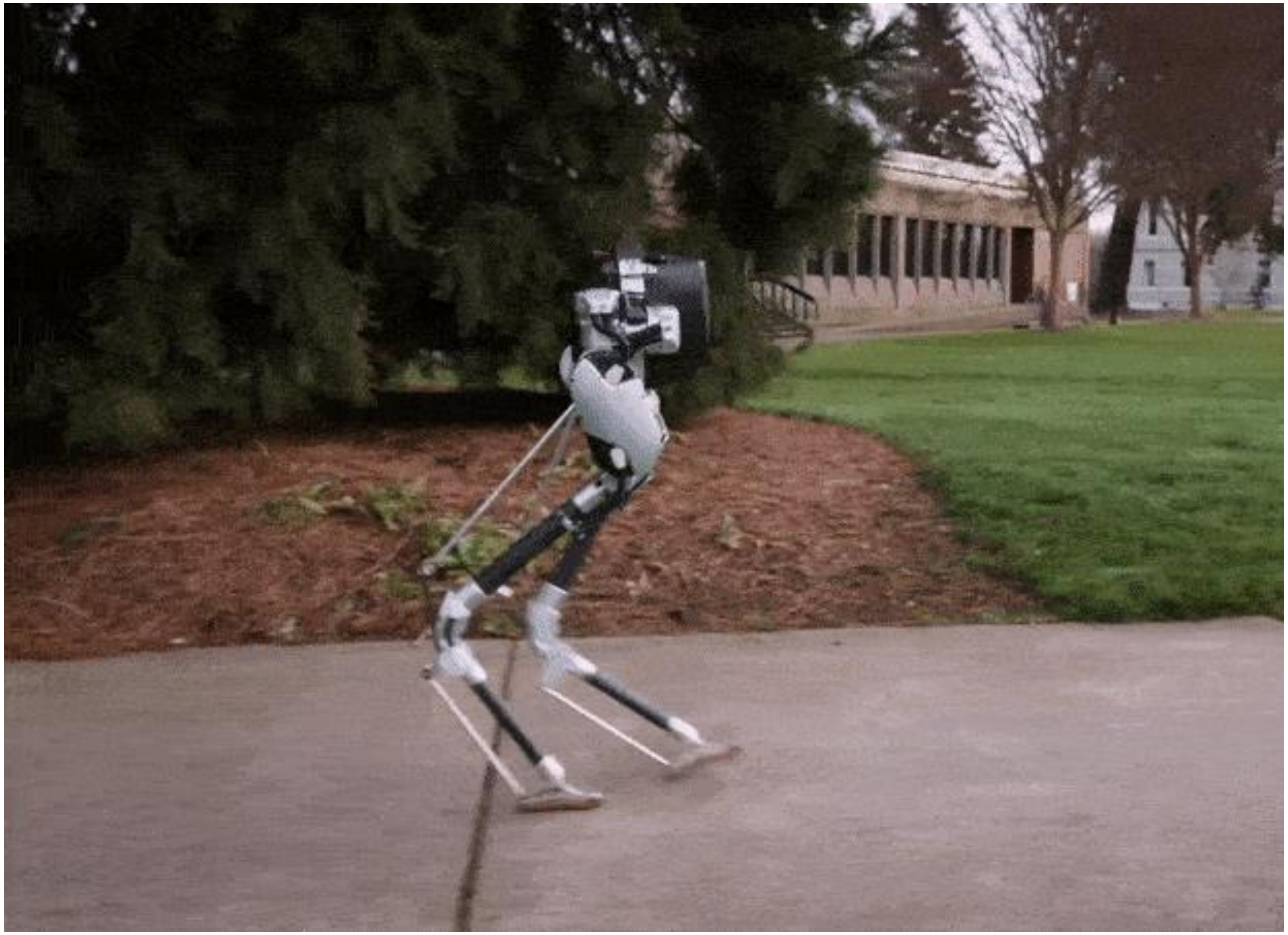


MIT Direct Drive Mark IV
(planar five-bar linkage)



UMinnesota Direct Drive Arm
(spatial five-bar linkage)

