

# Binomial random variables

56 marks

Suppose  $X \sim \text{Binomial}(n, p)$ , then

$$Pr(X = x) = \binom{n}{x} p^x (1-p)^{n-x} \quad x = 0, 1, \dots, n.$$

from which it follows that  $E(X) = np$  and  $Var(X) = np(1-p)$ .

Once  $x$  is observed, the unknown proportion can be estimated (e.g. via maximum likelihood) as the numerical value of  $\hat{p} = x/n$ ; the corresponding random estimator would be  $\tilde{p} = X/n$ .

In this question, you are going to develop your intuition about simple proportions as random variables through a bit of mathematics, some introductory R programming, and a little simulation and visualization.

- a. The **odds** (in favour of the event) are defined by the ratio  $p/(1-p)$  (e.g. even odds are 1, or 1:1, when  $p = 1/2$ ; odds are 9:1 in favour when  $p = 9/10$ ; odds less than one are often inverted to be described as the odds *against* the event).

Suppose we are interested in comparing the binomial probabilities  $Pr(X = a)$  to  $Pr(X = b)$ . For example, the ratio,  $Pr(X = a)/Pr(X = b)$ , tells us how many times more (or less) likely it is to observe  $x = a$  than  $x = b$ .

- i. (1 mark) Express the ratio  $Pr(X = a)/Pr(X = b)$  as a function of the odds.

$$\frac{Pr(X = a)}{Pr(X = b)} = \frac{\binom{n}{a} p^a (1-p)^{n-a}}{\binom{n}{b} p^b (1-p)^{n-b}} = \frac{\binom{n}{a}}{\binom{n}{b}} \frac{p^{a-b}}{(1-p)^{-n+a+n-b}} = \frac{\binom{n}{a}}{\binom{n}{b}} \frac{p^{a-b}}{(1-p)^{a-b}} = \frac{\binom{n}{a}}{\binom{n}{b}} \text{odds}^{a-b}$$

- ii. (4 marks) Here, in two different ways, you will write a function of the odds which calculates  $Pr(X = a)/Pr(X = b)$  for a given  $n$ .

For the first way, write `prob_ratio1()` to do the calculation using the function `choose()`:

```
prob_ratio1 <- function (n, a, b, odds = 1) {  
  choose_n_a <- choose(n,a)  
  choose_n_b <- choose(n,b)  
  ans <- choose_n_a/choose_n_b*odds^(a-b)  
  return(ans)  
}  
# print(prob_ratio1(5,3,4,5))
```

For the second way, write `prob_ratio1()` to do the calculation using the function `dbinom()`:

```
prob_ratio2 <- function (n, a, b, odds = 1) {  
  p <- odds/(1+odds)  
  p_a <- dbinom(a,size=n,prob=p)  
  p_b <- dbinom(b,size=n,prob=p)  
  return(p_a/p_b)  
}  
# print(prob_ratio2(5,3,4,5))
```

Both are calculating the same values.

iii. \*(2 marks)\* Report the following values for `prob_ratio1()` and `prob_ratio2()`

```
# using choose()
prob_ratio1(50, a = 5, b = 45)

## [1] 1

prob_ratio1(50, a = 5, b = 45, odds = 9)

## [1] 6.765496e-39

# and using dbinom()
prob_ratio2(50, a = 5, b = 45)

## [1] 1

prob_ratio2(50, a = 5, b = 45, odds = 9)

## [1] 6.765496e-39
```

b. Extreme proportions like  $\hat{p} \approx 0$  or  $\hat{p} \approx 1$  often generate a great deal of interest in an analysis.

For example,  $p$  might be the proportion of people in some population who perhaps die from some exposure to some toxin, or are cured of a disease by some treatment, or maybe just say they would vote for a particular party or candidate. In any of these cases it can be surprising (even alarming) to see either  $\hat{p} \approx 0$  or  $\hat{p} \approx 1$ , so much so that some explanation seems in order.

