Exploring models

Summary, explainability, and prediction

R.W. Oldford

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Modelling

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2. "to find unanticipated aspects in the data, and **to suggest unthought-of models** for the data's **summarization and exposure**;
3. "to employ the data **to assess** the (always incomplete) **adequacy** of a **contemplated model**;
4. "to provide both incentives and guidance for further analysis of the data; and
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Here, model is generally (though not exclusively) to be understood in a more formal mathematical sense. Tukey and Wilk also cautioned against taking them too seriously. In their words, "**Models must be used but must never be believed.**"

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G.E.P. Box (1976) "Science and Statistics", *Journal of the American Statistical Association*, *71*, pp. 791-799.

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- we make **inferences** from the model about $\mu()$ using the **estimator** $\widetilde{\mu}(x_1, \ldots, x_p)$ and its distribution.
- when $\mu()$ is expressed in terms of a finite number of unknown parameters, say $\theta_1, \ldots, \theta_k$, we say that it is a **parametric model** with parameter estimates $\widehat{\theta}_1, \ldots, \widehat{\theta}_k$ and corresponding estimators $\widetilde{\theta}_1, \ldots, \widetilde{\theta}_k$.

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Response models - examples

Regression models

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\begin{aligned}
Y &= \mu(x_1, \ldots, x_q) + R \\
E(R) &= 0 \\
R &\sim F_R(r; \sigma)
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Such models include the **linear model** whereby

$$\mu(x_1, \ldots, x_q) = \theta_0 + \theta_1 x_1 + \cdots + \theta_p x_p.$$

Here linear refers to the mean model being linear in the unknown parameters θ_i.

(There are non-linear regression models as well.)

Generalizing the linear model

A slight generalization is to instead model a function of the conditional mean, as in the so-called **generalized linear model** where now there is a known function $g(\mu)$ called the **link** function and we model

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$$\mu(x_1, \ldots, x_q) = \theta + h_1(x_1) + \cdots + h_p(x_p)$$

where $h_i(x_i)$ are arbitrary functions, each of only a single explanatory variate (x_i). This is called an **additive model** (being additive in functions of the explanatory variates).

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These are only a few of the many models that are possible.

Response models - in R a consistent interface

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$$y \sim x1 + x2 + x3$$

specifies a linear model with `y` as the response and the variates named `x1`, `x2`, and `x3` as the explanatory variates (or **predictors**). The variates `x1`, `x2`, and `x3` are sometimes called the **terms** of the model.

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  - $y \sim x1 + x2 + x3$ fits the linear model having conditional mean of Y being

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- terms are joined together with binary operators `+`, `-`, `:`, `*`, and `/`, where for terms `a` and `b` we understand that
  - `+ b` indicates adding a separate term `b` to the model,
  - `- b` indicates removing the term `b` from the model,
  - `a:b` indicates an *interaction* term between `a` and `b` be added,
  - `a*b` is a short-hand equivalent to `a + b + a:b`,
  - `a/b` indicates `b` *nested* within `a` and is equivalent to `a + a:b`

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- `poly(x, p)` specifies a polynomial in `x` of degree `p` (uses orthogonal polynomials)

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  - an indication of the statistical significance of each term in the model

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Once estimated from the available data, there are common interfaces we expect to have with the fitted model. Suppose the fitted model has been assigned to the variable `myfit`, then common interactions we might expect include

- `summary(myfit)` should return (and print) a **statistical summary** of the data such as
  - an overall measure of the quality of the fit
  - an indication of the statistical significance of each term in the model
- `plot(fit)`

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Response models - in R a consistent interface

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  - often also produces prediction intervals for a new observation and confidence intervals for the conditional mean

Response models - in R a consistent interface

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- `plot(fit)` should produce one or more plots that summarize the fit and provide some diagnostic tools for assessing its quality
- `predict(fit , ... )` provide predictions of the response (typically its estimated conditional mean) at any collection of variate values
  - requires a data set of new values for every variate named in the model formula
  - often also produces prediction intervals for a new observation and confidence intervals for the conditional mean
- `str(fit)` reveals the structure of the fitted model. Here we expect to also find `myfit$residuals` containing the residuals, or deviations, of the observed responses from their fitted conditional mean

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Facebook data - fitting linear models

Linear models are fitted in R using the `lm()` function.

```
fit1 <- lm(log10(Impressions) ~ Paid, data = facebook)
summary(fit1)
```

```
##
## Call:
## lm(formula = log10(Impressions) ~ Paid, data = facebook)
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -1.25955 -0.32022 -0.09619  0.28444  2.03001
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept)  4.01543    0.02655 151.236  < 2e-16 ***
## Paid         0.21142    0.05031   4.203 3.13e-05 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.5038 on 497 degrees of freedom
##   (1 observation deleted due to missingness)
## Multiple R-squared:  0.03432,    Adjusted R-squared:  0.03238
## F-statistic: 17.66 on 1 and 497 DF,  p-value: 3.128e-05
```

Facebook data - contents of linear fits

Extracting contents

```r
fit1$coefficients
```

```
## (Intercept)        Paid
##   4.0154262   0.2114186
```

```r
head(model.matrix(fit1))
```

```
##   (Intercept) Paid
## 1           1    0
## 2           1    0
## 3           1    0
## 4           1    1
## 5           1    0
## 6           1    0
```

```r
head(fit1$residuals)
```

```
##          1          2          3          4          5          6
## -0.3086231  0.2646283 -0.3746467  0.7175934  0.1179211  0.3036590
```

```r
# And prediction (based on the estimated mean don't forget)
predict(fit1, newdata = data.frame(Paid = c(0,1)))
```

```
##        1        2
## 4.015426 4.226845
```

```r
# The predicted mean increase in Impressions for paid advertising
diff(10^predict(fit1, newdata = data.frame(Paid = c(0,1))))
```

```
##        2
## 6497.921
```

Facebook data - linear model with a factor

Recall that Category took values Product, Inspiration, Action

```r
fit2 <- lm(log10(Impressions) ~ Category, data = facebook)
summary(fit2)
```

```
##
## Call:
## lm(formula = log10(Impressions) ~ Category, data = facebook)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -1.3727 -0.3074 -0.1079  0.2854  1.9168
##
## Coefficients:
##                     Estimate Std. Error t value Pr(>|t|)
## (Intercept)          4.12860    0.03482 118.583   <2e-16 ***
## CategoryInspiration -0.09723    0.05379  -1.807   0.0713 .
## CategoryProduct     -0.09449    0.05672  -1.666   0.0963 .
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.5105 on 497 degrees of freedom
## Multiple R-squared:  0.008645,   Adjusted R-squared:  0.004655
## F-statistic: 2.167 on 2 and 497 DF,  p-value: 0.1156
```

