TDA Introduction and some other concepts

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1 Introduction

This document is for me to keep track of the definitions and ideas I have about topological signal processing. This will probably get long, so maybe we can chunk it later. I'm not sure if I want to start all the way with the definition of a topological space. But I think I can start with the ideas of homology and cohomology.

2 Definitions and Preliminaries

2.1 Topological Preliminaries

Definition 2.1 (Topology)**.** Given a set X, a *topology* on X is a set $\mathcal{T} \subset \mathcal{P}(X)$ of subsets of X. The elements of a topology are called *open sets*. To be a topology, certain requirements must be satisfied. Namely :

- $\phi, X \in \mathcal{T}$
- For any collection of open sets $\{U_\alpha\}_{\alpha \in I}$, $\bigcup_{\alpha \in I} U_\alpha$
- For any finite collection of open sets $\{U_\alpha\}_{\alpha \leq n}$,$\bigcap_\alpha U_\alpha$

2.2 Simplicial Complexes

The central objects of simplicial homology are simplicial complexes. The purpose of all of this is so that we can associate a topological space with a chain complex, and then we can do algebra with the chain complex to extract topological information about the space.

Definition 2.2 (Simplices and Simplicial Complexes)**.** A k-*simplex* is the convex hull of a set of $k+1$ points in some Euclidean space. We can think of a 0-simplex as a vertex. A 1-simplex is an edge, a 2-simplex is a triangle, and so on.

Note that

- a k-simplex has $k+1$ faces, which are the simplices of dimension $k-1$ that are contained in it.
- A *simplicial complex* is a collection of simplices such that the intersection of any two simplices is either empty or another simplex.
- The *dimension* of a simplicial complex is the maximum dimension of any of the simplices in the complex.

2.3 Simplicial Homology

In order to do algebra with simplicial complexes, we need to associate it to algebraic objects that we can manipulate. This is where the chain complex comes in.

Definition 2.3 (Chain Group)**.** Say X is a k-dimensional simplicial complex. The *chain group* $C_k(X, \mathbb{R})$ is the vector space (over $\mathbb{R}$) with basis elements given by the number of k-simplices in X.

Below is an example that we will use to illustrate the concept of a chain group.

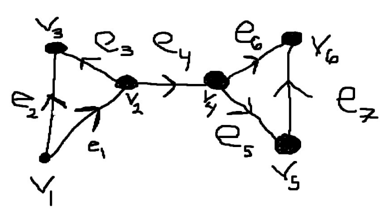

Figure 1: A simplicial complex, with 6 vertices and 7 edges.

Inspecting the image, we see that we have 6 vertices and 7 edges and no higher dimensional simplices. We can write down the chain groups associated with this complex: $C_1(X, \mathbb{R}) = \mathbb{R}^6, C_2(X, \mathbb{R}) = \mathbb{R}^7$

Note that $C_k(X, \mathbb{R}) = 0, k > 2$. For our purposes, we will use $\mathbb{R}$ for our base field for the associated vector space and will suppress writing down the associated field for the chain group. Writing $C_k(X)$ instead for brevity.

Definition 2.4 (Chain Maps and Chain Complexes)**.** Between chain groups, there exists a map $\delta_k : C_k(X) \to C_{k-1}(X)$ given by

$$\delta_k(v_{i_1}, \dots v_{i_k}) = \sum_{j=0}^{k} (-1)^j (v_{i_1}, \dots, \hat{v_{i_j}}, \dots, v_{i_k}) \tag{1}$$

This is called the *kth chain map.* A set pair of chain groups and chain maps form a *chain complex.* The associated chain maps have the property that $\delta_{k-1} \circ \delta_k = 0$.

As with any map between vector spaces, it is always an object of interest to find the kernel and image of the map. What we do instead is a bit different. We compare the kernel of the map to the image of the previous map. This is the idea behind the homology groups.

Definition 2.5 (Homology Groups)**.** Given a chain complex $C(X)$, we define the *kth homology group* to be

$$H_k(C) = ker(\delta_k)/Im(\delta_{k+1}) \tag{2}$$

2.3.1 A helpful example

Below, we have the result of the boundary map acting on the chain group defined by our edges. With blue indicating a positive value of the basis element of our chain group and red indicating a negative. We are interested in sums of chains whose boundary maps equal to zero. These would correspond to "holes" in the picture of our simplicial complex.