Suppose we have observed  $x$  from  $Binomial(n, p)$  and  $y$  from  $Binomial(m, p)$  – that is the same probability of occurrence but different sample sizes. Denote the observed proportions as  $\hat{p}_x = x/n$  and  $\hat{p}_y = y/m$ , respectively.

i. (2 marks) Give the mathematical expression for the ratio

$$\frac{Pr(\tilde{p}_x = 0)}{Pr(\tilde{p}_y = 0)}$$

and for the ratio

$$\frac{Pr(\tilde{p}_x = 1)}{Pr(\tilde{p}_y = 1)}$$

$$\frac{Pr(\tilde{p}_x = 0)}{Pr(\tilde{p}_y = 0)} = \frac{\binom{n}{0}p^0(1-p)^n}{\binom{m}{0}p^0(1-p)^m} = \frac{\binom{n}{0}}{\binom{m}{0}}(1-p)^{n-m} = (1-p)^{n-m}$$

$$\frac{Pr(\tilde{p}_x = 1)}{Pr(\tilde{p}_y = 1)} = \frac{\binom{n}{1}p^1(1-p)^0}{\binom{m}{1}p^1(1-p)^0} = \frac{\binom{n}{1}}{\binom{m}{1}}p^{n-m} = p^{n-m}$$

ii. (2 marks) Describe what happens to each of these ratios for  $n < m$  and  $m$  increases.

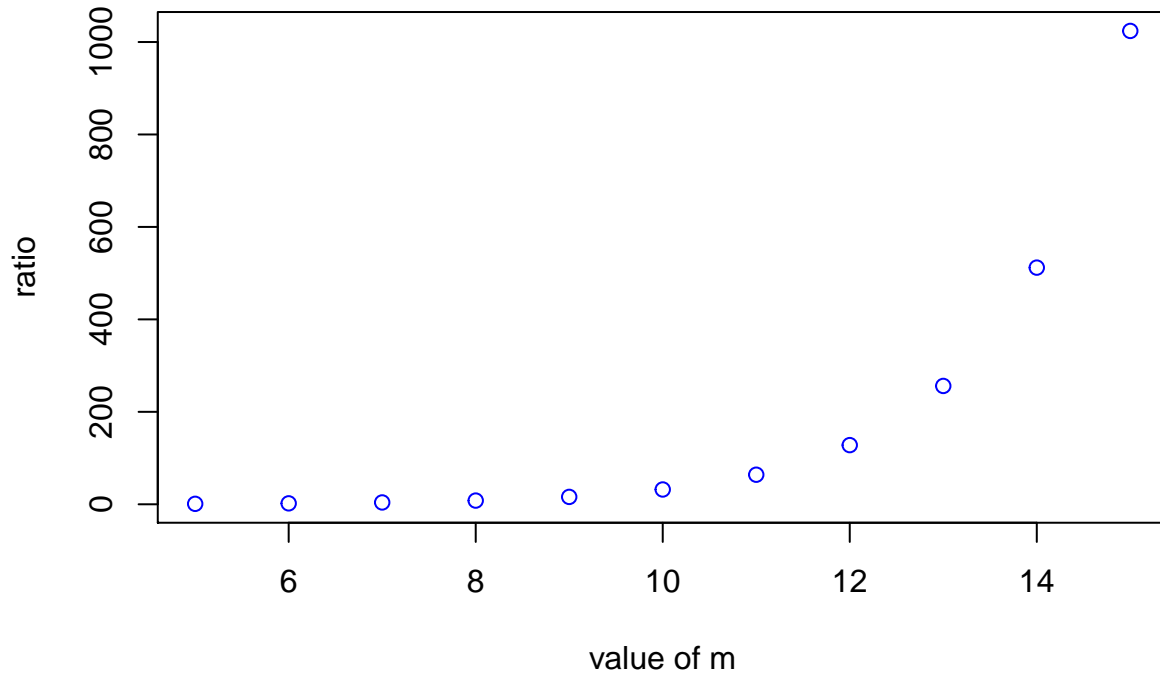
- As  $m$  increases, the first ratio will increase. The base  $1-p$  is between 0 and 1 and the exponent decreases from 0. The power will increase if  $m$  increases.
- As  $m$  increases, the second ratio will increase. The base  $p$  is between 0 and 1 and the exponent decreases from 0. The power will increase if  $m$  increases.

iii. (3 marks) Suppose  $p = 1/2$ ,  $n = 5$  and  $m \in \{5, 6, 7, \dots, 14, 15\}$ . Using the `plot()` function (with appropriate title and axis labels), plot the curve of the pair  $(n, m)$  for all values.

```

ms <- seq(from=5, to=15)
get_ratio <- function(p,n,ms){
  ans = c()
  for (i in 1:length(ms)){
    ans <- c(ans, p^(n-ms[i]))
  }
  return(ans)
}
ratios <- get_ratio(1/2,5,ms)
plot(ms,ratios,xlab="value of m",ylab="ratio",col="blue")