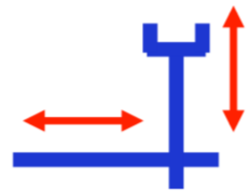
Robots with parallelogram structure



Cassie -- Agility Robotics

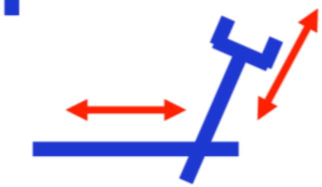
Analysis of inertial couplings

- Cartesian robot



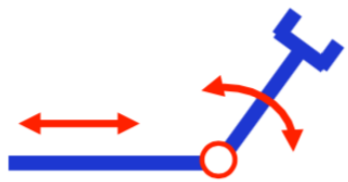
$$D = \begin{bmatrix} d_{11} & \\ & d_{22} \end{bmatrix}$$

- Cartesian "skew" robot



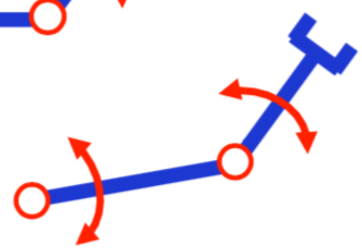
$$D = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix}$$

- PR robot



$$D(q) = \begin{bmatrix} d_{11} & d_{12}(q_2) \\ d_{21}(q_2) & d_{22} \end{bmatrix}$$

- 2R robot



$$D(q) = \begin{bmatrix} d_{11}(q_2) & d_{12}(q_2) \\ d_{21}(q_2) & d_{22} \end{bmatrix}$$

$$\boldsymbol{D}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{G}(\boldsymbol{q}) = \boldsymbol{B}\boldsymbol{u}$$

- dynamic model turns out to be **linear** if

$\boldsymbol{G} \equiv \mathbf{0}$

$\boldsymbol{D} = \text{Constant} \rightarrow \boldsymbol{C} \equiv \mathbf{0}$

$r_2^{\text{CoM2}} = 0$ in PR

$r_2^{\text{CoM2}} = 0$ in 2R

COM of link 2 on joint 2 axis

Analysis of gravity term

- static balancing
  - distribution of masses (including motors)
- mechanical compensation
  - articulated system of springs
  - closed kinematic chains
- absence of gravity
  - applications in space
  - constant (motion on horizontal plane)

$\Rightarrow \boldsymbol{G(q) \approx 0}$

Properties of Dynamic Equations

- **Skew Symmetry (反对称性)**

$$\boldsymbol{D}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{\tau}$$

$$d_{kj} \qquad c_{kj} = \sum_{i=1}^{n} \frac{1}{2}\left\{\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k}\right\}\dot{q}_i$$

$\boldsymbol{N}(\boldsymbol{q},\dot{\boldsymbol{q}}) \equiv \dot{\boldsymbol{D}}(\boldsymbol{q}) - 2\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})$ **is skew symmetric, that is** $n_{jk} = -n_{kj}$

By the chain rule, $\dot{d}_{kj} = \sum_{i=1}^{n} \frac{\partial d_{kj}}{\partial q_i}\dot{q}_i$

$$n_{kj} = \dot{d}_{kj} - 2c_{kj}$$

$$= \sum_{i=1}^{n}\left\{\frac{\partial d_{kj}}{\partial q_i} - \left\{\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k}\right\}\right\}\dot{q}_i$$

$$= \sum_{i=1}^{n}\left\{\frac{\partial d_{ij}}{\partial q_k} - \frac{\partial d_{ki}}{\partial q_j}\right\}\dot{q}_i \qquad n_{jk} = -n_{kj}$$

Properties of Dynamic Equations

- **Passivity (无源性)**

There exists a constant $\beta \geq 0$, such that $\int_0^T \dot{q}^{\mathrm{T}}(t)\tau(t)\mathrm{d}t \geq -\beta, \forall T > 0$

1. The term $\dot{q}^{\mathrm{T}}\tau$ has units of power. Thus, $\int_0^T \dot{q}^{\mathrm{T}}(t)\tau(t)\mathrm{d}t$ is the energy produced by the system over the time interval $[\,0,\ T\,]$.
2. Passivity means that the amount of energy dissipated by the system has a lower bound given by $-\beta$.
3. The word passivity comes from circuit theory where a passive system according to the above definition is one that can be built from passive components (resistors, capacitors, inductors).
4. Likewise a passive mechanical system can be built from masses, springs, and dampers.

