Lecture 2C.2

Newton-Euler Equations

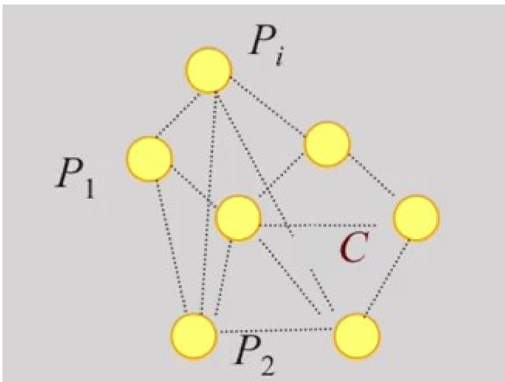
Most of us are familiar with Newton's equations of motion for a single particle of mass *m*:

$$\mathbf{F} = m\mathbf{a}$$

If we know the resultant force acting on a single particle of mass, *m*, then the acceleration is simply obtained from Newton's second law.

What does this equation look like when we consider a system of particles or a rigid -body?

Let's start with a system of particles. Before we can define Newton's Second Law for a system of particles, we must first define the centre of mass. The image below shows a set of particles. P_1, P_2, and so on.



Let's assume that we can write position vectors $\mathbf{p_1}$, $\mathbf{p_2}$, $\mathbf{p_i}$, for each of these particles. Let's denote the mass of the i_{th} particle by M_i.

We can now compute the average position vector, by weighting each position vector with the appropriate mass.

$$r_c = \frac{1}{m} \sum_{i=1,N} m_i p_i$$

In this equation you see here, $m_i p_i$, is essentially the weighted sum of all the position vectors. We divide that by the total mass, *m*, to obtain a new position vector. This position vector defines the centre of mass.

It turns out that the centre of mass for a system of particles, S, behaves in exactly the same way as a single particle, with mass m (equal to the total mass of the system) would have behaved, if it had been located at the centre of mass.

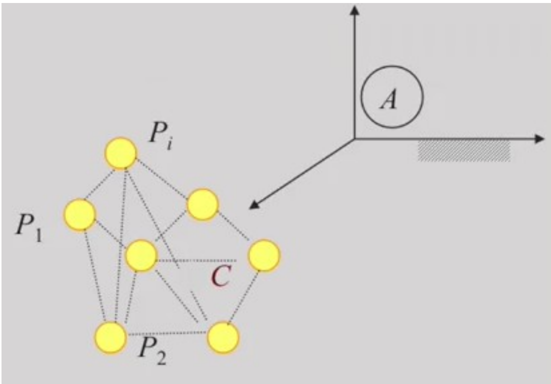
So instead of writing F = ma for a system of particles, we write F = m × the acceleration of the centre of mass:

$$F = \sum_{i=1}^{N} F_i = m \frac{{}^{A}d\,{}^{A}v^{C}}{dt}$$

We sum the individual forces to get F, and equate that to the total mass, m, times the derivative of the velocity of the centre of mass.

The notation probably requires some explanation.

1. The superscript A refers to the fact that we're computing these quantities in an inertial frame A.



2. The superscript C, refers to the fact that we're computing the velocity of the centre of mass.

Remember that the right-hand side is essentially the rate-of-change of linear momentum. If L the Linear Momentum is L, then the total force, F, is given by:

$$F = \frac{{}^{A}dL}{dt}$$

This is Newton's second law for a system of particles. Again the derivative of each of these quantities, must be performed in an inertial frame in order for this equation to be valid. This is also true for a rigid body.

One way to think of a rigid-body is to consider it as simply an infinite set of particles, which are all glued together rigidly. So if this is valid for a set of particles, it must also be valid for an infinite collection of particles.

So that was Newton's second law. What are the rotational laws of motion for a rigid-body?

Let's derive the rotational equations of motion for a rigid body. For linear motion we considered the linear momentum of the rigid body and it's derivative. For rotational motion, the analogous quantity to consider is the angular momentum.

The rate of change of angular momentum of a rigid-body B, relative to the centre-of-mass, C, in an inertial frame {A} is equal to the resultant moment of all external forces acting on the body relative to C:

$$\frac{{}^{A}d\,{}^{A}H_C^B}{dt} = M_C^B$$

Where H is the angular momentum of a rigid body with origin C (the centre-of-mass), in an inertial frame {A}. We want to calculate the rate-of-change of angular momentum of the rigid-body B. That equals the net moment applied to the rigid-body. Once again, the differentiation must be done in an inertial frame.

In order to actually perform the calculation on the left-hand side, we have to replace H with something that we can easily measure. It turns out that the angular momentum is nothing but the body’s inertia times its angular velocity.

$${}^{A}H_C^B = I_C\,{}^{A}\omega^B$$

These computations have to be done in three dimensions. The angular momentum and the angular velocity are 3-dimensional vectors. The quantity in between is the inertia tensor, whose components can be written as a 3x3 matrix. Here, the subscript C denotes the fact that we measured the components with the centre-of-mass, C, as the origin. The angular velocity is also obtained in the inertial frame, and the leading superscript A captures this. The trailing superscript B, tells us that this is the angular velocity of the rigid body B that we are measuring.

Finally, M is the net moment. Take all the external forces, compute their moments, and add these moments to all external couples or torques. Once again, the trailing subscript C reminds us that we're computing moments with the centre-of-mass, C, as the origin.