```



iv. \*(2 marks)\* In plain English, express when  $\widehat{p}$  is most likely to be 1 and when you should

c. For the binomial definition, as given above:

i. (2 marks) Mathematically derive  $E(\tilde{p})$  and the **standard deviation**  $SD(\tilde{p})$ .

$$\tilde{p} = \frac{X}{n}$$

$$E(\tilde{p}) = E\left(\frac{1}{n} \cdot X\right) = \frac{1}{n} E(X)$$

$$E(X) = n \cdot p$$

$$E(\tilde{p}) = \frac{1}{n} E(n \cdot p) = \frac{1}{n} \cdot n \cdot E(p) = p$$

$$Var(\tilde{p}) = Var\left(\frac{X}{n}\right) = \frac{1}{n^2} Var(X)$$

$$Var(X) = E(X^2) - E^2(X) = np(1-p)$$

$$Var(\tilde{p}^2) = Var\left(\frac{X}{n}\right) = \left(\frac{1}{n}\right)^2 Var(X) = \frac{1}{n^2} np(1-p) = \frac{p(1-p)}{n}$$

$$SD(\tilde{p}) = \sqrt{Var(\tilde{p})} = \sqrt{\frac{p(1-p)}{n}}$$

- ii. (2 marks) Write a function that calculates the standard deviation of  $\tilde{p}$  for any pair of values of  $n$  and  $p$  as follows:

```
sd_p_wig <- function (n, p) {
  # Your code here
  return(sqrt(p*(1-p)/n))
}
```

- d. Chebyshev's inequality relates the nearness of a random variable  $Y$  to its expectation  $\mu$  as a function of its standard deviation  $\sigma$  (provided both exist and are finite) as follows:

$$Pr(|Y - \mu| \geq k\sigma) \leq \frac{1}{k^2}$$

for any constant  $k > 1$ .

- i. (2 marks) Write this inequality when  $Y = \tilde{p}$ , the binomial proportion estimator.

$$\mu = E(\tilde{p}) = \frac{X}{n} = p \quad Y = \tilde{p} \quad \sigma = SD(\tilde{p}) = \sqrt{\frac{p(1-p)}{n}}$$

$$Pr(|Y - \mu| \geq k\sigma) \leq \frac{1}{k^2} = Pr\left(|\tilde{p} - p| \geq k\sqrt{\frac{p(1-p)}{n}}\right) \leq \frac{1}{k^2}$$

- ii. (4 marks) Suppose that  $k = 5$ . As a function of  $p$ , mathematically express the sample size  $n$  needed to ensure by Chebyshev's inequality that for our estimator  $\tilde{p}$  we have