Properties of Dynamic Equations

● Passivity (无源性)

To prove the passivity property, let H be the total energy of the system, that is, the sum of the kinetic and potential energies,

$$H = \frac{1}{2}\dot{q}^{\mathrm{T}} D(q)\dot{q} + P(q)$$

The derivative $\dot{H}$ satisfies

$$\dot{H} = \dot{q}^{\mathrm{T}} D(q)\ddot{q} + \frac{1}{2}\dot{q}^{\mathrm{T}}\dot{D}(q)\dot{q} + \dot{q}^{\mathrm{T}}\frac{\partial P}{\partial q} \qquad D(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \tau$$

$$= \dot{q}^{\mathrm{T}}\{\tau - C(q,\dot{q})\dot{q} - g(q)\} + \frac{1}{2}\dot{q}^{\mathrm{T}}\dot{D}(q)\dot{q} + \dot{q}^{\mathrm{T}} g(q) \qquad \frac{\partial P}{\partial q} = g(q)$$

$$= \dot{q}^{\mathrm{T}}\tau + \frac{1}{2}\dot{q}^{\mathrm{T}}\{\dot{D}(q) - 2C(q,\dot{q})\}\,\dot{q}$$

skew symmetry：$X^{\mathrm{T}} S X = 0$

$$= \dot{q}^{\mathrm{T}}\tau$$

Integrating both sides of the above equation with respect to time gives,

$$\int_0^T \dot{q}^{\mathrm{T}}(\xi)\tau(\xi)\mathrm{d}\xi = \int_0^T \dot{H}(t)\mathrm{dt} = H(T) - H(0) \geq -H(0)$$

Properties of Dynamic Equations

- **Bounds of Inertial Matrix (惯性矩阵的界)**

The inertia matrix for an *n*-link rigid robot is symmetric and positive definite. For a fixed value of the generalized coordinate *q*, let $0 < \lambda_1(q) \leq \cdots \leq \lambda_n(q)$ denote the *n* eigenvalues of $D(q)$. As a result, it can easily be shown that

$$\lambda_1(q)I_{n\times n} \leq D(q) \leq \lambda_{\mathrm{n}}(q)I_{n\times n}$$

If all of the joints are revolute, then the inertia matrix contains only terms involving sine and cosine functions and, hence, is bounded as a function of the generalized coordinates.

As a result, one can find constants λ_m and λ_M that provide uniform (independent of q) bounds in the inertia matrix

$$\lambda_m I_{n\times n} \leq D(q) \leq \lambda_M I_{n\times n} < \infty$$

Properties of Dynamic Equations

Linearity in the Parameters

The robot equations of motion are defined in terms of certain parameters, such as link masses, moments of inertia, etc. The equations of motion are linear in these inertia parameters.

There exists an $n \times l$ function, $Y(q, \dot{q}, \ddot{q})$ and an l-dimensional vector Θ such that the Euler–Lagrange equations can be written as：

$$D(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) = Y(q, \dot{q}, \ddot{q})\Theta$$

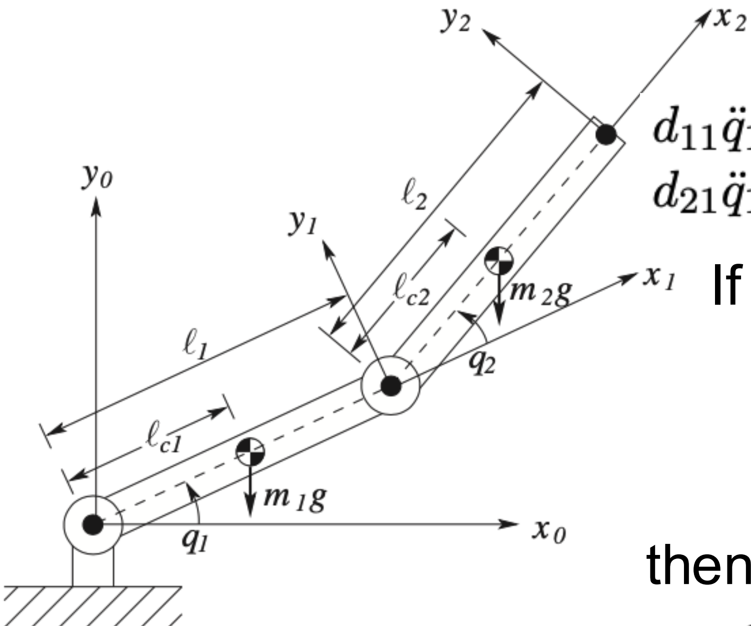
where function $Y(q, \dot{q}, \ddot{q})$ is called the **regressor (回归量)** and $\Theta \in \mathbb{R}^l$ is the **parameter vector (参数向量)**.

