Lecture 1B.Supplemental_3

Supplementary Material - Rates of Convergence

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In this segment, we'll examine the rate of convergence of the error function to 0.

Recall the control problem we looked at in the lecture. We wanted to determine the appropriate input that will cause the error between a desired state and the actual state of a dynamical system to eventually become 0. Mathematically, we want: the error function to go to 0 as t approaches infinity:

$$e(t) = x^{des}(t) - x(t) \to 0 \qquad \text{as} \quad t \to \infty$$

However, we specified a second criteria that was a bit stricter. Not only do we want the error function to eventually reach 0, we want it to reach 0 at a certain rate. More formally we say that a function exponentially converges to 0 if there exists constants alpha and beta, and a constant time t_0, such that for all t greater than t_0, the norm of the error function is always less than an exponential function defined by alpha and beta

$$\|e(t)\| \leq \alpha e^{-\beta t}$$

Note than many values of alpha, beta and t_0 may satisfy this criteria for a given error function. We only need to find one such function.

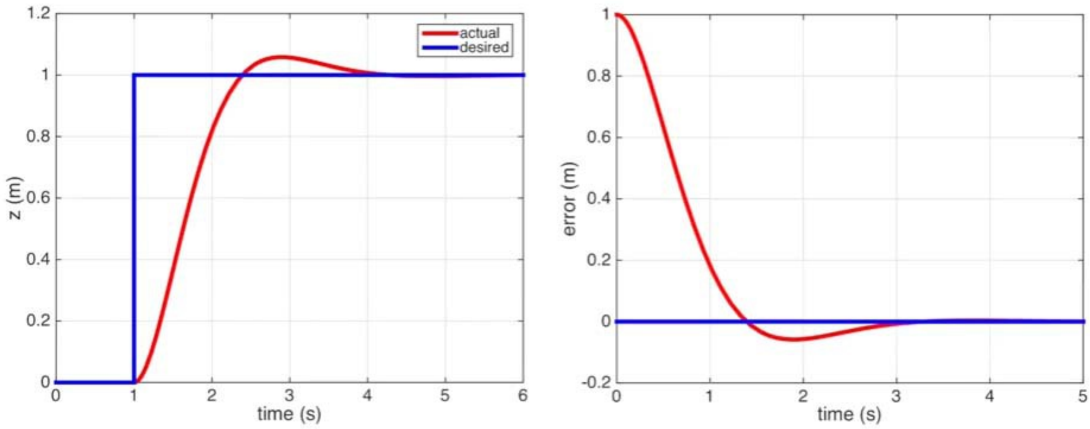
On the other hand, not all values of alpha, beta, and t_0 will satisfy this criteria. Again, we only need to find one set of values that do fulfil the criteria.

Throughout this course we will use a PD or PID controller to control a quadrotor.

$$u(t) = \ddot{x}^{des}(t) + K_v \dot{e}(t) + K_p e(t)$$

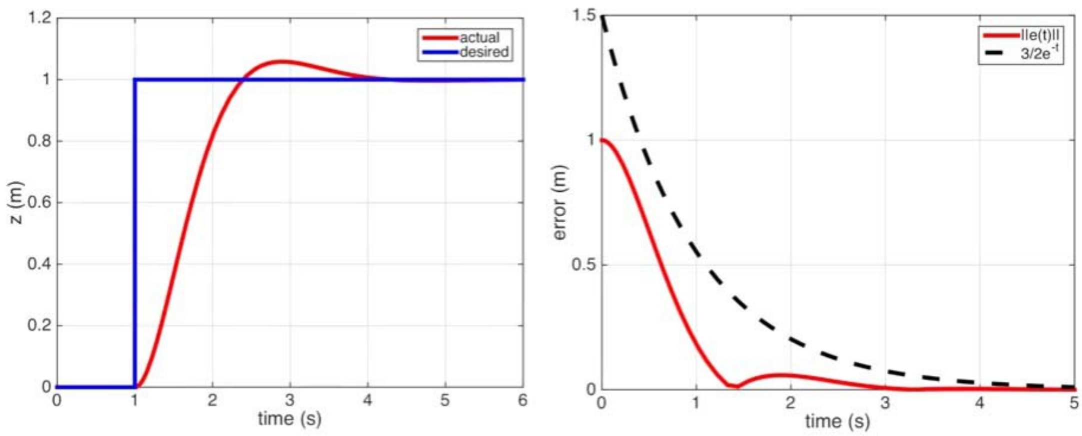
In the lecture, we discussed the behaviour of this controller under different values of K_v and K_p in the context of controlling the height of a quadrotor. For a well tuned PD controller we saw a response curve that looked like the curve on the left in the figure below:

Example 1: PD Controller



We can calculate the error from such a curve by subtracting the actual height from the desired height at each point in time. The result is the curve illustrated on the right. The curve clearly converges to 0, indicating that our system eventually reaches its desired height.

