Supplementary Material - Solving for Coefficients of Minimum Jerk Trajectories

Presented by Lucy Tang, PhD student.

In this segment, we'll talk more about how to find the coefficients of a minimum-jerk trajectory.

Recall that we can find the minimum-jerk trajectory by solving the following minimization problem.

$$x^*(t) = \arg\min_{x(t)} \int_0^T L(\dddot{x}, \ddot{x}, \dot{x}, x, t)dt = \arg\min_{x(t)} \int_0^T L(\dddot{x}^2)dt$$

By solving the Euler-Lagrange equation:

$$\frac{\delta L}{x} - \frac{d}{dt}\left(\frac{\delta L}{\delta \dot{x}}\right) + \frac{d^2}{dt^2}\left(\frac{\delta L}{\delta \ddot{x}}\right) - \frac{d^3}{dt^3}\left(\frac{\delta L}{\delta \dddot{x}}\right) = 0$$

we find that a necessary condition for the minimum-jerk trajectory is that its 6^{th}-derivative must be 0. This leads us to conclude that a minimum-jerk trajectory takes the form of a 5^{th}-order polynomial, with six unknown coefficients. Our goal is to find the values of these coefficients:

$$x(t) = c_5 t^5 + c_4 t^4 + c_3 t^3 + c_2 t^2 + c_1 t + c_0$$

To do this we define six boundary conditions:

	Position	Velocity	Acceleration
t = 0	a	0	0
t = T	B	0	0

Let's write out each of these boundary conditions explicitly. The position of the trajectory is given by the function x(t). At time 0, this function evaluates to c_0. According to the boundary condition, this must be equal to a.

At time T this function evaluates to the expression below. According to the boundary condition, this must be equal to b. Note that in this expression, the constant T is exponentiated to various powers. However, because T is known, we can evaluate these exponentials explicitly and they become constants. The unknown coefficients are not exponentiated, therefore this expression is linear in terms of the coefficients:

Position constraints: $x(t) = c_5t^5 + c_4t^4 + c_3t^3 + c_2t^2 + c_1t + c_0$

$$x(0) = c_0 = a$$

$$x(T) = c_5(T)^5 + c_4(T)^4 + c_3(T)^3 + c_2(T)^2 + c_1(T) + c_0 = b$$

We can rewrite the position constraints at t = 0 in matrix form:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_5 \\ c_4 \\ c_3 \\ c_2 \\ c_1 \\ c_0 \end{bmatrix} = a$$

Here, we are multiplying a 1x6 matrix with a 6x1 matrix, which will result in a 1x1 matrix, or simply, a constant. Further, if we carry out the matrix multiplication, c_0 will multiplied by 1 and all the other coefficients will be multiplied by 0. Thus this matrix expression is equivalent to the constraint $c_0 = a$.

We can similarly represent the position constraint at time T in matrix form:

$$\begin{bmatrix} T^5 & T^4 & T^3 & T^2 & T & 1 \end{bmatrix} \begin{bmatrix} c_5 \\ c_4 \\ c_3 \\ c_2 \\ c_1 \\ c_0 \end{bmatrix} = b$$

Again, because T is known beforehand, the matrix on the left contains only constants. We see that the second matrix term on the left-hand side, which contains our unknown coefficients, is the same as the second matrix term of the left-hand side of the position constraint at t = 0.

Next, we consider the velocity constraints. We can find the function for the trajectory's velocity by differentiating the position function x(t). The conditions at t = 0 and t = T give us the following constraints:

Velocity constraints: $\dot{x}(t) = 5c_5t^4 + 4c_4t^3 + 3c_3t^2 + 2c_2t + c_1$

$$\dot{x}(0) = c_1 = 0$$

$$\dot{x}(T) = 5c_5(T)^4 + 4c_4(T)^3 + 3c_3(T)^2 + 2c_2(T) + c_1 = 0$$

As before, we can write the constraints in matrix form. We can write the constraint at t = 0 in the following way:

$\dot{x}(0) = c_1 = 0$

We can do the same for the constraint at t = T:

$\dot{x}(T) = 5_{c_5}(T)^4 + 4_{c4}(T)^3 + 3_{c_3}(T)^2 + 2_{c_2}(T) + c_1 = 0$

Giving us the matrix equations:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} c_5 \\ c_4 \\ c_3 \\ c_2 \\ c_1 \\ c_0 \end{bmatrix} = 0$$

$$\begin{bmatrix} 5T^4 & 4T^3 & 3T^2 & 2T & 1 & 0 \end{bmatrix} \begin{bmatrix} c_5 \\ c_4 \\ c_3 \\ c_2 \\ c_1 \\ c_0 \end{bmatrix} = 0$$

Again, we see that the matrix containing the unknown coefficients remains the same in both of these constraints.