In general, a given rigid body is described by 10 parameters, namely, the total mass, the six independent entries of the inertia tensor, and the three coordinates of the center of mass.

An n-link robot then has a maximum of $10n$ dynamics parameters. However, since the link motions are constrained by joint interconnections, there are actually fewer than $10n$ independent parameters.

Properties of Dynamic Equations

- **Linearity in the Parameters**



$$\begin{aligned} d_{11}\ddot{q}_1 + d_{12}\ddot{q}_2 + c_{121}\dot{q}_1\dot{q}_2 + c_{211}\dot{q}_2\dot{q}_1 + c_{221}\dot{q}_2^2 + g_1 &= \tau_1 \\ d_{21}\ddot{q}_1 + d_{22}\ddot{q}_2 + c_{112}\dot{q}_1^2 + g_2 &= \tau_2 \end{aligned}$$

If we group the inertia terms as

$$\begin{aligned} \Theta_1 &= m_1\ell_{c1}^2 + m_2(\ell_1^2 + \ell_{c2}^2) + I_1 + I_2 \\ \Theta_2 &= m_2\ell_1\ell_{c2} \\ \Theta_3 &= m_2\ell_{c2}^2 + I_2 \end{aligned}$$

then we can write the inertia matrix elements as

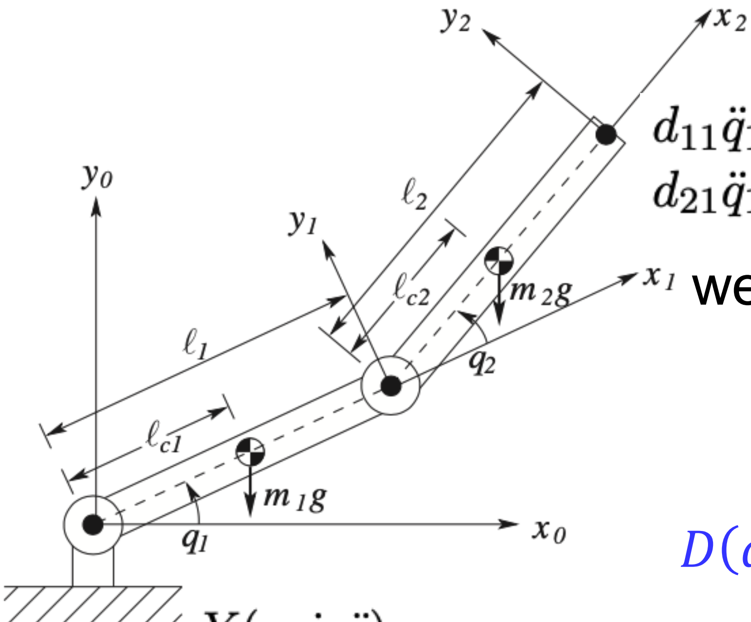
$$\begin{aligned} d_{11} &= \Theta_1 + 2\Theta_2\cos(q_2) \\ d_{12} &= d_{21} = \Theta_3 + \Theta_2\cos(q_2) \\ d_{22} &= \Theta_3 \end{aligned}$$

No additional parameters are required in the Christoffel symbols. However, the gravitational torques generally require additional parameters. Setting

$$\begin{aligned} \Theta_4 &= m_1\ell_{c1} + m_2\ell_1 \\ \Theta_5 &= m_2\ell_{c2} \end{aligned}$$