For exponential convergence, we are concerned with the magnitude of the error function and want to see how quickly it goes to 0. In the figure below, the red curve on the right plots the absolute value of the error over time:

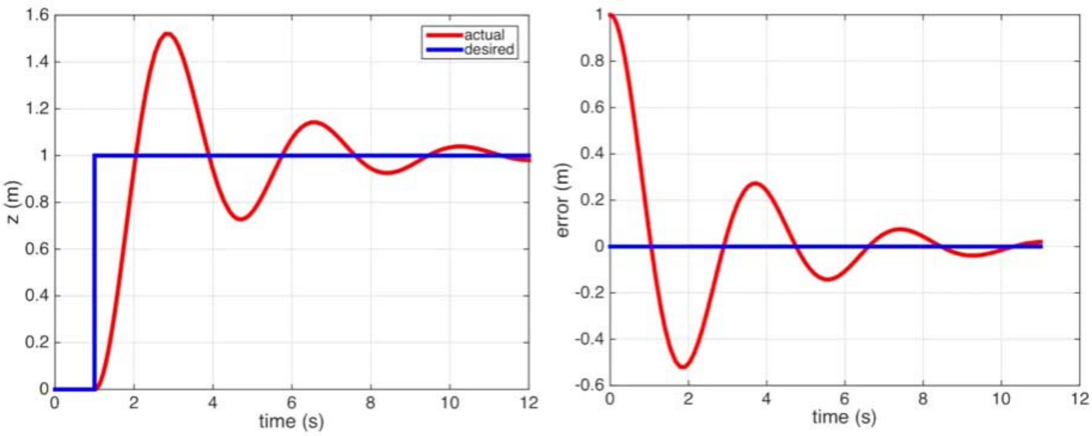


We can see that the error is bounded by an exponential function (the dashed curve). This exponential curve has the constants $t_0 = 0$, $\alpha = 3/2$, and $\beta = 1$. After time 0, the magnitude of the error remains less than the function:

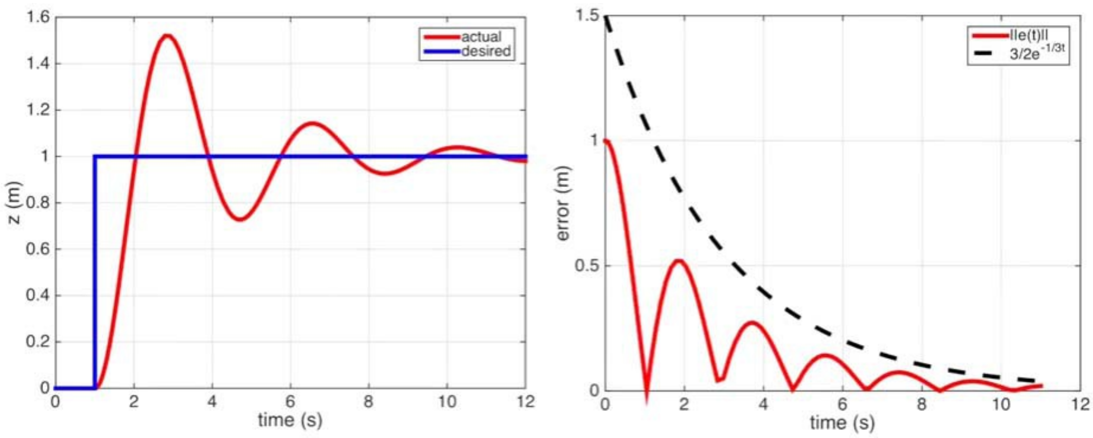
$$\frac{3}{2}e^{-t} \quad \text{for all time.}$$

Consider now the response we saw for a controller with a high proportional gain:

Example 2: High K_p



In this case, the error function oscillates significantly before approaching the desired value of 0. Again, we want to plot the absolute value of the error function:



It turns out that even though the error oscillates, we can still find an exponential curve that serves as an upper bound for the absolute value of the error function. Thus, this oscillatory error function also converges exponentially to 0.

These examples exemplify how for a PD controller, even though the error function may have different characteristics, it will always converge exponentially to 0 provided that K_p and K_v are positive.