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Facebook data - contents of linear model with a factor

Extracting contents

```r
fit2$coefficients
```

```
##         (Intercept) CategoryInspiration     CategoryProduct
##          4.12860385         -0.09722799         -0.09449137
```

```r
head(model.matrix(fit2))
```

```
##   (Intercept) CategoryInspiration CategoryProduct
## 1           1                   0               1
## 2           1                   0               1
## 3           1                   1               0
## 4           1                   0               1
## 5           1                   0               1
## 6           1                   0               1
```

```r
head(fit2$residuals)
```

```
##           1           2           3           4           5           6
## -0.32730938  0.24594205 -0.39059638  0.91032577  0.09923478  0.28497275
```

```r
# And prediction on original scale of Impressions
10^predict(fit2, newdata = data.frame(Category = factor(levels(facebook$Category))))
```

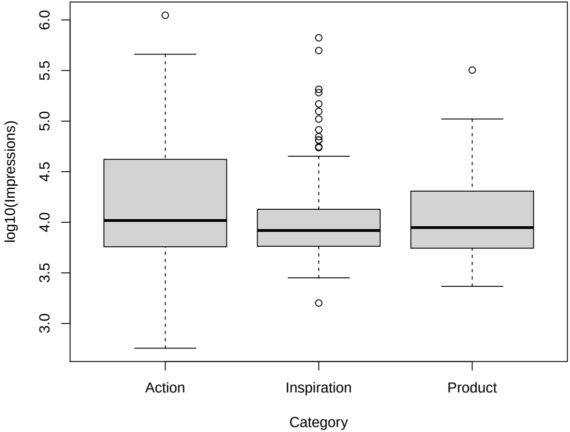
```
##        1        2        3
## 13446.33 10749.19 10817.14
```

Conclusions?

Facebook data - other uses of formula

Formulas are also used by other functions (e.g. `boxplot()`)

```
boxplot(log10(Impressions) ~ Category, data = facebook, col = "lightgrey")
```

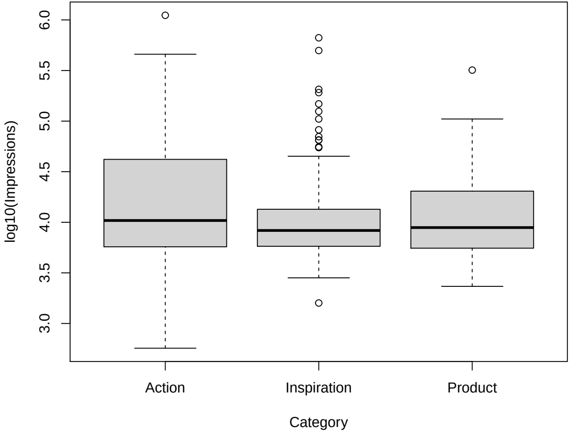


Comments?

Facebook data - other uses of formula

Formulas are also used by other functions (e.g. `boxplot()`)

```
boxplot(log10(Impressions) ~ Category, data = facebook, col = "lightgrey")
```

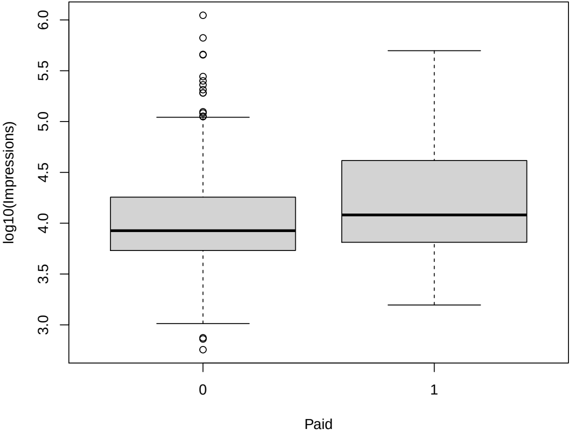


Comments? How is this "model" different from the one constructed by `lm()`'?

Facebook data - other uses of formula

How about `log10(Impressions)` as a function of `Paid`?

```
boxplot(log10(Impressions) ~ Paid, data = facebook, col = "lightgrey")
```

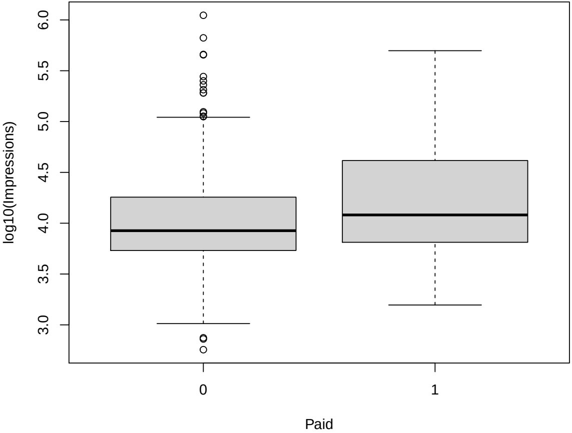


Comments?

Facebook data - other uses of formula

How about `log10(Impressions)` as a function of `Paid`?

```
boxplot(log10(Impressions) ~ Paid, data = facebook, col = "lightgrey")
```

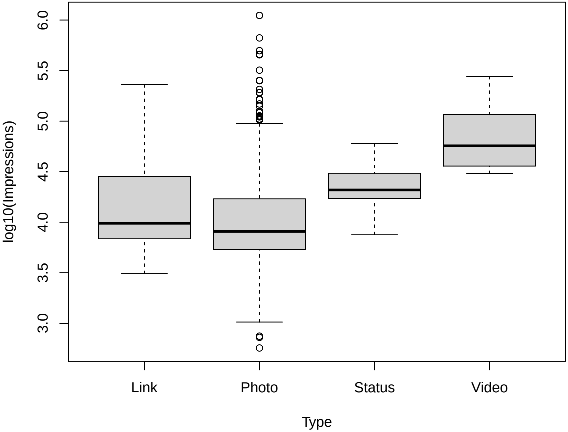


Comments? How is this model formula interpreted?

Facebook data - other uses of formula

How about `log10(Impressions)` as a function of `Type`?

```
boxplot(log10(Impressions) ~ Type, data = facebook, col = "lightgrey")
```



Comments?

