Linear Algebra

Serob Tigranyan

May 21, 2025

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1 Task 1

Given the following basis vectors:

$$e_1 = (1, 1, 3)$$
$$e_2 = (1, 1, 0)$$
$$e_3 = (1, 2, 0)$$

And vector x:

$$x = (5, 4, 6)$$

1.1 Linear Independence

We need to check if the given basis vectors are valid:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 3 & 0 & 0 \end{bmatrix}$$

Determine the Rank by first finding the determinant:

$$D(A) = 0 + 6 + 0 - 3 - 0 - 0 = 3$$

Since our determinant is non-zero, then we have a full rank which is 3 which further means that the basis vectors are linearly independent.

$$rank(A) = 3$$

1.2 Vector expressed in given basis

We need to express vector x in our basis e_1, e_2, e_3.

$$x = \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3$$

Therefore we have:

$$(5, 4, 6) = \lambda(1, 1, 3) + \lambda(1, 1, 0) + \lambda(1, 2, 0)$$
$$(5, 4, 6) = (\lambda_1 + \lambda_2 + \lambda_3; \lambda_1 + \lambda_2 + 2\lambda_3; 3\lambda_1)$$

We form a linear system of equations:

$$\begin{cases} \lambda_1 + \lambda_2 + \lambda_3 = 5 \\ \lambda_1 + \lambda_2 + 2\lambda_3 = 4 \\ 3\lambda_1 = 6 \end{cases}$$

Now we solve the system:

$$\begin{cases} \lambda_1 + \lambda_2 + \lambda_3 = 5 \\ \lambda_1 + \lambda_2 + 2\lambda_3 = 4 \\ \lambda_1 = 2 \end{cases}$$

$$\begin{cases} \lambda_2 = 3 - \lambda_3 \\ 2 + (3 - \lambda_3) + 2\lambda_3 = 4 \\ \lambda_1 = 2 \end{cases}$$

$$\begin{cases} \lambda_2 = 3 - \lambda_3 \\ \lambda_3 = -1 \\ \lambda_1 = 2 \end{cases}$$

$$\begin{cases} \lambda_2 = 4 \\ \lambda_3 = -1 \\ \lambda_1 = 2 \end{cases}$$

Therefore our vector x expressed in our basis vectors e_1, e_2, e_3 is:

$$x_e = (2, 4, -1)$$

2 Task 2

Given the vectors a_1 and a_2 we need to find if they're orthogonal and if so, then find their orthogonal basis vectors:

$$a_1 = (4, 4, 1, -1)$$

$$a_2 = (1, -1, 1, 1)$$

2.1 Orthogonality

To check if these two are orthogonal we perform scalar multiplication on both of them:

$$(a_1 \cdot a_2) = 4 - 4 + 1 - 1 = 0 \Rightarrow a_1 \perp a_2$$

Our vectors are indeed orthogonal, now we need to find their orthogonal basis vectors.

2.1.1 Orthogonal Basis Vectors

To find the orthogonal basis vectors we pick arbitrary vector:

$$\gamma = (x, y, z, w)$$

So that:

$$\gamma \perp a_1$$

$$\gamma \perp a_2$$

We form a linear system of equations from this like so:

$$\begin{cases} (\gamma \cdot a_1) = 0 \\ (\gamma \cdot a_2) = 0 \end{cases} \Rightarrow \begin{cases} 4x + 4y + z - w = 0 \\ x - y + z + w = 0 \end{cases}$$

Now we try and solve the system:

$$\begin{cases} 4x + 4y + z - w = 0 \\ x - y + z + w = 0 \end{cases}$$

To solve the system we'll need to find the rank:

$$A = \begin{bmatrix} 4 & 4 & 1 & -1 \\ 1 & -1 & 1 & 1 \end{bmatrix}$$

Find the deltas of matrix A:

$$\Delta_1 = 4 \neq 0$$

$$\Delta_2 = \begin{bmatrix} 4 & 4 \\ 1 & -1 \end{bmatrix} = -4-4 = -8 \neq 0 \Rightarrow rank(A) = 2$$

Now we need the identity matrix based on our rank:

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Now we solve our system based on the identity matrix.
Because our identity matrix is 2x2, we move two elements from the left to right:

$$\begin{cases} 4x+4y = -z+w \\ x-y = -z-w \end{cases}$$

Now solve by placing the first row elements to be on the right side of the equations:

$$\begin{cases} 4x+4y = -1 \\ x-y = -1 \end{cases}$$

$$\begin{cases} 4x+4y = -1 \\ x = -1+y \end{cases}$$

$$\begin{cases} y = \frac{3}{8} \\ x = -1+y \end{cases}$$

$$\begin{cases} y = \frac{3}{8} \\ x = -\frac{5}{8} \end{cases}$$

Now the second row of elements to be on the right side of the equations:

$$\begin{cases} 4x+4y = 1 \\ x-y = -1 \end{cases} \Rightarrow \begin{cases} y = \frac{5}{8} \\ x = -\frac{3}{8} \end{cases}$$

Now we have to basis vectors γ_1 and γ_2:

$$\gamma_1 = \left(-\frac{5}{8}; \frac{3}{8}; 1; 0\right)$$

$$\gamma_2 = \left(-\frac{3}{8};\frac{5}{8};0;1\right)$$

Now we check if they're orthogonal:

$$(\gamma_1 \cdot \gamma_2) = \left(\frac{5}{8}\cdot\frac{3}{8}+\frac{3}{8}\cdot\frac{5}{8}+0+0\right) = \frac{15}{32} \neq 0$$

Unfortunately our vectors are not orthogonal therefore we must find the fourth β.
We do this by picking any of the previous vectors to serve as a_3, in this case $a_3 = \gamma_1$:

$$\beta = (x, y, z, w) \neq 0$$

$$\beta \perp a_1, \beta \perp a_2, \beta \perp a_3$$

Form a linear system of equations:

$$\begin{cases} (\beta \cdot a_1) = 0 \\ (\beta \cdot a_2) = 0 \\ (\beta \cdot a_3) = 0 \end{cases} \Rightarrow \begin{cases} 4x+4y+z-w=0 \\ x-y+z+w=0 \\ -\frac{5x}{8}+\frac{3y}{8}+z=0 \end{cases}$$

To solve we'll need to turn this into a matrix and find the rank again:

$$A = \begin{bmatrix} 4 & 4 & 1 & -1 \\ 1 & -1 & 1 & 1 \\ -\frac{5}{8} & \frac{3}{8} & 1 & 0 \end{bmatrix}$$

Find the rank:

$$\Delta_3 = \begin{bmatrix} 4 & 4 & 1 \\ 1 & -1 & 1 \\ -\frac{5}{8} & \frac{3}{8} & 1 \end{bmatrix} = -\frac{49}{4} \neq 0 \Rightarrow rank(A) = 3$$

Now form a linear system of equations again:

$$\begin{cases} 4x+4y+z=w \\ x-y+z=-w \\ -\frac{5}{8}+\frac{3y}{8}+z=0 \end{cases}$$

We can solve this system of equations using Cramer's Rule:

$$\begin{bmatrix} 4 & 4 & 1 \\ 1 & -1 & 1 \\ -\frac{5}{8} & \frac{3}{8} & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$D(A) = -\frac{49}{4}$$

$$x = D\left(\begin{bmatrix} 1 & 4 & 1 \\ -1 & -1 & 1 \\ 0 & \frac{3}{8} & 1 \end{bmatrix}\right) = \frac{9}{4}$$

$$y = D\left(\begin{bmatrix} 4 & 1 & 1 \\ 1 & -1 & 1 \\ -\frac{5}{8} & 0 & 1 \end{bmatrix}\right) = -\frac{25}{4}$$

$$z = D\left(\begin{bmatrix} 4 & 4 & 1 \\ 1 & -1 & -1 \\ -\frac{5}{8} & \frac{3}{8} & 0 \end{bmatrix}\right) = \frac{15}{4}$$

Therefore we get our vector β which is our fourth basis vector a_4:

$$a_4 = \beta = \left(-\frac{9}{49}; \frac{25}{49}; -\frac{15}{49}; 1\right)$$

We conclude with our orthogonal basis vectors:

$$a_1 = (4; 4; 1; -1)$$

$$a_2 = (1; -1; 1; 1)$$

$$a_3 = (-\frac{5}{8}; \frac{3}{8}; 1; 0)$$

$$a_4 = (-\frac{9}{49}; \frac{25}{49}; -\frac{15}{49}; 1)$$