$$Pr\left(|\tilde{p} - p| \geq \frac{1}{50}\right) \leq \frac{1}{25}.$$

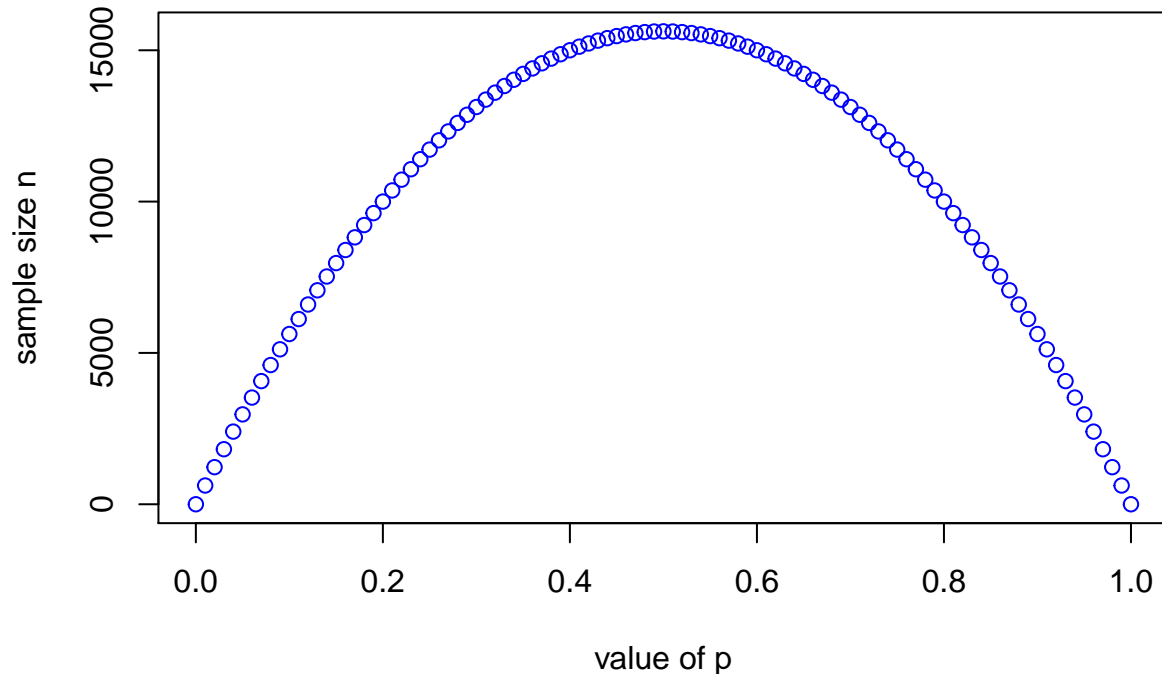
$$k\sqrt{\frac{p(1-p)}{n}} = 1/50$$

$$k = 5$$

$$\sqrt{\frac{p(1-p)}{n}} = \frac{1}{250}n = 250^2 p(1-p)$$

Using the function `plot()`, with appropriate title and x and y axis labels, plot the curve of  $n$  as a function of  $p$  for  $p = \text{seq}(0,1, 0.01)$ .

```
p <- seq(0,1,0.01)
get_n <- function(p_list){
  ans = c()
  for (i in 1:length(p_list)){
    ans <- c(ans, ((250^2)*p[i]*(1-p[i])))
  }
  return(ans)
}
n_list <- get_n(p)
plot(p,n_list,xlab="value of p",ylab="sample size n",col="blue")
```



- iii. (4 marks) Suppose that  $k = 5$ , and  $n = 2500$ . As a function of  $p$ , mathematically express the bound  $B$  given by Chebyshev's inequality, so that for our estimator  $\tilde{p}$  we have

$$Pr(|\tilde{p} - p| \geq B) \leq \frac{1}{25}.$$

Using the function `plot()`, with appropriate title and x and y axis labels, plot the curve of  $B$  as a function of  $p$  for `p = seq(0,1, 0.01)`. iv (2 marks) In simple English, summarize what the largest  $B$  says about how well  $p$  is likely to be estimated when  $n = 2500$  according to Chebyshev's inequality.

$$Pr(|\tilde{p} - p| \geq B) \leq \frac{1}{25}$$

$$Pr(|\tilde{p} - p| \geq k \sqrt{\frac{p(1-p)}{n}}) \leq \frac{1}{25}$$

$$B = k \sqrt{\frac{p(1-p)}{n}} \quad k = 5 \quad n = 2500$$

$$B = \frac{\sqrt{p(1-p)}}{10}$$

d. The functions `plot()`, `lines()`, and `abline()` can be used to plot some data, to add curves, and to add a straight line to a plot. To learn more, type `help("plot.default")`, `help("lines")`, and `help("abline")`.

Use these plotting functions, and appropriate arguments (including meaningful titles, and axis labels),

- i. (3 marks) As a function of `p = seq(0, 1, 0.01)` for a fixed `n = 10`. Add a vertical **dashed** line in **red** at the value of  $p$  which maximizes the standard deviation.

Hand in your code and plot.

