What is the shape of data?

A brief introduction to topological data analysis

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- Data analysis is the process of studying and modeling data to extract useful information.
- Typically, we use statistics and machine learning to analyze data. We can improve these methods by using tools from topology.

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- We can think of manifolds as "things that locally look like" $\mathbb{R}^n$. Most objects in the real world are manifolds.
- If we take enough samples, we can approximate the shape of the manifold our data is living in. This allows us to answer topological questions about our data.

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- We construct a simplicial complex from the data. This is a combinatorial object that encodes the shape of the data.
- We compute topological invariants of the simplicial complex. This would be something like the Euler characteristic, homology, or persistent homology of the complex.
- This new topological data provides us with a new descriptor of our original dataset. This can then be used in conjunction with other methods to improve our analysis.

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- A k-simplex is the convex hull of $k+1$ linearly independent points in $\mathbb{R}^n$.
- A simplicial complex K is a collection of simplices that satisfies two conditions:

1. If $\sigma \in K$, then every face of σ is also in K.
2. If $\sigma_1, \sigma_2 \in K$, then $\sigma_1 \cap \sigma_2$ is either empty or a face of both σ_1 and σ_2.

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- Let $C_0, C_1, \ldots, C_k, \ldots$ be groups isomorphic to the integers, with C_i having one copy of $\mathbb{Z}$ per k-simplex in our simplicial complex.
- Notice that we can recover a set of $k-1$-simplices from a k-simplex by looking at its boundary.

Simplicial Holemology 2 : Electric Boogaloo

Objects of Interest

- hi

Topological Preliminaries

- A topological space is a pair (X, τ) consisting of a set X with a collection of subsets τ, called the topology on X, such that:
  1. $\emptyset, X \in \tau$
  2. If $U_i \in \tau$ for $i \in I$, then $\bigcup_{i \in I} U_i \in \tau$
  3. If $U_1, U_2 \in \tau$, then $U_1 \cap U_2 \in \tau$
- The Euclidean topology on $\mathbb{R}^n$ is the collection of all open subsets of $\mathbb{R}^n$. We define an open set as a set $B_r(x) = \{y \in \mathbb{R}^n : d(x, y) < r\}$. Where d is the Euclidean metric.

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- A topological space is second countable if it has a countable basis.

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- In the Euclidean topology, the topology is actually generated by the collection of open balls. These are a basis for the Euclidean topology.

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- A homeomorphism is a continuous map $f : X \to Y$ such that there exists a continuous map $g : Y \to X$ such that $g \circ f = id_X$ and $f \circ g = id_Y$. (Bicontinuous, Bijective)

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- If there exists a homeomorphism between two topological spaces, we say that they are homeomorphic.

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- A topological manifold is a second countable Hausdorff space that is locally homeomorphic to $\mathbb{R}^n$.
- The last condition means that for every point $x \in X$, there exists an open set $U \in \tau$ such that $x \in U$ and U is homeomorphic to an open set $\hat{U} \subseteq \mathbb{R}^n$.
- To make this a smooth manifold, we require that the homeomorphism is actually a diffeomorphism. (Smooth map with smooth inverse)