Supplementary Material - Linearisation of Quadrotor Equations of Motion

Presented by Lucy Tang, PhD student.

In lecture, we claimed that in the linearised-equations of motion for the quadrotor, the second-derivative of position is proportional to u_1, and the fourth-derivative of position is proportional to u_2. In this segment, we'll derive these relationships explicitly.

Recall the quadrotor's equations-of-motion, which come from the linear and angular momentum balances:

Linear momentum balance:

$$m\ddot{\mathbf{r}} = \begin{bmatrix} 0 \\ 0 \\ -mg \end{bmatrix} + R \begin{bmatrix} 0 \\ 0 \\ F_1 + F_2 + F_3 + F_4 \end{bmatrix}$$

Angular momentum balance:

$$I \begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} L(F_2 - F_4) \\ L(F_3 - F_1) \\ M_1 - M_2 + M_3 - M_4 \end{bmatrix} - \begin{bmatrix} p \\ q \\ r \end{bmatrix} \times I \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

We can use u_1 to represent the total thrust applied and u_2 to represent the moment vector. We could then rewrite the equations-of-motion in terms of u_1 and the components of u_2:

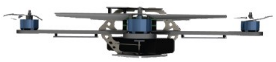
Linear momentum balance:

$$m\ddot{\mathbf{r}} = \begin{bmatrix} 0 \\ 0 \\ -mg \end{bmatrix} + R \begin{bmatrix} 0 \\ 0 \\ u_1 \end{bmatrix}$$

Angular momentum balance:

$$I \begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} u_{2x} \\ u_{2y} \\ u_{2z} \end{bmatrix} - \begin{bmatrix} p \\ q \\ r \end{bmatrix} \times I \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

In the equilibrium hover configuration, the position and yaw angle of the quadrotor can be at some arbitrary value r_0 and φ_0 respectively. However, the angles θ and ϕ, as well as $\dot{r}$, $\dot{\theta}$, $\dot{\phi}$ and $\dot{\varphi}$ are all zero. We want to derive expressions for the equations of motion when the quadrotor is near this equilibrium-configuration.



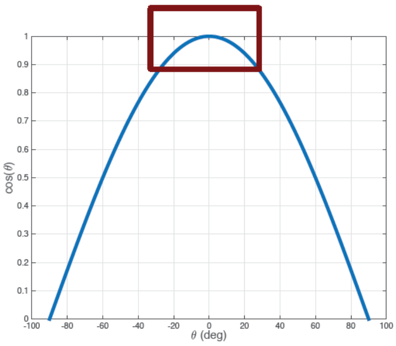
$$\mathbf{r} = \mathbf{r}_0, \theta = \phi = 0, \psi = \psi_0$$
$$\dot{\mathbf{r}} = 0, \dot{\theta} = \dot{\phi} = \dot{\psi} = 0$$

First, let's consider the value of cos(θ) near the equilibrium-configuration when θ = 0. Around θ = 0, the function cos(θ) can be approximated using the Taylor expansion which is shown here:

$$\cos(\theta) \approx \cos(\theta)\Big|_{\theta=0} + \frac{d\cos(\theta)}{d\theta}\Big|_{\theta=0}\theta$$

We can consider all the terms after the first two terms in this series to be negligibly small. The value of cos(θ) at θ = 0 is 1. The derivative of cos(θ) is -sin θ. Since sin θ, at θ = 0 is 0. The approximation of cos(θ) near θ = 0 is simply 1.

What is the value of $\cos(\theta)$ near $\theta = 0$?



Can be approximated with the Taylor Series:

$$\cos(\theta) \approx \cos(\theta)|_{\theta=0} + \frac{d\cos(\theta)}{d\theta}|_{\theta=0}\theta + \text{higher order terms}$$
$$\approx 1 - \sin(\theta)|_{\theta=0}\theta$$
$$\approx 1$$

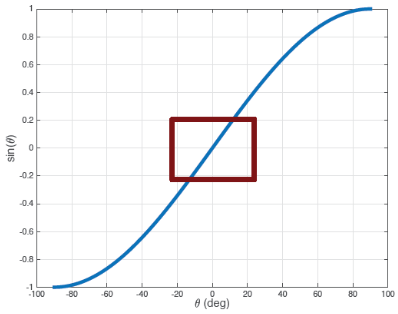
We can confirm this qualitatively by looking at the value of the cosine function, near θ = 0. Looking at the plot of the cosine function for angles between -30° to 30°, we see that the cosine values are indeed close to 1.

