Lecture 2C.4

Quadrotor Equations of Motion

Now let's return to the Quadrotor dynamics. These equations tell us the net force and the net moment:

$$F = F_1 + F_2 + F_3 + F_4 - mg\vec{a}_3$$

$$M = (r_1 \times F_1) + (r_2 \times F_2) + (r_3 \times F_3) + (r_4 \times F_4) + M_1 + M_2 + M_3 + M_4$$

If we combine the net force and net moment with the Newton-Euler Equations we get these two sets of equations:

$$m\ddot{r} = \begin{bmatrix} 0 \\ 0 \\ -mg \end{bmatrix} + R\begin{bmatrix} 0 \\ 0 \\ F_1 + F_2 + F_3 + F_4 \end{bmatrix}$$

$$I\begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} L(F_2 - F_4) \\ L(F_3 - F_1) \\ M_1 - M_2 + M_3 - M_4 \end{bmatrix} - \begin{bmatrix} p \\ q \\ r \end{bmatrix} \times I\begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

On the right side of the first equation, we have the total thrust which is $\mathbf{u_1}$ (this thrust vector is known in the body-fixed frame). The matrix R is rotating this thrust vector to an inertial frame.

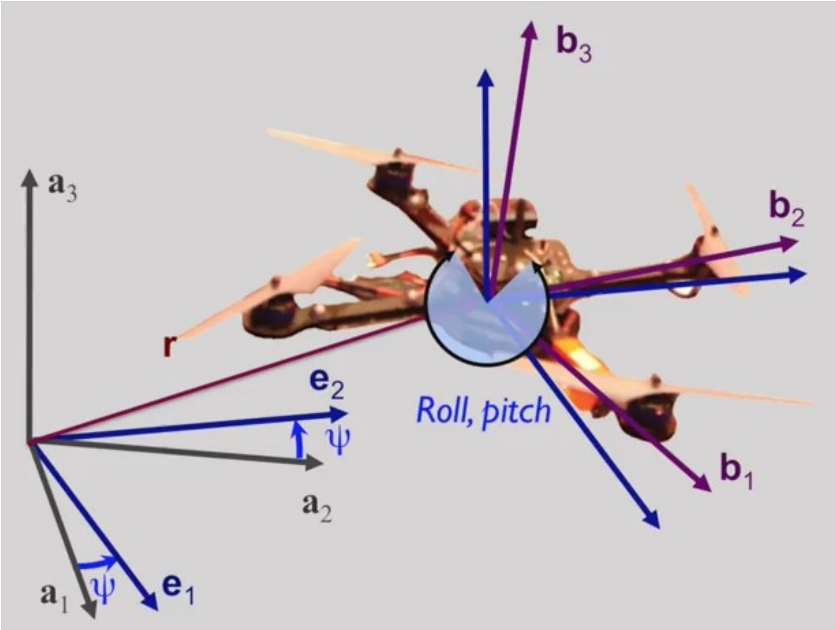
At the bottom you see the net moment, also known in the body fixed frame.

The equations as they've been written have components in the inertial frame on the top, and in the body-fixed frame at the bottom. These are the Newton-Euler equations, and these are the equations we'll use to develop controllers and planners for our vehicles.

A reasonable question to ask is: how do we actually calculate these parameters. Or in an online setting, how do we estimate these parameters.

In fact, the parameters we really need to know are those corresponding to the geometry, such as the length, L, for example, and those corresponding to the physical properties, like the mass, m and inertia, I. These parameters appear linearly in these equations, so, if we have a system that allows us to measured positions, velocities, and accelerations, it's actually not that hard to estimate the lengths, masses and inertias.

