Linear Algebra

Serob Tigranyan

May 22, 2025

Contents

1 Task 1

Given the following basis vectors:

$$e_1 = (1, 1, 3)$$

$$e_2 = (1, 1, 0)$$

$$e_3 = (1, 2, 0)$$

And vector x:

$$x = (5, 4, 6)$$

1.1 Linear Independence

We need to check if the given basis vectors are valid:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 3 & 0 & 0 \end{bmatrix}$$

Determine the Rank by first finding the determinant:

$$D(A) = 0 + 6 + 0 - 3 - 0 - 0 = 3$$

Since our determinant is non-zero, then we have a full rank which is 3 which further means that the basis vectors are linearly independent.

$$rank(A) = 3$$

1.2 Vector expressed in given basis

We need to express vector x in our basis e_1, e_2, e_3.

$$x = \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3$$

Therefore we have:

$$(5, 4, 6) = \lambda(1, 1, 3) + \lambda(1, 1, 0) + \lambda(1, 2, 0)$$

$$(5, 4, 6) = (\lambda_1 + \lambda_2 + \lambda_3; \lambda_1 + \lambda_2 + 2\lambda_3; 3\lambda_1)$$

We form a linear system of equations:

$$\begin{cases} \lambda_1 + \lambda_2 + \lambda_3 = 5 \\ \lambda_1 + \lambda_2 + 2\lambda_3 = 4 \\ 3\lambda_1 = 6 \end{cases}$$

Now we solve the system:

$$\begin{cases} \lambda_1 + \lambda_2 + \lambda_3 = 5 \\ \lambda_1 + \lambda_2 + 2\lambda_3 = 4 \\ \lambda_1 = 2 \end{cases}$$

$$\begin{cases} \lambda_2 = 3 - \lambda_3 \\ 2 + (3 - \lambda_3) + 2\lambda_3 = 4 \\ \lambda_1 = 2 \end{cases}$$

$$\begin{cases} \lambda_2 = 3 - \lambda_3 \\ \lambda_3 = -1 \\ \lambda_1 = 2 \end{cases}$$

$$\begin{cases} \lambda_2 = 4 \\ \lambda_3 = -1 \\ \lambda_1 = 2 \end{cases}$$

Therefore our vector x expressed in our basis vectors e_1, e_2, e_3 is:

$$x_e = (2, 4, -1)$$

2 Task 2

Given the vectors a_1 and a_2 we need to find if they're orthogonal and if so, then find their orthogonal basis vectors:

$$a_1 = (4, 4, 1, -1)$$

$$a_2 = (1, -1, 1, 1)$$

2.1 Orthogonality

To check if these two are orthogonal we perform scalar multiplication on both of them:

$$(a_1 \cdot a_2) = 4 - 4 + 1 - 1 = 0 \Rightarrow a_1 \perp a_2$$

Our vectors are indeed orthogonal, now we need to find their orthogonal basis vectors.

2.1.1 Orthogonal Basis Vectors

To find the orthogonal basis vectors we pick arbitrary vector:

$$\gamma = (x, y, z, w)$$

So that:

$$\gamma \perp a_1$$

$$\gamma \perp a_2$$

We form a linear system of equations from this like so:

$$\begin{cases} (\gamma \cdot a_1) = 0 \\ (\gamma \cdot a_2) = 0 \end{cases} \Rightarrow \begin{cases} 4x + 4y + z - w = 0 \\ x - y + z + w = 0 \end{cases}$$

Now we try and solve the system:

$$\begin{cases} 4x + 4y + z - w = 0 \\ x - y + z + w = 0 \end{cases}$$

To solve the system we'll need to find the rank:

$$A = \begin{bmatrix} 4 & 4 & 1 & -1 \\ 1 & -1 & 1 & 1 \end{bmatrix}$$

Find the deltas of matrix A:

$$\Delta_1 = 4 \neq 0$$

$$\Delta_2 = \begin{bmatrix} 4 & 4 \\ 1 & -1 \end{bmatrix} = -4 - 4 = -8 \neq 0 \Rightarrow rank(A) = 2$$

Now we need the identity matrix based on our rank:

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Now we solve our system based on the identity matrix.
Because our identity matrix is 2x2, we move two elements from the left to right:

$$\begin{cases} 4x + 4y = -z + w \\ x - y = -z - w \end{cases}$$

Now solve by placing the first row elements to be on the right side of the equations:

$$\begin{cases} 4x + 4y = -1 \\ x - y = -1 \end{cases}$$

$$\begin{cases} 4x + 4y = -1 \\ x = -1 + y \end{cases}$$

$$\begin{cases} y = \frac{3}{8} \\ x = -1 + y \end{cases}$$

$$\begin{cases} y = \frac{3}{8} \\ x = -\frac{5}{8} \end{cases}$$

Now the second row of elements to be on the right side of the equations:

$$\begin{cases} 4x + 4y = 1 \\ x - y = -1 \end{cases} \Rightarrow \begin{cases} y = \frac{5}{8} \\ x = -\frac{3}{8} \end{cases}$$

Now we have to basis vectors γ_1 and γ_2:

$$\gamma_1 = \left(-\frac{5}{8}; \frac{3}{8}; 1; 0\right)$$

$$\gamma_2 = \left(-\frac{3}{8}; \frac{5}{8}; 0; 1\right)$$

Now we check if they're orthogonal:

$$(\gamma_1 \cdot \gamma_2) = \left(\frac{5}{8} \cdot \frac{3}{8} + \frac{3}{8} \cdot \frac{5}{8} + 0 + 0\right) = \frac{15}{32} \neq 0$$

Unfortunately our vectors are not orthogonal therefore we must find the fourth β.
We do this by picking any of the previous vectors to serve as a_3, in this case $a_3 = \gamma_1$:

$$\beta = (x, y, z, w) \neq 0$$

$$\beta \perp a_1, \beta \perp a_2, \beta \perp a_3$$

Form a linear system of equations:

$$\begin{cases} (\beta \cdot a_1) = 0 \\ (\beta \cdot a_2) = 0 \\ (\beta \cdot a_3) = 0 \end{cases} \Rightarrow \begin{cases} 4x + 4y + z - w = 0 \\ x - y + z + w = 0 \\ -\dfrac{5x}{8} + \dfrac{3y}{8} + z = 0 \end{cases}$$

To solve we'll need to turn this into a matrix and find the rank again:

$$A = \begin{bmatrix} 4 & 4 & 1 & -1 \\ 1 & -1 & 1 & 1 \\ -\frac{5}{8} & \frac{3}{8} & 1 & 0 \end{bmatrix}$$

Find the rank:

$$\Delta_3 = \begin{bmatrix} 4 & 4 & 1 \\ 1 & -1 & 1 \\ -\frac{5}{8} & \frac{3}{8} & 1 \end{bmatrix} = -\frac{49}{4} \neq 0 \Rightarrow rank(A) = 3$$

Now form a linear system of equations again:

$$\begin{cases} 4x + 4y + z = w \\ x - y + z = -w \\ -\dfrac{5}{8} + \dfrac{3y}{8} + z = 0 \end{cases}$$

We can solve this system of equations using Cramer's Rule:

$$\begin{bmatrix} 4 & 4 & 1 \\ 1 & -1 & 1 \\ -\frac{5}{8} & \frac{3}{8} & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$D(A) = -\frac{49}{4}$$

$$x = \left| \begin{bmatrix} 1 & 4 & 1 \\ -1 & -1 & 1 \\ 0 & \frac{3}{8} & 1 \end{bmatrix} \right| = \frac{9}{4}$$

$$y = \left| \begin{bmatrix} 4 & 1 & 1 \\ 1 & -1 & 1 \\ -\frac{5}{8} & 0 & 1 \end{bmatrix} \right| = -\frac{25}{4}$$

$$z = \left| \begin{bmatrix} 4 & 4 & 1 \\ 1 & -1 & -1 \\ -\frac{5}{8} & \frac{3}{8} & 0 \end{bmatrix} \right| = \frac{15}{4}$$

Therefore we get our vector β which is our fourth basis vector a_4:

$$a_4 = \beta = \left(-\frac{9}{49}; \frac{25}{49}; -\frac{15}{49}; 1 \right)$$

We conclude with our orthogonal basis vectors:

$$a_1 = (4; 4; 1; -1)$$

$$a_2 = (1; -1; 1; 1)$$

$$a_3 = (-\frac{5}{8}; \frac{3}{8}; 1; 0)$$

$$a_4 = (-\frac{9}{49}; \frac{25}{49}; -\frac{15}{49}; 1)$$

3 Task 3

Given the basis vectors:

$$\{a_1 = (2,3,5); a_2 = (0,1,2); a_3 = (1,0,0)\}$$

We need to find the matrix that transforms these basis vectors to:

$$\{e_1, e_2, e_3\} \Rightarrow \{(1,1,3); (1,1,0); (1,2,0)\}$$

First we will form a linear system of equations:

$$\begin{cases} e_1 = x_1 a_1 + x_2 a_2 + x_3 a_3 \\ e_2 = y_1 a_1 + y_2 a_2 + y_3 a_3 \\ e_3 = z_1 a_1 + z_2 a_2 + z_3 a_3 \end{cases}$$

Now we solve each equations separately by forming a yet another system of equations for each entry of e_1 and solve them with Cramer's rule.
I will skip Cramer's parts because we already know it from the first semester.

