

Exploring models

Summary, explainability, and prediction

R.W. Oldford

Modelling

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They suggested that objectives in data analysis that are comparable to those of experimentation are¹

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2. “to find unanticipated aspects in the data, and **to suggest unthought-of models** for the data’s **summarization and exposure**;
3. “to employ the data **to assess** the (always incomplete) **adequacy** of a **contemplated model**;
4. “to provide both incentives and guidance for further analysis of the data; and
5. “to keep the investigator usefully stimulated while he absorbs the feeling of his data and considers what to do next.”

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Here, model is generally (though not exclusively) to be understood in a more formal mathematical sense. Tukey and Wilk also cautioned against taking them too seriously. In their words, “**Models must be used but must never be believed.**”

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G.E.P. Box (1976) “Science and Statistics”, *Journal of the American Statistical Association*, 71, pp. 791-799.

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- ▶ when $\mu()$ is expressed in terms of a finite number of unknown parameters, say $\theta_1, \dots, \theta_k$, we say that it is a **parametric model** with parameter estimates $\hat{\theta}_1, \dots, \hat{\theta}_k$ and corresponding estimators $\tilde{\theta}_1, \dots, \tilde{\theta}_k$.

Response models - examples

Regression models

$$\begin{aligned} Y &= \mu(x_1, \dots, x_q) + R \\ E(R) &= 0 \\ R &\sim F_R(r; \sigma) \end{aligned}$$

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Such models include the **linear model** whereby

$$\mu(x_1, \dots, x_q) = \theta_0 + \theta_1 x_1 + \dots + \theta_p x_p.$$

Here linear refers to the mean model being linear in the unknown parameters θ_j .

(There are non-linear regression models as well.)

Generalizing the linear model

A slight generalization is to instead model a function of the conditional mean, as in the so-called **generalized linear model** where now there is a known function $g(\mu)$ called the **link** function and we model

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These are only a few of the many models that are possible.

Response models - in R a consistent interface

Providers of response models in R try to have a consistent modelling interface.

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specifies a linear model with y as the response and the variates named x1, x2, and x3 as the explanatory variates (or **predictors**). The variates x1, x2, and x3 are sometimes called the **terms** of the model.

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- ▶ the intercept term θ_0 is always implicitly assumed to be part of the model; it can be removed by adding a -1 term to the model.
- ▶ that is
 - ▶ $y \sim x1 + x2 + x3$ fits the linear model having conditional mean of Y being

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specifies that the additive term `x1` enters the model as a usual linear model term, while `s(x2)` and `s(x3)` indicate that the model terms for `x2` and `x3` are to be separate smooth additive functions for each of `x2` and `x3` respectively.

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- ▶ terms are joined together with binary operators `+`, `-`, `:`, `*`, and `/`, where for terms `a` and `b` we understand that
 - ▶ `a + b` indicates adding a separate term `b` to the model,
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 - ▶ `a:b` indicates an *interaction* term between `a` and `b` be added,
 - ▶ `a*b` is a short-hand equivalent to `a + b + a:b`,
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- ▶ `poly(x, p)` specifies a polynomial in `x` of degree `p` (uses orthogonal polynomials)

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 - ▶ an indication of the statistical significance of each term in the model

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Once estimated from the available data, there are common interfaces we expect to have with the fitted model. Suppose the fitted model has been assigned to the variable `myfit`, then common interactions we might expect include

- ▶ `summary(myfit)` should return (and print) a **statistical summary** of the data such as
 - ▶ an overall measure of the quality of the fit
 - ▶ an indication of the statistical significance of each term in the model
- ▶ `plot(fit)`

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 - ▶ requires a data set of new values for every variate named in the model formula
 - ▶ often also produces prediction intervals for a new observation and confidence intervals for the conditional mean
- ▶ `str(fit)` reveals the structure of the fitted model. Here we expect to also find `myfit$residuals` containing the residuals, or deviations, of the observed responses from their fitted conditional mean

Facebook data - fitting linear models

Linear models are fitted in R using the `lm()` function.

```
fit1 <- lm(log10(Impressions) ~ Paid, data = facebook)
summary(fit1)
```

```
##
## Call:
## lm(formula = log10(Impressions) ~ Paid, data = facebook)
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -1.25955 -0.32022 -0.09619  0.28444  2.03001
##
## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept)  4.01543     0.02655 151.236 < 2e-16 ***
## Paid         0.21142     0.05031   4.203 3.13e-05 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.5038 on 497 degrees of freedom
## (1 observation deleted due to missingness)
## Multiple R-squared:  0.03432,    Adjusted R-squared:  0.03238
## F-statistic: 17.66 on 1 and 497 DF,  p-value: 3.128e-05
```