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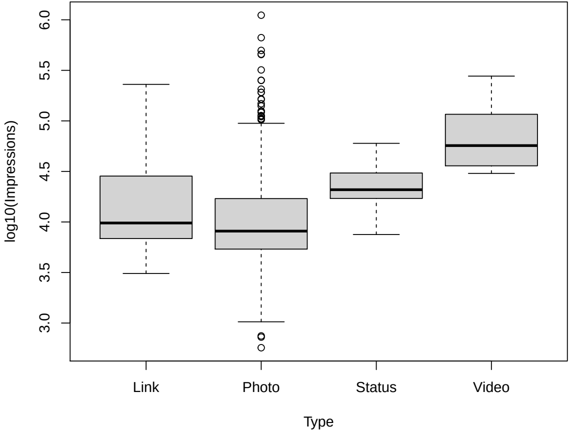
Facebook data - other uses of formula

How about `log10(Impressions)` as a function of `Type`?

```
boxplot(log10(Impressions) ~ Type, data = facebook, col = "lightgrey")
```

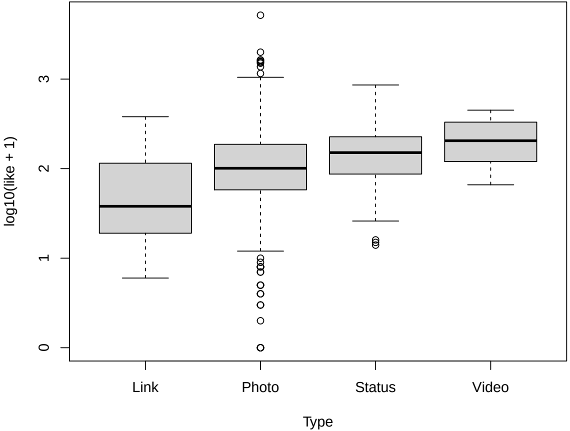


Comments? How is this model formula interpreted?

Facebook data - other uses of formula

How about `log10(Impressions)` as a function of `Type`?

```
boxplot(log10(like +1) ~ Type, data = facebook, col = "lightgrey")
```

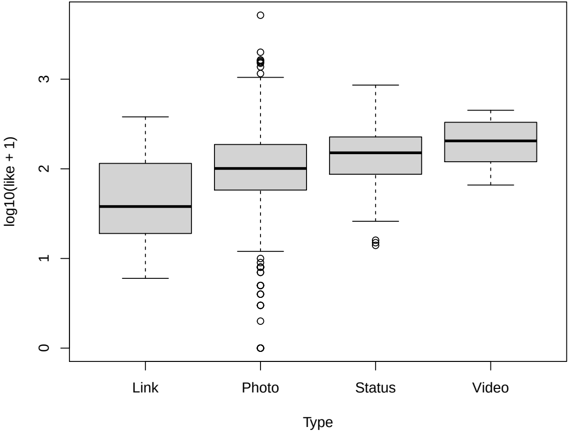


Comments?

Facebook data - other uses of formula

How about `log10(Impressions)` as a function of `Type`?

```
boxplot(log10(like +1) ~ Type, data = facebook, col = "lightgrey")
```

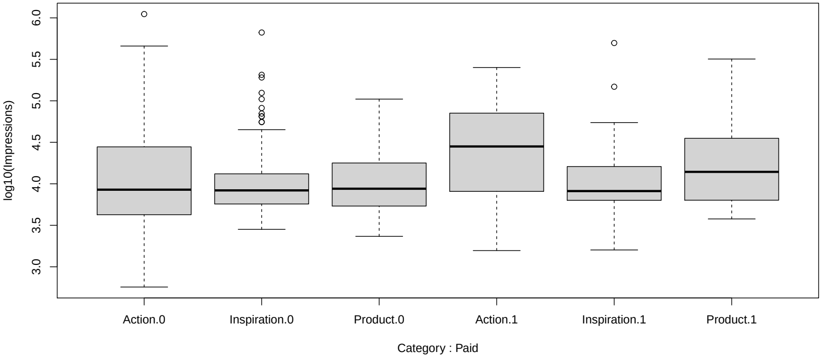


Comments? How is this model formula interpreted?

Facebook data - other uses of formula

This works well when explanatory variates are categorical.

```
boxplot(log10(Impressions) ~ Category + Paid, data = facebook, col = "lightgrey")
```

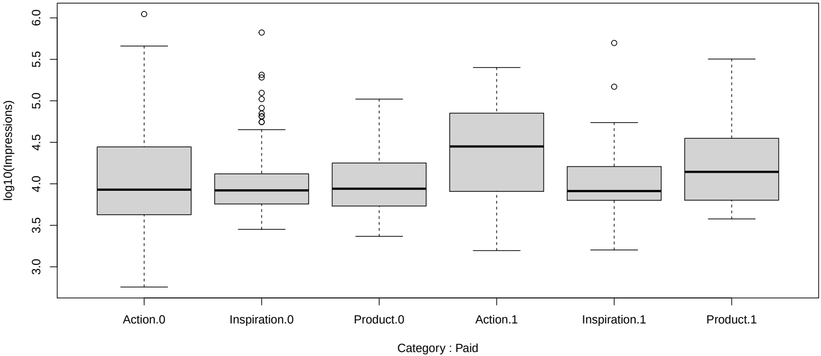


Comments?

Facebook data - other uses of formula

This works well when explanatory variates are categorical.

```r
boxplot(log10(Impressions) ~ Category + Paid, data = facebook, col = "lightgrey")
```



Comments?

Note the labels on the horizontal axis.

Facebook data - other uses of formula

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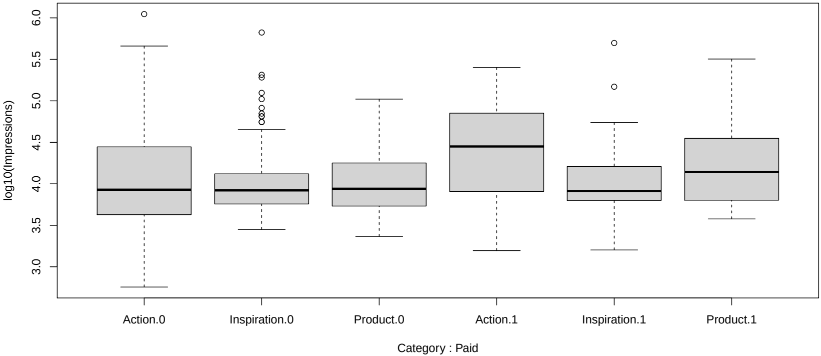
```
boxplot(log10(Impressions) ~ Category + Paid, data = facebook, col = "lightgrey")
```



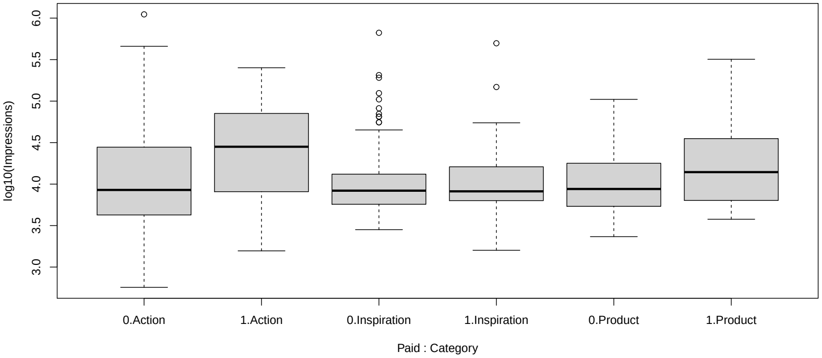
Comments?