Properties of Dynamic Equations

- **Linearity in the Parameters**



$$
\begin{aligned}
d_{11}\ddot{q}_1 + d_{12}\ddot{q}_2 + c_{121}\dot{q}_1\dot{q}_2 + c_{211}\dot{q}_2\dot{q}_1 + c_{221}\dot{q}_2^2 + g_1 &= \tau_1 \\
d_{21}\ddot{q}_1 + d_{22}\ddot{q}_2 + c_{112}\dot{q}_1^2 + g_2 &= \tau_2
\end{aligned}
$$

we can write the gravitational terms g_1 and g_2 as

$$
\begin{aligned}
g_1 &= \Theta_4 g\cos(q_1) + \Theta_5 g\cos(q_1+q_2) \\
g_2 &= \Theta_5 g\cos(q_1+q_2)
\end{aligned}
$$

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = Y(q,\dot{q},\ddot{q})\Theta$$

$$Y(q,\dot{q},\ddot{q}) =$$

$$
\begin{bmatrix}
\ddot{q}_1 & \cos(q_2)(2\ddot{q}_1+\ddot{q}_2) - \sin(q_2)(\dot{q}_1^2 + 2\dot{q}_1\dot{q}_2) & \ddot{q}_2 & g\cos(q_1) & g\cos(q_1+q_2) \\
0 & \cos(q_2)\ddot{q}_1 + \sin(q_2)\dot{q}_1^2 & \ddot{q}_1+\ddot{q}_2 & 0 & g\cos(q_1+q_2)
\end{bmatrix}
$$

$$
\Theta = \begin{bmatrix} \Theta_1 \\ \Theta_2 \\ \Theta_3 \\ \Theta_4 \\ \Theta_5 \end{bmatrix} = \begin{bmatrix} m_1\ell_{c1}^2 + m_2(\ell_1^2 + \ell_{c2}^2) + I_1 + I_2 \\ m_2\ell_1\ell_{c2} \\ m_2\ell_{c2}^2 + I_2 \\ m_1\ell_{c1} + m_2\ell_1 \\ m_2\ell_{c2} \end{bmatrix}
$$

Generalized Forces

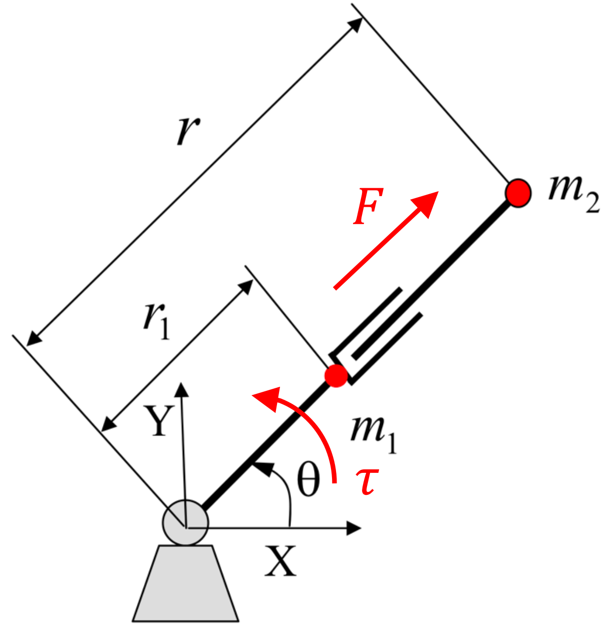
How to obtain the generalized forces f_1 and f_2 ?

The virtual work:

$$\delta W = \sum_i (f_i \delta q_i)$$

Except for q_i, let the virtual displacement of the remaining joints be zero. Then f_i can be written as

$$f_i = \frac{\delta W}{\delta q_i}$$



$$\delta q_1 = \delta\theta \qquad \delta W = \tau\delta\theta \quad \Rightarrow \quad f_1 = \frac{\delta W}{\delta q_1} = \frac{\tau\delta\theta}{\delta\theta} = \tau$$

