Shafarevich Chapter 1 Section 2 Exercises

Mauricio Montes

March 24, 2024

Exercise 1

The set $X \subset \mathbb{A}^2$ is defined by the equation $f : x^2 + y^2 = 1$ and $g : x = 1$. Find the ideal $\mathfrak{U}_X$. Is it true that $\mathfrak{U}_X = (f, g)$?

Proof. We have that $X = V(f) \cap V(g)$. These sets intersect at exactly one point, namely $(1, 0)$. As seen by example 1.7 in the section, we have that $\mathbb{A}^2 = \mathbb{A}^2[X] = \mathbb{A}^2[x, y]/\mathfrak{U}_X$ this means that $\mathfrak{U}_X = (x, y)$. However, we can also see that $y \notin (f, g)$, so it must be that $\mathfrak{U}_X \neq (f, g)$.

□

Exercise 2

Let $X \subset \mathbb{A}^2$ be the algebraic plane curve defined by $y^2 = x^3$. Prove that an element of $k[X]$ can be written uniquely in the form $P(x) + Q(x)y$, where $P(x), Q(x) \in k[x]$.

Proof. Suppose we have a generic element $f(x, y) \in k[X]$. We can write $f(x, y) = \sum_{i=0}^{N} a_i x^{N-i} y^i$. We can break this sum up into two sums, one for the even and one for the odd powers of i. We can then write $f(x, y) = P(x) + Q(x)y$ where the even powers of i are in $P(x)$ and the odd powers of i are in $Q(x)$. This is because of the fact that any even power of i will yield an even power of y that can then be converted to an x^3 term. The odd powers will then get converted to x^3 terms, except for one last y term. i

□

Exercise 3

Let X be the curve of the previous exercise and $f(t) = (t^2, t^3)$. Prove that f is not an isomorphism.

Proof. We can see that f is not an isomorphism because it is not bijective at its inverse. We can see that if $g(x,y) = \frac{y}{x}$, then $g(f(t)) = \frac{t^3}{t^2} = t$. However, $f(g(x,y)) = f(\frac{y}{x}) = (\frac{y^2}{x^2}, \frac{y^3}{x^3})$. Which is not defined whenever $x = 0$. Thus f is not an isomorphism.

□

Exercise 6

Consider the regular map $f : \mathbb{A}^2 \to \mathbb{A}^2$ given by $f(x,y) = (x, xy)$. Find the image $f(\mathbb{A}^2)$; is it open in $\mathbb{A}^2$? Is it dense? Is it closed?

Proof. The image of f is $\mathbb{A}^2 \setminus \{(0,y) \mid y \in \mathbb{A}^2\}$. This set is dense. This is because the closure of the image is $\mathbb{A}^2$. This is because if we consider the ideal of all polynomials vanishing on $f(\mathbb{A}^2)$, we get that these polynomials will be vanishing on a dense set (consider a small disk away from the origin), and thus our ideal must be the zero ideal. Thus the closure of the image is $\mathbb{A}^2$. It is not closed, however, since the closure of a closed set would be the set itself. It is not open since the complement of the image is not complement of a closed set.

□

Exercise 9

Prove that the map $f(x,y) = (\alpha x, \beta y + P(x))$ is an automorphism of $\mathbb{A}^2$, where $\alpha, \beta \in k$ are nonzero elements, and $P(x)$ is a polynomial. Prove that maps of this type form a group B.

Proof. This map has inverse given by $g(x,y) = \left(\frac{x}{\alpha}, \frac{y-P(x)}{\beta}\right)$ Composing both ways, we see we get an automorphism of $\mathbb{A}^2$. To see how maps of this type form a group. The associated operation would be composition. The identity element is the identity map $id(x,y) = (x,y)$. The inverse of a map f is the map g given by the inverse of f. The operation of function composition is associative. Thus we have a group.

$\square$

Exercise 11

Suppose that X consists of two points. Prove that the coordinate ring $k[X]$ is isomorphic to the direct sum of two copies of k.

Proof. We know that $k[X] = k[x,y]/\mathfrak{U}_X$. Since X consists of two points, we have that $\mathfrak{U}_X = (x-a)(x-b)$ for some $a, b \in k, a \neq b$. We can then write $k[X] = k[x,y]/(x-a)(x-b)$. We can then write $k[X] = k[x,y]/(x-a)(x-b) = k[x,y]/(x-a) \oplus k[x,y]/(x-b) = k \oplus k$. This follows from the fact that $(x-a)$ and $(x-b)$ are coprime, and thus the Chinese Remainder Theorem applies.

$\square$

Exercise 15

Prove that if $X = \cup U_\alpha$ is an open covering of a closed set X, by open set U_α, then there exists a finite subcovering such that $X = \cup_{i=1}^{n} U_{\alpha_i}$.

Proof. Since X is closed, it is the solution to a finite set of polynomials. Hence, it has a corresponding ideal $\mathfrak{U}_X \subseteq k[T]$, that is finitely generated and is the intersection of a finite number of maximal ideals. These have corresponding closed sets V_i that we can take the complement of to get open sets U_i. We can then write $X \subseteq U_i$. Not sure how to finish this proof. Can take the complement of the V_i, but there's no guarantee that those complements are any of the U_α.

$\square$

Exercise 16

Prove that the Frobenius map is an injective map. Is it an isomorphism when $X = \mathbb{A}^1$?

Proof. To see why this map is injective. Notice that the kernel of the homomorphism from X to itself is only the zero element. This is because the

Frobenius map is given by $f(x_1, \ldots, x_n) = (x_1, \ldots, x_n)^p$. When $X = \mathbb{A}^1$, the Frobenius map is an isomorphism. Something about how taking things mod p is an isomorphism? Not sure.

$\square$