

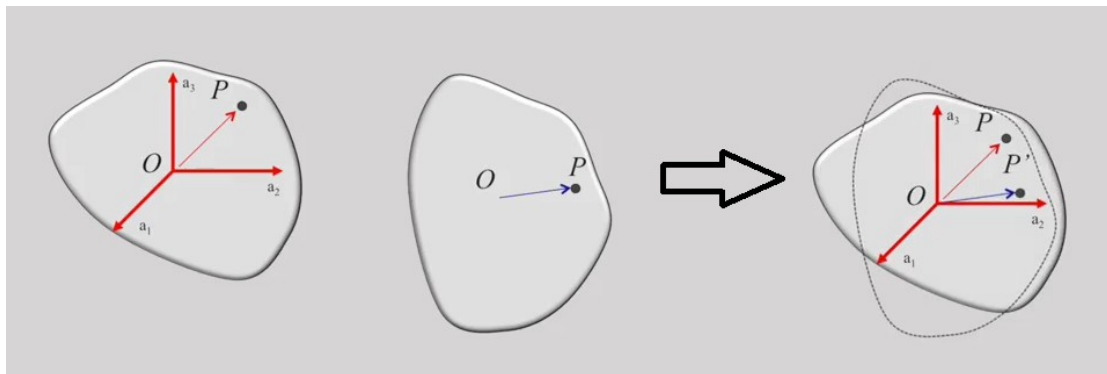
Axis-Angle Representations for Rotations

We've seen two different approaches to parameterising rotations. We've called these **coordinate systems**. For example, using rotation matrices, we can represent rotations with a set of nine numbers or nine coordinates. Those coordinates are redundant, so in reality they are not really coordinates. We've done the same thing with Euler angles where we get a set of three numbers or three coordinates. We're now going to look at a different way of looking at rotations. This involves explicitly representing the axis of rotation and the angle of rotation.

The key idea comes from a theorem by Euler. Euler said, that any displacement of a rigid body, such that a point on the body, O, remains fixed is equivalent to a rotation about a fixed axis through the point O.

What Euler is saying, is that any rotation is equivalent to a rotation about a fixed axis through a point O.

Let's see how to verify that. In order to do that, we're going to consider displacements where the point O is fixed. On the left we have a rigid-body in two positions and orientations. We're going to translate the rigid body from the second position and orientation, so that the two origins are identical as shown on the right:



Now the point P has moved to the new point P'. As we've seen before, the vector \vec{OP} can be written as a linear combination of the basis vectors \mathbf{a}_1 , \mathbf{a}_2 , \mathbf{a}_3 :

$$\vec{OP} = p_1\mathbf{a}_1 + p_2\mathbf{a}_2 + p_3\mathbf{a}_3$$

Similarly, the point P' can also be written as a linear combination of \mathbf{a}_1 , \mathbf{a}_2 , and \mathbf{a}_3 :

$$\vec{OP'} = q_1\mathbf{a}_1 + q_2\mathbf{a}_2 + q_3\mathbf{a}_3$$

So now we have two 3x1 vectors, the vector \mathbf{p} and the vector \mathbf{q} , that are related by the rotation matrix:

$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$$

Or, $\mathbf{q} = \mathbf{R}\mathbf{p}$

What we'd like to do, is to ask if there's an axis that remains fixed under this rotation. In other words, is there a point \mathbf{P} , such that, after the rotation through \mathbf{R} , \mathbf{P} stays fixed, i.e. the point \mathbf{P} is invariant to the rotation?

This is the essence of the proof. So, going back to the basic equation where \mathbf{p} gets rotated to \mathbf{q} , we want to see if there is a point \mathbf{p} that maps onto itself. If there is such a point, then point \mathbf{p} would define the axis of rotation

Well if there was such a point \mathbf{p} , we could write:

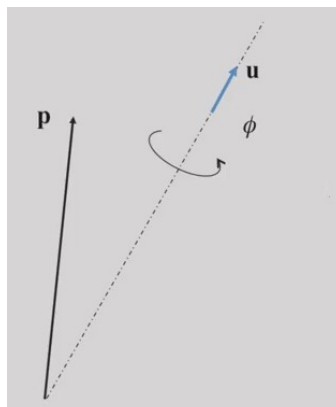
$$\mathbf{p} = \mathbf{R}\mathbf{p}$$

This is nothing but a statement that the eigenvalue of the rotation matrix \mathbf{R} is one. We can solve the eigenvalue problem and verify that if $\mathbf{R}\mathbf{p} = \lambda\mathbf{p}$, then $\lambda = 1$ is a solution to this eigenvalue problem.

This is easy to verify and is a good exercise to try.

