Assigment 2

The assignment is divided into programming and mathematical questions. Both of them are given in this notebook.

Programming questions: I am giving you a template that you can use to write your code. Description of the questions is integrated in the comments.

Upload your code on Learn dropbox and submit pdfs of the code and answers to the mathematical questions on Crowdmark.

--

Load modules

In [1]:
```
# !pip install numpy, scipy, scikit-image, skimage, matplotlib

import matplotlib.pyplot as plt

from skimage.color import rgb2gray
from skimage import data
from skimage.transform import resize

# Numpy is useful for handling arrays and matrices.
import numpy as np
```

Load image

In [2]:
```
img = data.astronaut()
img = rgb2gray(img)*255 # convert to gray and change scale from (0,1)

n = img.shape[0]

plt.figure(1, figsize=(10, 10))
plt.imshow(img, cmap='gray', vmin=0, vmax=255)
plt.show()
```

Compute the differences operators here. Use your code from Assignment 1.

In [3]:
```
# You will need these three methods to construct sparse differences op
# If you do not use sparse operators you might have scalability proble
from scipy.sparse import diags
from scipy.sparse import kron
from scipy.sparse import identity
m = img.shape[0] # ROWS
n = img.shape[1] # COLS


I = diags([1], [0],shape=(n,n),dtype='int8')
J = diags([-1,1],[0,1],shape=(m,m),dtype='int8')
Dh = kron(J, I)
Dv = kron(I, J)

# Use your code from Assignment 1.
# Make sure that you compute the right D_h and D_v matrices.
```

Add noise to the image

In [4]:
```
mean_ = 0
standard_deviation = 30
dimensions = (n,n)

noise = np.random.normal(mean_,standard_deviation,dimensions)

noisy_image = img + noise

plt.figure(1, figsize=(10, 10))
plt.imshow(noisy_image, cmap='gray', vmin=0, vmax=255)
plt.show()
```

Question 1: implement gradient descent using the Lipschitz constant as the step-size for the denoising problem. Use eigsh method from scipy.sparse.linalg to compute the Lipschitz constant. Marks: 10

In [5]:
```
from scipy.sparse import csr_matrix # sparse matrix
from scipy.sparse.linalg import norm # compute the norm of a sparse mat
from scipy import real
from scipy.sparse.linalg import spsolve
from scipy.sparse.linalg import eigsh
from numpy.linalg import norm
import math
import time

# find lipschitz constant
# !!! we comment this because computing L is time-consuming.
# I_mn_mn = identity(m*n)
# A = lambda_*(Dh.transpose().dot(Dh) + Dv.transpose().dot(Dv)) + I_mn_
# eigv = eigsh(A.transpose().dot(A), 1, which='LM', return_eigenvectors
# L = math.sqrt(eigv)
# eigv =  1088.98015996, L = 32.999699391964164
L = 32.999699391964164
def gradient_descent(x0, epsilon, lambda_, max_iterations):

# x0: is the initial guess for the x variables
# epsilon: is the termination tolerance parameter
# lambda_: is the regularization parameter of the denoising problem.
# max_iterations: is the maximum number of iterations that you allow th

# Write your code here.
    counter = 0
    x = x0
    xs = list()     # xs is list of the x calcualted in each iteration
    xs.append(x)
    D = Dh + 1J * Dv
    # D_conjugate_transpose = D.conjugate().transpose()
    # lecture 5 Slide 31
    Dct = D.conjugate().transpose()
    gradient_f_x = lambda_ * real(Dct.dot(D.dot(x))) + x - x0  # x0 is
    start = time.time()
    while counter < max_iterations and norm(gradient_f_x,2) > epsilon:
        x = x - (1/L)*gradient_f_x              # lecture 5 slide 16, c
        xs.append(x.flatten('F'))
        gradient_f_x = lambda_ * real(Dct.dot(D.dot(x))) + x - x0   # u
        counter += 1                                                 # u
    end = time.time()
    print('Iterations: ', counter)
    print('running time: %s seconds'% str(end-start)[0:4])
    return xs
```

Call Gradient Descent

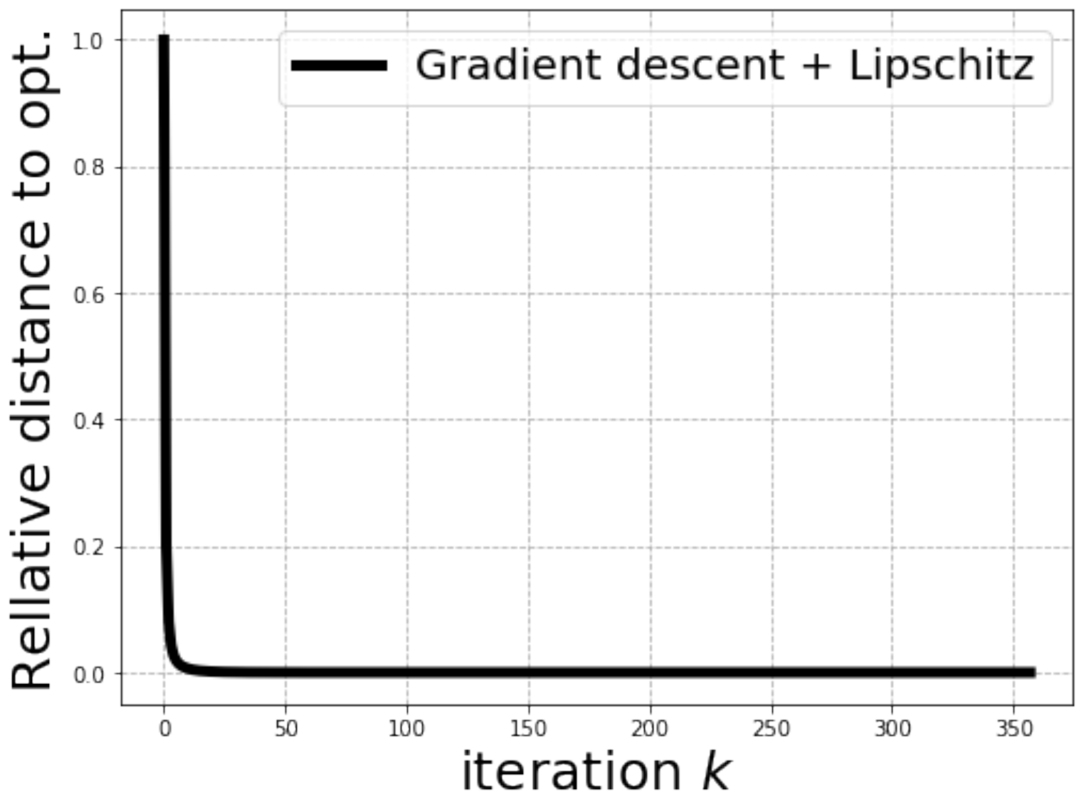
In [6]:
```
# Initialize parameters of gradient descent.
lambda_ = 4
epsilon = 1.0e-2
max_iterations = 2000

# Set x0 equal to the vectorized noisy image.
# Write your code here.
x0 = noisy_image.flatten('F')
list_of_x = gradient_descent(x0, epsilon, lambda_, max_iterations)
```

```
Iterations:  357
running time: 3.59 seconds
```