Facebook data - contents of linear fits

Extracting contents

```
fit1$coefficients
```

```
## (Intercept)      Paid  
## 4.0154262 0.2114186
```

```
head(model.matrix(fit1))
```

```
## (Intercept) Paid  
## 1          1    0  
## 2          1    0  
## 3          1    0  
## 4          1    1  
## 5          1    0  
## 6          1    0
```

```
head(fit1$residuals)
```

```
##          1          2          3          4          5          6  
## -0.3086231 0.2646283 -0.3746467 0.7175934 0.1179211 0.3036590
```

And prediction (based on the estimated mean don't forget)

```
predict(fit1, newdata = data.frame(Paid = c(0,1)))
```

```
##          1          2  
## 4.015426 4.226845
```

The predicted mean increase in Impressions for paid advertising

```
diff(10^predict(fit1, newdata = data.frame(Paid = c(0,1))))
```

```
##          2  
## 6497.921
```

Facebook data - linear model with a factor

Recall that Category took values Product, Inspiration, Action

```
fit2 <- lm(log10(Impressions) ~ Category, data = facebook)
summary(fit2)
```

```
##
## Call:
## lm(formula = log10(Impressions) ~ Category, data = facebook)
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -1.3727 -0.3074 -0.1079  0.2854  1.9168
##
## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept)      4.12860    0.03482  118.583   <2e-16 ***
## CategoryInspiration -0.09723    0.05379   -1.807    0.0713 .
## CategoryProduct    -0.09449    0.05672   -1.666    0.0963 .
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.5105 on 497 degrees of freedom
## Multiple R-squared:  0.008645,    Adjusted R-squared:  0.004655
## F-statistic: 2.167 on 2 and 497 DF,  p-value: 0.1156
```

Facebook data - contents of linear model with a factor

Extracting contents

```
fit2$coefficients
```

```
##          (Intercept) CategoryInspiration    CategoryProduct
##          4.12860385          -0.09722799          -0.09449137
```

```
head(model.matrix(fit2))
```

```
##   (Intercept) CategoryInspiration CategoryProduct
## 1           1              0              1
## 2           1              0              1
## 3           1              1              0
## 4           1              0              1
## 5           1              0              1
## 6           1              0              1
```

```
head(fit2$residuals)
```

```
##           1           2           3           4           5           6
## -0.32730938  0.24594205 -0.39059638  0.91032577  0.09923478  0.28497275
```

And prediction on original scale of Impressions

```
10~predict(fit2, newdata = data.frame(Category = factor(levels(facebook$Category))))
```

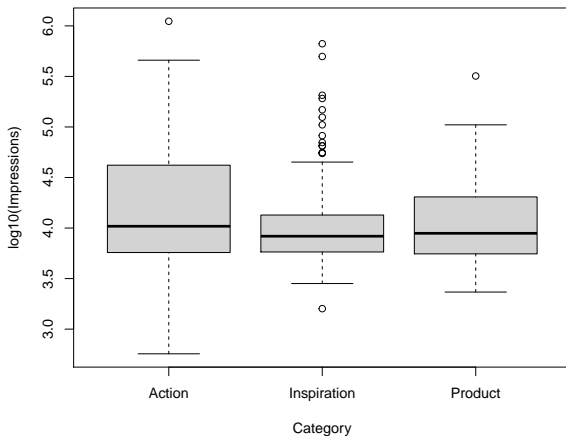
```
##           1           2           3
## 13446.33 10749.19 10817.14
```

Conclusions?

Facebook data - other uses of formula

Formulas are also used by other functions (e.g. `boxplot()`)

```
boxplot(log10(Impressions) ~ Category, data = facebook, col = "lightgrey")
```

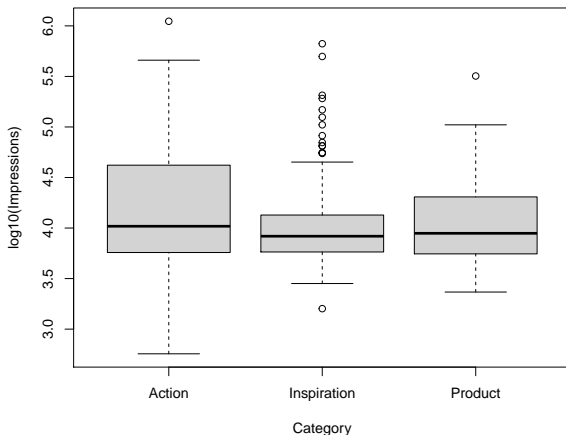


Comments?