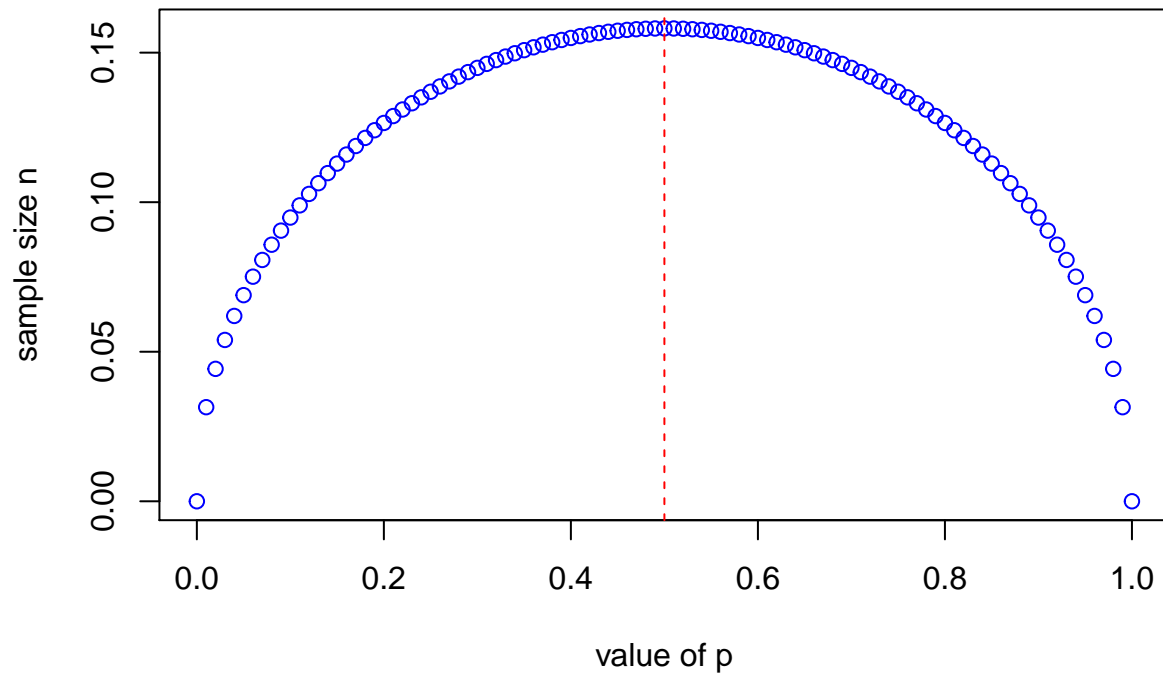
```
p_list <- seq(0, 1, 0.01)
n <- 10
get_sd <- function(n,list_p){
  ans = c()
  for (i in 1:length(p_list)){
```

```

    ans <- c(ans, sqrt(p[i]*(1-p[i])/n))
  }
  return(ans)
}
sd_list <- get_sd(n,p_list)
max_index = which.max(sd_list)

plot(p_list,sd_list,xlab="value of p",ylab="sample size n",col="blue")
abline(v=p_list[max_index],lty=2,col="red")

```



ii. (4 marks) As a function of  $n \in \{5, 10, 15, \dots, 50\}$  for the different values of  $p \in \{0.1, 0.3, 0.5, 0.8\}$ .

- Use a different colour and line type for each curve (i.e. value of `p`).
- Use `lwd = 2` for all curves.
- Use `legend()` to add a legend to the "topright" corner of the plot, identify each curve by its `p` value.
- Hand in your code and plot.

```

n_list = seq(5,50,5)
p_list = c(0.1,0.3,0.5,0.8)
colors = c("red","green","pink","maroon")

get_sd <- function(n_list,p_list){
  ans_list = c()
  for(i in 1:length(p_list)){
    ans <- c(sapply(n_list, function(n) sqrt(p_list[i]*(1-p_list[i])/n)))
    ans_list <- c(ans_list,ans)
    print(typeof(ans))
  }
  return(ans_list)
}

vals<-get_sd(n_list, p_list)