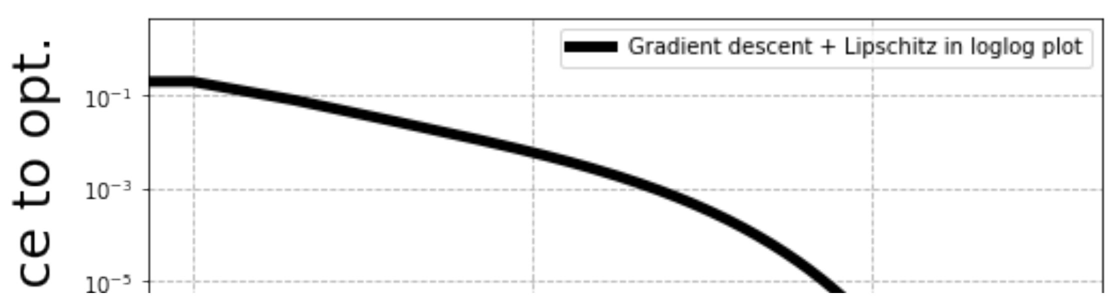
Plot $$(f(x_k) - f(x^*))/(f(x_0) - f(x^*))$$ vs the iteration counter k, where $$x^*$$ is the minimizer of the denoising problem, which you can compute by using spsolve, similarly to Assignment 1.

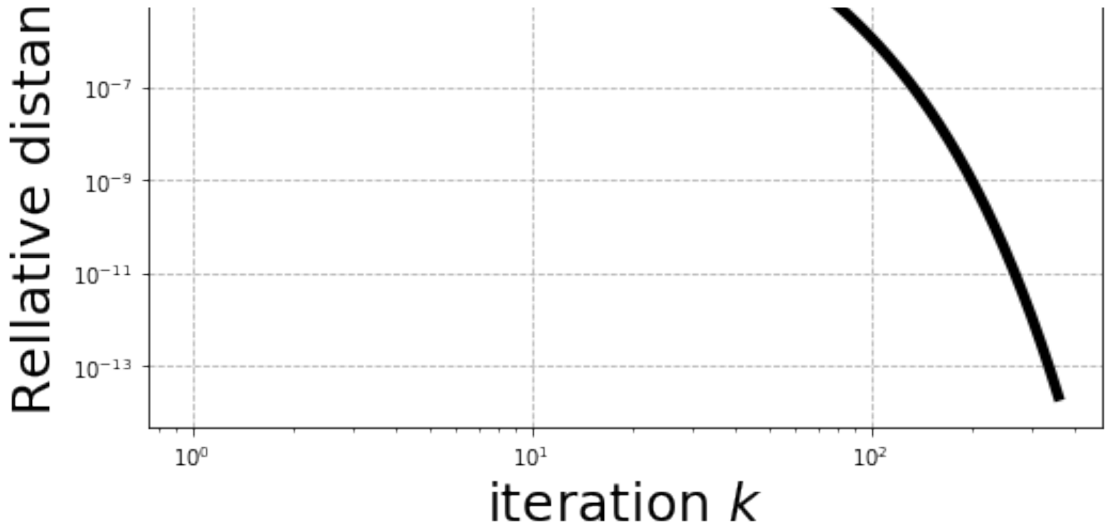
In [7]:
```
# Plot the rellative objective function vs number of iterations.

# Write your code here.

# Here is an example code
# helper function

def denoising(x):
    return lambda_/2 * math.pow(norm(D.dot(x), 2),2) + 1/2 * math.pow(

store_data_for_plotting = list()
D = Dh + 1J * Dv
A = lambda_*real(Dh.transpose().dot(Dh) + Dv.transpose().dot(Dv)) + ic
x_minimizer = spsolve(A,x0)
f_minimizer = denoising(x_minimizer)
f_x0 = denoising(x0)
denominator = f_x0-f_minimizer
for x in list_of_x:
    numerator = denoising(x.flatten('F')) - f_minimizer
    store_data_for_plotting.append(numerator / denominator)

fig = plt.figure(figsize=(8, 6))
plt.plot(store_data_for_plotting, label=("Gradient descent + Lipschitz
```

```
plt.legend(prop={'size': 20},loc="upper right")
plt.xlabel("iteration $k$", fontsize=25)
plt.ylabel("Rellative distance to opt.", fontsize=25)
plt.grid(linestyle='dashed')
plt.show()
plt.savefig('1.jpg')

fig = plt.figure(figsize=(8, 6))
plt.loglog(store_data_for_plotting,label=("Gradient descent + Lipschit
plt.legend(prop={'size': 10},loc="upper right")
plt.xlabel("iteration $k$", fontsize=25)
plt.ylabel("Rellative distance to opt.", fontsize=25)
plt.grid(linestyle='dashed')
plt.show()
plt.savefig('2.jpg')
```




```
<Figure size 432x288 with 0 Axes>
```



