Lecture 1B.Supplemental_2

Supplementary Material - Dynamical Systems

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Dynamical systems:	systems where the effects of actions do not appear immediately

In this segment, we'll formally define a dynamical system and look at mathematical models of some example dynamical systems. A dynamical system is a system in which the effects of input actions do not immediately affect the system. For example, if you turn on the thermostat in a cold room, the temperature in the room will not instantly rise to the set temperature. It will take some time for the room to actually heat up. Similarly, if you push the gas pedal in your car, it takes time for your car to speed up to the desired velocity.

State:	a collection of variables that completely characterizes the motion of a system.

Every dynamical system is defined by its state, which is a collection of variables that completely characterizes the motion of a system. The most common states, are the positions and velocities of the physical components of the system.

The states of a system are commonly denoted by the variable x. As we've seen in the lectures, we use the notation x(t) to describe the values of a system's states over time. This function x(t) is often characterized by a set of governing, ordinary, differential equations. The **order** of a system refers to the highest derivative that appears in the governing equations.

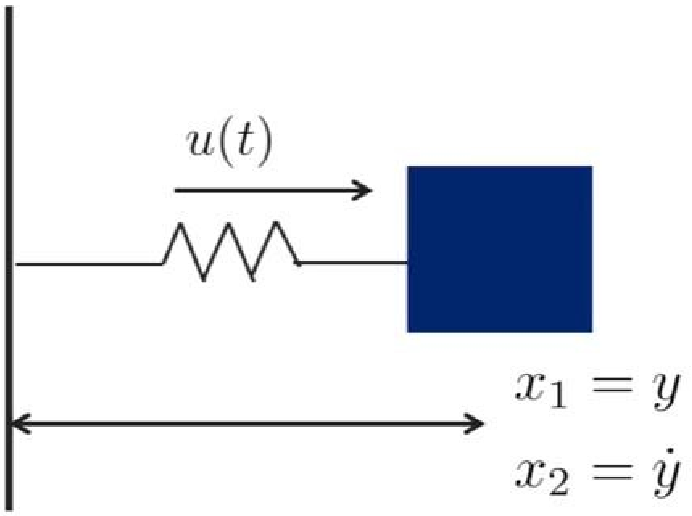
In the lecture, we analyzed the dynamical system:

$$\ddot{x} = u(t)$$

The second derivative of x is the highest derivative that appears in this equation, so this represents a **second-order system**.

Let’s consider a few more examples of dynamical systems and see how they are modelled. An example of a one-dimensional dynamical system is a mass on a string:

Example 1: Mass-Spring System



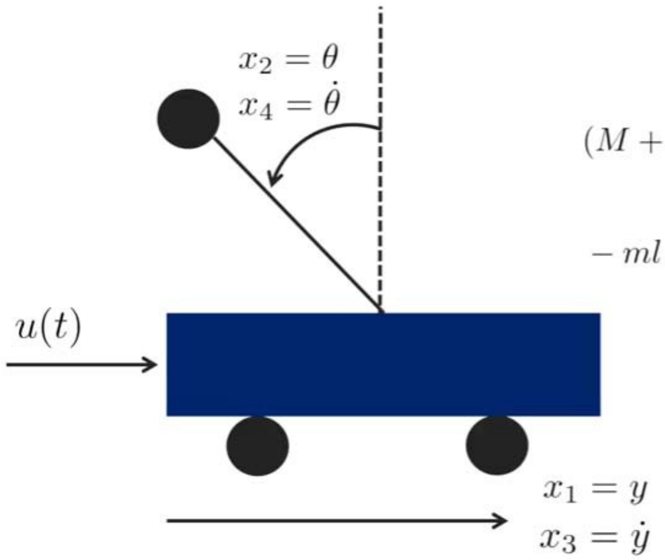
The mass can only move backwards and forwards in the y direction. The position of the mass is governed by the following ordinary differential equation:

$$m\ddot{y}(t) + ky(t) = u(t)$$

The highest derivative of y to appear in the equation is the second derivative, making this a second-order system. The state of the system is the position and velocity of the mass along the y axis. The input to this system is an additional force on the mass.

An inverted pendulum on a cart is an example of a two dimensional dynamical system:

Example 2: Pendulum on a Cart



$$\begin{aligned}(M+m)\ddot{y}(t) - ml\cos(\theta(t))\ddot{\theta}(t) \\ + ml\sin(\theta(t))\dot{\theta}(t)^2 = u(t) \\ -ml\cos(\theta(t))\ddot{y}(t) + (J+ml^2)\ddot{\theta}(t) \\ - mgl\sin(\theta(t)) = 0\end{aligned}$$

Here, the cart is allowed to drive along the y direction while the pendulum is simultaneously allowed to fall. The motion of this system can be modelled by the following set of coupled ordinary differential equations:

$$(M+m)\ddot{y}(t) - ml\cos(\theta(t)\ddot{\theta}(t) + ml\sin(\theta(t))\dot{\theta}(t)^2 = u(t)$$

$$-ml\cos(\theta(t))\ddot{y}(t) + (J+ml^2)\ddot{\theta}(t) - mgl\sin(\theta(t)) = 0$$

We see that, once again, the second derivatives of the cart position and pendulum angle are the highest-order derivatives to appear in the equations, making this a second-order system.

This system has four states. The first two are the positions of the cart and the pendulum and the last two are their velocities. The input is an additional force on the cart itself. We assume that we cannot directly apply a force to the pendulum.

Finally, consider the Quadrotor. In this weeks lecture, we only looked at the motion of the Quadrotor in the Z direction. As a result, we were able to model it as a one dimensional system. However, it turns out that to completely characterise the motion of the Quadrotor, we need to know its x-y-z position and angular orientation, as well as its linear and angular velocities.

Example 3: Quadrotor

$$x_1 = x \qquad x_7 = \dot{x}$$
$$x_2 = y \qquad x_8 = \dot{y}$$
$$x_3 = z \qquad x_9 = \dot{z}$$
$$x_4 = \phi \qquad x_{10} = p$$
$$x_5 = \theta \qquad x_{11} = q$$
$$x_6 = \psi \qquad x_{12} = r$$

We'll talk more about the dynamic equations of the quadrotor in the coming weeks.