Lecture 2C.3

Principal Axes and Principal Moments of Inertia

Before we can write down the rotational equations of motion we have to define principle axes and moments of inertia.

A principal axis of inertia is defined by a direction. Let's say U is that direction.

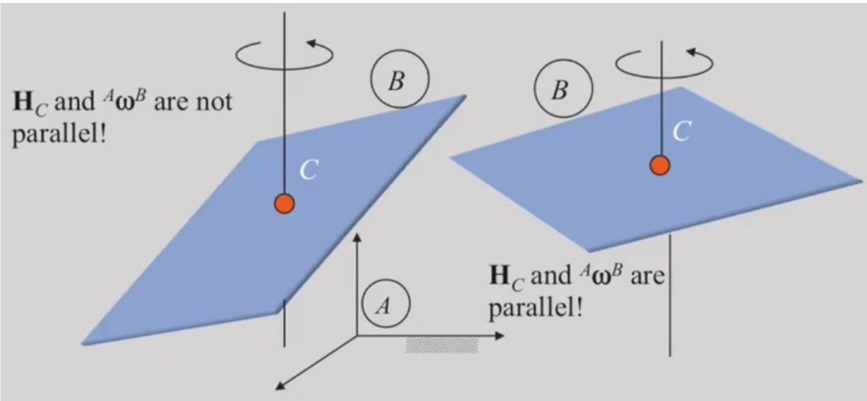
- **u** is a unit vector along a principal axis if **I**.**u** is parallel to **u**.

There's a theorem that says that says we can always find three such independent principle axes. In other words there are three independent axes such that **I** times a unit vector along that axis will give a vector that's parallel to that axis.

The moment of inertia, **I**, is essentially a scaling term. If we take I .u and it differs from u by a scalar factor, that scalar factor is the moment of inertia. These moments of inertia are called **principal moments of inertia**.

- The moment-of-inertia with respect to a principal axis, **u**.**I**.**u**, is called a **principal moment of inertia**.

Here's a simple example that illustrates what the moments of inertia tell us:



We have a rigid frame, {A}, and a parallel plate that's spinning about a vertical axis. The plate is shown in two different configurations.

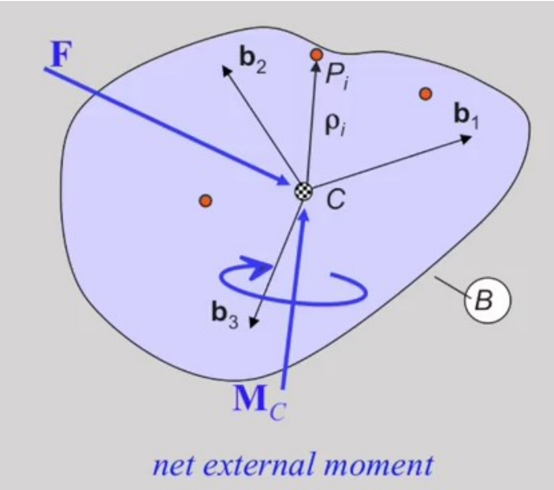
On the right side, the axis is perpendicular to the plate. In fact this configuration is symmetric. If we compute the angular momentum, we find that the angular-momentum vector and the angular-velocity vector are parallel.

On the left side, the configuration is symmetric, but the axis is not perpendicular to the plate. In this configuration, the angular-momentum vector and the angular-velocity vector are not parallel. This is because the axis of rotation does not coincide with any of the principal axes.

Now we can deal with Euler's Equations which will tell us the rotational equations of motion. Once again, it comes down to the basic observation that the rate-of-change of angular-momentum is equal to the net moment applied to the rigid body.

We take C, the centre-of-mass, as the origin for all our calculations.

In the figure below, b_1, b_2 and b_3 are a set of body-fixed unit vectors that define a body-fixed frame. We'll now specify that these vectors point along principal axes and we'll write our angular velocity as linear combinations of b_1, b_2, and b_3, and the components are ω_1, ω_2, and ω_3:



$$^{A}\omega^{B} = \omega_1 b_1 + \omega_2 b_2 + \omega_3 b_3$$

There are two key aspects to what we are doing:

1. We're taking C, the centre-of-mass, as the origin.
2. We're requiring that the body-fixed frame in which we'll do our calculations will be along the three principal axes.

We can take the basic equation we derived in the last lecture:

$$\frac{^{A}d\,{}^{A}H_C^B}{dt} = M_C^B$$

And break the left-hand side two terms:

$$\frac{^{B}d\,H_C}{dt} + {}^{A}\omega^{B} \times H_C = M_C$$

The first term involves the derivative in a body fixed frame. The second term involves a "correction factor", which is a vector that takes into account the fact that the differentiation is done in the body fixed frame. This correction factor is simply the angular velocity of the moving body-fixed frame crossed with the angular momentum.

This "correction factor", as we've referred to it, is actually a well-known fact in mechanics. Anytime you differentiate a vector in a moving frame, its derivative is different from the derivative in a fixed frame. That difference is obtained by simply factoring in the cross product of the angular velocity with that vector.

The first term on the left-hand side can be written in terms of inertia matrix times the angular velocity vector. Because we have chosen principle axes, it turns out that the off-diagonal elements in the inertia tensor are zero. Therefore, the first term which involves $I \times \omega$ (the inertia tensor times the angular-velocity vector) consists of three terms:

The diagonal terms I_{11}, I_{22}, I_{33}, which multiply with ω_1, ω_2, and ω_3, yield the three terms that we see on the right hand side:

$$\frac{{}^{B}d\ H_C}{dt} = I_{11}\dot{\omega}_1 b_1 + I_{22}\dot{\omega}_2 b_2 + I_{33}\dot{\omega}_3 b_3$$

The $\omega \times H$ term can also be written in component form, and that gives us the following matrix equation:

$$\begin{bmatrix} I_{11} & 0 & 0 \\ 0 & I_{22} & 0 \\ 0 & 0 & I_{33} \end{bmatrix} \begin{bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{bmatrix} + \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \begin{bmatrix} I_{11} & 0 & 0 \\ 0 & I_{22} & 0 \\ 0 & 0 & I_{33} \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = \begin{bmatrix} M_{C,1} \\ M_{C,2} \\ M_{C,3} \end{bmatrix}$$

These are Euler's Equations of Motion. They are quite compact.

The first term is essentially the derivative of the angular momentum in a body-fixed frame, and the second term is the correction. The term on the right-hand side, is the net moment.

In what follows we will use **p**, **q** and **r** to denote the components of the angular velocity vector along $\mathbf{b_1}$, $\mathbf{b_2}$, and $\mathbf{b_3}$.