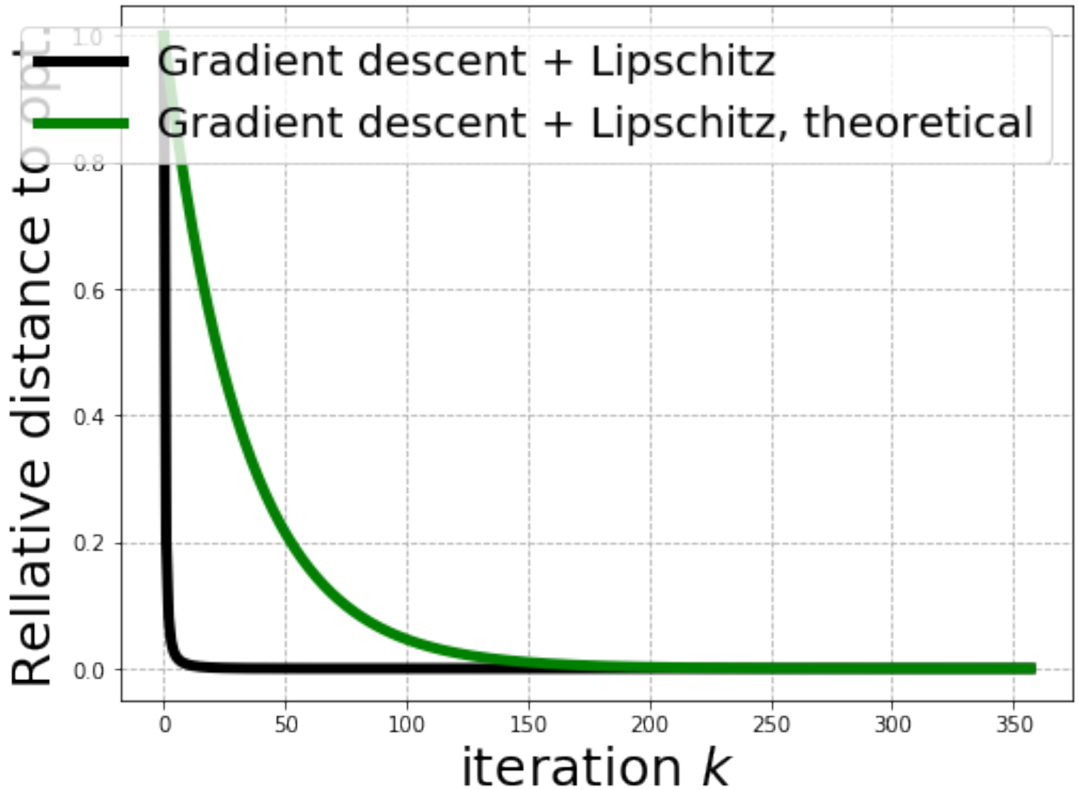


<Figure size 432x288 with 0 Axes>

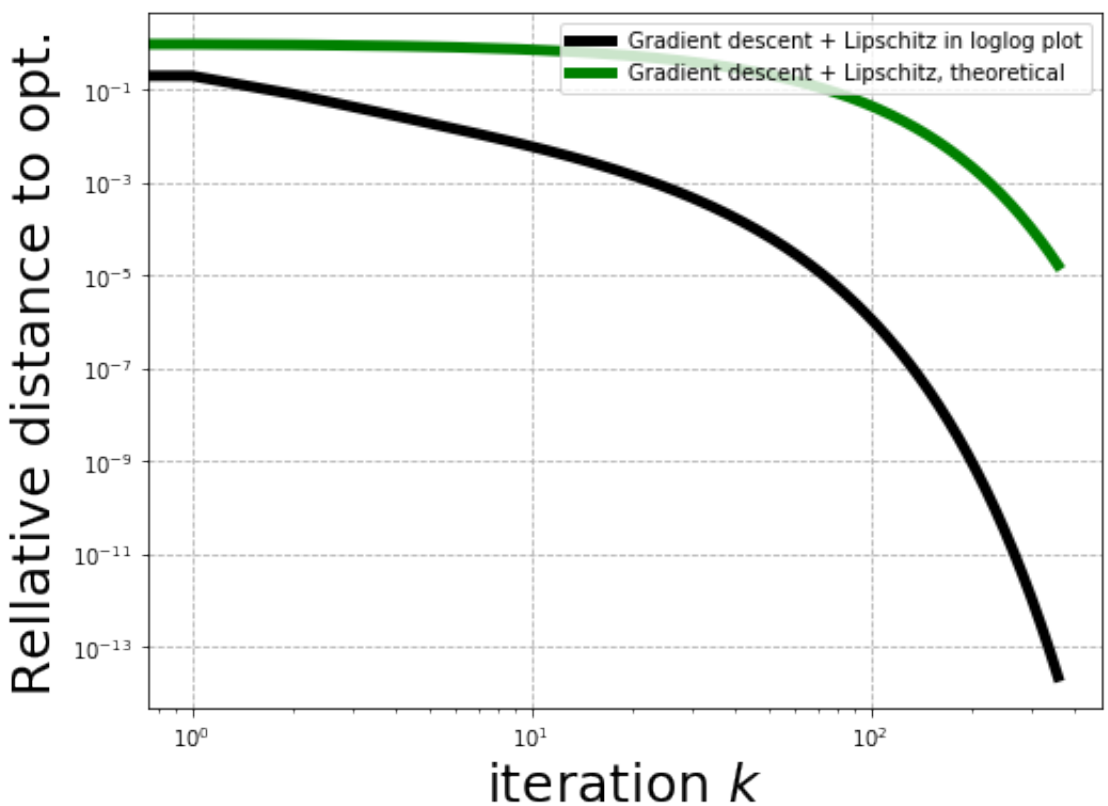
In [8]:
```
## Question 2: is there a "gap" between the practical convergence rate
```

In [9]:
```
# the theoretical convergence rate is O(log(1/ε))
# There is a gap between theoretical and practical convergence rate. A
# rate is larger than the theoretical one. The convexity might be a re
# http://www.mit.edu/~rakhlin/6.883/lectures/recitation_1.pdf
# According to the lemma 4(b) in page 3, provides intuition as to why

# generate theoretical data for plotting
store_data_for_plotting_q2 = [(1- 1/L)**x for x in range(len(store_dat

fig = plt.figure(figsize=(8, 6))
plt.plot(store_data_for_plotting, label=("Gradient descent + Lipschitz

plt.plot(store_data_for_plotting_q2, label=("Gradient descent + Lipsch

plt.legend(prop={'size': 20},loc="upper right")
plt.xlabel("iteration $k$", fontsize=25)
plt.ylabel("Rellative distance to opt.", fontsize=25)
plt.grid(linestyle='dashed')
plt.show()
plt.savefig('3.jpg')

fig = plt.figure(figsize=(8, 6))

plt.loglog(store_data_for_plotting,label=("Gradient descent + Lipschit
plt.loglog(store_data_for_plotting_q2,label=("Gradient descent + Lipsc
plt.legend(prop={'size': 10},loc="upper right")
plt.xlabel("iteration $k$", fontsize=25)
plt.ylabel("Rellative distance to opt.", fontsize=25)
plt.grid(linestyle='dashed')
plt.show()
```

```
plt.savefig('4.jpg')
```



