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南方科技大学
SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY

# Robot Modeling & Control ME331

## Section 12： Dynamics II

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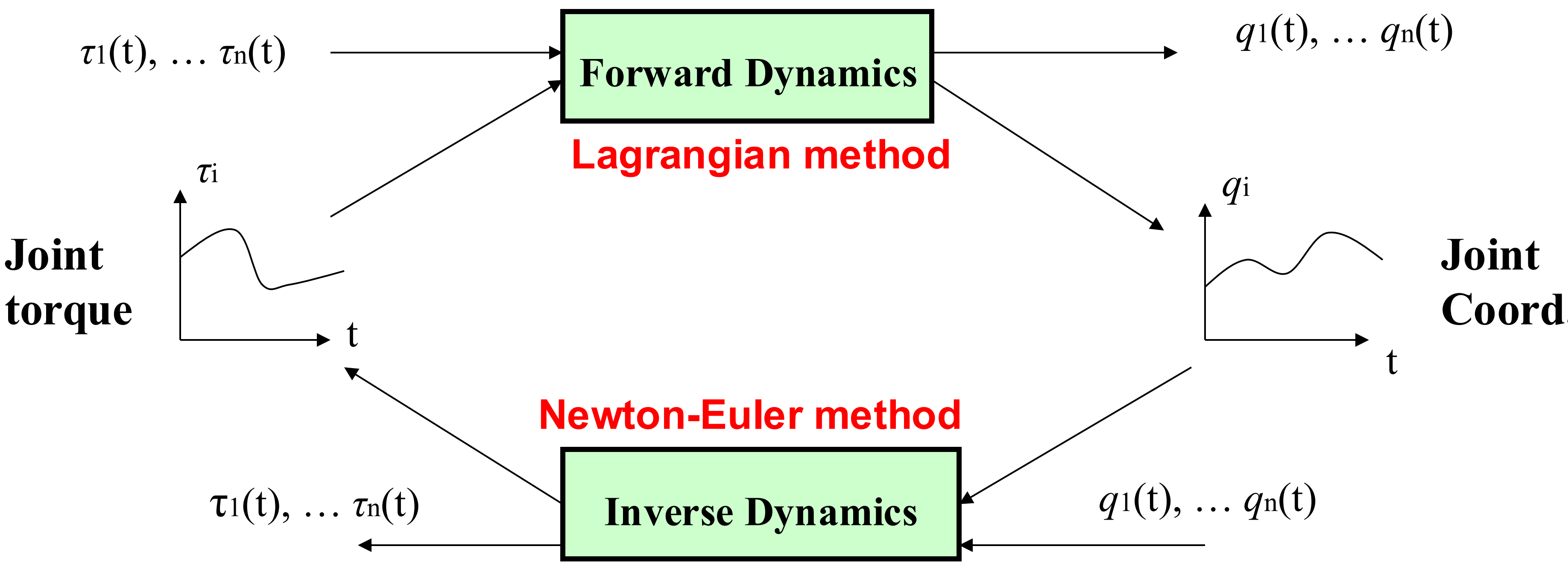
# Outline

- Review
  - Holonomic Constraints and Virtual Work
- The Euler-Lagrange Equations
  - D'Alembert's Principle
- Kinetic and Potential Energy
  - The Inertia Tensor
  - Kinetic Energy for an *n*-Link Robot
  - Potential Energy for an n-Link Robot
- Equations of Motion
- Some Common Configurations

# Dynamics

- **Mathematical equations describing the dynamic behavior of the robot**
  - For computer simulation
  - Design of suitable controller
  - Evaluation of robot structure

**Force ⟺ Motion**



# The Euler–Lagrange Equations

## Holonomic Constraints and Virtual Work

A constraint on the $k$ coordinates $r_1$ ,..., $r_k$ is called **holonomic** (完整的) if it is an equality constraint of the form

$$g_i(r_1,\ldots,r_k) = 0, \quad i = 1,\cdots,l$$

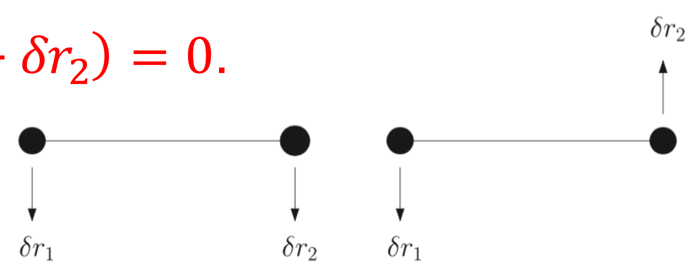
It may be possible to express the coordinates of the $k$ particles in terms of $n$ **generalized coordinates** $q_1, \ldots, q_n$.

The coordinates of the various particles, subjected to the set of constraints, can be expressed in the form

$$r_i = r_i(q_1,\ldots,q_n), \quad i = 1,\cdots,k$$

**Virtual displacements** : any set of infinitesimal displacements, $\delta r_1,\ldots,\delta r_k$, that are consistent with the constraints.

$$\|r_1 - r_2\| = l \quad \Longrightarrow \quad (r_1 - r_2)^T(\delta r_1 - \delta r_2) = 0.$$




The set of all virtual displacements is

$$\delta r_i = \sum_{j=1}^{n} \frac{\partial r_i}{\partial q_j} \delta q_j$$

# The Euler–Lagrange Equations

## Holonomic Constraints and Virtual Work

**Discussion of constrained systems in equilibrium.**

$$\sum_{i=1}^{k} F_i^T \delta r_i = 0$$

where $F_i$ is the total force on particle $i$, and consists of (i) the externally applied force $f_i$ , (ii) the constraint force $f_i^a$.

$$f_1^a = c(r_1 - r_2) \qquad f_2^a = -c(r_1 - r_2)$$

$$f_1^{a^T} \delta r_1 + f_2^{a^T} \delta r_2 = c(r_1 - r_2)^T (\delta r_1 - \delta r_2) = 0.$$

$$\sum_{i=1}^{k} f_i^{a^T} \delta r_i = 0$$

$$\sum_{i=1}^{k} f_i^T \delta r_i = 0$$

**Principle of virtual work**: *The work done by external forces corresponding to any set of virtual displacements is zero.*

