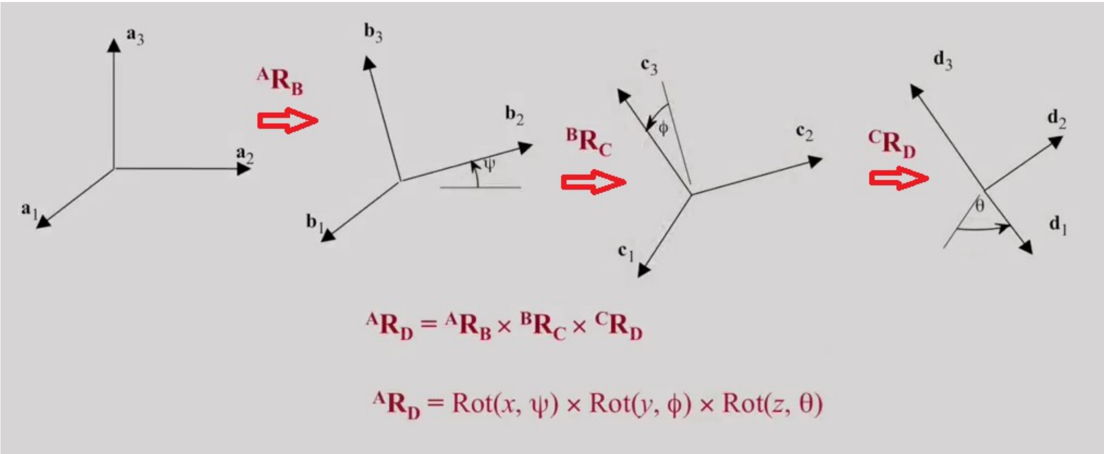
Lecture 2A.3

Euler Angles

The answer to the question that we posed at the end of the previous lecture is three. In fact, Euler showed that it's possible to cover the entire group of rotations using just three coordinates, and these coordinates are called Euler Angles.

To show how Euler angles work, I want to think about three successive rotations. The first rotation, ${}^{A}R_{B}$, going from frame {A} to frame {B}, the second, ${}^{B}R_{C}$, going from {B} to {C}, and the third, ${}^{C}R_{D}$, going from {C} to {D}:



A key idea behind Euler angles is that to describe the rotation, {A} to {D}, we can break it up into three successive rotations, {A} to {B}, {B} to {C}, and then {C} to {D}. We do this by simply multiplying the intermediate rotations.

So in the diagram above, the first rotation about $\mathbf{a}_1$ is through the angle, φ. This is a rotation about the x axis, and it allows us to get to $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$. The second rotation is about the vector $\mathbf{b}_2$, and that gets you to $\mathbf{c}_1$, $\mathbf{c}_2$, and $\mathbf{c}_3$. This rotation is through the angle, ϕ. Finally, the third rotation, is about the vector $\mathbf{c}_3$ and through the angle, θ.

So by simply multiplying rotations about the x-axis through φ, the y-axis through ϕ, and the z-axis through θ, we end up with the net rotation, which is from frame {A} to frame {D}. Again, remember that the nomenclature here refers both to displacements and to transformations.

So a rotation with a superscript A and a subscript D, can serve two purposes. It can transform vectors, written in terms of $\mathbf{d}_1$, $\mathbf{d}_2$, $\mathbf{d}_3$, into a vector whose components are written in terms of $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$. It can also refer to the fact that a rigid-body has been displaced from its initial orientation at frame {A}, into its new orientation at frame {B}.

The three angles that we have seen here are the **roll**, **pitch**, and **yaw** angles.

$$ {}^{A}\mathbf{R}_{D} = \underbrace{Rot(x, \psi)}_{roll} \times \underbrace{Rot(y, \phi)}_{pitch} \times \underbrace{Rot(z, \theta)}_{yaw} $$

Imagine a vehicle whose axis is oriented along $\mathbf{a}_1$. The first rotation is a roll rotation. The second rotation is a pitching motion about the second axis, which is $\mathbf{b}_2$. The third angle is a yaw rotation, and that is about the third axis which is $\mathbf{c}_3$.

Euler said that any rotation can be described by three successive rotations about linearly independent axes. So if we have three Euler angles, like the angles just described, we can deduce from them a 3x3 rotation matrix.