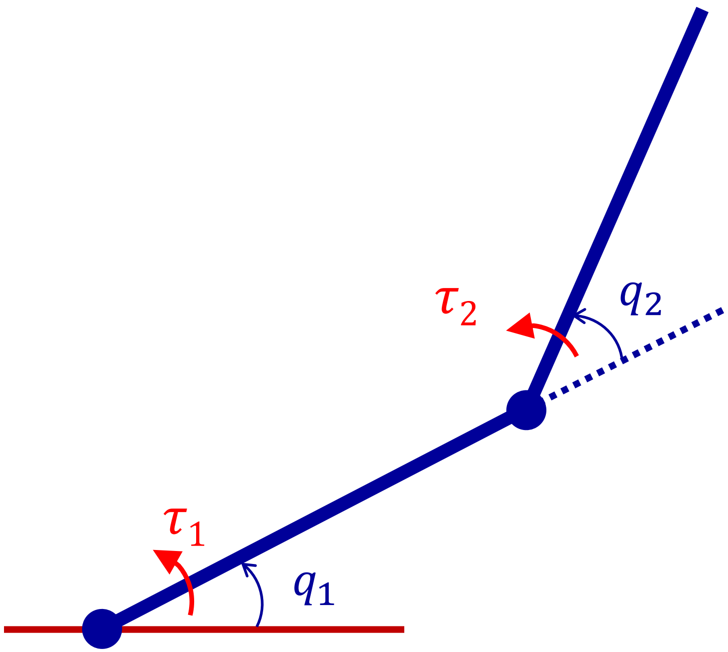
$$\delta q_2 = \delta r \qquad \delta W = F\delta r \quad \Rightarrow \quad f_2 = \frac{\delta W}{\delta q_2} = \frac{F\delta r}{\delta r} = F$$

The form of generalized forces depends on the the selection of generalized coordinates!

Generalized Forces

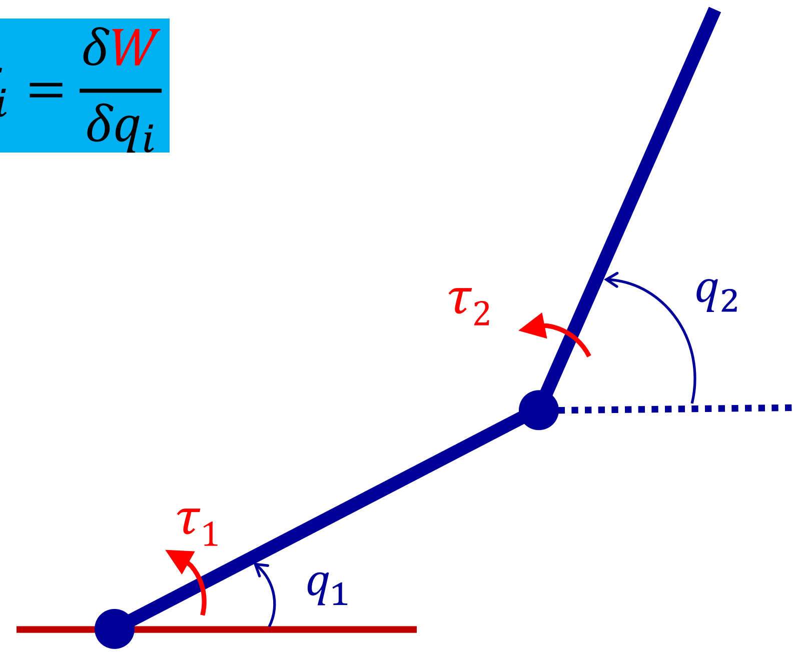
Example: Derive the Generalized forces for the following two cases.

$$f_i = \frac{\delta W}{\delta q_i}$$



$f_1 = ?$

$f_2 = ?$



$f_1 = ?$

$f_2 = ?$

Generalized Forces

- **Actuation torques（驱动转矩）**

$$\boldsymbol{D}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{B\tau}$$

- **Viscous damping（阻尼力）**

$$\boldsymbol{D}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{B\tau} - \boldsymbol{F}_v\dot{\boldsymbol{q}}$$

where $\boldsymbol{F}_v$ is the coefficient of damping matrix（n×n, diagonal）.