Note the labels on the horizontal axis.How is this model formula interpreted?

Facebook data - other uses of formula

What has changed here?

```
boxplot(log10(Impressions) ~ Paid + Category, data = facebook, col = "lightgrey")
```

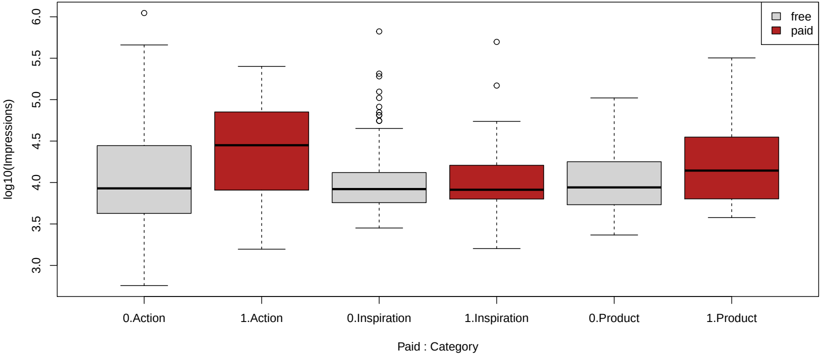


Note the labels on the horizontal axis.

Facebook data - other uses of formula

What has changed here?

```
boxplot(log10(Impressions) ~ Paid + Category, data = facebook,
            col = rep(c("lightgrey", "firebrick"), 3))
legend("topright", legend = c("free", "paid"), fill = c("lightgrey", "firebrick"))
```



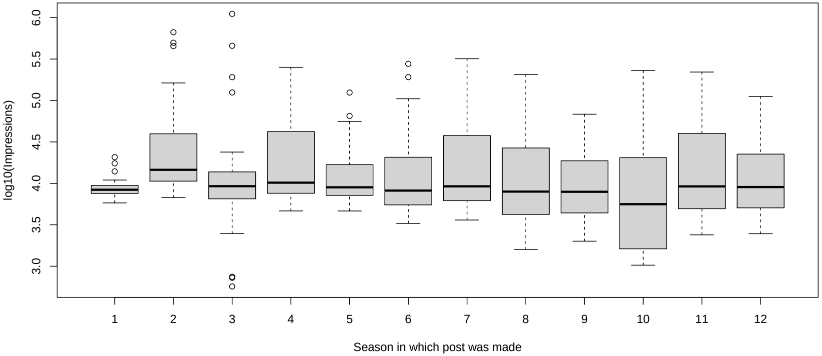
Note the labels on the horizontal axis.

Comments?

Facebook data - other uses of formula

By month

```
boxplot(log10(Impressions) ~ Post.Month,
        xlab = "Season in which post was made",
        data = facebook, col = "lightgrey"
        )
```

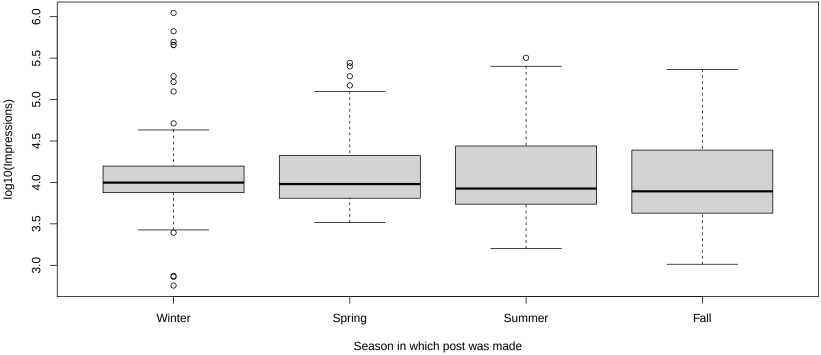


Comments?

Facebook data - other uses of formula

A numeric variate could be made categorical using `cut()`

```
boxplot(log10(Impressions) ~ cut(Post.Month, 4,
                                 labels = c("Winter", "Spring", "Summer", "Fall")),
        xlab = "Season in which post was made",
        data = facebook, col = "lightgrey")
```

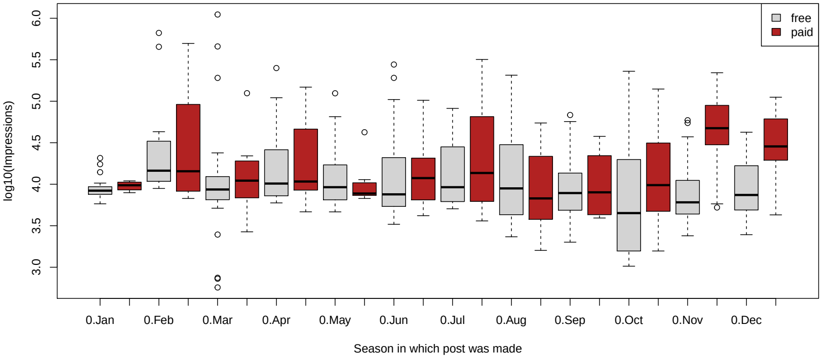


Comments?

Facebook data - other uses of formula

How about?

```
boxplot(log10(Impressions) ~ Paid + cut(Post.Month, 4,
                                        labels = c("Winter", "Spring",
                                                   "Summer", "Fall")),
        xlab = "Season in which post was made",
        data = facebook, col = rep(c("lightgrey", "firebrick"), 4))
legend("topright", legend = c("free", "paid"), fill = c("lightgrey", "firebrick"))
```

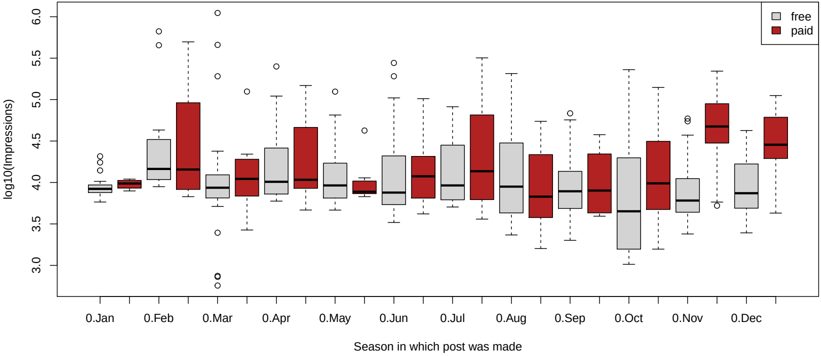


Comments?