```

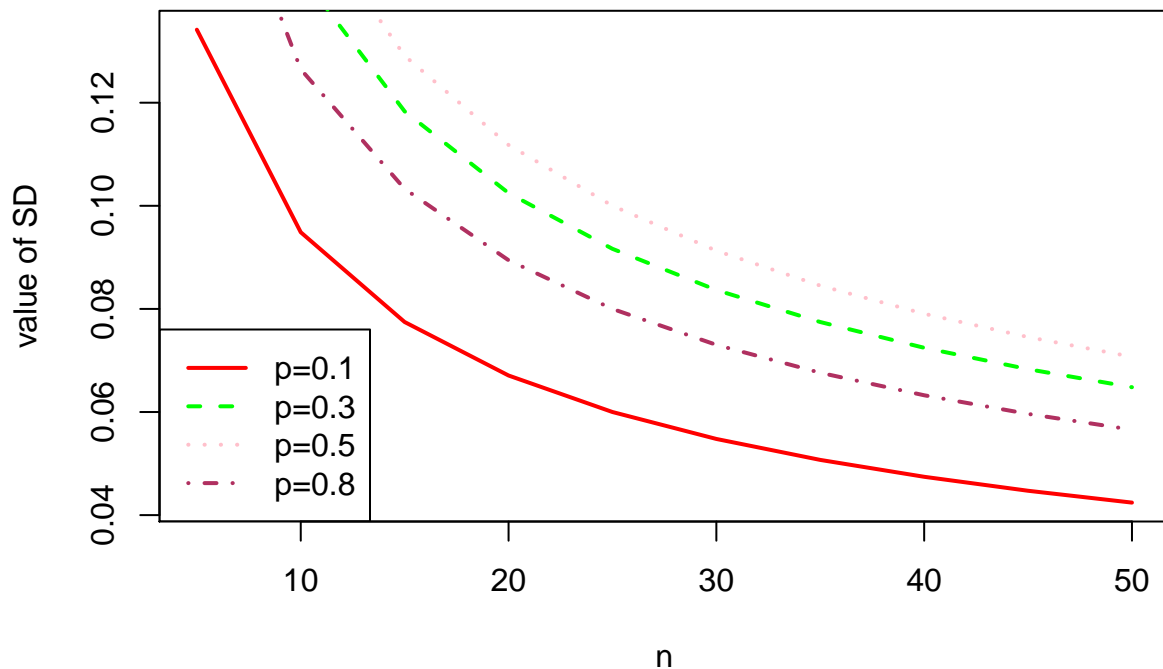
```
## [1] "double"
## [1] "double"
## [1] "double"
## [1] "double"

dim(vals)<-c(10,4)

plot(n_list,vals[,1],lwd = 2,lty=1, type="l",col=colors[1],xlab="n",ylab="value of SD")
for(i in 2:length(p_list)){
  print(length(n_list))
  print(vals[,i])
  lines(n_list,vals[,i],lwd = 2,lty=i, type ="l",col=colors[i])
}

## [1] 10
## [1] 0.20493902 0.14491377 0.11832160 0.10246951 0.09165151 0.08366600
## [7] 0.07745967 0.07245688 0.06831301 0.06480741
## [1] 10
## [1] 0.22360680 0.15811388 0.12909944 0.11180340 0.10000000 0.09128709
## [7] 0.08451543 0.07905694 0.07453560 0.07071068
## [1] 10
## [1] 0.17888544 0.12649111 0.10327956 0.08944272 0.08000000 0.07302967
## [7] 0.06761234 0.06324555 0.05962848 0.05656854

legend("bottomleft",
  legend = lapply(p_list, function(x) paste("p=",x,sep="")),
  #legend = p_list,
  col = colors,
  lty=c(1,2,3,4),
  lwd=2
)
```



iii. (2 marks) Comment on your findings about the dependency of the standard deviation of the binomial proportion estimator  $\tilde{p}$  as a function of  $n$  and  $p$ .

e. The R function `rbinom()` can be used to generate pseudo-random values  $x$  from a binomial distribution. In this question, you will examine the samples of the sample **proportions**,  $x/n$ , drawn from binomials with the same value of  $p$  but different values of  $n$ .

- i. (3 marks) Using `rbinom()`, for every  $n \in \{5, 10, 15, \dots, 45, 50\}$  generate 100 proportions, each one based on an independent pseudo-random value from  $\text{Binomial}(n, 0.5)$ .

You might find the `rep()` function useful in constructing  $n$ .

Show your code.