<Figure size 432x288 with 0 Axes>



```
<Figure size 432x288 with 0 Axes>
```

Question 3: implement gradient descent with line-search for the denoising problem. Marks: 15

In [10]:

```
D = Dh + 1j * Dv
Dct = D.conjugate().transpose()
# Write a line-search function here.
# I am giving you a hint about what the input could be, but feel free
# def line_search(x,D,Dx,vec_image,grad_reg,grad_fit,lambda_):

# D: represents the complex forward differences matrix from the Lectu
# Dx: represents the matrix-vector product of D times x.
# vec_image: is the vectorized noisy image
# grad_reg: is the gradient of the regularization term ||Dx||_2^2
# grad_fit: is the gradient of the least-squares term ||x-vec_image||_
# lambda_: is the regularization parameter of the denoising problem.

# Write your code here.
def line_search(x, gradient_f_x):
    alpha_=1
    diff = x - alpha_ * gradient_f_x
    LHS = denoising(diff)
    RHS = denoising(x)
    while LHS >= RHS:
        alpha_ = alpha_ / 2
        diff = x - alpha_ * gradient_f_x
        LHS = denoising(diff)
#         RHS does not change
#         RHS = denoising(x)    #!!! this step significantly changes t
    return alpha_

# Write gradient descent + line-search here.
# I am giving you a hint about what the input could be, but feel free
def gradient_descent_ls(x0, epsilon, lambda_, max_iterations):

# x0: is the initial guess for the x variables
# epsilon: is the termination tolerance parameter
# lambda_: is the regularization parameter of the denoising problem.
# max_iterations: is the maximum number of iterations that you allow t

# Write your code here.
    counter = 0
    x = x0
    xs = [x]
    gradient_f_x = lambda_ * real(Dct.dot(D.dot(x))) + x - x0
    start = time.time()
```

```
    while counter < max_iterations and norm(gradient_f_x,2) > epsilon:
        alpha_ = line_search(x, gradient_f_x)
        x = x - alpha_ * gradient_f_x
        gradient_f_x = lambda_ * real(Dct.dot(D.dot(x))) + x - x0
        xs.append(x)
        counter += 1
    end = time.time()
    print('Iterations: ', counter)
    print('running time: %s seconds'% str(end-start)[0:4])
    return xs
```

Call Gradient Descent with line-search

In [11]:

```
# Initialize parameters of gradient descent
lambda_ = 4
epsilon = 1.0e-2
max_iterations = 2000

# Set x0 equal to the vectorized noisy image.
# Write your code here.
x0 = noisy_image.flatten('F')
list_of_x_q3 = gradient_descent_ls(x0, epsilon, lambda_, max_iteratior
```

```
Iterations:  92
running time: 6.05 seconds
```

