Algebraic Geometry HW 4

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Exercise 1 Section 3

Solution: Given the closed set $X \subset \mathbb{A}^3$ defined by the equations $f : x^2 - y^2 - z^2 + 1 = 0$ and $g : x^2 + y^2 + z^2 = 0$. We wish to decompose X into irreducible components.

The equations decompose in the following way: Note that moving z to the other side in f allows us to rewrite g as $2x^2 + 1 = 0$. Meaning that $x = \pm\frac{i}{\sqrt{2}}$. Replacing this relation in g, we get that $y^2 + z^2 = \frac{1}{2}$.

This means that X is the union of the sets $X_1 = \{x = \pm\frac{i}{\sqrt{2}}\}, X_2 = \{y^2 + z^2 = \frac{1}{2}\}$

(I don't think so really. Union is supposed to be a product of these polynomials, so I need another formulation)

Exercise 4 Section 3

Solution: We wish to decompose X given by $y^2 = xz, z^2 = y^3$ into irreducible components. Substituting in the first equation in the second, we get that $z^2 = (xz)y$. Meaning $z = xy$. Plugging this back into our first equation. We see that $y^2 = xz = x(xy) = x^2y$. Meaning $y = 0$ or $y = x^2$. Which then implies that $x^6 = z^2$ and $z = x^3$ or $z = -x^3$. This means that X is the union of the sets $X_1 = \{y = 0\}, X_2 = \{y = x^2, z = x^3\}, X_3 = \{y = x^2, z = -x^3\}$. These irreducible components all come from clear birational maps of $\mathbb{A}^1$ and so are all isomorphic to $\mathbb{A}^1$.

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Solution: As per the hint, studying the inclusion of X in the affine variety $Y = \mathbb{A}^2$. We have an induced isomorphism on the coordinate rings $\phi : k[Y] = k[x, y] \to k[X]$. To see why this is an isomorphism, consider $f = \frac{g}{h} \in k[X]$. The condition of f being regular is that h does not vanish on X. However, if h is not a unit in $k[X]$, then there are infinitely many solutions to $h = 0$ in X. So then h must be a unit if $f \in k[X]$. But then this means that $f \in k[x, y] = k[Y]$. So thus ϕ is an isomorphism. However! The inclusion map is not an isomorphism. Thus, $\mathbb{A}^2 - \mathbf{0}$ is not an affine variety.

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Solution:

1 Exercise 6 Section 4

Solution: Supposing there was such a map ϕ, then we would have an induced homomorphism $\phi^* : k[\mathbb{A}^n] \to k[\mathbb{P}^1]$. However, the coordinate ring of the left hand side is $k[x_1, \ldots, x_n]$. But the right hand side is just k. This homomorphism is a surjection, so it is achieved by sending sending variables to constants, i.e. quotienting out by $(x_i - c_i)$ for some $c_i \in k$. Ideal variety correspondence then tells us that the map ϕ is then given by $C = (c_1, \ldots, c_n)$.