Facebook data - other uses of formula

How about?

```
boxplot(log10(Impressions) ~ Paid + cut(Post.Month, 4,
                                         labels = c("Winter", "Spring",
                                                    "Summer", "Fall")),
        xlab = "Season in which post was made",
        data = facebook, col = rep(c("lightgrey", "firebrick"), 4))
legend("topright", legend = c("free", "paid"), fill = c("lightgrey", "firebrick"))
```

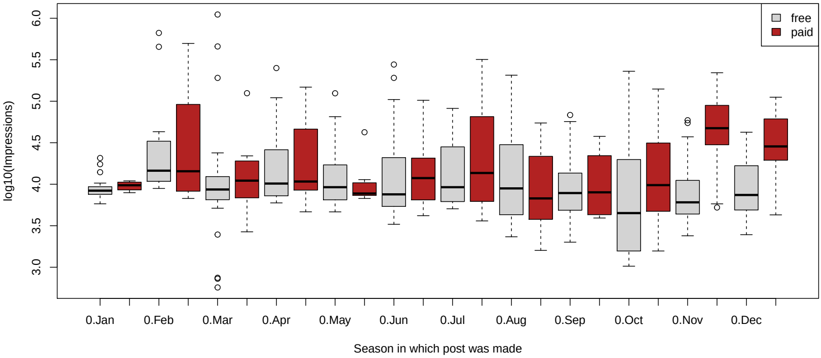


Comments? Huh? What happened here?

Facebook data - other uses of formula

How about?

```
boxplot(log10(Impressions) ~ Paid + cut(Post.Month, 4,
                                        labels = c("Winter", "Spring",
                                                   "Summer", "Fall")),
        xlab = "Season in which post was made",
        data = facebook, col = rep(c("lightgrey", "firebrick"), 4))
legend("topright", legend = c("free", "paid"), fill = c("lightgrey", "firebrick"))
```

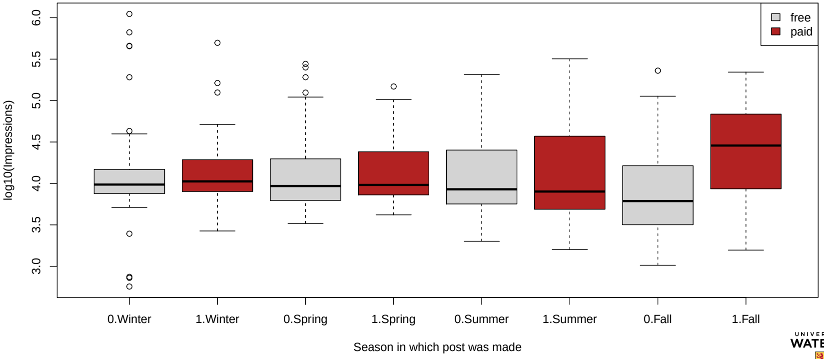


Comments? Huh? What happened here? A bug?

Facebook data - other uses of formula

How about?

```
facebook2 <- data.frame(facebook,
                        season = cut(facebook$Post.Month, 4,
                                     labels = c("Winter", "Spring",
                                                "Summer", "Fall")))
boxplot(log10(Impressions) ~ Paid + season,
        xlab = "Season in which post was made",
        data = facebook2, col = rep(c("lightgrey", "firebrick"), 4))
legend("topright", legend = c("free", "paid"), fill = c("lightgrey", "firebrick"))
```

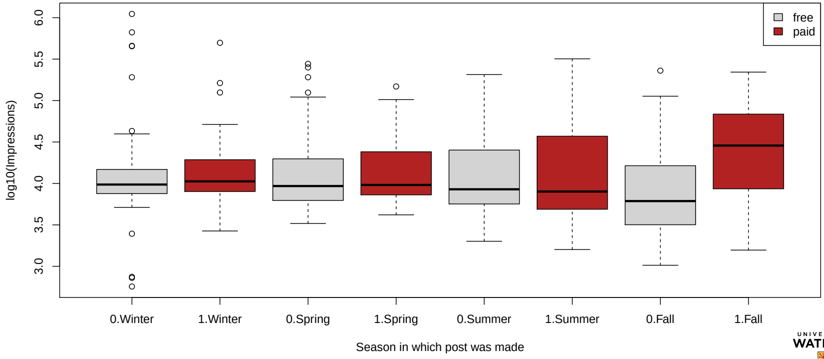


Comments?

Facebook data - other uses of formula

How about?

```
facebook2 <- data.frame(facebook,
                        season = cut(facebook$Post.Month, 4,
                                     labels = c("Winter", "Spring",
                                                "Summer", "Fall")))
boxplot(log10(Impressions) ~ Paid + season,
        xlab = "Season in which post was made",
        data = facebook2, col = rep(c("lightgrey", "firebrick"), 4))
legend("topright", legend = c("free", "paid"), fill = c("lightgrey", "firebrick"))
```

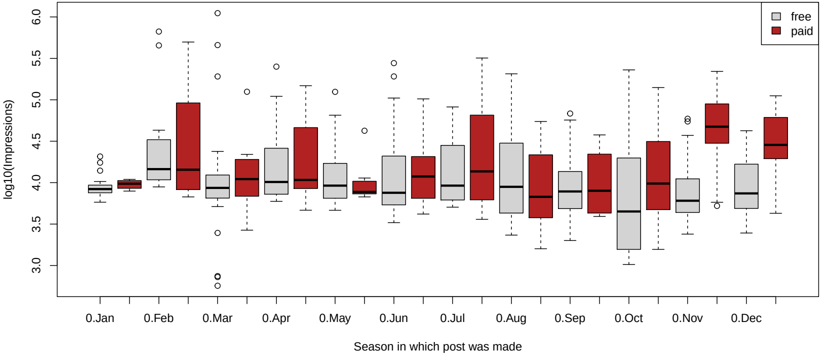


Comments?