Plot

$$(f(x_k) - f(x^*))/(f(x_0) - f(x^*))$$

vs the iteration counter k, where

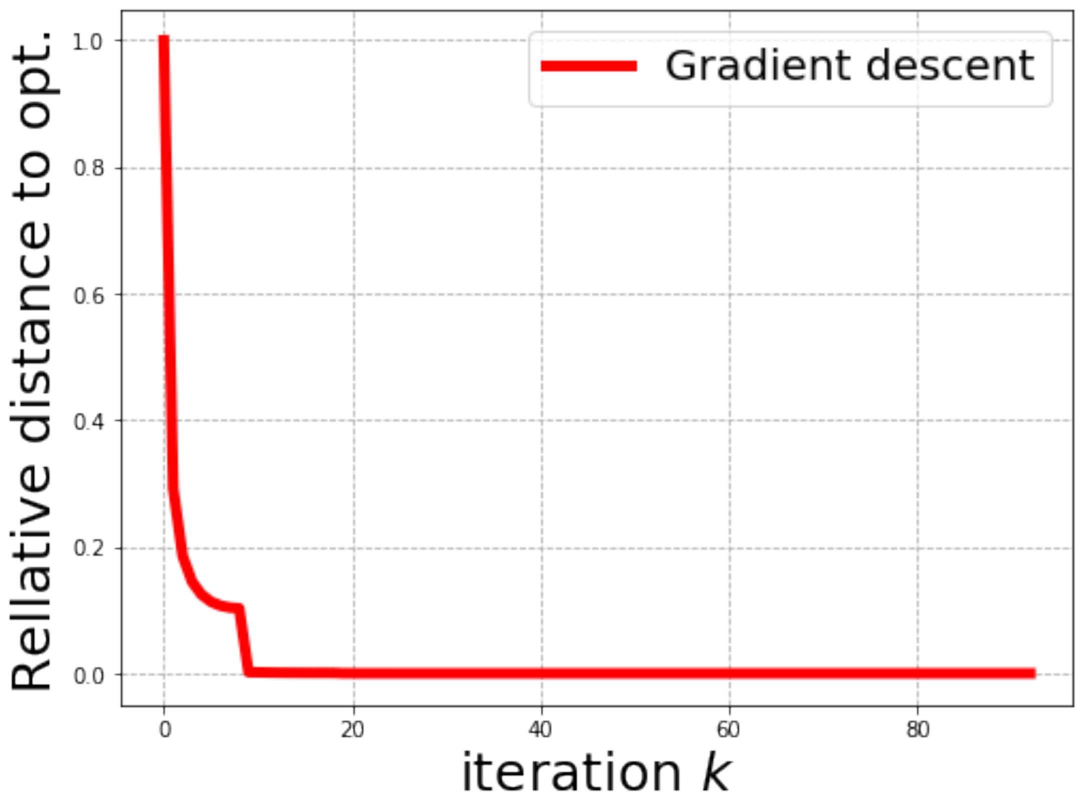
$$x^*$$

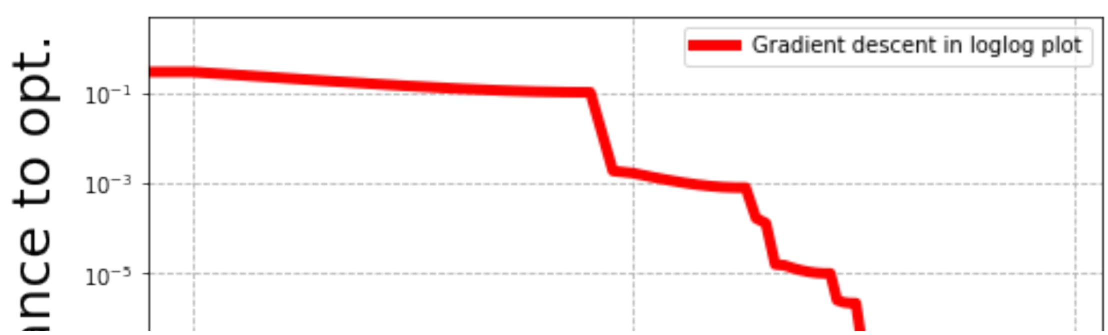
is the minimizer of the denoising problem, which you can compute by using spsolve, similarly to Assignment 1.

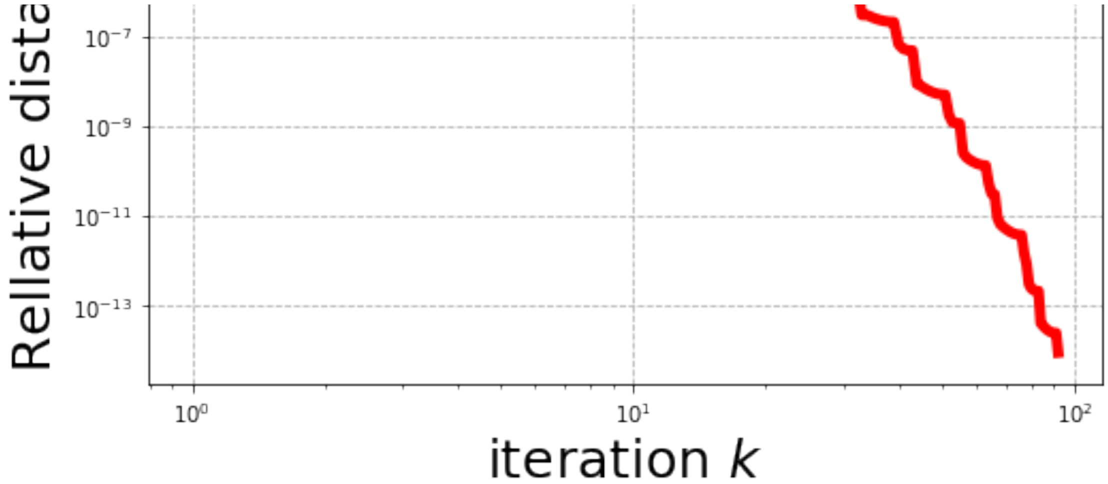
In [12]:

```
# Plot the rellative objective function vs number of iterations.

# Write your code here.

# Here is an example code
store_data_for_plotting_q3 = []
denominator = f_x0-f_minimizer
for x in list_of_x_q3:
    numerator = denoising(x.flatten('F')) - f_minimizer
    store_data_for_plotting_q3.append(numerator / denominator)

# Here is an example code
```

```
fig = plt.figure(figsize=(8, 6))
plt.plot(store_data_for_plotting_q3, label=("Gradient descent"), linew

plt.legend(prop={'size': 20},loc="upper right")
plt.xlabel("iteration $k$", fontsize=25)
plt.ylabel("Rellative distance to opt.", fontsize=25)
plt.grid(linestyle='dashed')
plt.savefig('5.jpg')

fig = plt.figure(figsize=(8, 6))
plt.loglog(store_data_for_plotting_q3,label=("Gradient descent in logl
plt.legend(prop={'size': 10},loc="upper right")
plt.xlabel("iteration $k$", fontsize=25)
plt.ylabel("Rellative distance to opt.", fontsize=25)
plt.grid(linestyle='dashed')
plt.show()
plt.savefig('6.jpg')
```








```
&lt;Figure size 432x288 with 0 Axes&gt;
```

Question 4: What is the advantage of using line-search to compute the step-size at each iteration instead of using constant step-sizes equal to 1/L? Where L is the Lipschitz constant. Is gradient descent with line-search faster than gradient descent with constant step-sizes in terms of running time? Is gradient descent with line-search faster than gradient descent with constant step-sizes in terms of running time when you add computation of the Lipschitz constant in the running time? Is gradient descent with line-search faster than gradient descent with constant step-sizes in terms of number of required iterations? Marks: 10

In [13]:

However, in practice knowing L is not necessary. (Lecture04.pdf Slide
lculating L is time consuming as it requires solving the eigenvalues o
ing line-search to compute the step-size saves the time as the worst c

Yes. Gradient descent with line-search faster than gradient descent wi

Yes. Gradient descent with line-search faster than gradient descent wi

Gradient descent satisfies the termination condition after 357 iterat

Questions 5: implement gradient descent with Armijo line-search for the denoising problem. Marks: 10