# The Euler–Lagrange Equations

## D'Alembert's Principle

**Principle of virtual work**:

$$\sum_{i=1}^{k} f_i^T \delta r_i = 0 \quad (\text{for static system})$$

**D'Alembert's principle:**

$$\sum_{i=1}^{k} f_i^T \delta r_i - \sum_{i=1}^{k} \dot{p}_i^T \delta r_i = 0 \quad (\text{for dynamic system})$$

Recall $\delta r_i = \sum_{j=1}^{n} \frac{\partial r_i}{\partial q_j} \delta q_j$. The virtual work done by the forces $f_i$ is given by

$$\sum_{i=1}^{k} f_i^T \delta r_i = \sum_{i=1}^{k} \sum_{j=1}^{n} f_i^T \frac{\partial r_i}{\partial q_j} \delta q_j = \sum_{j=1}^{n} \psi_j \delta q_j$$

where

$$\psi_j = \sum_{i=1}^{k} f_i^T \frac{\partial r_i}{\partial q_j}$$

is called the $j^{th}$ **generalized force**.

# The Euler–Lagrange Equations

## D'Alembert's Principle

$$\sum_{i=1}^{k} f_i^T \delta r_i - \sum_{i=1}^{k} \dot{p}_i^T \delta r_i = 0 \quad \text{(for dynamic system)}$$

Now, let us study the **second summation**. Since $p_i = m_i \dot{r}_i$, it follows that

$$\sum_{i=1}^{k} \dot{p}_i^T \delta r_i = \sum_{i=1}^{k} m_i \ddot{r}_i^T \delta r_i = \sum_{i=1}^{k} \sum_{j=1}^{n} m_i \ddot{r}_i^T \frac{\partial r_i}{\partial q_j} \delta q_j$$

Next, using the product rule of differentiation, we have

$$\frac{d}{dt}\left[m_i \dot{r}_i^T \frac{\partial r_i}{\partial q_j}\right] = m_i \ddot{r}_i^T \frac{\partial r_i}{\partial q_j} + m_i \dot{r}_i^T \frac{d}{dt}\left[\frac{\partial r_i}{\partial q_j}\right]$$

Rearranging the above and summing over all $i = 1, \cdots, n$ yields

$$\sum_{i=1}^{k} m_i \ddot{r}_i^T \frac{\partial r_i}{\partial q_j} = \sum_{i=1}^{k} \left\{ \frac{d}{dt}\left[m_i \dot{r}_i^T \frac{\partial r_i}{\partial q_j}\right] - m_i \dot{r}_i^T \frac{d}{dt}\left[\frac{\partial r_i}{\partial q_j}\right] \right\}$$

Differentiating equation $r_i = r_i(q_1, \ldots, q_n)$ using the chain rule gives

$$v_i = \dot{r}_i = \sum_{j=1}^{n} \frac{\partial r_i}{\partial q_j} \dot{q}_j \quad \Rightarrow \quad \frac{\partial v_i}{\partial \dot{q}_j} = \frac{\partial r_i}{\partial q_j}$$

# The Euler–Lagrange Equations

## D'Alembert's Principle

Now, let us study the **second summation**. Since $p_i = m_i \dot{r}_i$, it follows that

$$\sum_{i=1}^{k} \dot{p}_i^T \delta r_i = \sum_{i=1}^{k} \sum_{j=1}^{n} m_i \ddot{r}_i^T \frac{\partial r_i}{\partial q_j} \delta q_j$$

$$\sum_{i=1}^{k} m_i \ddot{r}_i^T \frac{\partial r_i}{\partial q_j} = \sum_{i=1}^{k} \left\{ \frac{d}{dt}\left[ m_i \dot{r}_i^T \frac{\partial r_i}{\partial q_j} \right] - m_i \dot{r}_i^T \frac{d}{dt}\left[ \frac{\partial r_i}{\partial q_j} \right] \right\}$$

$$\frac{\partial v_i}{\partial \dot{q}_j} = \frac{\partial r_i}{\partial q_j}$$

$$\frac{d}{dt}\left[ \frac{\partial r_i}{\partial q_j} \right] = \sum_{l=1}^{n} \frac{\partial^2 r_i}{\partial q_j \partial q_l} \dot{q}_l = \frac{\partial}{\partial q_j} \sum_{l=1}^{n} \frac{\partial r_i}{\partial q_l} \dot{q}_l = \frac{\partial v_i}{\partial q_j}$$

$$\text{Recall } \sum_{i=1}^{k} m_i \ddot{r}_i^T \frac{\partial r_i}{\partial q_j} = \sum_{i=1}^{k} \left\{ \frac{d}{dt}\left[ m_i \dot{r}_i^T \frac{\partial r_i}{\partial q_j} \right] - m_i \dot{r}_i^T \frac{d}{dt}\left[ \frac{\partial r_i}{\partial q_j} \right] \right\}$$

$$= \sum_{i=1}^{k} \left\{ \frac{d}{dt}\left[ m_i v_i^T \frac{\partial v_i}{\partial \dot{q}_j} \right] - m_i \dot{r}_i^T \frac{\partial v_i}{\partial q_j} \right\}$$

# The Euler–Lagrange Equations

## D'Alembert's Principle

$$\sum_{i=1}^{k} m_i \ddot{r}_i^T \frac{\partial r_i}{\partial q_j} = \sum_{i=1}^{k} \left\{ \frac{d}{dt} \left[ m_i v_i^T \frac{\partial v_i}{\partial \dot{q}_j} \right] - m_i \dot{r}_i^T \frac{\partial v_i}{\partial q_j} \right\}$$

If we define the **kinetic energy** $K$ to be the quantity

$$K = \sum_{i=1}^{k} \frac{1}{2} m_i v_i^T v_i$$

Then the above equation can be compactly expressed as

$$\sum_{i=1}^{k} m_i \ddot{r}_i^T \frac{\partial r_i}{\partial q_j} = \frac{d}{dt} \frac{\partial K}{\partial \dot{q}_j} - \frac{\partial K}{\partial q_j}$$

