Lecture 2A.5

Angular Velocity

Now that we're reasonably comfortable with the concept of rotations & displacements in general, it's time to start thinking about the rate of change of rotations. That leads us to the concept of an angular velocity vector. What does it mean to differentiate a rotation and get a velocity? We know the analogue for position vectors - differentiate a position vector to get a velocity. Now we want to take a 3x3 rotation matrix R(t) and differentiate that.

The first thing we should recognise is that this matrix is not just a bunch of numbers it's actually orthogonal & there are two relationships which govern the orthogonality:

$$R^T(t)R(t) = I$$
$$R(t)R^T(t) = I$$

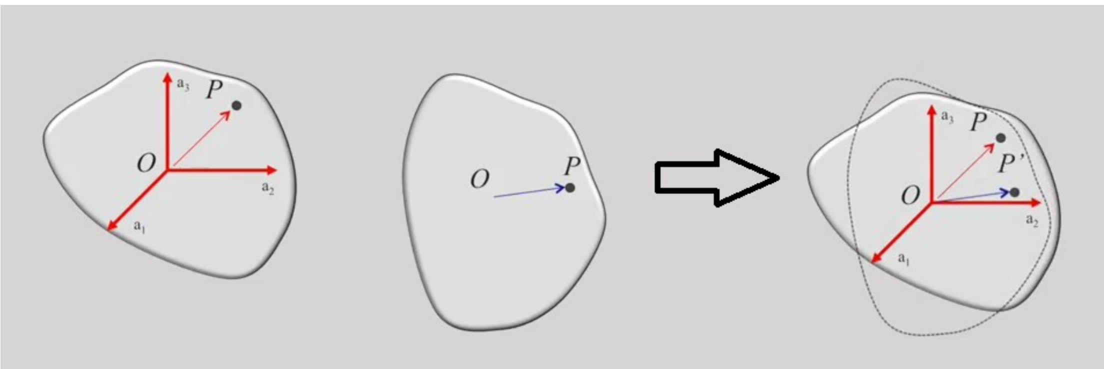
Differentiating both sides of each equation, using the product rule, gives two identities that relate the derivative of the rotation matrix and the transpose of a rotation matrix:

$$\dot{R}^T R + R^T \dot{R} = 0$$
$$R\dot{R}^T + \dot{R}R^T = 0$$

This tells us that $R^T \dot{R}$ (where $\dot{R}$ is a derivative of R) and $\dot{R}R^T$ are both skew-symmetric. In other words, there are only three independent elements in these skew-symmetric matrices.

So rather than think in terms of the derivative of a rotation matrix, we will think in terms of the derivative, pre-multiplied by R^T, or post-multiplied by R^T.

Let's return to our canonical example of a rigid-body with a generic position vector, p, and its rotated position, which we call **p'**. Once again, we'll ensure that the origins are the same, and we'll consider the displacement from P to P':

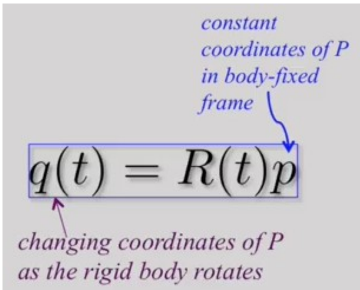


Recall that we use the position vector **p** to denote the coordinates of the point P, with respect to A_1, A_2, A_3, and the vector **q** to denote the coordinates of the point P' with respect to A_1, A_2, A_3. Remember that the vector **p** is a 3x1 vector that consists of coordinates of P in a body fixed frame.

$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$$

$$\mathbf{Q}(t)=R(t)\mathbf{p}$$

Now if the rigid-body stays rigid, then vector **p** doesn't change. The only thing that changes as the body rotates is the vector **q**. As the rigid-body rotates, the rotation matrix changes as a function of time, and the vector **q** changes as a function of time, but the vector **p** stays constant.



If we differentiate both sides of this equation (and take advantage of the fact that **p** remains constant) we obtain:

$$\dot{q} = \dot{R}p$$

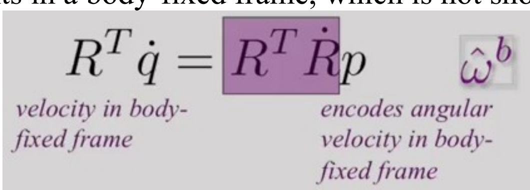
The only derivatives that appear are the derivatives of **q** and R.

As we've seen before, **p** is the position in a body fixed frame. $\dot{q}$, on the other hand, is the velocity in an *inertial* frame.

