Supplementary Material - Minimum Velocity Trajectories

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In lecture we posed the question of why is the minimum-velocity curve also the shortest-distance curve? In this segment, we'll discuss the answer to this question.

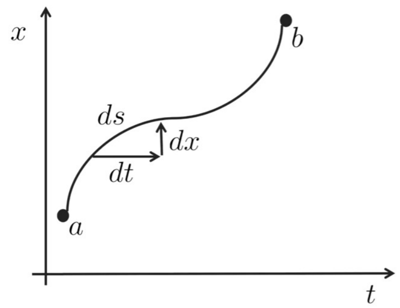
Recall that to find the minimum-velocity curve, we solved for the trajectory:

$$x^*(t) = \arg\min_{x(t)} \int_0^T \dot{x}^2 dt$$

We were able to use the Euler-Lagrange equation to find that the general form of the equation is:

$$x(t) = c_1 t + c_0$$

Now, let's find the minimum-distance trajectory. Given two points, a and b, we want to find the trajectory that is the shortest in total length. We can find the length of a trajectory between a and b by integrating infinitesimal segments ds along the curve:



Each infinitesimal segments ds has a corresponding change in dt and a change in dx. We can find the length of the segment ds using the distance function:

$$ds = \sqrt{dt^2 + dx^2}$$

We can rewrite this function by factoring-out a factor of dt from under the square root, and then making use of the fact that, by definition, $\dot{x} = \dfrac{dx}{dt}$:

$$ds = \sqrt{1 + \dot{x}^2 dt}$$

To find the total length of the curve we just integrate dx along the entire curve.

$$\text{Length of curve} = \int \mathrm{ds} = \int_0^T \sqrt{1+\dot{x}^2 dt}$$

We can now mathematically represent the problem of finding the minimum-distance trajectory in the familiar form:

$$x^*(t) = \arg\min_{x(t)} \int_0^T \sqrt{1+\dot{x}^2 dt}$$

Finding the function, x*(t) that minimizes the integral of a cost-function with respect to t. In this case, the cost function, L, is:

$$L(\dot{x}, x, t) = \sqrt{1+\dot{x}^2 dt}$$

Again, the necessary condition for the optimal trajectory is given by the Euler-Lagrange equation. To find this condition, we need to evaluate the Euler-Lagrange Equation for our cost-function, L.

We start by evaluating each term in the equation. The partial-derivative of L with respect to x is 0, because x does not appear in the cost-function:

$$\frac{\delta L}{\delta x} = 0$$

The partial derivative of L with respect to $\dot{x}$ is:

$$\frac{\delta L}{\delta \dot{x}} = \frac{\dot{x}}{\sqrt{1+\dot{x}^2}}$$

Note that this is the partial-derivative of L with respect to $\dot{x}$. We are not taking any derivatives with respect to time yet.

To use the Euler-Lagrange equation we need to find the time-derivative of $\delta L/\delta \dot{x}$. However, it turns out that we don't need to explicitly calculate this. Substituting the terms we've found into the Euler-Langrange Equation, we get the following expression for the necessary condition for the minimum-distance trajectory:

$$\frac{d}{dt}\left(\frac{\dot{x}}{\sqrt{1+\dot{x}^2}}\right) = 0$$

We can directly integrate this expression with respect to time to get an expression for the velocity of the trajectory:

$$\frac{\dot{x}}{\sqrt{1+\dot{x}^2}} = K \quad \Rightarrow \quad \dot{x} = \sqrt{\frac{K^2}{1-K^2}} = c_1$$

Here, k is an arbitrary constant. We can solve this equation for $\dot{x}$. We see that $\dot{x}$ is a function of only the constant K. Therefore, $\dot{x}$ is a constant and doesn't vary with time. We can re-label this constant as c_1. Integrating the expression $\dot{x} = c_1$ gives the position function of the minimum-distance trajectory:

$$x(t) = c_1 t + c_0$$

c_1 and c_0 are arbitrary constants. We see that this is the equation for the minimum-velocity curve. Thus, given the same set of boundary-conditions, the minimum-velocity trajectory will be the same as the minimum-distance trajectory.