Recall the **second summation**:

$$\sum_{i=1}^{k} \dot{p}_i^T \delta r_i = \sum_{i=1}^{k} \sum_{j=1}^{n} m_i \ddot{r}_i^T \frac{\partial r_i}{\partial q_j} \delta q_j = \sum_{j=1}^{n} \left\{ \frac{d}{dt} \frac{\partial K}{\partial \dot{q}_j} - \frac{\partial K}{\partial q_j} \right\} \delta q_j$$

# The Euler–Lagrange Equations

## D'Alembert's Principle

$$\sum_{i=1}^{k} f_i^T \delta r_i - \sum_{i=1}^{k} \dot{p}_i^T \delta r_i = 0 \ \text{(for dynamic system)}$$

$$\sum_{i=1}^{k} f_i^T \delta r_i = \sum_{j=1}^{n} \psi_j \delta q_j$$

$$\sum_{i=1}^{k} \dot{p}_i^T \delta r_i = \sum_{j=1}^{n} \left\{ \frac{d}{dt}\frac{\partial K}{\partial \dot{q}_j} - \frac{\partial K}{\partial q_j} \right\} \delta q_j$$

$$\sum_{j=1}^{n} \left\{ \frac{d}{dt}\frac{\partial K}{\partial \dot{q}_j} - \frac{\partial K}{\partial q_j} - \psi_j \right\} \delta q_j = 0$$

Now, since the virtual displacements $\delta q_j$ are independent, we can conclude that each coefficient in the above Equation is zero, that is,

$$\frac{d}{dt}\frac{\partial K}{\partial \dot{q}_j} - \frac{\partial K}{\partial q_j} = \psi_j, \qquad j = 1, \cdots n$$

# The Euler–Lagrange Equations

## D'Alembert's Principle

$$\frac{d}{dt}\frac{\partial K}{\partial \dot{q}_j} - \frac{\partial K}{\partial q_j} = \psi_j, \qquad j = 1, \cdots n$$

If the generalized force $\psi_j$ is the sum of an externally applied generalized force and another one due to a potential field, then a further modification is possible.

Suppose there exist functions $\tau_j$ and a potential energy function $P(q)$ such that

$$\psi_j = -\frac{\partial P}{\partial q_j} + \tau_j$$

Then $\frac{d}{dt}\frac{\partial K}{\partial \dot{q}_j} - \frac{\partial K}{\partial q_j} = \psi_j$ can be written in the form

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_j} - \frac{\partial \mathcal{L}}{\partial q_j} = \tau_j$$

where $\mathcal{L} = K - P$ is the Lagrangian.

We have recovered the **Euler–Lagrange Equation**.

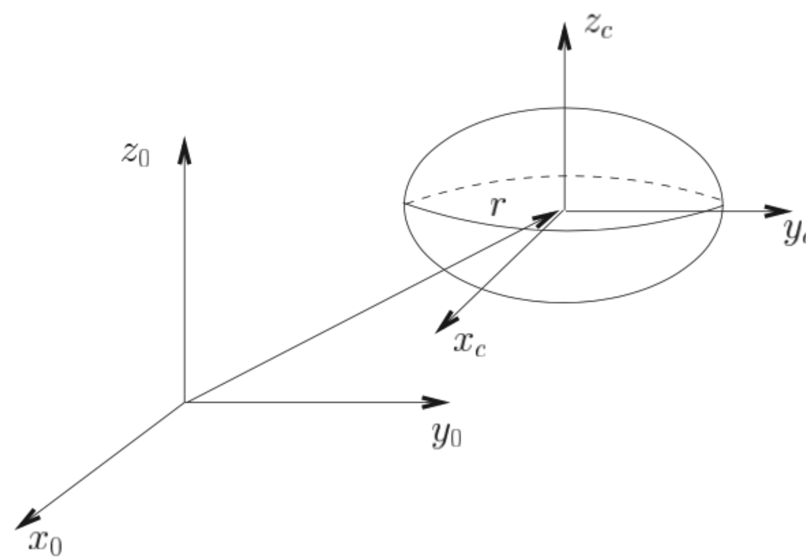
# Kinetic and Potential Energy

The kinetic energy of a rigid object is the sum of two terms:

1) the translational kinetic energy obtained by concentrating the entire mass of the object at the center of mass;

2) the rotational kinetic energy of the body about the center of mass.

We attach a coordinate frame at the center of mass (called the **body-attached frame**) as shown in the right figure.




The kinetic energy of the rigid body is then given as

$$K = \frac{1}{2} m v^T v + \frac{1}{2} \omega^T \mathcal{J} \omega$$

where $m$ is the total mass of the object, $v$ and $\omega$ are the linear and angular velocity vectors, respectively, and $\mathcal{J}$ is a symmetric $3 \times 3$ matrix called the **inertia tensor.**

# Kinetic and Potential Energy

## The Inertial Tensor

$$K = \frac{1}{2} m v^T v + \frac{1}{2} \omega^T \mathcal{J} \omega$$

where the above linear and angular velocity vectors, $v$ and $\omega$, respectively, are expressed in the inertial frame.

In this case, we know that $\omega$ is found from the skew-symmetric matrix

$$S(\omega) = \dot{R} R^T$$

where $R$ is the orientation transformation from the body-attached frame and the inertial frame.

It is therefore necessary to express the inertia tensor, $\mathcal{J}$, also in the inertial frame in order to compute the triple product $\omega^T \mathcal{J} \omega$.

If we denote as $I$ the inertia tensor expressed instead in the body-attached frame, then the two matrices are related via a similarity transformation according to

$$\mathcal{J} = R I R^T$$

This is an important observation because $I$ is a constant matrix independent of the motion of the object and easily computed.