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```
boxplot(log10(Impressions) ~ Category, data = facebook, col = "lightgrey")
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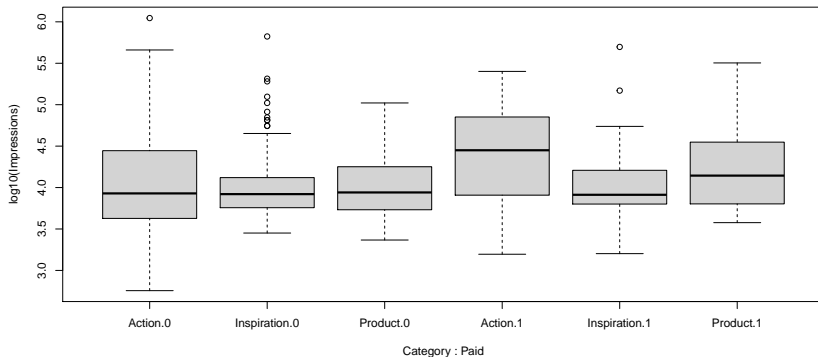


Comments? How is this “model” different from the one constructed by `lm()`’?

Facebook data - other uses of formula

This works well when explanatory variates are categorical.

```
boxplot(log10(Impressions) ~ Category + Paid, data = facebook, col = "lightgrey")
```

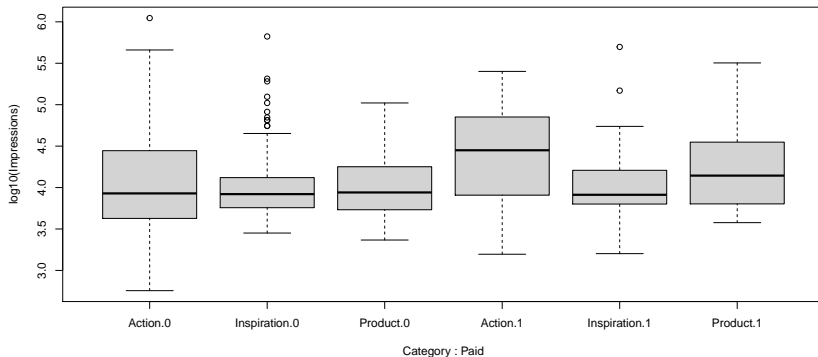


Comments?

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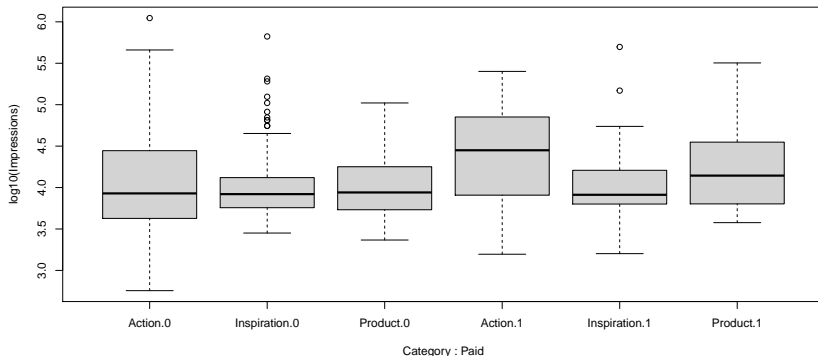
Comments?

Note the labels on the horizontal axis.

