Derivation of the Slope and Intercept in Linear Regression

In this document, we will explore how to derive the slope (m) and intercept (b) in simple linear regression by minimizing the sum of squared errors (SSE). The example will demonstrate how to compute the slope using partial derivatives and the step-by-step process to reach the closed-form solution for both m and b.

Example Data Set

Consider the following data points (x, y):

x	2	4	6	8
y	4	8	10	12

Our goal is to find the line of best fit ($y = mx + b$) for this data set.

Step 1: Calculate the Mean of x and y

The mean of x is:

$\bar{x} = (2 + 4 + 6 + 8) / 4 = 5$

The mean of y is:

$\bar{y} = (4 + 8 + 10 + 12) / 4 = 8.5$

Step 2: Derivation of the Slope (m)

We use the formula for the slope (m):

$m = \sum(x_i - \bar{x})(y_i - \bar{y}) / \sum(x_i - \bar{x})^2$

Let’s calculate the numerator ($\Sigma(xi - \bar{x})(yi - \bar{y})$) and denominator ($\Sigma(xi - \bar{x})^2$):

The table below shows the steps for calculating the necessary values for slope (m):

x	y	$x - \bar{x}$	$y - \bar{y}$	$(x - \bar{x})(y - \bar{y})$	$(x - \bar{x})^2$
2	4	-3	-4.5	13.5	9
4	8	-1	-0.5	0.5	1
6	10	1	1.5	1.5	1
8	12	3	3.5	10.5	9

Now, summing the columns:

$\Sigma(x_i - \bar{x})(y_i - \bar{y}) = 13.5 + 0.5 + 1.5 + 10.5 = 26$
$\Sigma(x_i - \bar{x})^2 = 9 + 1 + 1 + 9 = 20$

Thus, the slope m is:

$m = 26 / 20 = 1.3$

Step 3: Calculate the Intercept (b)

Now that we know the slope (m), we can calculate the intercept (b) using the formula:

$b = \bar{y} - m * \bar{x}$

Substituting the values of $\bar{x} = 5$, $\bar{y} = 8.5$, and $m = 1.3$:

$b = 8.5 - (1.3 * 5) = 8.5 - 6.5 = 2$

Thus, the intercept b is 2.

Final Equation

The final linear regression equation that best fits the data is:

$y = 1.3x + 2$

Step-by-Step Derivation Using Partial Derivatives

We can derive the formula for m by minimizing the sum of squared errors (SSE). The SSE is:

$SSE = \Sigma(y_i - (mx_i + b))^2$

We take the partial derivatives of SSE with respect to m and b and set them to zero to find the values of m and b that minimize SSE.

1. Derivative with Respect to b

The derivative of SSE with respect to b is:

$\partial/\partial b \; SSE = -2\, \Sigma(y_i - mx_i - b) = 0$

This leads to the equation:

$\Sigma y_i = m\, \Sigma x_i + b * n$

Dividing by n and rearranging, we get:

$b = \bar{y} - m * \bar{x}$

2. Derivative with Respect to m

The derivative of SSE with respect to m is:

$$\partial/\partial m \; SSE = -2 \, \Sigma x_i(y_i - m x_i - b) = 0$$

To solve for m, substitute $b = \bar{y} - m * \bar{x}$ into this equation:

Substitute the expression for b into the equation:

$$\Sigma x_i \, (y_i - m x_i - (\bar{y} - m \, \bar{x})) = 0$$

This simplifies to:

$$\Sigma x_i \, (y_i - \bar{y} - m \, (x_i - \bar{x})) = 0$$

Now distribute x_i over the terms in parentheses:

$$\Sigma x_i \, (y_i - \bar{y}) - m \, \Sigma x_i(x_i - \bar{x}) = 0$$

Now, recognize that $\Sigma x_i(x_i - \bar{x})$ is equal to $\Sigma(x_i - \bar{x})^2$. Solving for m, we get:

$$m = \Sigma(x_i - \bar{x})(y_i - \bar{y}) \,/\, \Sigma(x_i - \bar{x})^2$$