Shafarevich Chapter 1 Section 2 Exercises

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Exercise 1

The set $X \subset \mathbb{A}^2$ is defined by the equation $f : x^2 + y^2 = 1$ and $g : x = 1$. Find the ideal $\mathfrak{U}_X$. Is it true that $\mathfrak{U}_X = (f, g)$?

Proof. We have that $X = V(f) \cap V(g)$. These sets intersect at exactly one point, namely $(1, 0)$. As seen by example 1.7 in the section, we have that $\mathbb{A}^2 = \mathbb{A}^2[X] = \mathbb{A}^2[x, y]/\mathfrak{U}_X$ this means that $\mathfrak{U}_X = (x, y)$. However, we can also see that $y \notin (f, g)$, so it must be that $\mathfrak{U}_X \neq (f, g)$.

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Exercise 2

Let $X \subset \mathbb{A}^2$ be the algebraic plane curve defined by $y^2 = x^3$. Prove that an element of $k[X]$ can be written uniquely in the form $P(x) + Q(x)y$, where $P(x), Q(x) \in k[x]$.

Proof. Suppose we have a generic element $f(x, y) \in k[X]$. We can write $f(x, y) = \sum_{i=0}^{N} a_i x^{N-i} y^i$. We can break this sum up into two sums, one for the even and one for the odd powers of i. We can then write $f(x, y) = P(x) + Q(x)y$ where the even powers of i are in $P(x)$ and the odd powers of i are in $Q(x)$. This is because of the fact that any even power of i will yield an even power of y that can then be converted to an x^3 term. The odd powers will then get converted to x^3 terms, except for one last y term. i

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Exercise 3

Let X be the curve of the previous exercise and $f(t) = (t^2, t^3)$. Prove that f is not an isomorphism.

Proof. We can see that f is not an isomorphism because it is not bijective at its inverse. We can see that if $g(x, y) = \frac{y}{x}$, then $g(f(t)) = \frac{t^3}{t^2} = t$. However, $f(g(x, y)) = f(\frac{y}{x}) = (\frac{y^2}{x^2}, \frac{y^3}{x^3})$. Which is not defined whenever $x = 0$. Thus f is not an isomorphism.

□

Exercise 6

Consider the regular map $f : \mathbb{A}^2 \to \mathbb{A}^2$ given by $f(x, y) = (x, xy)$. Find the image $f(\mathbb{A}^2)$; is it open in $\mathbb{A}^2$? Is it dense? Is it closed?

Proof. The image of f is $\mathbb{A}^2 \setminus \{(0, y) \mid y \in k\}$. This set is not open in $\mathbb{A}^2$ - consider the preimage of a disk centered at the origin. This set is dense, however. This is because the closure of the image is $\mathbb{A}^2$. This is because if we consider the ideal of all polynomials vanishing on $f(\mathbb{A}^2)$, we get that these polynomials will be vanishing on a dense set (consider a small disk away from the origin), and thus our ideal must be the zero ideal. Thus the closure of the image is $\mathbb{A}^2$. It is not closed, however, since the closure of a closed set would be the set itself.

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