Facebook data - other uses of formula

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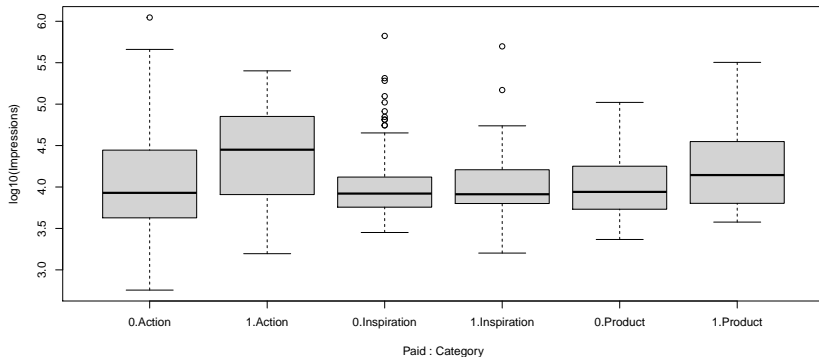
Comments?

Note the labels on the horizontal axis. How is this model formula interpreted?

Facebook data - other uses of formula

What has changed here?

```
boxplot(log10(Impressions) ~ Paid + Category, data = facebook, col = "lightgrey")
```



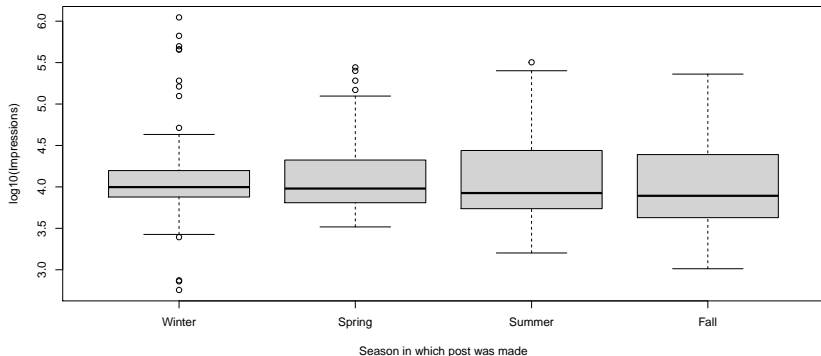
Note the labels on the horizontal axis.

Comments?

Facebook data - other uses of formula

A numeric variate could be made categorical using `cut()`

```
boxplot(log10(Impressions) ~ cut(Post.Month, 4,  
                                labels = c("Winter", "Spring", "Summer", "Fall")),  
        xlab = "Season in which post was made",  
        data = facebook, col = "lightgrey")
```

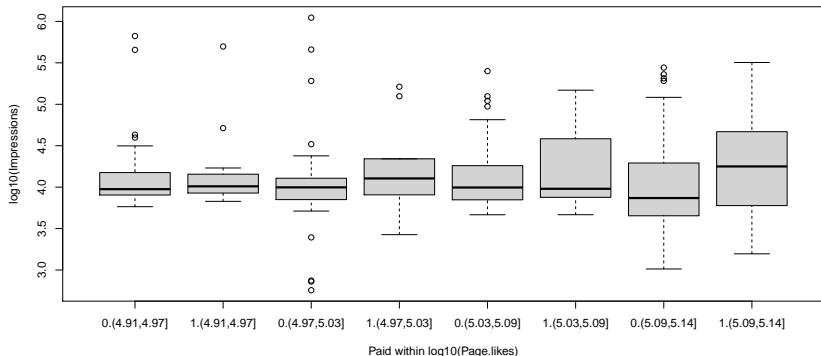


Comments?

Facebook data - other uses of formula

This one is ratio-scaled (as opposed to ordinal)

```
boxplot(log10(Impressions) ~ Paid + cut(log10(Page.likes), 4),  
        xlab = "Paid within log10(Page.likes)",  
        data = facebook, col = "lightgrey")
```

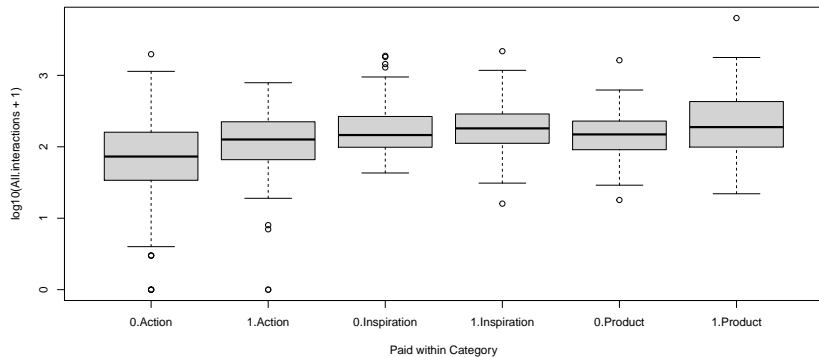


Comments?

