

What is the shape of data?

A brief introduction to topological data analysis

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- ▶ This means that we can stretch, bend, and deform objects, but we cannot tear or glue them.
- ▶ Data analysis is the process of studying and modeling data to extract useful information.
- ▶ Typically, we use statistics and machine learning to analyze data. We can improve these methods by using tools from topology.

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- ▶ We can think of manifolds as "things that locally look like" \mathbb{R}^n . Most objects in the real world are manifolds.
- ▶ If we take enough samples, we can approximate the shape of the manifold our data is living in. This allows us to answer topological questions about our data.

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- ▶ We compute topological invariants of the simplicial complex. This would be something like the Euler characteristic, homology, or persistent homology of the complex.
- ▶ This new topological data provides us with a new descriptor of our original dataset. This can then be used in conjunction with other methods to improve our analysis.

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- ▶ We can remedy this by building a simplicial complex from our data to approximate the manifold.
- ▶ A k -simplex is the convex hull of $k + 1$ linearly independent points in \mathbb{R}^n .
- ▶ A simplicial complex K is a collection of simplices that satisfies two conditions:
 1. If $\sigma \in K$, then every face of σ is also in K .
 2. If $\sigma_1, \sigma_2 \in K$, then $\sigma_1 \cap \sigma_2$ is either empty or a face of both σ_1 and σ_2 .

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- ▶ Let $C_0, C_1, \dots, C_k, \dots$ be groups isomorphic to the integers, with C_i having one copy of \mathbb{Z} per k -simplex in our simplicial complex.
- ▶ Notice that we can recover a set of $k - 1$ -simplices from a k -simplex by looking at its boundary.

Simplicial Holography 2 : Electric Boogaloo

Objects of Interest

► hi

Topological Preliminaries

- ▶ A topological space is a pair (X, τ) consisting of a set X with a collection of subsets τ , called the topology on X , such that:
 1. $\emptyset, X \in \tau$
 2. If $U_i \in \tau$ for $i \in I$, then $\bigcup_{i \in I} U_i \in \tau$
 3. If $U_1, U_2 \in \tau$, then $U_1 \cap U_2 \in \tau$
- ▶ The Euclidean topology on \mathbb{R}^n is the collection of all open subsets of \mathbb{R}^n . We define an open set as a set $B_r(x) = \{y \in \mathbb{R}^n : d(x, y) < r\}$. Where d is the Euclidean metric.

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- ▶ A basis for a topological space (X, τ) is a collection \mathcal{B} of open sets such that every open set in τ can be written as a union of sets in \mathcal{B} .
- ▶ A topological space is second countable if it has a countable basis.

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- ▶ In the Euclidean topology, the topology is actually generated by the collection of open balls. These are a basis for the Euclidean topology.

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- ▶ If there exists a homeomorphism between two topological spaces, we say that they are homeomorphic.

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- ▶ A topological manifold is a second countable Hausdorff space that is locally homeomorphic to \mathbb{R}^n .
- ▶ The last condition means that for every point $x \in X$, there exists an open set $U \in \tau$ such that $x \in U$ and U is homeomorphic to an open set $\hat{U} \subseteq \mathbb{R}^n$.
- ▶ To make this a smooth manifold, we require that the homeomorphism is actually a diffeomorphism. (Smooth map with smooth inverse)