In [14]:
```
# Create a line-search function
def line_search_Armijo(x, grad, gamma_):

# D: represents the complex forward differences matrix from the Lectur
# Dx: represents the matrix-vector product of D times x.
# vec_image: is the vectorized noisy image
# grad_reg: is the gradient of the regularization term ||Dx||_2^2
# grad_fit: is the gradient of the least-squares term ||x-vec_image||_
# lambda_: is the regularization parameter of the denoising problem.
# gamma: parameter of Armijo line-search as was defined in the lecture
# second_grad_22: second_order_gradient_of fx

# Write your code here.

    alpha_ = 1
    diff = x - alpha_ * grad
    # store the value for reuse
#     deno_x = denoising(x)
    LHS = denoising(diff)
    RHS = denoising(x) - alpha_ * gamma_ * norm(grad) ** 2
    while LHS > RHS:
        alpha_ /= 2
        diff = x - alpha_ * grad
        LHS = denoising(diff)
        RHS = denoising(x) - alpha_ * gamma_ * norm(grad) ** 2
    return alpha_


def gradient_descent_Armijo(x0, epsilon, lambda_, max_iterations, gamm

# x0: is the initial guess for the x variables
# epsilon: is the termination tolerance parameter
# lambda_: is the regularization parameter of the denoising problem.
# max_iterations: is the maximum number of iterations that you allow t
# gamma: parameter of Armijo line-search as was defined in the lecture

# Write your code here.

#   second_grad_22 = lambda_*(Dh.transpose().dot(Dh) + Dv.transpose().
    counter = 0
    D = Dh + 1j * Dv
    Dct = D.conjugate().transpose()
    # list
    x = x0
    xs = [x]
```

```
    gradient_f_x = lambda_ * real(Dct.dot(D.dot(x))) + x - x0
#   grad_22 = math.pow(norm(gradient_f_x,2),2)
    start = time.time()
    while norm(gradient_f_x, 2) > epsilon and counter < max_iterations
#       x = line_search_als(x, gradient_f_x, gamma_)
        alpha_ = line_search_Armijo(x, gradient_f_x, gamma_)
        x = x - alpha_ * gradient_f_x
        gradient_f_x = lambda_ * real(Dct.dot(D.dot(x))) + x - x0
        xs.append(x)
        counter += 1
    end = time.time()
    print('Iterations: ', counter)
    print('running time: %s seconds'% str(end-start)[0:4])
    return xs
```

Call Gradient Descent with Armijo line search

In [15]:

```
# # Initialize parameters of gradient descent
# lambda_ = 4
# epsilon = 1.0e-2
# max_iterations = 2000
# gamma_ = 0.4

# # Set x0 equal to the vectorized noisy image.
# # Write your code here.

# x0 = noisy_image.flatten('F')
# list_of_x_q5 = gradient_descent_ar(x0, epsilon, lambda_, max_iterati
# Initialize parameters of gradient descent
lambda_ = 4
epsilon = 1.0e-2
max_iterations = 2000
gamma_ = 0.3   # terminates in 82 iteration
gamma_ = 0.4   # terminates in 98 iteration
gamma_ = 0.5   # terminates in 93 iteration
# Set x0 equal to the vectorized noisy image.

# Write your code here.
list_of_x_q5 = gradient_descent_Armijo(x0, epsilon, lambda_, max_itera
```

```
Iterations:  93
running time: 8.58 seconds
```