# Kinetic and Potential Energy

## The Inertial Tensor

Let the mass density of the object be represented as a function of position, $\rho(x, y, z)$. Then the inertia tensor in the body attached frame is computed as

$$I = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$$

where

$$I_{xx} = \iiint (y^2 + z^2)\, \rho(x, y, z) dxdydz \qquad I_{xy} = I_{yx} = -\iiint xy\, \rho(x, y, z) dxdydz$$

$$I_{yy} = \iiint (x^2 + z^2)\, \rho(x, y, z) dxdydz \qquad I_{xz} = I_{zx} = -\iiint xz\, \rho(x, y, z) dxdydz$$

$$I_{zz} = \iiint (x^2 + y^2)\, \rho(x, y, z) dxdydz \qquad I_{yz} = I_{zy} = -\iiint yz\, \rho(x, y, z) dxdydz$$

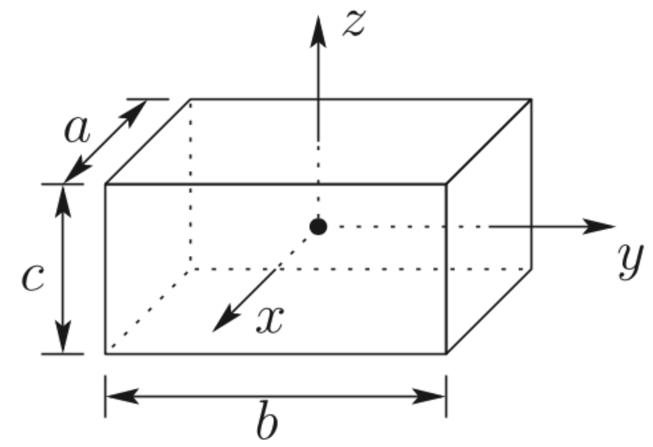
**principal moments of inertia (主惯性矩)**

**cross products of inertia (惯性积)**

# Kinetic and Potential Energy

## The Inertial Tensor：Example

Consider the rectangular solid of length a width b and height c and suppose that the density is constant, $\rho(x, y, z) = \rho$.




If the body frame is attached at the geometric center of the object, then by symmetry, the cross products of inertia are all zero.

It is a simple exercise to compute

$$I_{xx} = \int_{-c/2}^{c/2}\int_{-b/2}^{b/2}\int_{-a/2}^{a/2}(y^2 + z^2)\rho(x, y, z)dxdydz = \rho\frac{abc}{12}(b^2 + c^2) = \frac{m}{12}(b^2 + c^2)$$

Likewise, a similar calculation shows that

$$I_{yy} = \frac{m}{12}(a^2 + c^2) \qquad I_{zz} = \frac{m}{12}(a^2 + b^2)$$

# Kinetic and Potential Energy

## Kinetic Energy for an $n$-Link Robot

Consider a manipulator with $n$ links. The linear and angular velocities of any point on any link can be expressed in terms of the Jacobian matrix and the derivatives of the joint variables.

For Jacobian matrices $J_{v_i}$ and $J_{\omega_i}$, of dimension $3 \times n$, we have

$$v_i = J_{v_i}(q)\dot{q}, \qquad \omega_i = J_{\omega_i}(q)\dot{q}$$

Suppose the mass of link $i$ is $m_i$ and that the inertia matrix of link $i$, evaluated around a coordinate frame parallel to frame $i$ but whose origin is at the center of mass, equals $I_i$. Then the overall kinetic energy of the manipulator equals

$$\begin{aligned} K &= \frac{1}{2}\sum_{i=1}^{n}\{m_i {v_i}^T v_i + {\omega_i}^T \mathcal{I}_i \omega_i\} \\ &= \frac{1}{2}\dot{q}^T\left[\sum_{i=1}^{n}\{m_i J_{v_i}(q)^T J_{v_i}(q) + J_{\omega_i}(q)^T R_i(q)\, I_i R_i(q)^T J_{\omega_i}(q)\}\right]\dot{q} \\ &= \frac{1}{2}\dot{q}^T D(q)\dot{q} \end{aligned}$$

# Kinetic and Potential Energy

## Kinetic Energy for an $n$-Link Robot

The overall kinetic energy of the manipulator equals

$$
\begin{aligned}
K &= \frac{1}{2}\sum_{i=1}^{n}\{m_i v_i{}^T v_i + \omega_i{}^T \mathcal{I}_i \omega_i\} \\
&= \frac{1}{2}\dot{q}^T\left[\sum_{i=1}^{n}\left\{m_i J_{v_i}(q)^T J_{v_i}(q) + J_{\omega_i}(q)^T R_i(q) I_i R_i(q)^T J_{\omega_i}(q)\right\}\right]\dot{q} \\
&= \frac{1}{2}\dot{q}^T D(q)\dot{q}
\end{aligned}
$$

where $D(q)$ is an $n \times n$ configuration dependent matrix called the **inertia matrix**.

- The inertia matrix is symmetric and positive definite for any manipulator.
- Symmetry of $D(q)$ is easily seen from Equation.
- Positive definiteness can be inferred from the fact that the kinetic energy is always nonnegative and is zero if and only if all of the joint velocities are zero.

# Kinetic and Potential Energy

## Potential Energy for an $n$-Link Robot

In the case of rigid dynamics, the only source of potential energy is gravity. The potential energy of the $i^{th}$ link can be computed by assuming that the mass of the entire object is concentrated at its center of mass and is given by

$$P_i = m_i g^T r_{ci}$$

where $g$ is the vector giving the direction of gravity in the inertial frame and the vector $r_{ci}$ gives the coordinates of the center of mass of link $i$.

The total potential energy of the $n$-link robot is therefore

$$P = \sum_{i=1}^{n} P_i = \sum_{i=1}^{n} m_i g^T r_{ci}$$

In the case that the robot contains elasticity, for example if the joints are flexible, then the potential energy will include terms containing the energy stored in the elastic elements.

Note that the potential energy is a function only of the generalized coordinates and not their derivatives.

# Equations of Motion

We specialize the Euler–Lagrange equations to the case when two conditions hold.

1) The kinetic energy is a quadratic function of the vector $\dot{q}$ of the form

$$K = \frac{1}{2}\dot{q}^T D(q)\dot{q} = \frac{1}{2}\sum_{i,j} d_{ij}(q)\,\dot{q}_i\dot{q}_j$$

where $d_{ij}(q)$ are the entries of the $n \times n$ inertia matrix $D(q)$, which is symmetric and positive definite for each $q \in \mathbb{R}^n$.

