Topological Signal Processing Definitions

Mauricio Montes

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1 Introduction

This document is for me to keep track of the definitions and ideas I have about topological signal processing. This will probably get long, so maybe we can chunk it later. I'm not sure if I want to start all the way with the definition of a topological space. But I think I can start with the ideas of homology and cohomology.

Update (4/3/24): We definitely need to backtrack. We're going to introduce the necessary machinery from topology and then we will define the simplices and go from there.

2 Definitions

2.1 Topology Preliminaries

Given a set X, a *topology* on X is a set $\mathcal{T} \subset \mathcal{P}(X)$ of subsets of X. The elements of a topology are called *open sets*. To be a topology, certain requirements must be satisfied. Namely :

- $\phi, X \in \mathcal{T}$
- For any collection of open sets $\{U_\alpha\}_{\alpha \in I}$, $\bigcup_{\alpha \in I} U_\alpha$
- For any finite collection of open sets $\{U_\alpha\}_{\alpha \leq n}$,$\bigcap_\alpha U_\alpha$

2.2 Simplicial Complexes

The central objects of simplicial homology are simplicial complexes.

- A k-**simplex** is the convex hull of a set of $k+1$ points in some Euclidean space. We can think of a 0-simplex as a vertex. A 1-simplex is an edge, a 2-simplex is a triangle, and so on.

- Note that a k-simplex has $k+1$ faces, which are the simplices of dimension $k-1$ that are contained in it.

- A **simplicial complex** is a collection of simplices such that the intersection of any two simplices is either empty or another simplex.

- The **dimension** of a simplicial complex is the maximum dimension of any of the simplices in the complex.

2.3 Simplicial Homology

In order to do algebra with simplicial complexes, we need to associate it to algebraic objects that we can manipulate. This is where the chain complex comes in.

- Say X is a k-dimensional simplicial complex. The *chain group* $C_k(X, \mathbb{R})$ is the vector space (over $\mathbb{R}$) with basis given by the number of k-simplices in X.

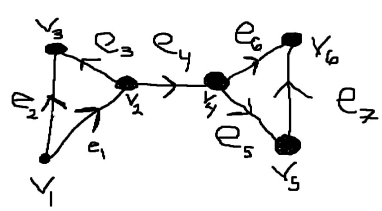

Figure 1: A simplicial complex, with 6 vertices and 7 edges.

Inspecting the image, we see that we have 6 vertices and 7 edges. No higher dimensional simplices. We can write down the chain groups associated with this complex: $C_1(X, \mathbb{R}) = \mathbb{R}^6, C_2(X, \mathbb{R}) = \mathbb{R}^7$

Note that $C_k(X, \mathbb{R}) = 0, k > 2$. For our purposes, we will use $\mathbb{R}$ for our base field for the associated vector space and will suppress writing down the associated field for the chain group. Writing $C_k(X)$ instead for brevity.

Between chain groups, there exists a map $\delta_k : C_k(X) \to C_{k-1}(X)$ given by

$$\delta_k(v_{i_1}, \ldots v_{i_k}) = \sum_{j=0}^{k} (-1)^j (v_{i_1}, \ldots, \hat{v_{i_j}}, \ldots, v_{i_k}) \tag{1}$$

This is called the *kth chain map*. A set pair of chain groups and chain maps form a *chain complex*. The associated chain maps have the property that $\delta_{k-1} \circ \delta_k = 0$. These chain maps decompose into their associated kernel and image maps.

Given a chain complex $C(X)$, we define the *kth homology group* to be $H_k(C) = ker(\delta_k)/Im(\delta_{k+1})$