Facebook data - other uses of formula

Or how about?

```r
boxplot(log10(Impressions) ~ Paid + cut(Post.Month, 12,
                                        labels = month.abb),
        xlab = "Season in which post was made",
        data = facebook, col = rep(c("lightgrey", "firebrick"), 6))
legend("topright", legend = c("free", "paid"), fill = c("lightgrey", "firebrick"))
```

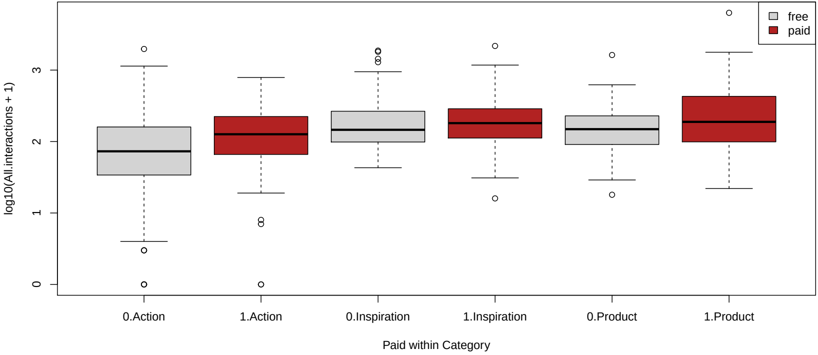


Comments?

Facebook data - other uses of formula

Change response.

```
boxplot(log10(All.interactions + 1) ~ Paid + Category,
        xlab = "Paid within Category",
        data = facebook, col = rep(c("lightgrey", "firebrick"), length(facebook$Category)))
legend("topright", legend = c("free", "paid"), fill = c("lightgrey", "firebrick"))
```



Comments?

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Facebook data - other uses of formula

Change response. Fitted model

```
fit3 <- lm(log10(All.interactions + 1) ~ Paid + Category,  data = facebook)
summary(fit3)
```

```
##
## Call:
## lm(formula = log10(All.interactions + 1) ~ Paid + Category, data = facebook)
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -1.95563 -0.23538  0.01645  0.26075  1.49121
##
## Coefficients:
##                     Estimate Std. Error t value Pr(>|t|)
## (Intercept)          1.80436    0.03615  49.907  < 2e-16 ***
## Paid                 0.15127    0.04857   3.115  0.00195 **
## CategoryInspiration  0.39403    0.05121   7.695 7.74e-14 ***
## CategoryProduct      0.36251    0.05417   6.692 5.96e-11 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.4859 on 495 degrees of freedom
##   (1 observation deleted due to missingness)
## Multiple R-squared:  0.1433, Adjusted R-squared:  0.1382
## F-statistic: 27.61 on 3 and 495 DF,  p-value: < 2.2e-16
```

Comments?

Facebook data - other uses of formula

Change response. A slightly different fitted model

```r
fit4 <- lm(log10(All.interactions + 1) ~ Paid * Category,  data = facebook)
summary(fit4)
```

```
##
## Call:
## lm(formula = log10(All.interactions + 1) ~ Paid * Category, data = facebook)
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -2.01793 -0.23100  0.02586  0.27246  1.51761
##
## Coefficients:
##                           Estimate Std. Error t value Pr(>|t|)
## (Intercept)                1.77795    0.03941  45.111  < 2e-16 ***
## Paid                       0.23998    0.07224   3.322  0.00096 ***
## CategoryInspiration        0.46570    0.06040   7.711 6.96e-14 ***
## CategoryProduct            0.37776    0.06302   5.994 3.95e-09 ***
## Paid:CategoryInspiration -0.25185    0.11299  -2.229  0.02627 *
## Paid:CategoryProduct     -0.04374    0.12234  -0.358  0.72083
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.4843 on 493 degrees of freedom
##   (1 observation deleted due to missingness)
## Multiple R-squared:  0.1524, Adjusted R-squared:  0.1438
## F-statistic: 17.72 on 5 and 493 DF,  p-value: 3.639e-16
```

Comments?

Facebook data - other uses of formula

Change response. A slightly different fitted model

```r
fit4$coefficients
```

```
##             (Intercept)                     Paid      CategoryInspiration
##              1.77795481               0.23997989               0.46569543
##         CategoryProduct Paid:CategoryInspiration     Paid:CategoryProduct
##              0.37776394              -0.25185230              -0.04374152
```

```r
head(model.matrix(fit4))
```

```
##   (Intercept) Paid CategoryInspiration CategoryProduct Paid:CategoryInspiration
## 1           1    0                   0               1                        0
## 2           1    0                   0               1                        0
## 3           1    0                   1               0                        0
## 4           1    1                   0               1                        0
## 5           1    0                   0               1                        0
## 6           1    0                   0               1                        0
##   Paid:CategoryProduct
## 1                    0
## 2                    0
## 3                    0
## 4                    1
## 5                    0
## 6                    0
```

Comments?

Facebook data - other uses of formula

Change explanatory variates to two factors

```
fit5 <- lm(log10(All.interactions + 1) ~ Type * Category,  data = facebook)
summary(fit5)
```

```
##
## Call:
## lm(formula = log10(All.interactions + 1) ~ Type * Category, data = facebook)
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -1.83783 -0.23716  0.01508  0.25688  1.60199
##
## Coefficients: (2 not defined because of singularities)
##                               Estimate Std. Error t value Pr(>|t|)
## (Intercept)                    1.73278    0.10895  15.904   <2e-16 ***
## TypePhoto                      0.10504    0.11469   0.916   0.3602
## TypeStatus                     0.36213    0.30167   1.200   0.2306
## TypeVideo                      0.65020    0.21397   3.039   0.0025 **
## CategoryInspiration            0.20172    0.49927   0.404   0.6864
## CategoryProduct               -0.03381    0.49927  -0.068   0.9460
## TypePhoto:CategoryInspiration  0.20383    0.50213   0.406   0.6850
## TypeStatus:CategoryInspiration -0.09280    0.62270  -0.149   0.8816
## TypeVideo:CategoryInspiration       NA         NA      NA       NA
## TypePhoto:CategoryProduct      0.39575    0.50315   0.787   0.4319
## TypeStatus:CategoryProduct     0.16441    0.57849   0.284   0.7764
## TypeVideo:CategoryProduct           NA         NA      NA       NA
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.4872 on 490 degrees of freedom
## Multiple R-squared:  0.1473, Adjusted R-squared:  0.1316
## F-statistic: 9.405 on 9 and 490 DF,  p-value: 3.024e-13
```

Comments?