Facebook data - other uses of formula

Change response.

```
boxplot(log10(All.interactions + 1) ~ Paid + Category,  
        xlab = "Paid within Category",  
        data = facebook, col = "lightgrey")
```



Comments?

Facebook data - other uses of formula

Change response. Fitted model

```
fit3 <- lm(log10(All.interactions + 1) ~ Paid + Category, data = facebook)
summary(fit3)
```

```
##
## Call:
## lm(formula = log10(All.interactions + 1) ~ Paid + Category, data = facebook)
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -1.95563 -0.23538  0.01645  0.26075  1.49121
##
## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept)    1.80436    0.03615  49.907 < 2e-16 ***
## Paid           0.15127    0.04857   3.115  0.00195 **
## CategoryInspiration 0.39403    0.05121   7.695 7.74e-14 ***
## CategoryProduct  0.36251    0.05417   6.692 5.96e-11 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.4859 on 495 degrees of freedom
## (1 observation deleted due to missingness)
## Multiple R-squared:  0.1433, Adjusted R-squared:  0.1382
## F-statistic: 27.61 on 3 and 495 DF,  p-value: < 2.2e-16
```

Comments?

Facebook data - other uses of formula

Change response. A slightly different fitted model

```
fit4 <- lm(log10(All.interactions + 1) ~ Paid * Category, data = facebook)
summary(fit4)
```

```
##
## Call:
## lm(formula = log10(All.interactions + 1) ~ Paid * Category, data = facebook)
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -2.01793 -0.23100  0.02586  0.27246  1.51761
##
## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept)      1.77795    0.03941  45.111 < 2e-16 ***
## Paid              0.23998    0.07224   3.322  0.00096 ***
## CategoryInspiration 0.46570    0.06040   7.711 6.96e-14 ***
## CategoryProduct    0.37776    0.06302   5.994 3.95e-09 ***
## Paid:CategoryInspiration -0.25185    0.11299  -2.229  0.02627 *
## Paid:CategoryProduct  -0.04374    0.12234  -0.358  0.72083
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.4843 on 493 degrees of freedom
## (1 observation deleted due to missingness)
## Multiple R-squared:  0.1524, Adjusted R-squared:  0.1438
## F-statistic: 17.72 on 5 and 493 DF,  p-value: 3.639e-16
```

Comments?

Facebook data - other uses of formula

Change response. A slightly different fitted model

```
fit4$coefficients
```

##	(Intercept)	Paid	CategoryInspiration
##	1.77795481	0.23997989	0.46569543
##	CategoryProduct	Paid:CategoryInspiration	Paid:CategoryProduct
##	0.37776394	-0.25185230	-0.04374152

```
head(model.matrix(fit4))
```

##	(Intercept)	Paid	CategoryInspiration	CategoryProduct	Paid:CategoryInspiration
## 1	1	0	0	1	0
## 2	1	0	0	1	0
## 3	1	0	1	0	0
## 4	1	1	0	1	0
## 5	1	0	0	1	0
## 6	1	0	0	1	0
##	Paid:CategoryProduct				
## 1	0				
## 2	0				
## 3	0				
## 4	1				
## 5	0				
## 6	0				

Comments?

Facebook data - other uses of formula

Change explanatory variates to two factors

```
fit5 <- lm(log10(All.interactions + 1) ~ Type * Category, data = facebook)
summary(fit5)
```

```
##
## Call:
## lm(formula = log10(All.interactions + 1) ~ Type * Category, data = facebook)
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -1.83783 -0.23716  0.01508  0.25688  1.60199
##
## Coefficients: (2 not defined because of singularities)
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept)      1.73278    0.10895   15.904 <2e-16 ***
## TypePhoto         0.10504    0.11469    0.916  0.3602
## TypeStatus        0.36213    0.30167    1.200  0.2306
## TypeVideo         0.65020    0.21397    3.039  0.0025 **
## CategoryInspiration  0.20172    0.49927    0.404  0.6864
## CategoryProduct   -0.03381    0.49927   -0.068  0.9460
## TypePhoto:CategoryInspiration  0.20383    0.50213    0.406  0.6850
## TypeStatus:CategoryInspiration -0.09280    0.62270   -0.149  0.8816
## TypeVideo:CategoryInspiration      NA         NA      NA      NA
## TypePhoto:CategoryProduct  0.39575    0.50315    0.787  0.4319
## TypeStatus:CategoryProduct  0.16441    0.57849    0.284  0.7764
## TypeVideo:CategoryProduct      NA         NA      NA      NA
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.4872 on 490 degrees of freedom
## Multiple R-squared:  0.1473, Adjusted R-squared:  0.1316
## F-statistic: 9.405 on 9 and 490 DF, p-value: 3.024e-13
```

Comments?