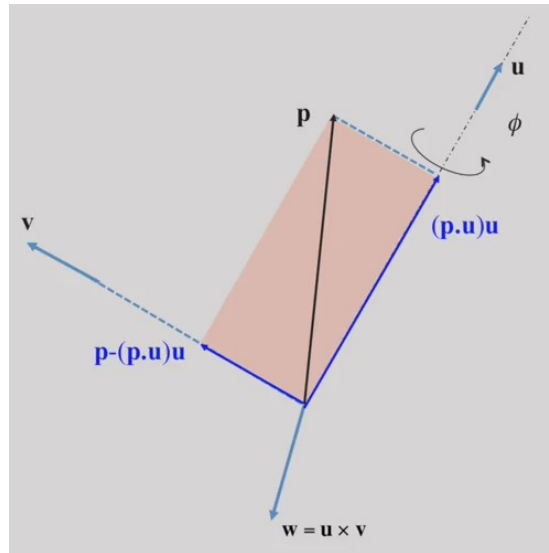
Now, let's consider the inverse problem. Suppose we have an axis of rotation and an angle of rotation. How do we calculate the rotation matrix armed with these two pieces of information? (And note that we already know how to do this if the axis of rotation is the x axis or the y axis or the z axis - in fact, we discussed formulae for these very simple cases).

So, we have an arbitrarily oriented axis given by the unit vector \mathbf{u} and an arbitrary angle of rotation given by the angle ϕ . Let's consider a generic vector \mathbf{p} , and see how the vector \mathbf{p} is rotated if the rigid body to which it's attached is rotated about the axis \mathbf{u} through an angle ϕ .



First, let's break this vector up into two components: One parallel to the vector \mathbf{u} , and the other perpendicular to the vector \mathbf{u} .

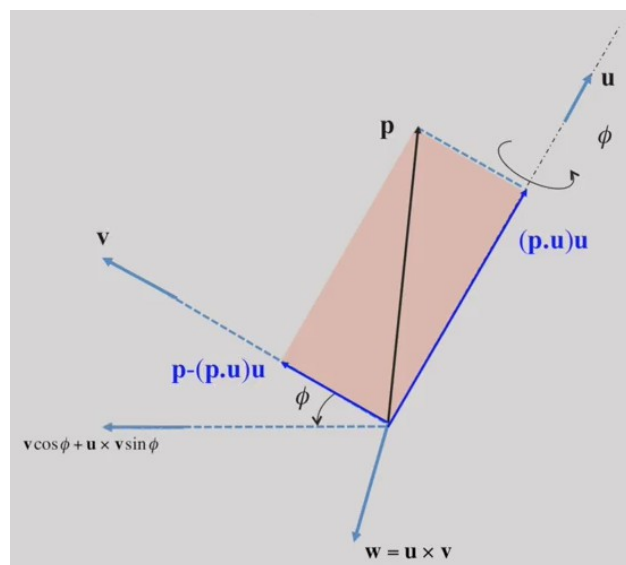
We can obtain the component parallel to the vector \mathbf{u} by projecting \mathbf{p} along \mathbf{u} to get $(\mathbf{p} \cdot \mathbf{u})\mathbf{u}$. Then, we can obtain the vector perpendicular to \mathbf{u} by subtracting that projection, giving $\mathbf{p} - (\mathbf{p} \cdot \mathbf{u})\mathbf{u}$:



Obviously, the component of vector \mathbf{p} that lies along the \mathbf{u} axis is never changed by a rotation about \mathbf{u} . In other words, the vector $(\mathbf{p} \cdot \mathbf{u})\mathbf{u}$ remains unchanged by the rotation R . But the perpendicular component, along \mathbf{v} will be rotated.

We can now use the three vectors, \mathbf{u} , \mathbf{v} and the orthogonal vector \mathbf{w} , where \mathbf{w} is given by $\mathbf{u} \times \mathbf{v}$ as our local coordinate system or local basis.

As we observed above, the vector along \mathbf{u} does not get rotated, but the vector along \mathbf{v} does. This is what it looks like:



The resulting vector is given by simple trigonometry as:

$$v \cos \phi + (u \times v) \sin \phi$$

So, after rotation, the vector \mathbf{P} consists of two new components. The one parallel to \mathbf{u} is the same as before, but the one perpendicular to \mathbf{u} has changed, and that is now given by $v \cos \phi + w \sin \phi$

Adding these components gives the new vector. We can derive an expression for the rotated \mathbf{p} . That is given by:

$$R\bar{\mathbf{p}} = \bar{\mathbf{p}} \cos \phi + UU^T (1 - \cos \phi) \bar{\mathbf{p}} + (\bar{\mathbf{u}} \times \bar{\mathbf{p}}) \sin \phi$$

Now, since \mathbf{p} was chosen as a generic vector, this expression holds for any vector \mathbf{p} .

We can to simplify this expression a little. Notice the term $\bar{\mathbf{u}} \times \bar{\mathbf{p}}$. This is obviously a cross product, but it can also be written as a matrix times \mathbf{p} . We will call this matrix $\hat{\mathbf{u}}$.