We can repeat this process to approximate the function sin(θ) near θ = 0. Again, we use the Taylor series. For sin(θ) we arrive at the following expression:

$$\sin(\theta) \approx \sin(\theta)\Big|_{\theta=0} + \frac{d\sin(\theta)}{d\theta}\Big|_{\theta=0}\theta$$

Since sin(θ), at θ = 0 is 0, and the derivative of sin(θ) is cos(θ), we can simplify the expression, and since the value of cos(θ) at θ = 0 is 1, the value of sin(θ) near θ = 0 is approximately θ:

What is the value of $\sin(\theta)$ near $\theta = 0$?



Can be approximated with the Taylor Series:

$$
\begin{aligned}
&\sin(\theta) \\
&\qquad \approx \sin(\theta)|_{\theta=0} + \frac{d\sin(\theta)}{d\theta}|_{\theta=0}\theta \\
&\qquad\qquad + \text{higher order terms} \\
&\qquad \approx 0 + \cos(\theta)|_{\theta=0}\theta \\
&\qquad \approx \theta
\end{aligned}
$$

This suggests that around θ = 0, we expect the sin function to look approximately linear. Examining the values of sin θ for angles from -30° to 30°, we see that the sin function indeed looks linear.

We will use these two approximations to linearise the equations-of-motion of the quadrotor. Again, we want to use these approximations to help direct simplified versions of the equations-of-motion that apply when the quadrotor is near the equilibrium hover configuration.

First, consider the linear momentum equation. We can explicitly write the rotation matrix in terms of the Euler angles:

$$
m\ddot{\mathbf{r}} = \begin{bmatrix} 0 \\ 0 \\ -mg \end{bmatrix} + \begin{bmatrix} c\psi c\theta - s\phi s\psi s\theta & -c\phi s\psi & c\psi s\theta + c\theta s\phi s\psi \\ c\theta s\psi + c\psi s\theta s\phi & c\phi c\psi & s\psi s\theta - c\psi c\theta s\phi \\ -c\phi s\theta & s\phi & c\phi c\theta \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ u_1 \end{bmatrix}
$$

We can then perform the matrix-multiplication to arrive at the following second order differential equations:

$$
\begin{aligned}
m\ddot{x} &= (c\psi s\theta + c\theta s\phi s\psi)\, u_1 \\
m\ddot{y} &= (s\psi s\theta - c\psi c\theta s\phi)\, u_1 \\
m\ddot{z} &= -mg + (c\phi c\theta)\, u_1
\end{aligned}
$$

At equilibrium, the pitch-angle θ and the roll-angle ϕ are both approximately 0. Therefore we can use the equation we derived earlier to approximate the sines and cosines of these angles.

$$\text{Sin}(\theta) \approx \theta,\ \ \sin(\phi) \approx \phi,\ \cos(\theta) \approx \sin(\phi) \approx 1$$

Substituting in these approximations reduces the differential equations to the ones shown below:

$$m\ddot{x} = (\theta\cos\varphi + \phi\sin\varphi)u_1$$

$$m\ddot{y} = (\theta\sin\varphi - \phi\cos\varphi)u_1$$

$$m\ddot{z} = -mg + u_1$$

We can clearly see that the second-derivative of position is proportional to the input, u_1.

Now consider the relationship between the angular-velocity components p, q, r and the first derivatives of the Euler angles. Again, we carry out the matrix-multiplication to arrive at three equations to relating p, q and r to $\dot{\phi}$, $\dot{\theta}$ and $\dot{\varphi}$:

$$p = \dot{\phi}c\theta - \dot{\psi}c\phi s\theta$$

$$q = \dot{\theta} + \dot{\psi}s\phi$$

$$r = \dot{\phi}s\theta + \dot{\psi}c\phi c\theta$$

We can then substitute in the approximations for the sines and cosines of θ and ϕ, giving us the equations shown here:

$$p = \dot{\phi} - \dot{\varphi}\theta$$

$$q = \dot{\theta} + \dot{\varphi}\phi$$

$$r = \dot{\phi}\theta + \dot{\varphi}$$

Next, we can approximate all terms that are the product of an angle and an angles derivative as 0. Near Hover, θand ϕ and all angular derivatives are close to 0. The product of two terms near 0 will be very, very small, and as a result we can approximate these terms to 0.