Facebook data - other uses of formula

Just the coefficients.

```
fit5$coefficients
```

```
##                      (Intercept)                         TypePhoto
##                       1.73278345                        0.10504178
##                       TypeStatus                         TypeVideo
##                       0.36213204                        0.65020444
##              CategoryInspiration                   CategoryProduct
##                       0.20171500                       -0.03381344
##   TypePhoto:CategoryInspiration TypeStatus:CategoryInspiration
##                       0.20382942                       -0.09279854
##   TypeVideo:CategoryInspiration       TypePhoto:CategoryProduct
##                               NA                        0.39574802
##      TypeStatus:CategoryProduct       TypeVideo:CategoryProduct
##                       0.16440892                                NA
```

Facebook data - other uses of formula

And now the corresponding model matrix.

```r
head(model.matrix(fit5))
```

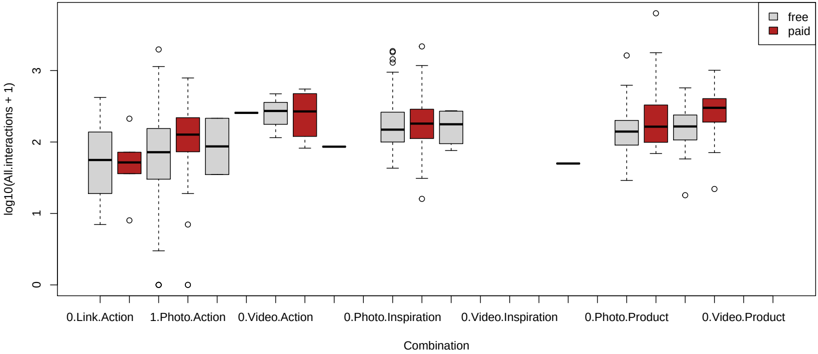
```
##   (Intercept) TypePhoto TypeStatus TypeVideo CategoryInspiration
## 1           1         1          0         0                   0
## 2           1         0          1         0                   0
## 3           1         1          0         0                   1
## 4           1         1          0         0                   0
## 5           1         1          0         0                   0
## 6           1         0          1         0                   0
##   CategoryProduct TypePhoto:CategoryInspiration TypeStatus:CategoryInspiration
## 1               1                             0                              0
## 2               1                             0                              0
## 3               0                             1                              0
## 4               1                             0                              0
## 5               1                             0                              0
## 6               1                             0                              0
##   TypeVideo:CategoryInspiration TypePhoto:CategoryProduct
## 1                             0                         1
## 2                             0                         0
## 3                             0                         0
## 4                             0                         1
## 5                             0                         1
## 6                             0                         0
##   TypeStatus:CategoryProduct TypeVideo:CategoryProduct
## 1                          0                         0
## 2                          1                         0
## 3                          0                         0
## 4                          0                         0
## 5                          0                         0
## 6                          1                         0
```

What does the intercept term represent?

Facebook data - other uses of formula

Or how about?

```
boxplot(log10(All.interactions + 1) ~ Paid + Type + Category,
        xlab = "Combination",
        data = facebook, col = rep(c("lightgrey", "firebrick"), length(facebook$Type) * len
legend("topright", legend = c("free", "paid"), fill = c("lightgrey", "firebrick"))
```

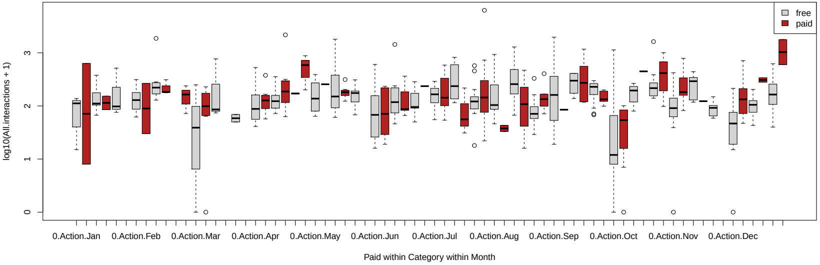


Comments?

Facebook data - other uses of formula

Or

```
boxplot(log10(All.interactions + 1) ~ Paid + Category + cut(Post.Month, 12, labels = month.abb),
        xlab = "Paid within Category within Month",
        data = facebook, col = rep(c("lightgrey", "firebrick"), 12 * length(facebook$Category)))
legend("topright", legend = c("free", "paid"), fill = c("lightgrey", "firebrick"))
```

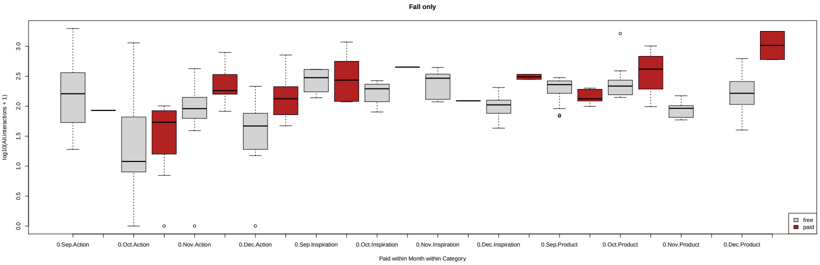


Comments?

Facebook data - other uses of formula

Focus only on Fall

```
Fall <- facebook$Post.Month %in% 9:12
with(facebook[Fall,], {
    boxplot(log10(All.interactions + 1) ~ Paid + cut(Post.Month, 4, labels = month.abb[9:12])+ Category ,
            xlab = "Paid within Month within Category", main = "Fall only",
            col = rep(c("lightgrey", "firebrick"), 4 * length(Category)))
    legend("bottomright", legend = c("free", "paid"), fill = c("lightgrey", "firebrick"))
}
)
```



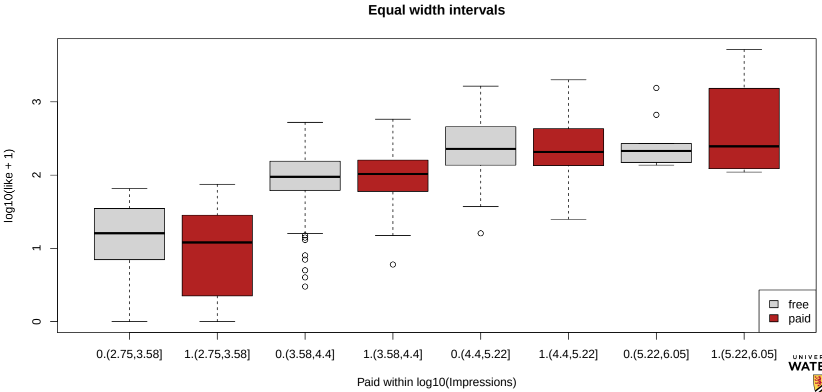
Comments?

What about a continuous explanatory variate?

Could use `cut()` on the continuous (ratio-scaled) variate to turn it into a categorical and proceed as before.