2) The potential energy $P = P(q)$ is independent of $\dot{q}$.

The Euler–Lagrange equations for such a system can be derived as follows. The Lagrangian can be written as

$$L = K - P = \frac{1}{2}\sum_{i,j} d_{ij}(q)\,\dot{q}_i\dot{q}_j - P(q)$$

The partial derivatives of the Lagrangian with respect to the $k^{th}$ joint velocity is given by

$$\frac{\partial L}{\partial \dot{q}_k} = \sum_j d_{kj}\,\dot{q}_j$$

# Equations of Motion

$$\frac{\partial L}{\partial \dot{q}_k} = \sum_j d_{kj}\, \dot{q}_j$$

and therefore

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} = \sum_j d_{kj}(q)\, \ddot{q}_j + \sum_j \frac{d}{dt} d_{kj}\, \dot{q}_j$$

$$= \sum_j d_{kj}(q)\, \ddot{q}_j + \sum_{i,j} \frac{\partial d_{kj}}{\partial q_i} \dot{q}_i \dot{q}_j$$

Similarly the partial derivative of the Lagrangian with respect to the $k^{th}$ joint position is given by

$$\frac{\partial L}{\partial q_k} = \frac{\partial \left( \frac{1}{2} \sum_{i,j} d_{ij}(q)\, \dot{q}_i \dot{q}_j - P(q) \right)}{\partial q_k} = \frac{1}{2} \sum_{i,j} \frac{\partial d_{ij}}{\partial q_k} \dot{q}_i \dot{q}_j - \frac{\partial P}{\partial q_k}$$

Thus, for each $k = 1, \cdots, n$, the Euler–Lagrange equations can be written

$$\sum_j d_{kj}(q)\, \ddot{q}_j + \sum_{i,j} \frac{\partial d_{kj}}{\partial q_i} \dot{q}_i \dot{q}_j - \frac{1}{2} \sum_{i,j} \frac{\partial d_{ij}}{\partial q_k} \dot{q}_i \dot{q}_j + \frac{\partial P}{\partial q_k} = \tau_k$$

# Equations of Motion

$$\sum_j d_{kj}(q)\,\ddot{q}_j + \sum_{i,j}\left\{\frac{\partial d_{kj}}{\partial q_i} - \frac{1}{2}\frac{\partial d_{ij}}{\partial q_k}\right\}\dot{q}_i\dot{q}_j + \frac{\partial P}{\partial q_k} = \tau_k$$

By interchanging the order of summation and taking advantage of symmetry, one can prove that

$$\sum_{i,j}\left\{\frac{\partial d_{kj}}{\partial q_i}\right\}\dot{q}_i\dot{q}_j = \frac{1}{2}\sum_{i,j}\left\{\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j}\right\}\dot{q}_i\dot{q}_j$$

Hence

$$\begin{aligned}\sum_{i,j}\left\{\frac{\partial d_{kj}}{\partial q_i} - \frac{1}{2}\frac{\partial d_{ij}}{\partial q_k}\right\}\dot{q}_i\dot{q}_j &= \frac{1}{2}\sum_{i,j}\left\{\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j}\right\}\dot{q}_i\dot{q}_j - \frac{1}{2}\sum_{i,j}\frac{\partial d_{ij}}{\partial q_k}\dot{q}_i\dot{q}_j \\ &= \sum_{i,j}\frac{1}{2}\left\{\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k}\right\}\dot{q}_i\dot{q}_j \\ &= \sum_{i,j} c_{ijk}\dot{q}_i\dot{q}_j\end{aligned}$$

where $c_{ijk} := \frac{1}{2}\left\{\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k}\right\}$ are known as **Christoffel symbols**.

# Equations of Motion

$$\sum_{j} d_{kj}(q)\,\ddot{q}_j + \sum_{i,j}\left\{\frac{\partial d_{kj}}{\partial q_i} - \frac{1}{2}\frac{\partial d_{ij}}{\partial q_k}\right\}\dot{q}_i\dot{q}_j + \frac{\partial P}{\partial q_k} = \tau_k$$

$$\sum_{i,j}\left\{\frac{\partial d_{kj}}{\partial q_i} - \frac{1}{2}\frac{\partial d_{ij}}{\partial q_k}\right\}\dot{q}_i\dot{q}_j = \sum_{i,j} c_{ijk}\dot{q}_i\dot{q}_j$$

where $c_{ijk} := \frac{1}{2}\left\{\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k}\right\}$ are known as **Christoffel symbols**.

For a fixed $k$, we have $c_{ijk} = c_{jik}$, which reduces the effort involved in computing these symbols by a factor of about one half.

Finally, if we define

$$g_k = \frac{\partial P}{\partial q_k}$$

then we can write the Euler–Lagrange equations as

$$\sum_{j=1}^{n} d_{kj}(q)\ddot{q}_j + \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ijk}(q)\dot{q}_i\dot{q}_j + g_k(q) = \tau_k, \qquad k = 1, \cdots, n$$

# Equations of Motion

$$\sum_{j=1}^{n} d_{kj}(q)\ddot{q}_j + \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ijk}(q)\dot{q}_i\dot{q}_j + g_k(q) = \tau_k, \qquad k = 1, \cdots, n$$

The first type involves the second derivative of the generalized coordinates.

The second type involves quadratic terms in the first derivatives of $\dot{q}$, where the coefficients may depend on $q$. These latter terms are further classified into those involving a product of the type $\dot{q}_i^2$ and those involving a product of the type $\dot{q}_i\dot{q}_j$ where $i \neq j$.

- ✓ Terms of the type $\dot{q}_i^2$ are called centrifugal (离心力).
- ✓ Terms of the type $\dot{q}_i\dot{q}_j$ are called Coriolis terms (科氏力).

The third type of terms are those involving only $q$ but not its derivatives. This third type arises from differentiating the potential energy.

