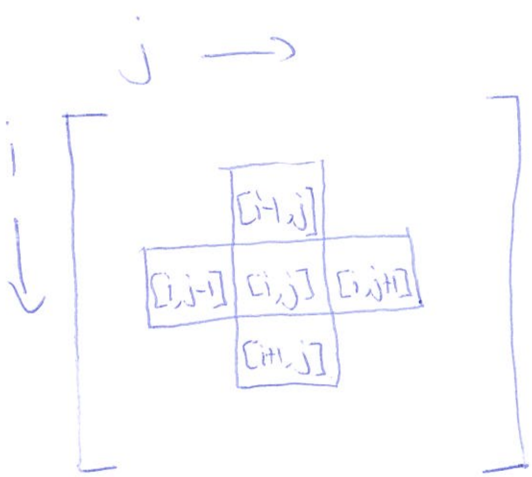
Most of the time we will use the square of the L_2 distance

$$\| z_{noisy} - z_{clean} \|_2^2$$

Horizontal forward differences



$$\partial x[i,j] = \text{pixel}[i, j+1] - \text{pixel}[i,j]$$

Horizontal forward differences operator

$$\begin{bmatrix} -1 & 1 & & & \\ & -1 & 1 & & \\ & & -1 & 1 & \\ & & & \ddots & \ddots \\ & & & & \end{bmatrix} \qquad \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$$

J n×4 matrix $\qquad$ I n×n matrix.

We define the matrix

$$\underset{n^2 \times n^2}{D_h} = J \otimes I$$

$\otimes$ denotes the Kronecker product

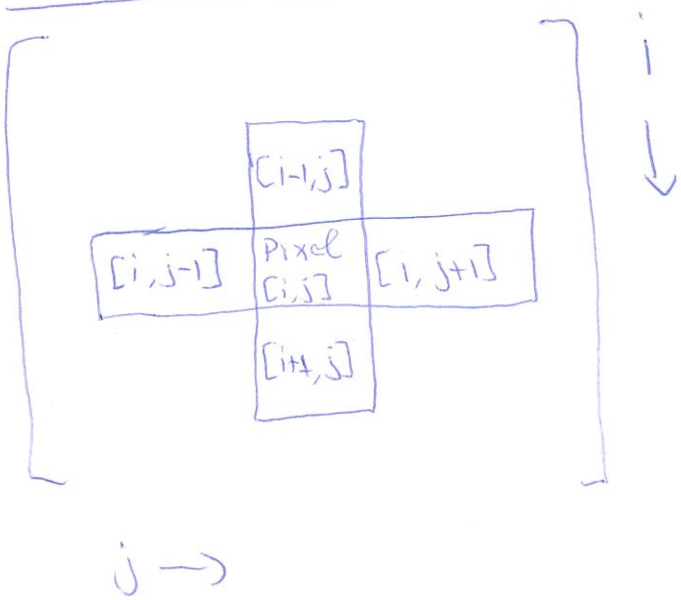
$$\begin{bmatrix} J_{11} \cdot I & , J_{12} \cdot I & \cdots & , J_{1n} \cdot I \\ J_{21} \cdot I & , J_{22} \cdot I & , \cdots & , J_{2n} \cdot I \\ \vdots & & & \\ J_{n1} \cdot I & , J_{n2} \cdot I & , \cdots & , J_{nn} \cdot I \end{bmatrix} = J \otimes I$$

For the particular matrices that we have we get that

$$J \otimes I = \begin{bmatrix} J_{11} \cdot I & , J_{12} \cdot I & & \\ & J_{22} \cdot I & , J_{23} \cdot I & \\ & & \ddots & \ddots \\ & & & \ddots \end{bmatrix}$$

$$J \otimes I = \begin{bmatrix} -I & +I & & & \\ & -I & +I & & \\ & & \ddots & \ddots & \\ & & & \ddots & \ddots \\ & & & & \end{bmatrix}$$

Vertical forward differences



$$dx[i,j] = \text{pixel}[i+1, j] - \text{pixel}[i,j]$$

Vertical forward differences operator

$$D_v = I \otimes J$$

$n^2 \times n^2$ matrix

We define the matrix

$$D_k = I \otimes J$$

$n^2 \times n^2$ matrix

$\otimes$ denotes the Kronecker product.

$$I \otimes J = \begin{bmatrix} I_{11} \cdot J, & I_{12} J, & \cdots, & I_{1n} \cdot J \\ I_{21} \cdot J, & I_{22} \cdot J, & \cdots, & I_{2n} \cdot J \\ \vdots & & & \\ I_{n1} \cdot J, & I_{n2} \cdot J, & \cdots, & I_{nn} \cdot J \end{bmatrix}$$

For the particular matrices that we have we get that

$$I \otimes J = \begin{bmatrix} I_{11} J, & 0 \cdot J, & \cdots, & 0 J \\ 0 \cdot J, & I_{22} \cdot J, & \cdots, & 0 \cdot J \\ \vdots & & & \\ 0 \cdot J, & \cdots & \cdots, & I_{nn} \cdot J \end{bmatrix}$$

$$= \begin{bmatrix} J & & & \\ & J & & \\ & & \ddots & \\ & & & J \end{bmatrix}$$

Measuring horizontal and forward differences

Let z be the vectorized version of a given image. Then

$D_h \cdot X$ measures the horizontal differences.

$D_v \cdot X$ measures the vertical differences.

matrix vector product.

Example

Say $n = 3$, then

$$D_h = \begin{array}{cccccc} 1 & -1 & & & & \\ & 1 & -1 & & & \\ & & 1 & -1 & & \\ & & & 1 & -1 & \\ & & & & 1 & - \end{array}$$

9x9 matrix

Example

Say $n = 2$ then

$$D_n = \begin{bmatrix} -1 & 0 & +1 & 0 \\ 0 & -1 & 0 & +1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -I & +I \\ 0 & -I \end{bmatrix}$$

4x4 matrix

where I is 2×2 identity matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$Z = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \overset{\text{vectorize}}{\rightsquigarrow} z = \begin{bmatrix} z_{11} \\ z_{21} \\ z_{12} \\ z_{22} \end{bmatrix}$$

$$D_n \cdot Z = \begin{bmatrix} -1 & 0 & +1 & 0 \\ 0 & -1 & 0 & +1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} z_{11} \\ z_{21} \\ z_{12} \\ z_{22} \end{bmatrix}$$

$$= \begin{bmatrix} -z_{11} + z_{12} \\ -z_{21} + z_{22} \\ -z_{12} \\ -z_{22} \end{bmatrix} \Big\} \text{ we assume the boundaries are zero.}$$

we ~~consider~~ assume the boundaries are zero.

$$D_v = \begin{bmatrix} -1 & +1 & 0 & 0 \\ 0 & -1 & & 0 \\ 0 & 0 & -1 & +1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$D_v \cdot Z = \begin{bmatrix} -1 & +1 & 0 & 0 \\ 0 & -1 & & 0 \\ 0 & 0 & -1 & +1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} Z_{11} \\ Z_{21} \\ Z_{12} \\ Z_{22} \end{bmatrix}$$

$$= \begin{bmatrix} -Z_{11} + Z_{21} \\ -Z_{21} \\ -Z_{12} + Z_{22} \\ -Z_{22} \end{bmatrix}$$

boundaries are assumed to be zero.

Complex differences operator

$$D = D_h + i \cdot D_v$$

D is a complex matrix.