Substituting these approximations into the equations, tells us that around hover, the angular velocity components are approximately the time derivatives of the Euler angles.

$$\mathrm{P} = \dot{\phi}$$

$$\mathrm{q} = \dot{\theta}$$

$$\mathrm{r} = \dot{\varphi}$$

Finally consider the angular-momentum equation:

$$I\begin{bmatrix}\dot{p}\\ \dot{q}\\ \dot{r}\end{bmatrix} = \begin{bmatrix}u_{2x}\\ u_{2y}\\ u_{2z}\end{bmatrix} - \begin{bmatrix}p\\ q\\ r\end{bmatrix} \times I\begin{bmatrix}p\\ q\\ r\end{bmatrix}$$

First, we approximate the off-diagonal inertia terms as close to 0.

$$I_{xy} \approx I_{yx} \approx I_{xz} \approx I_{zx} \approx I_{yx} \approx I_{zy} \approx 0$$

This allows us to simplify the inertia matrix:

$$\begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix} \begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} u_{2x} \\ u_{2y} \\ u_{2z} \end{bmatrix} - \begin{bmatrix} 0 & r & -q \\ -r & 0 & p \\ q & -p & 0 \end{bmatrix} \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

Performing the matrix-multiplication gives us the following set of equations:

$$I_{xx}\dot{p} = u_{2x} - I_{yy}qr + I_{zz}qr$$

$$I_{yy}\dot{q} = u_{2y} - I_{xx}pr - I_{zz}pr$$

$$I_{zz}\dot{r} = u_{2z} - I_{xx}pq + I_{yy}pq$$

We saw earlier that around hover, p, q and r are approximately $\dot{\phi}$, $\dot{\theta}$ and $\dot{\varphi}$ respectively and are therefore also close to 0. The pattern of any two angular-velocity-components can then be approximated as 0. This gives us the following set of equations:

$$I_{xx}\dot{p} = u_{2x}$$
$$I_{yy}\dot{q} = u_{2y}$$
$$I_{zz}\dot{r} = u_{2z}$$

Using again the approximation of the angular-velocity components as the Euler angle derivatives, we arrive at this set of second-order differential equations:

$$\ddot{\phi} = \frac{u_{2x}}{I_{xx}}$$

$$\ddot{\theta} = \frac{u_{2y}}{I_{yy}}$$

$$\ddot{\varphi} = \frac{u_{2z}}{I_{zz}}$$

Now let's go back to the linear-momentum equation in the x-direction:

$$m\ddot{x} = (\theta c\psi + \phi s\psi)\, u_1$$

We differentiate this equation twice:

$$m\dddot{x} = (\theta c\psi + \phi s\psi)\, \dot{u}_1 + \left(\dot{\theta} c\psi - \theta s\psi\dot{\psi} + \dot{\phi} s\psi + \phi c\psi\dot{\psi}\right) u_1$$

$$m\ddddot{x} = (\theta c\psi + \phi s\psi)\, \ddot{u}_1 + 2\left(\dot{\theta} c\psi - \theta s\psi\dot{\psi} + \dot{\phi} s\psi + \phi c\psi\dot{\psi}\right) \dot{u}_1 + \left(\ddot{\theta} c\psi - \dot{\theta} s\psi\dot{\psi} - \theta s\psi\ddot{\psi} - \theta c\psi\dot{\psi}^2 + \ddot{\phi} s\psi + \dot{\phi} c\psi\dot{\psi} + \phi c\psi\ddot{\psi} - \phi c\psi\dot{\psi}^2\right) u_1$$

and substitute the approximations of the angular-momentum equations-of-motion into the expression for the fourth derivative of x. Omitting terms that don't contain the second-derivative of an Euler angle, we arrive at the following expression for the fourth-derivative of x:

$$m\ddot{\ddot{x}} = ... + \left(\frac{u_{2y}}{I_{yy}}c\psi + \frac{u_{2z}}{I_{zz}}\theta(c\psi - s\psi) + \frac{u_{2x}}{I_{xx}}s\psi\right)u_1$$

Carrying out this procedure in the y- and z-directions yields similar equations. We can see that the fourth-derivative of position is indeed proportional to u_2.