Plot

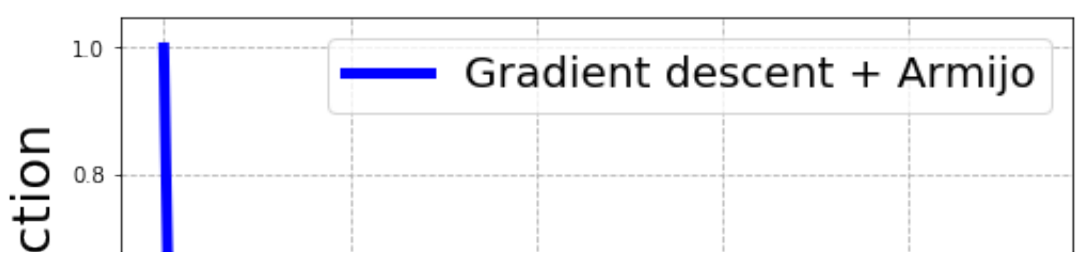
$$(f(x_k) - f(x^*))/(f(x_0) - f(x^*))$$

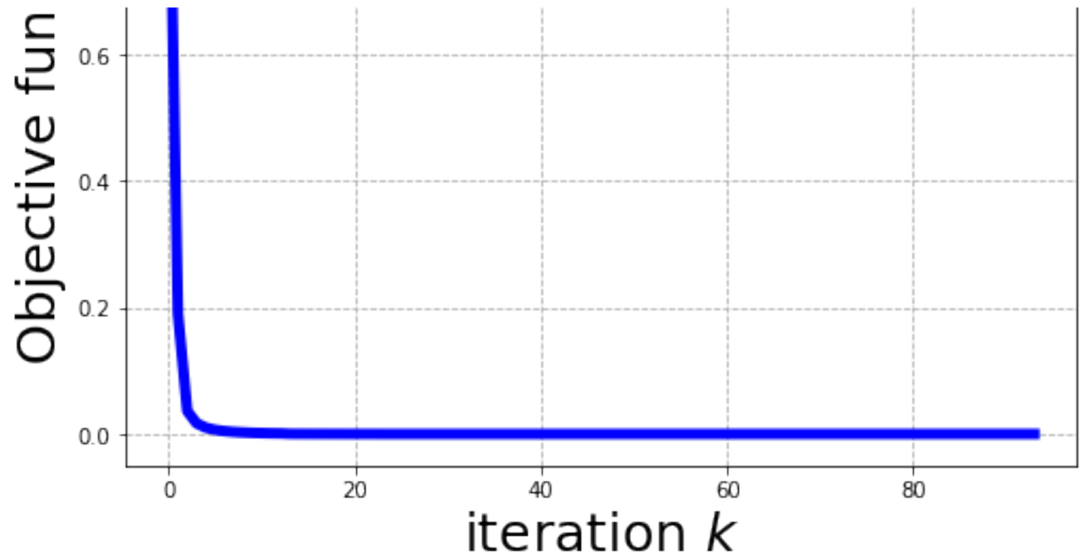
vs the iteration counter k, where

$$x^*$$

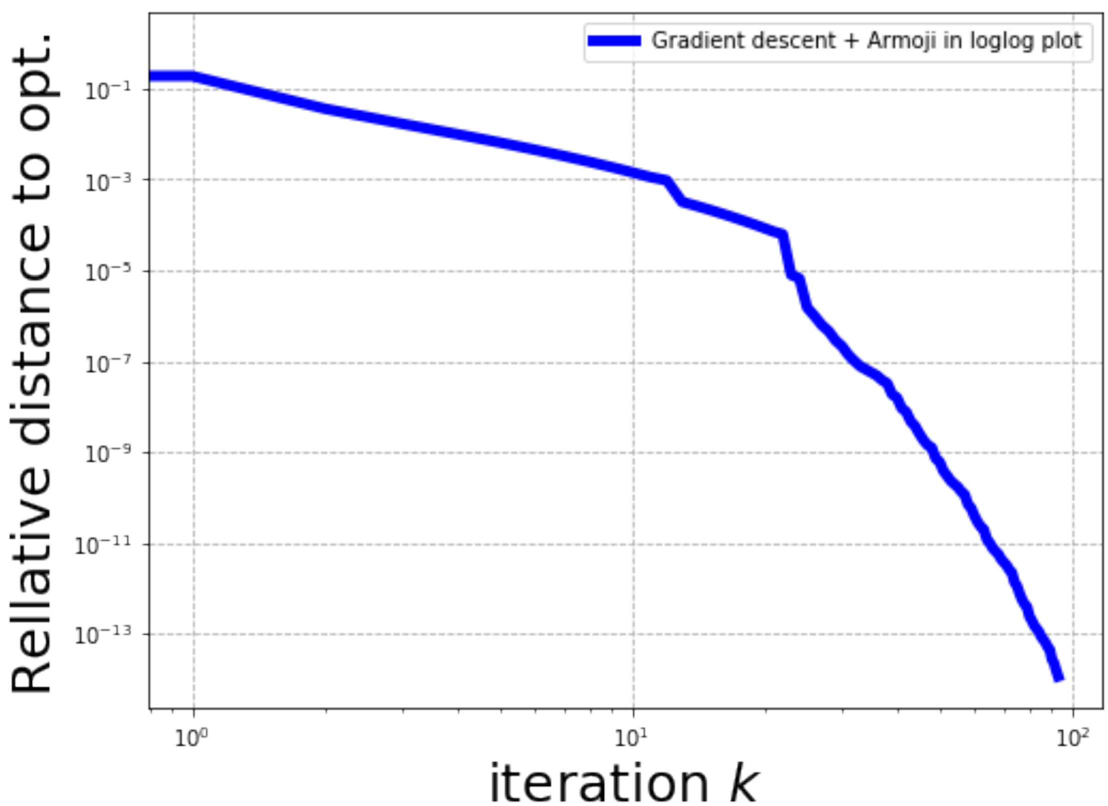
is the minimizer of the denoising problem, which you can compute by using spsolve, similarly to Assignment 1.

In [17]:
```
# Plot the rellative objective function vs number of iterations.

# Write your code here.

# Here is an example code
store_data_for_plotting_q5 = []
denominator = f_x0-f_minimizer
for x in list_of_x_q5:
    numerator = denoising(x.flatten('F')) - f_minimizer
    store_data_for_plotting_q5.append(numerator / denominator)

fig = plt.figure(figsize=(8, 6))
plt.plot(store_data_for_plotting_q5, label=("Gradient descent + Armijo

plt.legend(prop={'size': 20},loc="upper right")
plt.xlabel("iteration $k$", fontsize=25)
plt.ylabel("Objective function", fontsize=25)
plt.grid(linestyle='dashed')
plt.show()
plt.savefig('7.jpg')


fig = plt.figure(figsize=(8, 6))
plt.loglog(store_data_for_plotting_q5,label=("Gradient descent + Armoj
plt.legend(prop={'size': 10},loc="upper right")
plt.xlabel("iteration $k$", fontsize=25)
plt.ylabel("Rellative distance to opt.", fontsize=25)
plt.grid(linestyle='dashed')
plt.show()
plt.savefig('8.jpg')
```





<Figure size 432x288 with 0 Axes>



<Figure size 432x288 with 0 Axes>

Question 6: Is gradient descent with Armijo line-search faster than gradient descent with simple line-search in terms of running time? Is gradient descent with Armijo line-search faster than gradient descent with simple line-search in terms of number of required iterations? Explain any performance differences between the two approaches. Marks: 10

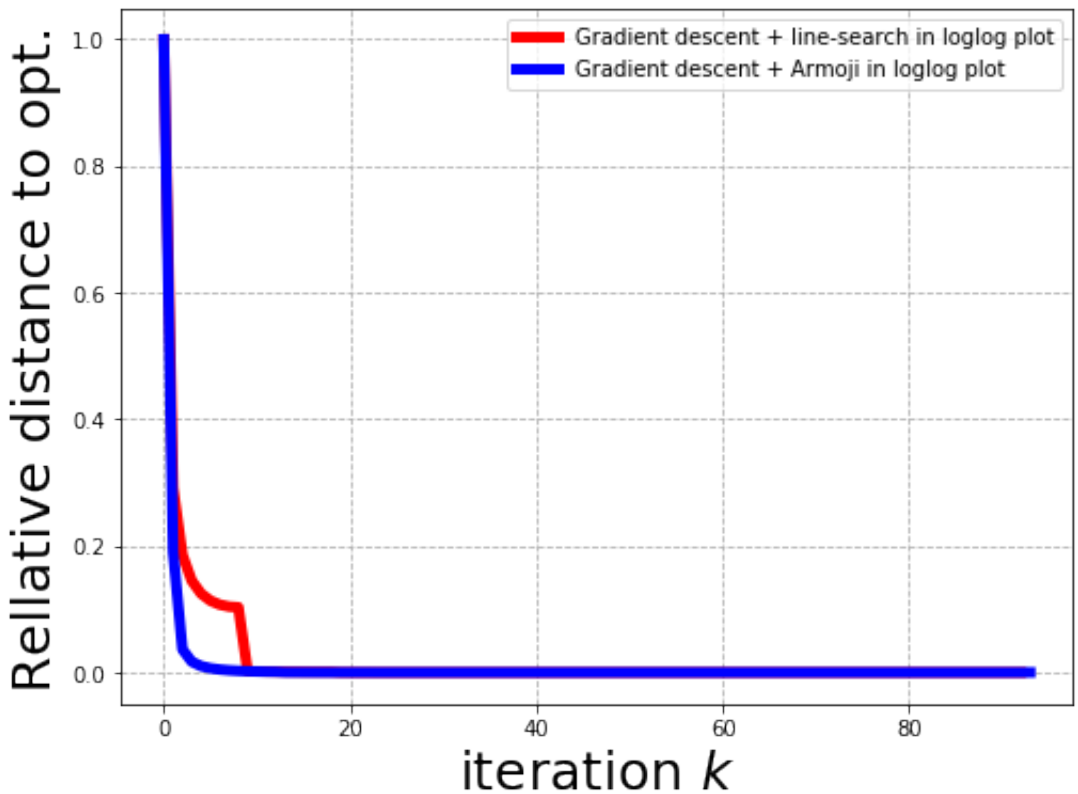
In [18]:

```
A: No. gradient descent with Armijo line-search slower than gradient de
It requires less iterations but takes less than a second longer time.
The reason could be that armijo line-search takes more steps of computa
Armoji line search: we need to update RHS in each iteration RHS = deno_
Simple line-search : no need to update RHS.

B: In our case gradient descent with Armijo line-search is slower.
but if we uncomment the code and compute RHS in every itertions. Armijo

C: Theoretically the armijo line search should be faster because:
In armijo line search
f(x_k - γαf(xk)) ≤ f(x_k) - αγ||▽f(x_k)k||_2^2 ==>
f(x_k+1) - f(x_k) <= - αγ||▽f(x_k)k||_2^2
Simple line search guarantees that
f(x_k+1) - f(x_k) <= 0

In Armijo line-search, the lower bound is a negative number. In simple
That is, in Armijo line-search, f(x) get more closer to f*, the minimiz
```