If we pre-multiply both sides by R^T we get a familiar form on the right hand side:

$$R^T \dot{q} = R^T \dot{R}p$$

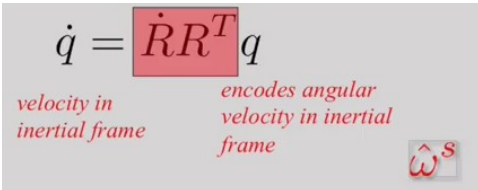
On the left-hand side, we have the velocity in the body-fixed frame, $\dot{q}$ - the velocity in the inertia frame has been transformed back to the body-fixed frame. And on the right-hand side, the familiar quantity $R^T \dot{R}$ encodes the angular velocity in a body-fixed frame. Let's call that $\hat{\omega}^b$ where the superscript b denotes that it's a body-fixed angular velocity. Again, it is the angular velocity of the rigid body, but we've chosen to write the components in a body-fixed frame, which is not shown here.



Another way of writing the equation relating the velocity in inertial frame and the position in the body-fixed frame is by rewriting **p** on the right-hand side in terms of R^T and **q**:

$$\dot{q} = \dot{R}R^T q$$

Let's take a look at the quantities on both sides of this equation. $\dot{q}$ is the velocity in the inertial frame. We've not transformed it back to the body-fixed frame. But on the right-hand side we see another familiar quantity, $\dot{R}R^T$, which is a skew-symmetric matrix. This encodes the angular velocity in the inertial frame. We call that $\hat{\omega}^s$ with the superscript s denoting the fact that it is not in the body-fixed frame but instead is a spatial angular velocity.



We've seen before that skew-symmetric matrices encode cross products. What we see in these two equations is essentially our ability to generate velocities by taking the cross product of an angular velocity vector with a position vector.

In the first equation, it's the angular velocity in the body-fixed frame which yields the velocity in the body-fixed frame. In the second equation, it's the angular velocity in the inertial frame which then yields the velocity in the inertial frame.

So we have two different representations of the same angular velocity vector. The first is written in terms of basis vectors on the body-fixed frame, and the second is written in terms of basis vectors in inertial frame.

Let's consider a simple rotation, a rotation about the z axis:

$$R = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This is easy to visualize, and it's also easy to write down. The transpose of R is given by:

$$R^T = \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Next, we'll differentiate R with the respect to Time:

$$\dot{R} = \begin{bmatrix} -\sin(\theta) & -\cos(\theta) & 0 \\ \cos(\theta) & -\sin(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \dot{\theta}$$

So $\dot{R}$ only depends on the derivative of θ, so as this angle changes R changes and we can write $\dot{R}$ as a function of $\dot{\theta}$ by simply pre-multiplying it by a matrix that has mostly zeroes except for cos(θ) and sin(θ).

If we do the same computations for this very simple matrix and derive the expressions for $R^T\dot{R}$, and $\dot{R}R^T$ we find these two matrices are the same. And this happens in this very special case where axis of rotation is constant.

$$R = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R^T\dot{R} = \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\sin(\theta) & -\cos(\theta) & 0 \\ \cos(\theta) & -\sin(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \dot{\theta}$$

$$= \dot{R}R^T = \dot{\theta} \begin{bmatrix} -\sin(\theta) & -\cos(\theta) & 0 \\ \cos(\theta) & -\sin(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{\theta} \qquad = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \dot{\theta}$$

In this particular case, this skew-symmetric matrix corresponds to the [0, 0, 1]' vector which is the z-axis. This is something we should have expected. If we rotate a rigid-body about the z-axis and the axis of rotation is constant, then clearly, the angular velocity vector will also be along the z axis.

What happens if our rotation is obtained by composing two rotations? For example, $R = R_z(\theta) R_x(\phi)$.

In this case the rotations are about the z-axis by θ, and about the x-axis by ϕ. Now if we do the computations, we differentiate R and pre-multiply by R^T to get the body-fixed angular velocity:

$$\begin{aligned} \hat{\omega}^b = R^T\dot{R} &= (R_z R_x)^T (\dot{R}_z R_x + R_z \dot{R}_x) \\ &= R_x^T R_z^T \dot{R}_z R_x + R_x^T \dot{R}_x \end{aligned}$$

We get two terms. One depends only on the rate of change of R_z and the second that depends only on the rate of change of R_x. The same thing is true if I use the spatial angular velocity which is $\dot{R}R^T$. I get two terms, one that depends on the rate of change of R_z, and the second that depends on the rate of change of R_x.

$$\begin{aligned}\hat{\omega}^s = \dot{R}R^T &= (\dot{R}_z R_x + R_z \dot{R}_x)(R_z R_x)^T \\ &= \dot{R}_z R_z^T + R_z \dot{R}_x R_x^T R_z^T\end{aligned}$$

We can see that these two expressions are different. In both case they consist of two terms. One depending on $\dot{\theta}$, and the second depending on $\dot{\phi}$, but because the axis of rotation is not fixed the two expressions are different.

The body-fixed angular velocity and the spatial angular velocity have different expressions.