Finally, we can repeat this process for the acceleration constraints. Differentiating the velocity function gets the acceleration function, and from this, we can write the conditions at t = 0 and t=T

Acceleration constraints: $\ddot{x}(t) = 20c_5t^3 + 12c_4t^2 + 6c_3t^2 + 2c_2$

$$\ddot{x}(0) = 2c_2 = 0$$

$$\ddot{x}(T) = 20c_5(T)^3 + 12c_4(T)^2 + 6c_3(T)^2 + 2c_2 = 0$$

We can now write these constraints in matrix form. As you probably expected, these expressions also contain the same matrix of unknown coefficients we saw earlier:

$$\begin{bmatrix} 0 & 0 & 0 & 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_5 \\ c_4 \\ c_3 \\ c_2 \\ c_1 \\ c_0 \end{bmatrix} = 0$$

$$\begin{bmatrix} 20T^3 & 12T^2 & 6T & 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_5 \\ c_4 \\ c_3 \\ c_2 \\ c_1 \\ c_0 \end{bmatrix} = 0$$

Because all six constraints can be written as a 1x6 matrix multiplied by the same 6x1 matrix, we can combine them into a single matrix expression:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ T^5 & T^4 & T^3 & T^2 & T & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 5T^4 & 4T^3 & 3T^2 & 2T & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 20T^3 & 12T^2 & 6T & 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_5 \\ c_4 \\ c_3 \\ c_2 \\ c_1 \\ c_0 \end{bmatrix} = \begin{bmatrix} a \\ b \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Here, each row of the 6x6 matrix on the left-hand side, in conjunction with the corresponding row of the 6x1 matrix on the right-hand side, represents one of the boundary conditions. Again, because T, a, and b are known beforehand the only unknown variables appear in the matrix of coefficients. We can now use one of many techniques to solve for this matrix of unknown coefficients.

Let's consider an example. Here, T = 1, a = 0, and b = 5. We can substitute these values into the previous expression to get the following matrix problem:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 5 & 4 & 3 & 2 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 20 & 12 & 6 & 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_5 \\ c_4 \\ c_3 \\ c_2 \\ c_1 \\ c_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

If we denote the 6x6 matrix on the left-hand side as A, and the 6x1 matrix on the right-hand side as b, we have a problem in the form Ax = b. Where x is the matrix of unknown coefficients. We can find x by inverting A and multiplying it by b:

$$x = A^{-1}b$$

This gives us the following matrix for x, which contains the values of the unknown coefficients in the order that we designated:

$$x = \begin{bmatrix} 30 \\ -75 \\ 50 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

This means that the minimum-jerk trajectory for this problem is:

$$x(t) = 30t^5 - 75t^4 + 50t^3$$

We can verify that this trajectory does, in fact, satisfy all the boundary conditions. Evaluating the function x(t) at 0 and 1, gives us a value of 0 and 5 respectively, as we specified in the boundary conditions:

	Position	Velocity	Acceleration
t = 0	a = 0 ✓	0	0
t = T = 1	b = 5 ✓	0	0

$$x(t) = 30t^5 - 75t^4 + 50t^3$$

$$x(0) = 0$$

$$x(1) = 30(1)^5 - 75(1)^4 + 50(1)^3 = 5$$

We can differentiate x(t) to get the velocity function $\dot{x}(t)$. This function is 0 at both 0 and 1:

	Position	Velocity	Acceleration
t = 0	a = 0 ✓	0 ✓	0
t = T = 1	b = 5 ✓	0 ✓	0

$$\dot{x}(t) = 150x^4 - 300t^3 + 150t^2$$

$$\dot{x}(0) = 0$$

$$\dot{x}(1) = 150 - 300 + 150 = 0$$

Finally, we can differentiate once more to get the acceleration function. This function is also zero at both 0 and 1:

	Position	Velocity	Acceleration
t = 0	a = 0 ✓	0 ✓	0 ✓
t = T = 1	b = 5 ✓	0 ✓	0 ✓

$$\ddot{x}(t) = 600x^3 - 900t^2 + 300t$$

$$\ddot{x}(0) = 0$$

$$\ddot{x}(1) = 600 - 900 + 300 = 0$$

It is important to notice that when transforming the condition expressions into matrix form, the order in which we place the unknown coefficients matters. We must be sure that all coefficients are matched up with the right constants. For example, in our example the matrix expression for the position constraint at t = T was this:

$$\begin{bmatrix} T^5 & T^4 & T^3 & T^2 & T & 1 \end{bmatrix} \begin{bmatrix} c_5 \\ c_4 \\ c_3 \\ c_2 \\ c_1 \\ c_0 \end{bmatrix} = b$$

If we reverse the order of the unknown coefficients in the second matrix term on the left-hand side, we must also reverse the order of the terms in the first matrix term, thus:

$$\begin{bmatrix} 1 & T & T^2 & T^3 & T^4 & T^5 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{bmatrix} = b$$

We can change the order of the unknown coefficients in the second matrix term however we want, as long as we make sure the corresponding coefficient in the first matrix term match up. When solving the complete problem it is important that we order the unknown coefficient in the second matrix term in the same manner for all constraints.