Facebook data - other uses of formula

Just the coefficients.

```
fit5$coefficients
```

```
##              (Intercept)              TypePhoto
##              1.73278345              0.10504178
##              TypeStatus              TypeVideo
##              0.36213204              0.65020444
##              CategoryInspiration          CategoryProduct
##              0.20171500              -0.03381344
## TypePhoto:CategoryInspiration TypeStatus:CategoryInspiration
##              0.20382942              -0.09279854
## TypeVideo:CategoryInspiration          TypePhoto:CategoryProduct
##              NA              0.39574802
##              TypeStatus:CategoryProduct          TypeVideo:CategoryProduct
##              0.16440892              NA
```

Facebook data - other uses of formula

And now the corresponding model matrix.

```
fit5$coefficients
```

```
##              (Intercept)              TypePhoto
##              1.73278345              0.10504178
##              TypeStatus              TypeVideo
##              0.36213204              0.65020444
##              CategoryInspiration      CategoryProduct
##              0.20171500              -0.03381344
## TypePhoto:CategoryInspiration TypeStatus:CategoryInspiration
##              0.20382942              -0.09279854
## TypeVideo:CategoryInspiration      TypePhoto:CategoryProduct
##              NA              0.39574802
##              TypeStatus:CategoryProduct      TypeVideo:CategoryProduct
##              0.16440892              NA
```

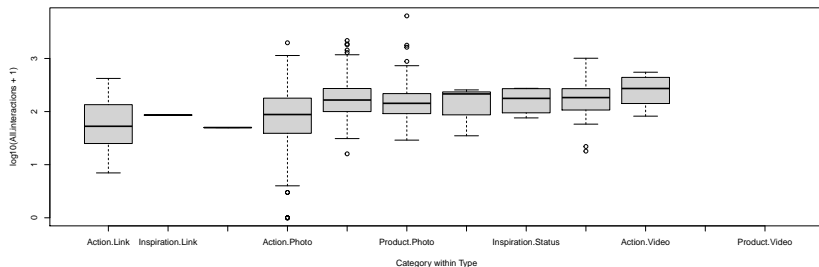
```
head(model.matrix(fit5))
```

```
##      (Intercept) TypePhoto TypeStatus TypeVideo CategoryInspiration
## 1              1          1          0          0              0
## 2              1          0          1          0              0
## 3              1          1          0          0              1
## 4              1          1          0          0              0
## 5              1          1          0          0              0
## 6              1          0          1          0              0
##      CategoryProduct TypePhoto:CategoryInspiration TypeStatus:CategoryInspiration
## 1              1              0              0
## 2              1              0              0
## 3              0              1              0
## 4              1              0              0
## 5              1              0              0
## 6              1              0              0
```

Facebook data - other uses of formula

Boxplots of All.interactions, same model.

```
boxplot(log10(All.interactions + 1) ~ Category + Type,  
        xlab = "Category within Type",  
        data = facebook, col = "lightgrey")
```

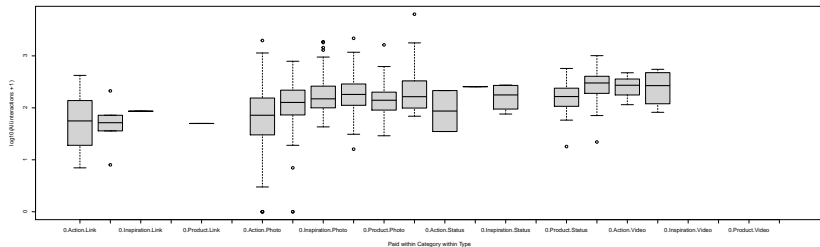


Comments?

Facebook data - other uses of formula

Boxplots of All.interactions and lots of combinations

```
boxplot(log10(All.interactions + 1) ~ Paid + Category + Type,  
        xlab = "Paid within Category within Type",  
        data = facebook, col = "lightgrey")
```



Comments?