MATH 7970 Homework 1

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February 17, 2024

1 Exercise 1.1.3

1.1 Problem

Prove that an irreducible cubic curve has at most one singular point, and that the multiplicity of a singular point is at most 2. If the singularity is a node then the cubic is projectively equivalent to the curve

$$y^2 = x^2 + x^3 \tag{1}$$

and if it is a cusp then the cubic is projectively equivalent to the curve

$$y^2 = x^3 \tag{2}$$

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1.2 Solution

Assume for sake of contradiction that the cubic curve has two singular points and consider the line L that passes through those two singular points. By Bezout's theorem, the sum of the multiplicities of the intersection points of L and the cubic is 3. However, the intersection of L and the cubic is between the two singular points, and the intersection point has multiplicity 2 for each singular point. Which is larger than 3, a contradiction. Therefore, the curve can only have at most one singular point.

2 Exercise 1.1.5

2.1 Problem

Prove that if the ground field has characteristic p, then every line through the origin is a tangent line to the curve $y = x^{p+1}$. Prove that over a field of characteristic 0, that there are at most a finite number of lines through a given point tangent to a given irreducible curve.

2.2 Solution

Let L be the line $y = mx$ and let $P = (0, 0)$ be the origin. Notice that for the curve $y = x^{p+1}$, this means $mx = x^{p+1}$, and so $x(x^p - m) = 0$. Which further simplifies to $x(x - m) = 0$, since we are in characteristic p. This is the equation of the tangent line to the curve at the origin. However, we cannot do this reduction in characteristic 0, and so the result does not hold. We only get finitely many solutions to $x^p - m = 0$ for a given m and p.

3 Exercise 1.1.9

3.1 Problem

For what values of m is the cubic $x_0^3 + x_1^3 + x_2^3 - mx_0x_1x_2 = 0$ singular in $\mathbb{P}^2$?

3.2 Solution

We can write down our system of equations for the partial derivatives as follows:

$$\begin{aligned}3x_0^2 - mx_1x_2 &= 0\\ 3x_1^2 - mx_0x_2 &= 0\\ 3x_2^2 - mx_0x_1 &= 0\end{aligned}$$

Solving this system of equations yields $x_0 = x_1 = x_2 = \frac{m}{3}$. Hence our cubic is nonsingular as long as not all of the x_i are equal to $\frac{m}{3}$ i.e. $m \neq 0$. Note, 0 is not included since this isn't a value in projective space.

4 Exercise 1.1.11

4.1 Problem

Prove that on the projective line and on a conic of $\mathbb{P}^2$, a rational function that is regular at every point is a constant.

4.2 Solution

Let f be a rational function on $\mathbb{P}^1$, written $\left(x, \frac{g(x)}{h(x)}\right)$, that is regular at every point on the projective line. Since it is a regular function at every point, it must be regular at the point at infinity. This means that $h(x)$ must have degree 0, and so f is a constant. So, we then have an equivalence between f and the point $(h(x)x, g(x))$. Thus, f is a constant.