It is common to write the above equation in matrix form as

$$\boldsymbol{D}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{\tau}$$

# Equations of Motion

$$\sum_{j=1}^{n} d_{kj}(q)\ddot{q}_j + \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ijk}(q)\dot{q}_i\dot{q}_j + g_k(q) = \tau_k, \qquad k = 1, \cdots, n$$

$$\boldsymbol{D}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{\tau}$$

where the $(k, j)^{th}$ element of the matrix $\boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})$ is defined as

$$c_{kj} = \sum_{i=1}^{n} c_{ijk}(q)\dot{q}_i = \sum_{i=1}^{n} \frac{1}{2}\left\{\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k}\right\}\dot{q}_i$$

and the gravity vector $\boldsymbol{g}(\boldsymbol{q})$ is given by

$$\boldsymbol{g}(\boldsymbol{q}) = [g_1(q), \cdots, g_n(q)]^T$$

In summary, the development in this section is very general and applies to any mechanical system whose kinetic energy is a quadratic function of the vector $\dot{q}$ and whose potential energy is independent of $\dot{q}$.

In the next section we apply this discussion to study specific robot configurations.

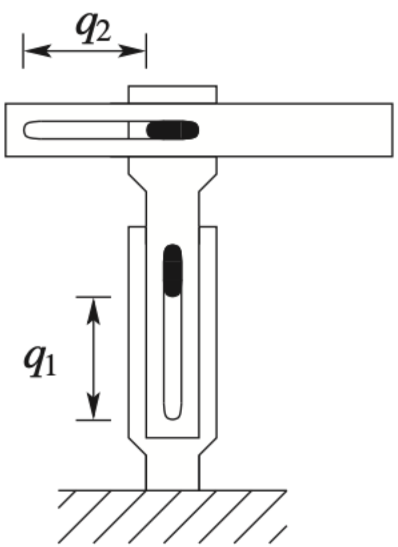
# Equations of Motion

- **Derive dynamic equations - Lagrangian method**
  - **Coordinate system, generalized coordinates, generalized forces**
  - **Calculate the position and velocity of CoM for each link**
  - **Calculate the total kinetic energy of robot** $K(q, \dot{q})$
  - **Calculate the total potential energy of robot** $P(q)$
  - **Lagrange function** $L = K - P$
  - **Substituting into dynamic equation**

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = f_i \quad \Longrightarrow \quad \frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}_i}\right) - \frac{\partial K}{\partial q_i} + \frac{\partial P}{\partial q_i} = f_i$$

# Some Common Configurations

## Two-Link Cartesian Manipulator




Denote the masses of the two links by $m_1$ and $m_2$, respectively, and denote the displacement of the two prismatic joints by $q_1$ and $q_2$ , respectively.

The corresponding generalized forces are the forces applied at each joint. Let us denote these by $f_i$, $i = 1,2$.

Since both joints are prismatic, the angular velocity Jacobian is zero and the kinetic energy of each link consists solely of the translational term. It follows that the velocity of the center of mass of link 1 is given by

$$v_{c1} = J_{v_{c1}} \dot{q}$$

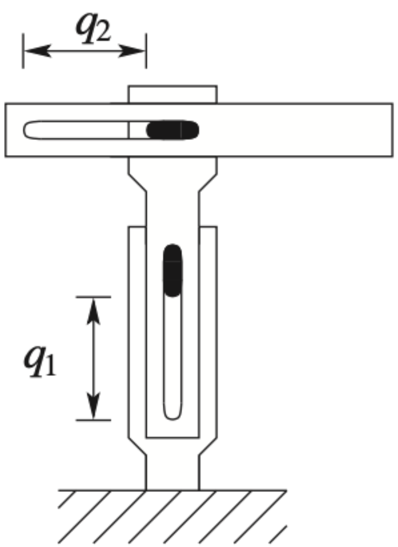
where

$$J_{v_{c1}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, \dot{q} = \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}$$

where the second column of the Jacobian is zero, because the velocity of the second link is unaffected by motion of the third link.

# Some Common Configurations

## Two-Link Cartesian Manipulator




Similarly,

$$v_{c2} = J_{v_{c2}}\dot{q}$$

where

$$J_{v_{c2}} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Hence, the kinetic energy is given by

$$K = \frac{1}{2}\dot{q}^T\{m_1 J_{v_{c1}}^T J_{v_{c1}} + m_2 J_{v_{c2}}^T J_{v_{c2}}\}\dot{q}$$

Comparing with $K = \frac{1}{2}\dot{q}^T D(q)\dot{q}$, we see that the inertia matrix $D$ is given simply by

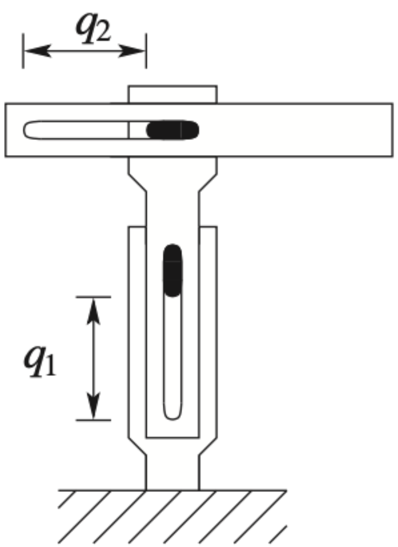
$$D = \begin{bmatrix} m_1 + m_2 & 0 \\ 0 & m_2 \end{bmatrix}$$

Next, the potential energy of link 1 is $m_1 g q_1$, while that of link 2 is $m_2 g q_1$. Hence, the overall potential energy is

$$P = g(m_1 + m_2)q_1$$

# Some Common Configurations

## Two-Link Cartesian Manipulator




Recall,

$$c_{kj} = \sum_{i=1}^{n} c_{ijk}(q)\dot{q}_i = \sum_{i=1}^{n} \frac{1}{2}\left\{\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k}\right\}\dot{q}_i$$

Since the inertia matrix is constant, all Christoffel symbols are zero. Furthermore, the components $g_k$ of the gravity vector are given by

$$g_1 = \frac{\partial P}{\partial q_1} = g(m_1 + m_2), \qquad g_2 = \frac{\partial P}{\partial q_2} = 0$$