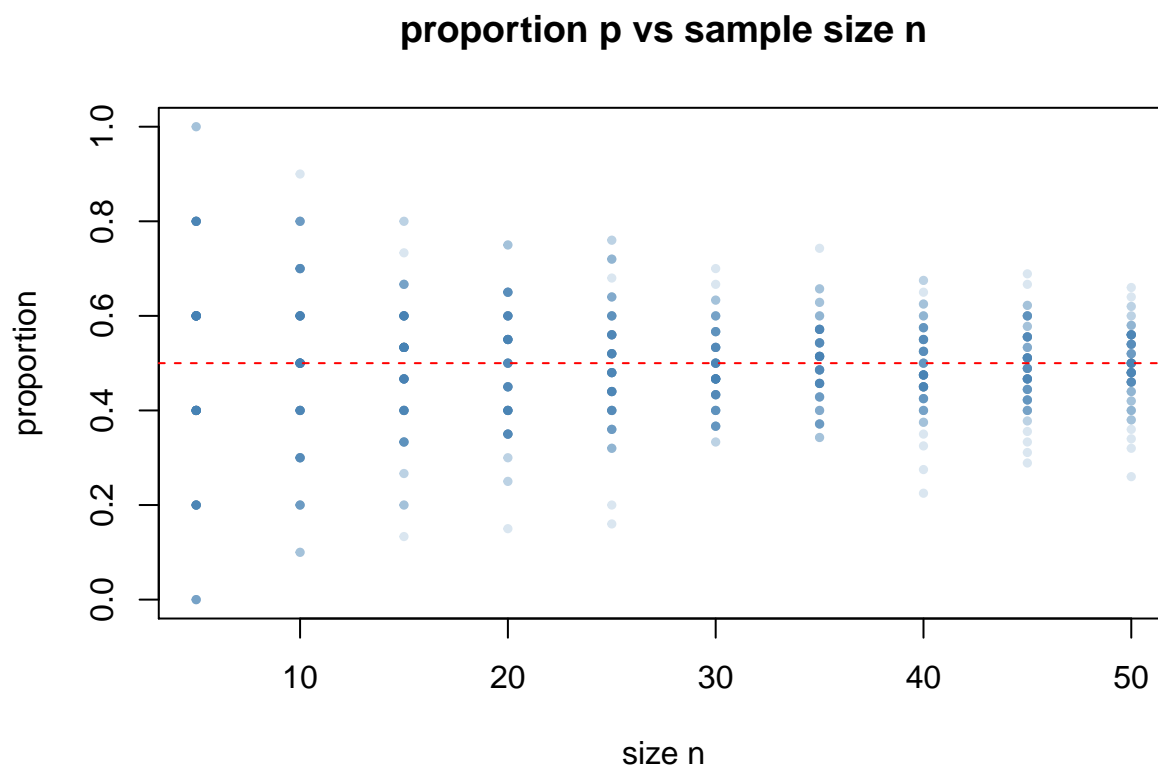
```
n_list = seq(5,50,5)
vals<- sapply(n_list, function(n) rbinom(100,n,0.5)/n)
```

- ii. (3 marks) Plot the pairs  $(n, \hat{p})$  as points (use plot arguments `ylim = c(0,1)`, `pch = 19`).

Note that there will be 100 proportions for every  $n$ .

Add a red dashed horizontal line at  $p = 0.5$ .

```
n_list = seq(5,50,5)
vals<- sapply(n_list, function(n) rbinom(100,n,0.5)/n)
vals_for_plot = unlist(vals)
n_for_plot = rep(seq(5,50,5),each = 100)
plot(n_for_plot, vals_for_plot, ylim = c(0,1), pch = 19, col =adjustcolor("steelblue", 0.2), cex = 0.5,
     xlab = "size n",ylab = "proportion",main="proportion p vs sample size n")
abline(h=0.5,lty=2,col="red")
```



- iii. (2 marks) Repeat the production of the above plot, complete with horizontal line, but instead of using  $n$  as the  $x$  variable in the plot, use `jitter(n, 2)`.

Show your code and resulting plot.