- **Coulomb friction（摩擦力）**

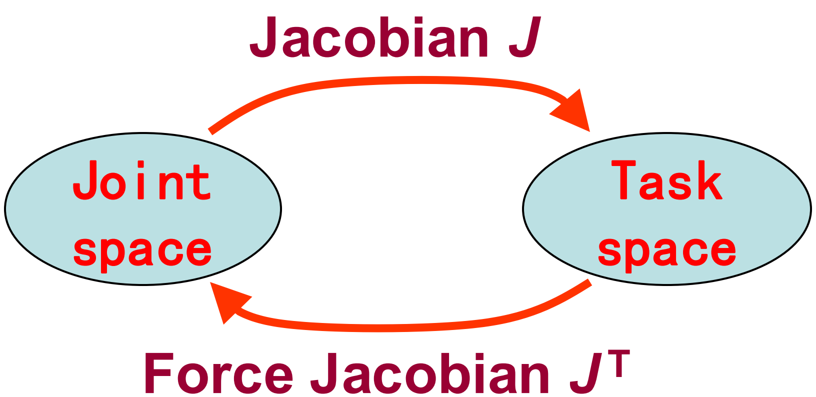
$$\boldsymbol{D}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{B\tau} - \boldsymbol{F}_v\dot{\boldsymbol{q}} - \boldsymbol{F}_s\mathrm{sgn}(\dot{\boldsymbol{q}})$$

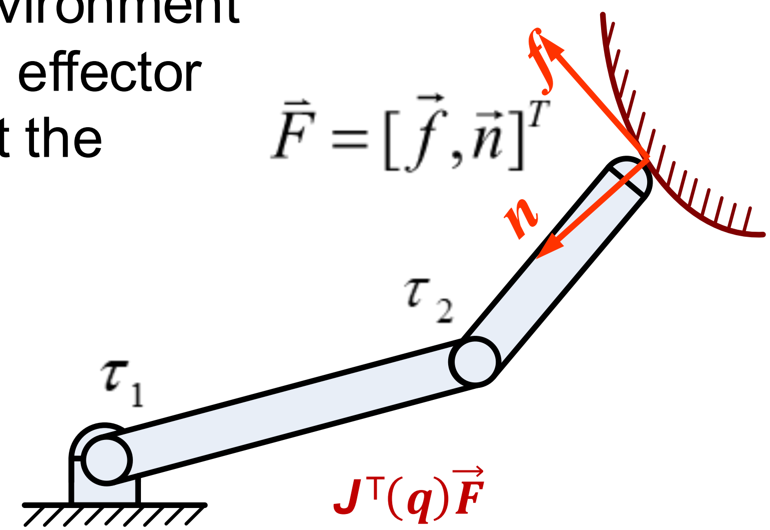
where $\boldsymbol{F}_s$ is the coefficient of friction matrix（n×n, diagonal）, and $\mathrm{sgn}(\dot{q})$ is the vector of sign function (n$\times$ 1)，determined by the sign of the joint velocity.

Generalized Forces

- **Contact forces / Impact forces（接触力 / 碰撞力）**

Interaction of the manipulator with the environment produces forces and moments at the end effector or tool. These, in turn, produce torques at the joints of the robot.





$$\boldsymbol{D}(\boldsymbol{q})\ddot{\boldsymbol{q}}+\boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}}+\boldsymbol{g}(\boldsymbol{q})=\boldsymbol{B}\boldsymbol{\tau}-\boldsymbol{F}_v\dot{\boldsymbol{q}}-\boldsymbol{F}_s\mathrm{sgn}(\dot{\boldsymbol{q}})+\boldsymbol{J}^{\mathrm{T}}(\boldsymbol{q})\vec{\boldsymbol{F}}$$

where $\vec{\boldsymbol{F}}$ represent the vector of forces and moments at the end effector,

$\boldsymbol{J}(\boldsymbol{q})$ is the transpose of the manipulator Jacobian。

Summary

- **Some Common Configurations**

  Two-Link Cartesian Manipulator

  Planar Elbow Manipulator

  Planar Elbow Manipulator with Remotely Driven Link

  Five-Bar Linkage

- **Analysis of inertial couplings**
- **Analysis of gravity term**
- **Properties of Dynamic Equations**

Midterm Test

Midterm Test

Time： April 21, 2025
10:20—12:10

Place： 一教 326

考试内容：

- 机器人基本概念
- 位姿描述与齐次变换
- 正运动学
- 逆运动学
- 速度运动学
- 动力学

考试方式：

- 半开卷—可看指定教材
- 5页内的A4纸笔记