Substituting into

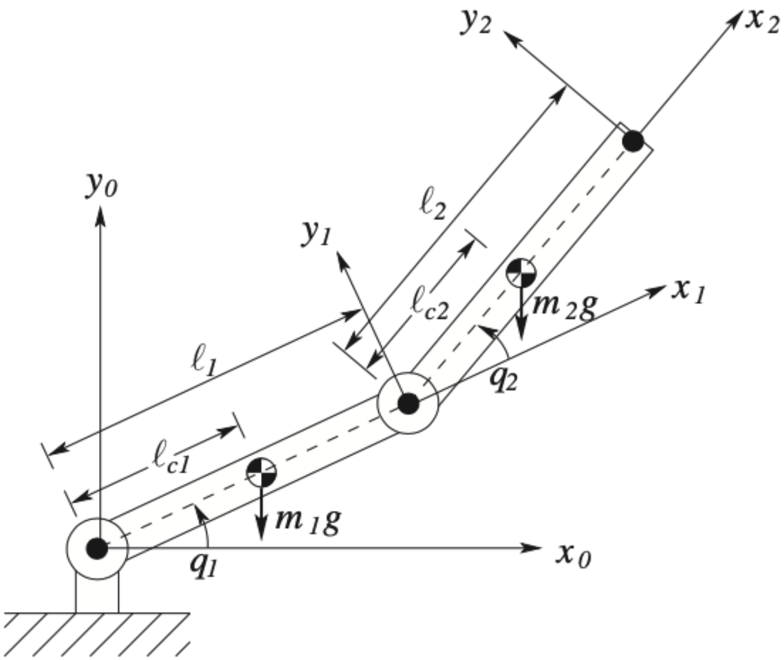
$$\sum_{j=1}^{2} d_{kj}(q)\ddot{q}_j + \sum_{i=1}^{2}\sum_{j=1}^{2} c_{ijk}(q)\dot{q}_i\dot{q}_j + g_k(q) = \tau_k, \qquad k = 1,2$$

gives the dynamical equations as

$$(m_1 + m_2)\ddot{q}_1 + g(m_1 + m_2) = f_1$$
$$m_2\ddot{q}_2 = f_2$$

# Some Common Configurations

## Planar Elbow Manipulator




$q_i$ : joint angle (generalized coordinate);
$m_i$: the mass of link $i$;
$l_i$ : the length of link $i$;
$l_{ci}$: the distance from the previous joint to the center of mass of link $i$;
$I_i$ : the moment of inertia of link $i$ about an axis coming out of the page, passing through the center of mass of link $i$.

We will use the Denavit–Hartenberg joint variables as generalized coordinates.

First, we consider the velocity terms.

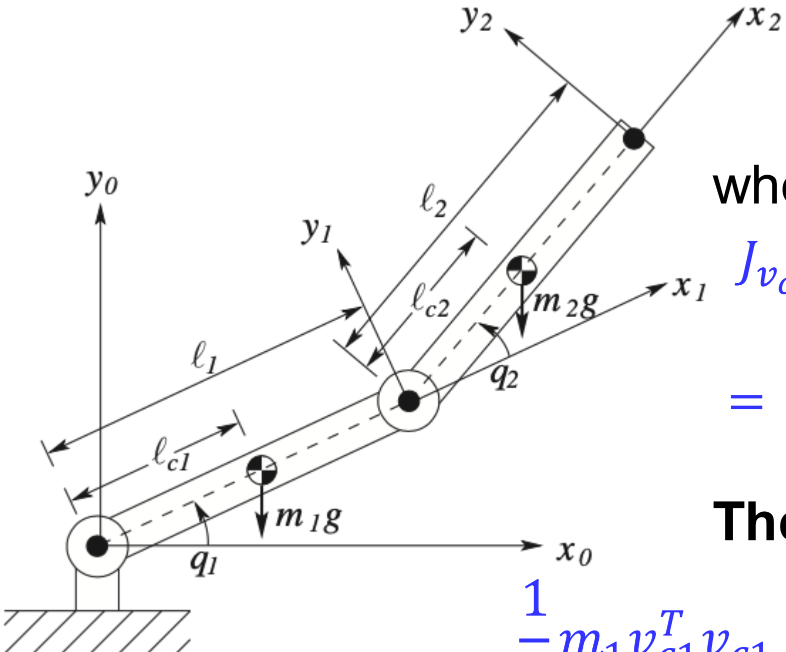
$$v_{c1} = J_{v_{c1}} \dot{q}$$

where

$$J_{v_{c1}} = \begin{bmatrix} -l_{c1} \sin q_1 & 0 \\ l_{c1} \cos q_1 & 0 \\ 0 & 0 \end{bmatrix}$$

# Some Common Configurations

## Planar Elbow Manipulator




Similarly,

$$v_{c2} = J_{v_{c2}}\dot{q}$$

where

$$J_{v_{c2}} = \begin{bmatrix} -l_1 \sin q_1 - l_{c2}\sin(q_1+q_2) & -l_{c2}\sin(q_1+q_2) \\ l_1\cos q_1 + l_{c2}\cos(q_1+q_2) & l_{c2}\cos(q_1+q_2) \\ 0 & 0 \end{bmatrix}$$

**The translational part of the kinetic energy is**

$$\frac{1}{2}m_1 v_{c1}^T v_{c1} + \frac{1}{2}m_2 v_{c2}^T v_{c2} = \frac{1}{2}\dot{q}^T\{m_1 J_{v_{c1}}^T J_{v_{c1}} + m_2 J_{v_{c2}}^T J_{v_{c2}}\}\dot{q}$$