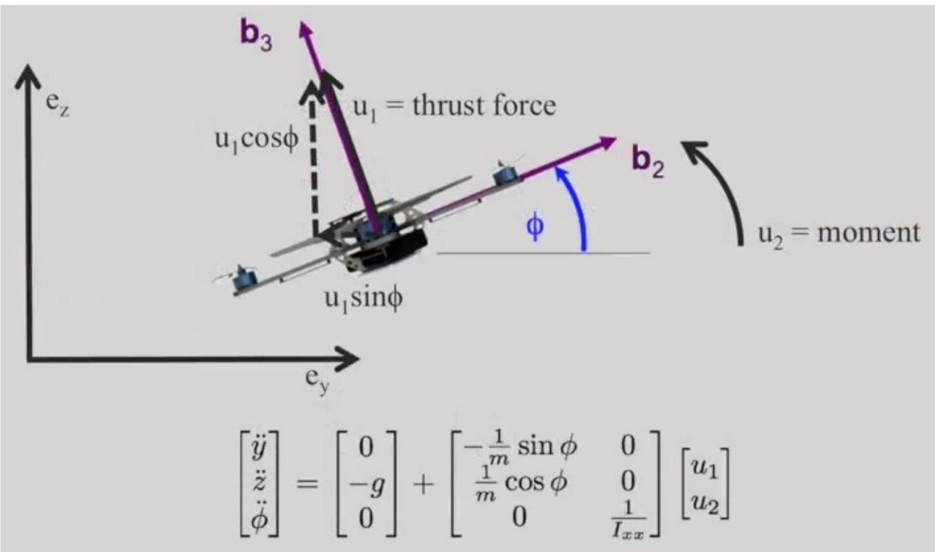
It is worth verifying that it's quite easy to calculate the angular velocity in the body-fixed frame. If you know the pitch, roll, and yaw angles, and also the rate-of-change of the pitch, roll, and yaw angles (on the right of the equation below), a simple transformation yields the angular velocity components p, q, and r along b_1, b_2, and b_3.



$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & -\cos\phi\sin\theta \\ 0 & 1 & \sin\phi \\ \sin\theta & 0 & \cos\phi\cos\theta \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\varphi} \end{bmatrix} \quad \begin{array}{l} \text{Roll} \\ \text{Pitch} \\ \text{Yaw} \end{array}$$

This model is quite complicated. It involves three components of position, velocity, and acceleration, and three components of rotations, angular-velocities, and angular -accelerations. To get a feel for the control problem, let's look fist at the Planar version of the model.

We start by looking at the equations of motion in the Y-Z plane. We will assume that the robot cannot move out of this plane or, in other words, that there is no motion in the x-direction. We will also assume that there are no yaw or pitch motions:



In this configuration, we come up with the three equations that you see here:

$$\begin{bmatrix} \ddot{y} \\ \ddot{z} \\ \ddot{\phi} \end{bmatrix} = \begin{bmatrix} 0 \\ -g \\ 0 \end{bmatrix} + \begin{bmatrix} -\frac{1}{m}\sin\phi & 0 \\ \frac{1}{m}\cos\phi & 0 \\ 0 & \frac{1}{I_{xx}} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

These equations describe the rates of change of velocity in the y and z directions, and the rate of change of the angular-velocity in the roll direction.

To describe these kinds of systems it is useful to define a state vector. In the three-dimensional case, we have six variables that describe the configuration of a robot, and a state vector that includes: the configuration and its derivative:

$$q = \begin{bmatrix} x \\ y \\ z \\ \phi \\ \theta \\ \varphi \end{bmatrix}, x = \begin{bmatrix} q \\ \dot{q} \end{bmatrix}$$

If Q is the six-dimensional configuration vector, q and $\dot{q}$ constitute a 12 dimensional description off the state. The space of all such state vectors is called a state space.

If we look at the equilibrium configuration, and that configuration is defined by the position x_0, y_0, z_0, the configuration by definition also has a zero roll angle and a zero pitch angle. It could, of course, have any yaw angle. The equilibrium configuration must also correspond to zero velocity. If we write it down in terms of a state space vector, we have the equilibrium configuration q_0 and a zero-velocity vector:

$$q_0 = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \\ 0 \\ 0 \\ \varphi \end{bmatrix}, x_e = \begin{bmatrix} q_e \\ 0 \end{bmatrix}$$

This gives a 12-dimensional vector, where the bottom six elements are all 0.

Similarly, for planar quadrotors, we have a three dimensional configuration space, a six-dimensional state-space, and equilibrium state that can be similarly defined:

Planar Quadrotor

State Vector

- q describes the configuration (position) of the system
- x describes the state of the system

$$q = \begin{bmatrix} y \\ z \\ \varphi \end{bmatrix}, \quad x = \begin{bmatrix} q \\ \hdashline \dot{q} \end{bmatrix}$$

Equilibrium at Hover

- q_e describes the equilibrium configuration of the system
- x_e describes the equilibrium state of the system

$$q_e = \begin{bmatrix} y_0 \\ z_0 \\ 0 \end{bmatrix}, \quad x_e = \begin{bmatrix} q_e \\ \hdashline 0 \end{bmatrix}$$