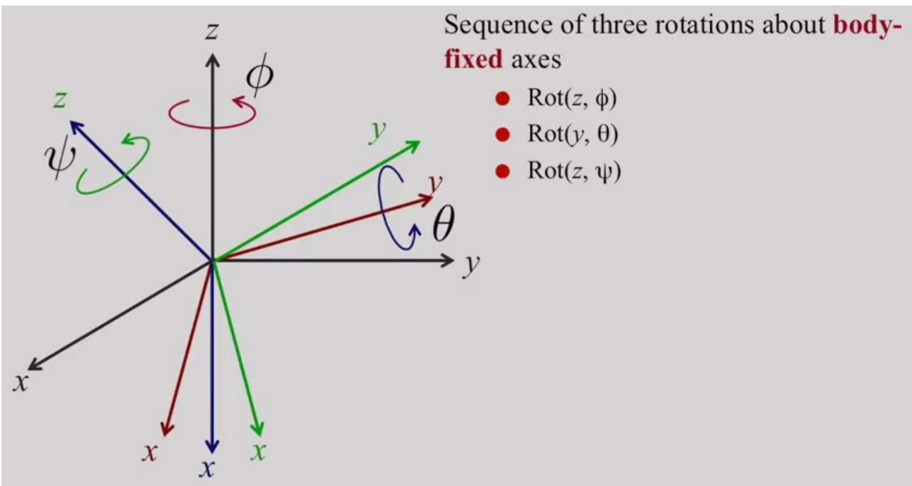
The question we want to ask now is whether the reverse is true. In other words, for every rotation matrix, is there a unique set of Euler angles? Or, to put it another way, is the map between the three Euler angles and the rotation matrix one-to-one?

Sadly, the answer is no. It's almost one to one, but there are points that are analogous to the North Pole and South Pole on the Earth's surface, at which the Euler angles are not well defined.

This set of Euler angles is often called the X-Y-Z Euler angles. That is because of the sequence in which these rotation matrices are applied, i.e. a rotation about the x axis followed by the y axis followed by the z axis. We can also have other types of Euler angles. This particular one is called Z-Y-Z Euler angles:

$$ R = Rot(z,\phi) \times Rot(y,\varphi) \times Rot(x,\theta) $$

Once again, the first rotation is about the z-axis, you rotate about the z-axis through ϕ. And then the second rotation is about the y-axis through θ. And then the third rotation is a rotation about the z-axis through φ.



Notice that every rotation occurs about a body fixed axis. Once the first rotation is complete, the second rotation about the y axis is now about a new y axis, because the first rotation rotated the original y-axis into a new position.

Z-Y-Z Euler angles have two rotations about z-axes. But are they the same z-axis? In other words, are the three axes really linearly independent? Clearly, in this case, the two z axes are not collinear, they are independent.

If this condition is satisfied, then the three angles are Euler angles, and they can parameterize the set of rotations.

Note that if an Euler angle is zero (e.g. θ = 0 is a special (**singular**) case) we might run into problems. In this case, θ = 0 would cause the two z axis to be collinear. As a result, the axes will not be linearly independent. In other words, the three axes about which were performing rotations would no longer be independent. So θ = 0 is analogous to being at the North Pole or the South Pole on the earth's surface.

To explore this further, we'll look inside the computations that are involved in going from a rotation matrix to the three Euler angles. Assume we have a known rotation matrix, containing a set of nine numbers, and we want to recover the three Euler angles: ϕ, θ, and φ.

$$\begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix}$$

In order to do this, we can write out the result of the multiplication of the three rotation matrices. The rotation about the z axis, followed by the rotation about the y axis, followed by the rotation about the z axis. The first rotation through ϕ, the second through θ, and the third through φ.

$$\begin{bmatrix} \cos\phi\cos\theta\cos\varphi - \sin\phi\sin\varphi & -\cos\phi\cos\theta\sin\varphi - \sin\phi\cos\varphi & \cos\phi\sin\theta \\ \sin\phi\cos\theta\cos\varphi + \cos\phi\sin\varphi & -\sin\phi\cos\theta\sin\varphi + \cos\phi\cos\varphi & \sin\phi\sin\theta \\ -\sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \end{bmatrix}$$

Equating these two matrices, we see that R_{33} equates to $\cos\theta$, so if we know R_{33} we can calculate θ. Now, if we know θ, we can look at the R_{31} element and calculate φ. In fact, we can get the same information from the R_{32} element, provided we know θ. Finally, we can use the R_{13} and R_{23} elements to calculate ϕ.