Let's perform a calculation, following our formula (and our picture), we see that

$$\partial(e_1) = v_2 - v_1$$
$$\partial(e_2) = v_3 - v_2$$
$$\partial(e_3) = v_3 - v_1$$

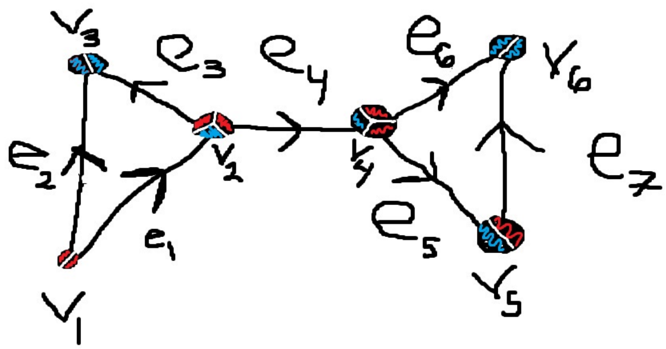

Figure 2: The boundary map of the edges of the simplicial complex

Since the map ∂ is linear (it is a map between vector spaces), we can write down

$$\partial(e_1 + e_2 - e_3) = \partial(e_1) + \partial(e_2) - \partial(e_3) = 0$$

This means that the element $(e_1 + e_2 - e_3) \in C_1(X)$ is actually inside $ker(\partial_1)$! Meaning that in the definition of homology, this is a nontrivial element! From our picture, we also see that $(e_5 + e_7 - e_6) \in ker(\partial_1)$. This gives 2 nontrivial elements in the kernel of ∂_1.

It is important to note that in the image, there is no higher dimensional 2-simplex that these 1-simplices are the boundary of, meaning that $Im(\partial_2) = 0$. Writing this all down, we mean to say:

$$H_1(X) = \mathbb{R}^2$$

We can augment this example by adding a 2-simplex to our simplicial complex. One that would fill in one of the holes in the complex. Then the homology group would be $H_1(X) = \mathbb{R}^1$. If we attached a 3-simplex to the complex, then $H_1(X) = \mathbb{R}^1$. But then $H_2(X) = \mathbb{R}^1$ as well! This is due to the creation of a "void" in the tetrahedron that we attached. This is the essence of simplicial homology.

Finally, a rapid fire interpretation / intuition of homology groups:

- $H_k(X) = 0$ for $k > dim(X)$
- $H_0(X)$ is the number of connected components of X
- $H_1(X)$ is the number of "holes" in X
- $H_2(X)$ is the number of "voids" in X
- If X, Y are simplicial complexes, and $f : X \to Y$ is a simplicial map, then f induces a map $f_* : H_k(X) \to H_k(Y)$

2.4 Simplicial Cohomology

As with anything, there is always a dual. Cohomology is the dual to homology in that all the maps go backwards. This is otherwise known as a contravariant functor. Similarly, the chain groups are not the same. We now have cochain groups, which can really be thought of as the dual vector space to our original chain group. We map real numbers to the k-simplices now. Let's set up all the definitions as in simplicial homology and compare :

[NEED COHOMOLOGY DEFINITIONS]

2.5 hodge laplacian

Fill me

2.6 Persistent Homology

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2.7 Sheaves

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2.8 Sheaf graph networks

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2.9 Sheaf Cohomology

Fill me

3 Further Directions and Questions

- Is there a theory about probability sheaves? Could one be written down such that the maps between sheaves would somehow communicate information about the probability states of network? (Does this question even make sense)
- What direct applications could this have towards signal processing? Is there an immediate impact besides the answer from Robinson "In theory we can"
- What is some nice data that TSP can be used on?
- The current methods from TDA are: Persistent Homology, Bottleneck/Wasserstein distance, 0 dimensional sublevel set persistence,
- Homology for lattice configurations?
- Homology of dynamical systems
- Given a coboundary map δ, we can construct a laplacian matrix associated with a sheaf.
- Chemical Reaction networks have a hypergraph structure. Is there a way to write down a sheaf and associated laplacian matrix with this hypergraph?
- Topological Entropy of a topological dynamical system? Can we interpret the CRN above as one? See Pierre Baudot