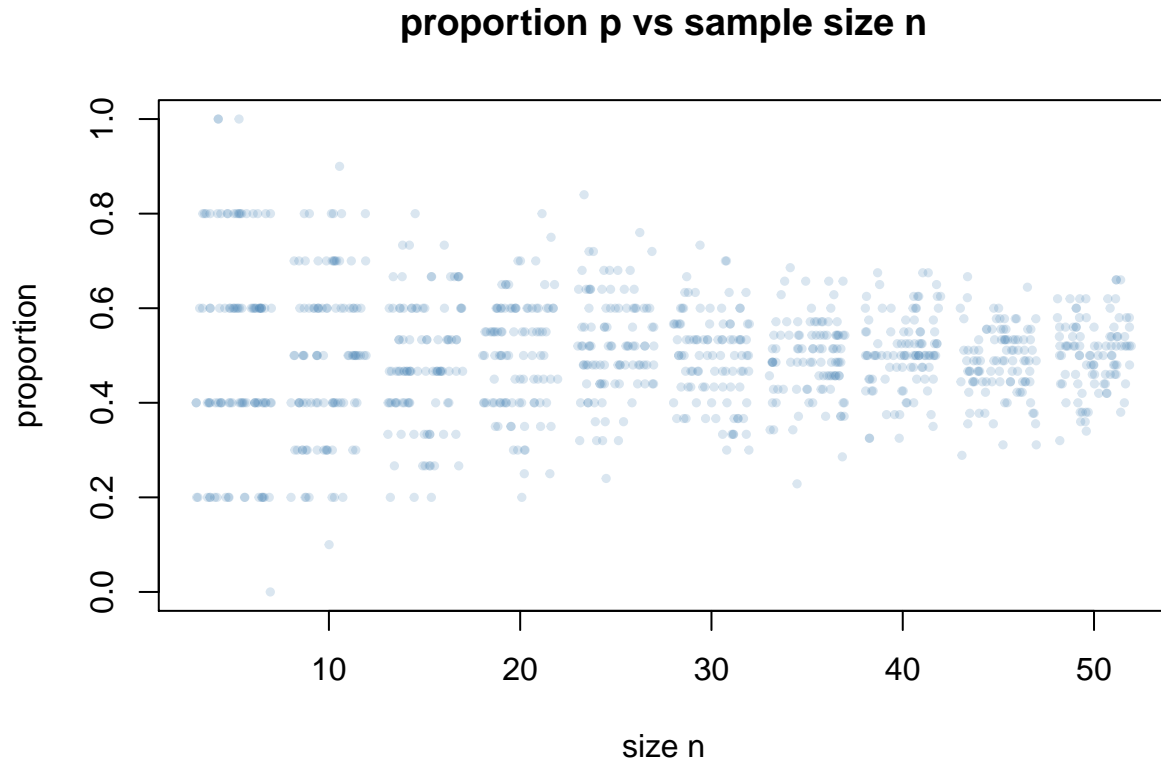


```

      Comment on the effect of `jitter()``
n_list = seq(5,50,5)
vals<- sapply(n_list, function(n) rbinom(100,n,0.5)/n)
vals1 = unlist(vals)
n1 = rep(seq(5,50,5),each = 100)
n2 = jitter(n_for_plot,2)

plot(jitter(n1,2), vals1, ylim = c(0,1), pch = 19, col =adjustcolor("steelblue", 0.2), cex = 0.5,
     xlab = "size n",ylab = "proportion",main="proportion p vs sample size n")

```



- The effect of jitter add random variation to the values of  $n$ .
  - Then there can be an estimations of the proportions for sample sizes other than  $[5, 10, \dots, 50]$
- iv. (1 mark) Based on the either of the above plots, what do you conclude about the distribution of binominal proportions as  $n$  increases?
- As  $n$  increases, the distribution of binominal proportions moves towards the real probability (which is 0.5 in this case).
- g. (6 marks) With supporting reference to any/all suitable discoveries you have made in the above questions, comment on each of the following:
- i. Which values for the true binomial probability  $p$  are hardest/easiest to estimate from a sample. Why?
    - $P = 0.5$  is hardest to estimate because from the plot,  $P = 0.5$  gives the largest standard deviation when keeping sample size  $n$  constant.
    - $P = 0.1$  is hardest to estimate because from the plot,  $P = 0.1$  gives the smallest standard deviation when keeping sample size  $n$  constant.
  - ii. Law of large numbers? What can you say about the effect of increasing sample size  $n$  on the quality and/or interpretation of your estimate  $\hat{p}$  of  $p$ ?

- The average of the results obtained from a large number of trials should be close to the expected value.(Wikipedia, [https://en.wikipedia.org/wiki/Law\\_of\\_large\\_numbers](https://en.wikipedia.org/wiki/Law_of_large_numbers))
  - Therefore, increasing the sample size  $n$  will make a more precise prediction of  $\hat{p}$  (closer to  $p$ )
- iii. Law of small numbers? What can you say about the effect of decreasing sample size  $n$  on the quality and/or interpretation of your estimate  $\hat{p}$  of  $p$ ?
- \* Small number of trials can give imprecise result(far away from the expectation).
  - \* Because there are chances that not all possibilities are covered with a small sample size,
  - \* likelihood of a Type II error skewing the results can be increased.