Solve for e_1:

$$(1,1,3) = x_1(2,3,5) + x_2(0,1,2) + x_3(1,0,0) \Rightarrow (1,1,3) = (2x_1 + x_3; 3x_1 + x_2; 5x_1 + 2x_2)$$

$$\begin{cases} 2x_1 + x_3 = 1 \\ 3x_1 + x_2 = 1 \\ 5x_1 + 2x_2 = 3 \end{cases} \Rightarrow \begin{cases} x_1 = -1 \\ x_2 = 4 \\ x_3 = 3 \end{cases}$$

Solve for e_2:

$$(1,1,0) = (2x_1 + x_3; 3x_1 + x_2; 5x_1 + 2x_2)$$

$$\begin{cases} 2x_1 + x_3 = 1 \\ 3x_1 + x_2 = 1 \\ 5x_1 + 2x_2 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = 2 \\ x_2 = -5 \\ x_3 = -3 \end{cases}$$

Solve for e_3:

$$(1,2,0) = (2x_1 + x_3; 3x_1 + x_2; 5x_1 + 2x_2)$$

$$\begin{cases} 2x_1 + x_3 = 1 \\ 3x_1 + x_2 = 2 \\ 5x_1 + 2x_2 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = 4 \\ x_2 = -10 \\ x_3 = -7 \end{cases}$$

From this we get our final system of equations:

$$\begin{cases} e_1 = -a_1 + 4a_2 + 3a_3 \\ e_2 = 2a_1 - 5a_2 - 3a_3 \\ e_3 = 4a_1 - 10a_2 - 7a_3 \end{cases}$$

$$A = \begin{bmatrix} -1 & 4 & 3 \\ 2 & -5 & -3 \\ 4 & -10 & -7 \end{bmatrix}$$

4 Task 4

We are given matrix A to find the Transition matrix α: Find the Transition matrix α:

$$a \to A = \begin{bmatrix} 1 & -3 & 3 \\ -2 & -6 & 13 \\ -1 & -4 & 8 \end{bmatrix}$$

$$\gamma_\alpha(x) = |x \cdot I - A| \Rightarrow \left| \begin{bmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{bmatrix} - \begin{bmatrix} 1 & -3 & 3 \\ -2 & -6 & 13 \\ -1 & -4 & 8 \end{bmatrix} \right| = x^3 - 3x^2 + 3x - 1 = (x-1)^3$$

$$(x-1)^3 = 0$$

$$x = 3$$

$$\alpha(a) = \lambda a$$

$$A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 3 \\ -2 & -6 & 13 \\ -1 & -4 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 - 3x_2 + 3x_3 \\ -2x_1 - 6x_2 + 13x_3 \\ -x_3 - 4x_2 + 8x_3 \end{bmatrix}$$

Now we form a linear system of equations to solve:

$$\begin{cases} -3x_2 + 3x_3 = 0 \\ -2x_1 - 7x_2 + 13x_3 = 0 \\ -x_1 - 4x_2 + 7x_3 = 0 \end{cases}$$

Now we form a matrix to solve the system:

$$\begin{bmatrix} 0 & -3 & 3 \\ -2 & -7 & 13 \\ -1 & -4 & 7 \end{bmatrix}$$

$$\Delta_1 = -3 \neq 0$$

$$\Delta_2 = \begin{bmatrix} 0 & -3 \\ -2 & -7 \end{bmatrix} = -6 \neq 0$$

$$\begin{bmatrix} 0 & -3 & 3 \\ -2 & -7 & 13 \\ -1 & -4 & 7 \end{bmatrix} = 0 \Rightarrow rank(A) = 0$$

Because the rank is 3, we ignore the third row in the system and solve for the first two:

$$\begin{cases} -3x_2 = -3x_3 \\ -2x_1 - 7x_2 = -13x_3 \end{cases} = \begin{cases} x_1 = 3 \\ x_2 = 1 \end{cases}$$

From this we get our vector:

$$(3,1,1)$$