Lecture 2C Supplemental_1

Supplementary Material - State-Space Form

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In the lecture, we saw the dynamical equations with a quadrotor written as matrix equations, which we call state-space form. In this segment, we'll demonstrate how to transform the ordinary differential equations representing a dynamical system into state-space form.

Recall that a dynamical system is a system where the effects of actions do not occur immediately. We have seen that the evolution of the states of these systems is governed by a set of ordinary differential equations. It's often helpful in control problems to rearrange these ordinary differential equations into ***state-space form***.

This means that we represent the differential equations in the form:

$$\dot{x} = f(x,u)$$

Where x is a matrix of states and u is a matrix of inputs. We can do this in a very systematic manner.

Suppose we have an ordinary differential equation governing a one-dimensional system whose position is represented by y.

1. First identify the order of the system n. Recall that the order is the highest derivative that appears in the differential equation.
2. We then define the states $x_1 = y(t), x_2 = \dot{y}(t), \ldots, x_n = y^{(n-1)}(t)$ where $y^{(n-1)}(t)$ is the (n-1)$_{st}$ derivative of y
3. Next, we create the state-vector $x = \begin{bmatrix} x_1 & x_2 & \ldots & x_n \end{bmatrix}^T = \begin{bmatrix} y(t) & \dot{y}(t) & \ldots & y^{(n-1)}(t) \end{bmatrix}^T$ which is a vector containing the previously defined states.
4. These states are governed by the following set of coupled first-order differential equations:

$$\frac{d}{dt}x_1 = \frac{d}{dt}y = \dot{y} = x_2$$

$$\frac{d}{dt}x_2 = \frac{d}{dt}\dot{y} = \ddot{y} = x_3$$

$$\ldots$$

$$\frac{d}{dt}x_n = \frac{d}{dt}y^{(n-1)} = g(y, \dot{y}, \ldots, y^{(n-1)}, u) = g(x_1, x_2, \ldots, x_n, u)$$

Because of the way we defined the states, the first equation simply states that the derivative of x_1 is x_2. The second equation states that the derivative of x_2 is x_3. The only non-trivial differential equation is the derivative of x_n which could be a function of all the other states plus the input u.

We get this function by rearranging the governing ordinary differential equation. Finally, we stack these first-order differential equations into a matrix:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ \dots \\ g(x_1, x_2, \dots, x_n, u) \end{bmatrix}$$

On the left hand side we have the matrix $\dot{x}$. On the right hand side we have a matrix whose components are functions of the state's x and the input u. Consider the Mass-Spring system we looked at earlier, and governed by the ordinary differential equation shown:

$$m\ddot{y}(t) + ky(t) = u(t)$$

The highest derivative that appears in this equation is the second derivative, making this a second order system. We need to define the states x_1=y and x_2=$\dot{y}$.

Next, we create the state vector x. In this case, x contains only two components, y and y dot which we have designated as x_1 and x_2:

$$x = [\mathrm{x}_1\ \mathrm{x}_2]^T = [y\ \dot{\mathrm{y}}]^T$$

We can now define the system of first-order differential equations. The first equation is trivial and simply states that the derivative of x_1 is x_2.

$$\frac{d}{dt}x_1 = \frac{d}{dt}y = \dot{y} = x_2$$

We get the second equation by solving for $\ddot{y}$ using the governing ordinary differential equation.

$$\frac{d}{dt}x_2 = \frac{d}{dt}\dot{y} = \ddot{y} = \frac{u(t) - ky(t)}{m} = \frac{u(t) - kx_1}{m}$$

We see that the derivative of x_2 is a function of x_1 and u.

We can write these two differential equations as a matrix:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ \dfrac{u(t) - kx_1}{m} \end{bmatrix}$$

We see that because k and m are constants, the system is actually linear in the states and input. As a result, we can write the equations in the following manner:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\dfrac{k}{m} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/m \end{bmatrix} u(t)$$

This equation is in the form $\dot{x}$ = Ax + Bu, which is the general form for a linear state-space equation. Again the matrix equation is linear in the states, x, and the inputs, u.

We can demonstrate how to extend this procedure to higher-order systems by using the Planar Quadrotor Model. From the lecture we know the Planar Quadrotor is governed by the following set of ordinary differential equations which are written in the terms of the variables y, z and ϕ:

$$m\ddot{y} = -\sin(\phi)u_1$$

$$m\ddot{z} = \cos(\phi)u_1 + mg$$

$$I_{xx}\ddot{\phi} = u_2$$

Here, the highest derivative appearing in any differential equation is the second derivative, so this is still a second order system.

Next, we essentially must carry out the required steps for each variable. Because n=2, we to need define states for the position and velocity of y, z, and ϕ. This gives us a system with six states:

$$x_1 = y, x_2 = z, x_3 = \phi, x_4 = \dot{y}, x_5 = \dot{z}, x_6 = \dot{\phi}$$

We place these six states into one state vector:

$$x = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{bmatrix}^T = \begin{bmatrix} y & z & \phi & \dot{y} & \dot{z} & \dot{\phi} \end{bmatrix}^T$$

We can now define the system of first-order differential equations. Again, the first three equations simply relate states to each other:

$$\frac{d}{dt}x_1 = \frac{d}{dt}y = \dot{y} = x_4$$

$$\frac{d}{dt}x_2 = \frac{d}{dt}z = \dot{z} = x_5$$

$$\frac{d}{dt}x_3 = \frac{d}{dt}\phi = \dot{\phi} = x_6$$

The last three equations come from rearranging the three governing ordinary differential equations:

$$\frac{d}{dt}x_4 = \frac{d}{dt}\dot{y} = \ddot{y} = \frac{\sin(\phi)u_1}{m} = \frac{\sin(x_3)u_1}{m}$$

$$\frac{d}{dt}x_5 = \frac{d}{dt}\dot{z} = \ddot{z} = \cos(\phi)u_1 - mg = \cos(x_3)u_1 - mg$$

$$\frac{d}{dt}x_6 = \frac{d}{dt}\dot{\phi} = \ddot{\phi} = \frac{u_2}{I_{xx}}$$

We can now place these equations into a single matrix equation:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \end{bmatrix} = \begin{bmatrix} x_4 \\ x_5 \\ x_6 \\ \dfrac{\sin(x_3)u_1}{m} \\ \cos(x_3)u_1 - mg \\ \dfrac{u_2}{I_{xx}} \end{bmatrix}$$

These equations are non-linear because of the functions sine and cosine of the state x_3.