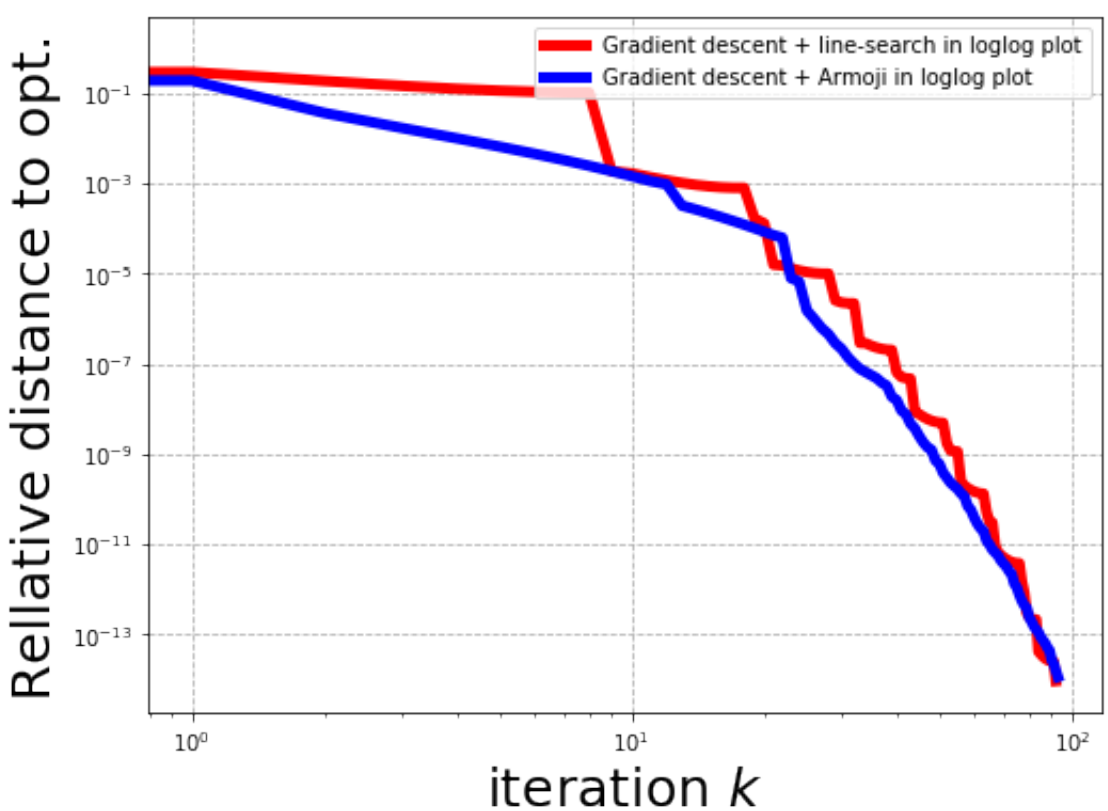
In [19]:

```
# plot the data for gradient descent with line-search and armijo line-
fig = plt.figure(figsize=(8, 6))
plt.plot(store_data_for_plotting_q3,label=("Gradient descent + line-se
plt.plot(store_data_for_plotting_q5,label=("Gradient descent + Armoji 
plt.legend(prop={'size': 10},loc="upper right")
plt.xlabel("iteration $k$", fontsize=25)
plt.ylabel("Rellative distance to opt.", fontsize=25)
plt.grid(linestyle='dashed')
plt.show()
plt.savefig('7.jpg')

fig = plt.figure(figsize=(8, 6))
plt.loglog(store_data_for_plotting_q3,label=("Gradient descent + line-
plt.loglog(store_data_for_plotting_q5,label=("Gradient descent + Armoj
plt.legend(prop={'size': 10},loc="upper right")
plt.xlabel("iteration $k$", fontsize=25)
plt.ylabel("Rellative distance to opt.", fontsize=25)
plt.grid(linestyle='dashed')
plt.show()
```

```
plt.savefig('8.jpg')
```




```
<Figure size 432x288 with 0 Axes>
```



<Figure size 432x288 with 0 Axes>

In [ ]:

Mathematical Questions

Question 7: prove that the denoising objective function is strongly convex. What is its strong convexity parameter? Marks: 5

Question 8: Prove that Armijo line-search will terminate after a finite number of steps. Hint: show that there exists a step-size

$$\alpha^* > 0$$

such that for any step-size smaller than

$$\alpha^*$$

the termination condition of Armijo line-search is satisfied. How many iterations will be required in worst-case for Armijo line-search to terminate? Marks 15

Question 9: what is the running time for gradient descent with Armijo line-search for the denoising problem to achieve

$$f(x_k) - f^* \leq \epsilon$$

?. The running time is computed by multiplying the worst-case iteration complexity times the FLOPS at each iteration. The FLOPS at each iteration is the number of additions, subtractions, multiplications and divisions that are performed during the current iteration. 10

Question 10: prove the convergence rate and iteration complexity for gradient descent with constant step-sizes (equal to 1/L) for strongly convex functions. Marks: 10

In [ ]:

In [ ]:

In [ ]:

In [ ]:

In [ ]:

In [ ]:

In [ ]:

In [ ]:

In [ ]:

In [ ]:

In [ ]:

In [ ]:

In [ ]:

In [ ]: