Supplementary Material - Minimum Velocity Trajectories from the Euler-Lagrange Equations

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In lecture, we saw that we can use the Euler-Lagrange equations to deduce the general form of a minimum velocity trajectory for a first-order system. In this segment, we'll go through the details of that calculation.

In lecture, we considered problems where we had to find the function x*(t) that minimizes the integral of a cost function L(T):

$$x^*(t) = \arg\min_{x(t)} \int_0^T L(\dot{x}, x, t)\,dt$$

When looking for the minimum-velocity trajectory, this cost function is $\dot{x}^2$. The Euler-Lagrange equation gives the necessary condition that must be satisfied by the optimal function x(t).

$$\frac{d}{dt}\left(\frac{\delta L}{\delta \dot{x}}\right) - \frac{\delta L}{x} = 0$$

Let's evaluate the Euler-Lagrange equation for the problem of finding the minimum-velocity trajectory. Again, the cost function for this problem is $L(\dot{x}, x, t) = \dot{x}^2$.

First let's find the individual terms in the Euler-Lagrange equation. Here, it is important to take care to differentiate with respect to the proper variable.

The partial derivative of L with respect to x is 0. This is because the term x does not appear in L.

The partial derivative of L with respect to $\dot{x}$ is $2\dot{x}$. Note that in this step we are not yet differentiating with respect to time.

Finally, we can evaluate the time-derivative of $\delta L/\delta\dot{x}$. Using the value of δL/δx found in the last step, we see that we need to take the time-derivative of $2\dot{x}$ which is $2\ddot{x}$.

Euler-Lagrange terms:

$$\left(\frac{\partial \mathcal{L}}{\partial x}\right) = 0 \quad \longleftarrow \text{No } x \text{ appears in } \mathcal{L}$$

$$\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = 2\dot{x}$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = \frac{d}{dt}(2\dot{x}) = 2\ddot{x}$$

Now we can substitute all these terms into the Euler-Lagrange equation. The equation becomes:

$$2\ddot{x} - 0 = 0$$

which is equivalent to simply:

$$\ddot{x} = 0$$

This is the condition we saw in the lecture for a minimum velocity trajectory. We can integrate this condition once with respect to t to get the velocity:

$$\dot{x} = c_1$$

Here, c_1 is an arbitrary constant. We can integrate the velocity to get the position function for the minimum velocity trajectory:

$$x(t) = c_1 t + c_0$$

Here c_0 is another arbitrary constant. In the lecture, we discussed how c_0 and c_1, can be found from the problem's boundary conditions.