Lecture 7 01/10/2019

proof: $f^*(y)$ is convex, where $f^*(y) = \max_{x \in \mathbb{R}^n} y^T x - f(x)$

Let's denote $g_x(y) := y^T x - f(x)$. Note that function $g_x(y)$ is linear w.r.t y. Thus $g_x(y)$ is convex w.r.t y. By convexity of $g_x(y)$ we have

$$g_x(tz + (1-t)s) \le t g_x(z) + (1-t) g_x(s) \quad \forall x \in \mathbb{R}^n$$

Let's take the maximum

$$\max_{x \in \mathbb{R}^n} g_x(tz + (1-t)s) \le \max_{x \in \mathbb{R}^n} t g_x(z) + (1-t) g_x(s)$$

$$\le t \max_{x \in \mathbb{R}^n} g_x(z) + (1-t) \max_{x \in \mathbb{R}^n} g_x(s)$$

By definition we have that

$$f^*(y) = \max_{x \in \mathbb{R}^n} g_x(y)$$

Thus using the above inequality we get

$$f^*(tz + (1-t)s) \le t f^*(z) + (1-t) f^*(s)$$

$\Rightarrow$ $f^*(y)$ is convex.

Convex conjugate of ℓ_1-norm

$$f^*(y) = \max_x \; x^T y - \|x\|_1$$

$$= \max_x \sum_{i=1}^n x_i y_i - \sum_{i=1}^n |x_i|$$

$$= \max_x \sum_{i=1}^n x_i y_i - |x_i|$$

$$= \sum_{i=1}^n \max_{x_i} x_i y_i - |x_i|$$

using separability of the function w.r.t x.)

Let's work with $\max_{x_i} x_i y_i - |x_i| = \max_{x_i} x_i y_i - x_i \,\mathrm{sign}(x_i)$

$$= \max_{x_i} x_i (y_i - \mathrm{sign}(x_i))$$

Say $y_i > 1$ then if $x_i = +\infty$ the maximum is $+\infty$

Say $y_i < -1$ then if $x_i = -\infty$ the maximum is $+\infty$.

Say $-1 \leq \cancel{x}_i \leq 1$ then maximum is attained for $x_i = 0$ which gives maximum value equal to zero. Thus

$$\max_{x_i} x_i y_i - |x_i| = \begin{cases} 0 & \text{if } |y_i| \leq 1 \\ +\infty & \text{otherwise} \end{cases}$$

Thus

$$f^*(y) = \sum_{i=1}^{n} \max_{x_i} x_i y_i - |x_i| = \begin{cases} 0 & \text{if } \|y\|_\infty \leq 1 \\ +\infty & \text{otherwise.} \end{cases}$$