$\hat{\mathbf{u}}$ is a skew-symmetric matrix, whose components are the same as the components of \mathbf{u} .

So, we can say that the rotation of a generic vector is given by the right hand side of the formula we just derived:

$$R\bar{\mathbf{p}} = \bar{\mathbf{p}} \cos \phi + UU^T (1 - \cos \phi) \bar{\mathbf{p}} + \hat{\mathbf{u}} \bar{\mathbf{p}} \sin \phi$$

Removing the vector \mathbf{p} from both sides essentially gives an expression that depends only on the axis of rotation and the angle of rotation. This is the rotation matrix:

$$Rot(\mathbf{u}, \phi) = I \cos \phi + UU^T (1 - \cos \phi) + \hat{\mathbf{u}} \sin \phi$$

It consists of three terms. One is the identity times cosine ϕ . The second is $UU^T (1 - \cos \phi)$, and the third is $\hat{\mathbf{u}} \sin \phi$, where $\hat{\mathbf{u}}$ is this skew-symmetric 3x3 matrix. This formula is called **Rodrigues' formula**.

So, given a generic vector \mathbf{u} , and a rotation angle, ϕ , we can derive the rotation matrix using this closed-form expression. We can verify that the expression makes sense by testing it on some simple cases we've seen before.

For example, take \mathbf{u} to be a unit vector along the x axis (or the y-, or z-axes). We can verify that the result is the same as $Rot(\mathbf{x}, \phi)$: a rotation about the x axis through ϕ .

Let's return to the question of whether the axis-angle representation is an *onto* representation? i.e. Is it 1-1?

It's *onto* if every rotation matrix can be represented by a combination of a vector \mathbf{u} and an angle ϕ . It's one-to-one if there's a *unique* representation. Because we know that for every rotation there's an axis of rotation, Euler's theorem tells us that it's *onto*.

What about the 1-1 property?

Well, we can immediately see that this does not hold. It is clear that a rotation about the axis \mathbf{u} through an angle ϕ is the same as a rotation about an axis pointing in the opposite direction through the angle $(2\pi - \phi)$ or $-\phi$. To verify it, just imagine rotating a body one way through an angle ϕ , and then the other way through an angle $-\phi$.

We can also substitute these values into Rodrigues' formula, and verify that we get the same rotation matrix.

But what if we restrict ϕ to the interval from zero to π ? If we do that, then $-\phi$ will not be in the same interval as ϕ . Likewise $2\pi - \phi$ will not be in the same interval as ϕ . But does this make the map one to one?

One way to explore this question is to look at Rodrigues formula and ask ourselves whether, for a given rotation matrix, we can extract \mathbf{u} and ϕ from the matrix? To do this, we can verify that if we take the trace of the rotation matrix, τ , which is essentially $R_{11} + R_{22} + R_{33}$, and we subtract one from it and divide by two, we will get $\cos\phi$:

$$\cos\phi = \frac{\tau - 1}{2}$$

This is obtained by directly applying Rodrigues' formula to get the trace. If this is true, we see from Rodrigues's formula that $\hat{\mathbf{u}}$ can be obtained thus:

$$\hat{\mathbf{u}} = \frac{1}{2\sin\phi}(\mathbf{R} - \mathbf{R}^T)$$

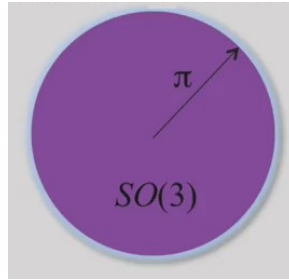
In this case we have obtained \mathbf{u} without actually solving for the eigenvector. So is the axis angle rotation 1 to 1?

1. (axis-angle) to rotation matrix map is many-to-1.
2. Restricting the angle to the interval $[0 - \pi]$ makes it 1-to-1, except for:
 - a. $\tau = 3$
 - b. $\tau = -1$

The case where $\tau = 3$ is equivalent to $\cos\phi = 1$, which means that $\phi = 0$. If $\phi = 0$, there is no unique axis of rotation.

If the trace, τ , is equal to -1 then $\phi = \pi$. In this case we could have two axes of rotation: either the axis we obtained using the formula above, \mathbf{u} , or the opposite, $-\mathbf{u}$.

Perhaps the best way to think about the rotation group, $SO(3)$, is by imagining a solid ball of radius π :



Any point on the ball's surface or inside the ball essentially represents a rotation whose angle is given by the radius. The vector that describes the point with the origin of the sphere as the origin of the vector tells you the axis of rotation.

Again, it seems like there's a one-to-one map between points on the surface of the ball or inside the ball and the set of rotations. This is true, again, except for points that are on the surface of the ball.

At any pair of points on the surface of the ball which are diametrically opposite each other essentially correspond to the same rotation.