$$\begin{aligned} &R_{33} = \cos\theta \\ &R_{32} = \sin\theta\sin\varphi \qquad &R_{31} = -\sin\theta\cos\varphi \\ &R_{13} = \cos\phi\sin\theta \qquad &R_{23} = \sin\phi\sin\theta \end{aligned}$$

If the R_{33} element is non-zero, then we can go ahead and calculate θ. We calculate it by taking the inverse-cosine function. Of course, the inverse-cosine function has some ambiguity, and because of that we don't know the sin of θ. It could either be positive or negative. But modulo that ambiguity we can determine θ.

$$\theta = \sigma\arccos(R_{33}), \qquad \sigma = \pm 1$$

Once we know θ, we can calculate φ and ϕ using the inverse tangent function. Notice that we have two pieces of information for both φ and ϕ, and because of that, we don't use the standard inverse-tangent function. We use something called the **atan2 function**. The atan2 function allows us overcome the ambiguity that exists in inverse-tangent functions. Unlike the traditional inverse-tangent function, which only uses one equation to solve for an angle, here, we use the fact that we have two equations for the same angle. The inverse tangent function atan2 allows us to do this.

$$\varphi = a\tan 2\left(\frac{R_{32}}{\sin\theta}, \frac{-R_{31}}{\sin\theta}\right)$$

$$\phi = a\tan 2\left(\frac{R_{23}}{\sin\theta}, \frac{R_{13}}{\sin\theta}\right)$$

You will see from these equations that we have two sets of Euler angles, and this is true for almost all rotation matrices. These equations were derived, assuming that the magnitude of R_{33} is not equal to one. $|R_{33}| < 1$.

So what happens if the magnitude of R_{33} is equal to one? R_{33} can either be plus one or minus one. In these cases, the angle θ will be equal to either zero or pi. So we have two alternatives, but in both these cases, we can see from the grouping of terms, that the rotation matrices are functions only of the sum of two angles, ϕ and φ.

If $R_{33} = 1$

$$R = \begin{bmatrix} \cos\phi\cos\varphi - \sin\phi\sin\varphi & -\cos\phi\sin\varphi - \sin\phi\cos\varphi & 0 \\ \sin\phi\cos\varphi + \cos\phi\sin\varphi & -\sin\phi\sin\varphi + \cos\phi\cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

If $R_{33} = -1$

$$R = \begin{bmatrix} -\cos\phi\cos\varphi - \sin\phi\sin\varphi & \cos\phi\sin\varphi - \sin\phi\cos\varphi & 0 \\ \cos\phi\sin\varphi - \sin\phi\cos\varphi & \sin\phi\sin\varphi + \cos\phi\cos\varphi & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

In other words, it's impossible to determine, either ϕ or φ uniquely. Given one, we can determine the other. But it's impossible to disambiguate between ϕ and φ, because the groupings only contain terms that combine them. So when $R_{33} = \pm 1$ we have an infinite set of Euler angles.

In most of our work, we use a different set of Euler angles. These are the Z-X-Y Euler angles. Again, they're so called because the three rotation matrices we consider are the rotation about the z axis through φ, followed by the rotation about the x axis through ϕ and then the rotation of the y axis in θ.

Sequence of three rotations about **body-fixed** axes:

Rot(z,φ)
Rot(x,φ)
Rot(y,θ)

Verify by multiplying the rotations that we get a rotation matrix that has the form shown:

$$R = \begin{bmatrix} \cos\varphi\cos\theta - \sin\phi\sin\varphi\sin\theta & -\cos\phi\sin\varphi & \cos\varphi\sin\theta + \cos\theta\sin\phi\sin\varphi \\ \cos\theta\sin\varphi + \cos\varphi\sin\phi\sin\theta & \cos\phi\cos\varphi & \sin\varphi\sin\theta - \cos\theta\sin\phi\cos\varphi \\ -\cos\theta & \sin\phi & \cos\phi\cos\theta \end{bmatrix}$$

We've seen that for every set of Euler angles, we might have at least two solutions for Euler's angles for a given rotation matrix. At some points we can have infinite solutions. What we really want is a second set of Euler angles to take care of the points at which we have infinite solutions.

This suggests that we might have many, many sets of Euler angles that we might want to consider so that we don't have any points at which we have infinite solutions. So a question that is worth asking is what is the minimum number of sets of Euler angles we need to cover all of the rotation group?