For example equal width intervals:

```
boxplot(log10(like +1) ~ Paid + cut(log10(Impressions), 4),
        xlab = "Paid within log10(Impressions)",
        data = facebook, main = "Equal width intervals",
        col = rep(c("lightgrey", "firebrick"), 2))
legend("bottomright", legend = c("free", "paid"), fill = c("lightgrey", "firebrick"))
```



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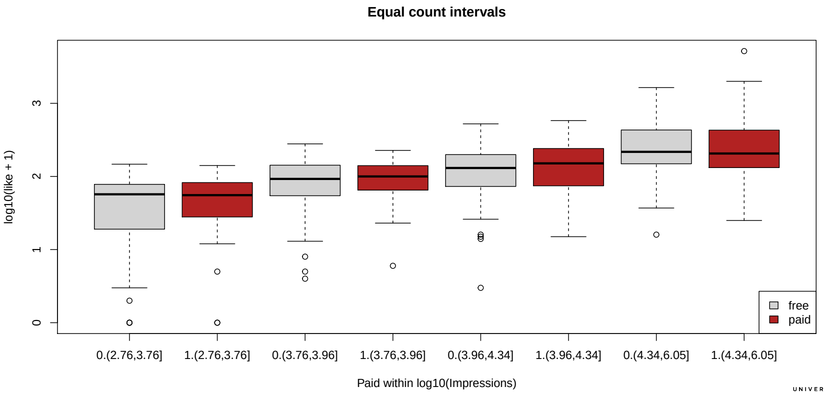
Comments?

What about a continuous explanatory variate?

Or perhaps four intervals of equal numbers

```
boxplot(log10(like +1) ~ Paid + cut(log10(Impressions),
                                    breaks = quantile(log10(Impressions))),
        xlab = "Paid within log10(Impressions)",
        data = facebook, main = "Equal count intervals",
        col = rep(c("lightgrey", "firebrick"), 2))
legend("bottomright", legend = c("free", "paid"), fill = c("lightgrey", "firebrick"))
```



Comments?

What about a continuous explanatory variate?

Alternatively, we could build a (perhaps complicated) linear model, say modelling the mean of response Y as a polynomial of explanatory variate x:

$$\mu(x) = \beta_0 + \beta_1 x + \beta_2 x^2 + \ldots + \beta_p x^p$$

(or as any other linear (in the coefficients) model).

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Such models can be fitted by least-squares.

Unfortunately, these models require a parametric form (e.g. a polynomial) be specified that will fit the data **everywhere** (i.e. globally for all x).

Alternatively, we could try fitting many simple functions of x **locally**, different at every value of x. Connecting the fitted values together produces an estimated $\mu(x)$

What about a continuous explanatory variate?

For example, while we might not be willing to have one line fit all points, we might be willing to have *different lines* fitted in separate (and contiguous) regions of x.

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For example, while we might not be willing to have one line fit all points, we might be willing to have *different lines* fitted in separate (and contiguous) regions of x. That is we could fit lines **locally** within each region of x.

We can fit locally by using **weighted least squares** which minimizes

$$\sum_{i=1}^{n} w_i(x)\left(y_i - \mu(x_i)\right)^2$$

where $w_i(x)$ depends on the location x where we are fitting $\mu(x)$.

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We could also make the weight function $w_i(x)$ to be 1 for those x_i near x and 0 for those far away. In this way, the weights determine the x_i values that contribute to fitting $\mu(x)$ and those which do not.

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For example, for the ith observation x_i when fitting at any point x we could have

$$w_i(x) = K\left(\frac{x_i - x}{a}\right)$$

for some $a > 0$ and function $K(t) \geq 0$ that is maximal at $(t = 0)$ and decreasing with $|t|$.

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The constant a controls the size of the region (i.e. which x_is) that will be used to fit at x. The smaller these regions were, the less structure we would be imposing on the underlying function $\mu(x)$.

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Locally weighted sum of squares fitting – `loess()`

In R there is a function called `loess` that fits a locally weighted sum of squares estimate that pays a little more attention to some of these problems.

```
loess(formula, data, ..., span = 0.75, ...)
```

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```
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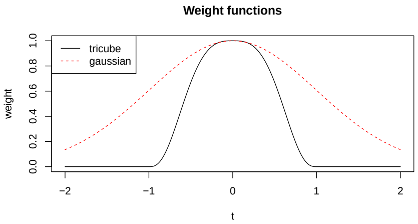
As its default weight function loess uses Tukey's tri-cube weight

$$K(t) = \begin{cases} (1 - |t|^3)^3 & t \in [-1, 1] \\ 0 & \text{otherwise.} \end{cases}$$



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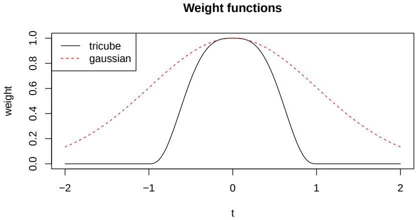
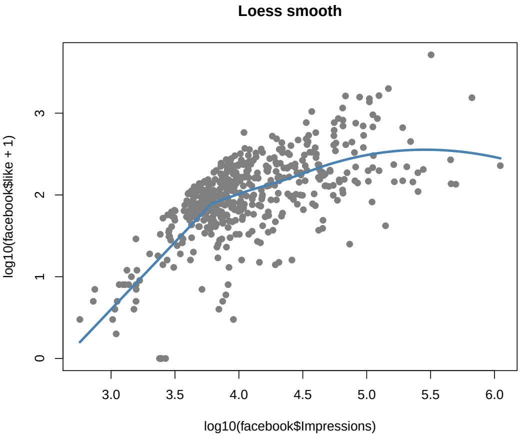


This is generally more resistant to outlying x_is than, say, a Gaussian weight function (i.e. $K(W) = \phi(w)$ the N(0,1) density).

N.B. `loess` is not restricted to fitting local lines. It can fit any degree polynomial locally (though typically only degree 1 or 2 is used in practice).

Locally weighted sum of squares fitting – `loess()`

```
fit <- loess(log10(like + 1) ~ log10(Impressions), data=facebook)
plot(log10(facebook$Impressions), log10(facebook$like + 1),
     col="grey50", pch=19, main = "Loess smooth")
# Get the values predicted by the loess
xmin <- min(facebook$Impressions)
xmax <- max(facebook$Impressions)
x <- seq(xmin, xmax, length.out = 200)
pred <- predict(fit, newdata = data.frame(Impressions = x))
lines(log10(x),pred, lwd=3, col="steelblue")
```



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Locally weighted sum of squares fitting – loess()

```r
fit2 <- loess(log10(like + 1) ~ log10(Impressions), data=facebook , span = 0.3)
plot(log10(facebook$Impressions), log10(facebook$like + 1),
     col="grey50", pch=19, main = "Loess smooth")
# Get the values predicted by the loess
xmin <- min(facebook$Impressions)
xmax <- max(facebook$Impressions)
x <- seq(xmin, xmax, length.out = 200)
pred <- predict(fit2, newdata = data.frame(Impressions = x))
lines(log10(x),pred, lwd=3, col="steelblue")
```