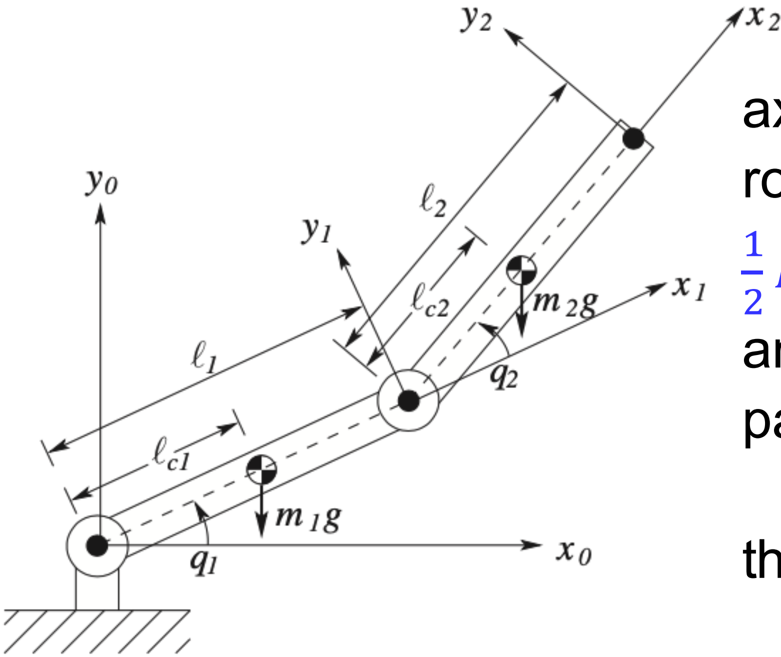
Next, we consider the angular velocity terms. Because of the particularly simple nature of this manipulator, many of the potential difficulties do not arise. First, it is clear that

$$\omega_1 = \dot{q}_1 k, \qquad \omega_2 = (\dot{q}_1 + \dot{q}_2)k$$

when expressed in the base inertial frame.

# Some Common Configurations

## Planar Elbow Manipulator




Moreover, since $\omega_i$ is aligned with the $z$-axes of each joint coordinate frame, the rotational kinetic energy reduces simply to $\frac{1}{2}I_i\omega_i^2$, where $I_i$ is the moment of inertia about an axis through the center of mass of link $i$ parallel to the $z_i$-axis.

Hence, **the rotational kinetic energy** of the overall system is

$$\frac{1}{2}\dot{q}^T\left\{I_1\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} + I_2\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\right\}\dot{q}$$

To form the inertia matrix $D(q)$, we merely have to add the translational kinetic energy and the rotational kinetic energy respectively. Thus

$$D(q) = m_1 J_{v_{c1}}^T J_{v_{c1}} + m_2 J_{v_{c2}}^T J_{v_{c2}} + \begin{bmatrix}I_1 + I_2 & I_2\\ I_2 & I_2\end{bmatrix}$$

Carrying out the above multiplications and using the standard trigonometric identities leads to

# Some Common Configurations

## Planar Elbow Manipulator

$$
\begin{aligned}
d_{11} &= m_1\ell_{c1}^2 + m_2(\ell_1^2 + \ell_{c2}^2 + 2\ell_1\ell_{c2}\cos q_2) + I_1 + I_2 \\
d_{12} &= d_{21} = m_2(\ell_{c2}^2 + \ell_1\ell_{c2}\cos q_2) + I_2 \\
d_{22} &= m_2\ell_{c2}^2 + I_2
\end{aligned}
$$

Now, we can compute the Christoffel symbols. This gives

$$
\begin{aligned}
c_{111} &= \frac{1}{2}\frac{\partial d_{11}}{\partial q_1} = 0 \\
c_{121} &= c_{211} = \frac{1}{2}\frac{\partial d_{11}}{\partial q_2} = -m_2\ell_1\ell_{c2}\sin q_2 = h \\
c_{221} &= \frac{\partial d_{12}}{\partial q_2} - \frac{1}{2}\frac{\partial d_{22}}{\partial q_1} = h \\
c_{112} &= \frac{\partial d_{21}}{\partial q_1} - \frac{1}{2}\frac{\partial d_{11}}{\partial q_2} = -h \\
c_{122} &= c_{212} = \frac{1}{2}\frac{\partial d_{22}}{\partial q_1} = 0 \\
c_{222} &= \frac{1}{2}\frac{\partial d_{22}}{\partial q_2} = 0
\end{aligned}
$$

# Some Common Configurations

## Planar Elbow Manipulator

Next, the potential energy of the manipulator is just the sum of those of the two links. For each link, the potential energy is just its mass multiplied by the gravitational acceleration and the height of its center of mass. Thus

$$\begin{aligned} P_1 &= m_1 g \ell_{c1} \sin q_1 \\ P_2 &= m_2 g(\ell_1 \sin q_1 + \ell_{c2} \sin(q_1 + q_2)) \end{aligned}$$

and so the total potential energy is

$$P = P_1 + P_2 = (m_1 \ell_{c1} + m_2 \ell_1) g \sin q_1 + m_2 \ell_{c2} g \sin(q_1 + q_2)$$

Therefore, the functions $g_k$ become

$$\begin{aligned} g_1 &= \frac{\partial P}{\partial q_1} = (m_1 \ell_{c1} + m_2 \ell_1) g \cos q_1 + m_2 \ell_{c2} g \cos(q_1 + q_2) \\ g_2 &= \frac{\partial P}{\partial q_2} = m_2 \ell_{c2} g \cos(q_1 + q_2) \end{aligned}$$

Finally, we can write down the dynamical equations of the system as

$$\begin{aligned} d_{11}\ddot{q}_1 + d_{12}\ddot{q}_2 + c_{121}\dot{q}_1\dot{q}_2 + c_{211}\dot{q}_2\dot{q}_1 + c_{221}\dot{q}_2^2 + g_1 &= \tau_1 \\ d_{21}\ddot{q}_1 + d_{22}\ddot{q}_2 + c_{112}\dot{q}_1^2 + g_2 &= \tau_2 \end{aligned}$$

# Summary

- The Euler-Lagrange Equations
  - D'Alembert's Principle
- Kinetic and Potential Energy
  - The Inertia Tensor
  - Kinetic Energy for an *n*-Link Robot
  - Potential Energy for an n-Link Robot
- Equations of Motion
- Some Common Configurations

# Homework 12

**Homework 12 is posted at http://bb.sustech.edu.cn**

**Due date: April 14, 2025**

**作业要求（Requirements）：**

1. **文件格式为以自己姓名学号作业序号命名的pdf文件；**

   **（File name: YourSID_ YourName_12.pdf）**

2. **作业里也写上自己的姓名和学号。**

   **（Write your name and SID in the homework）**

**Next class：Dynamics III**

**Midterm Test：April 21, 2025**