FEM BY HAND

ARDEN RASMUSSEN

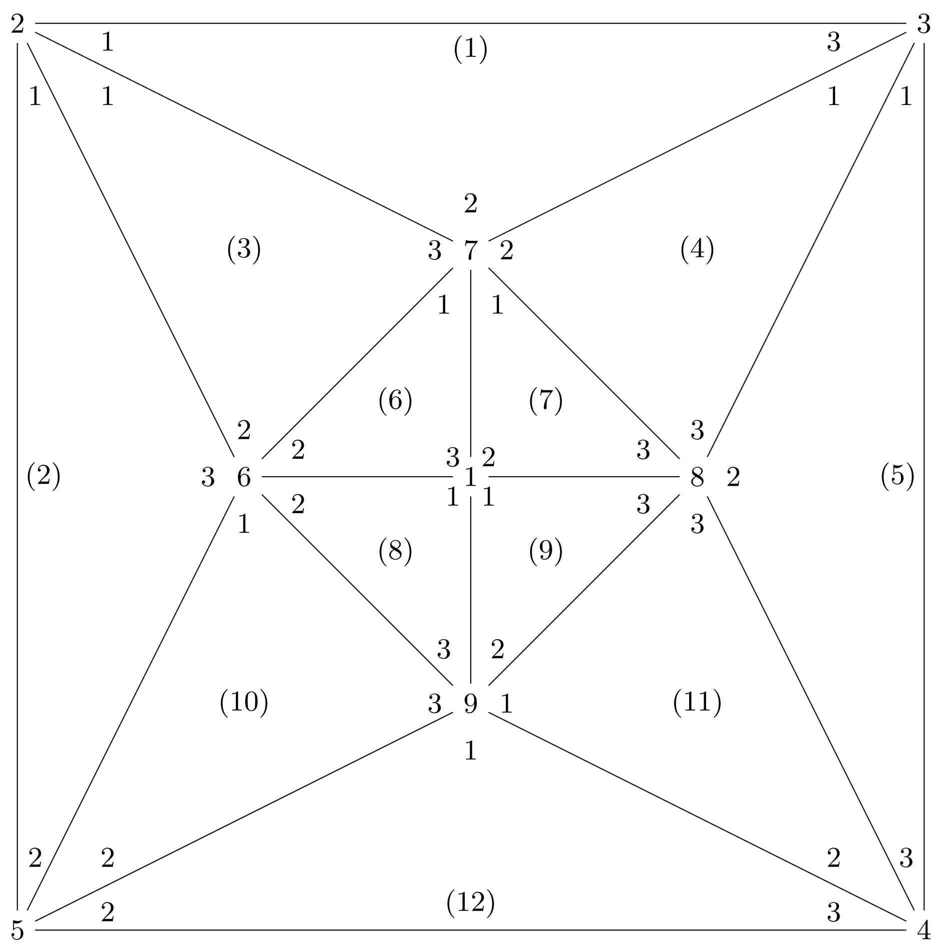

Figure 1

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We have the global matrix system exactly as below.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} & a_{19} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} & a_{27} & a_{28} & a_{29} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} & a_{37} & a_{38} & a_{39} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} & a_{47} & a_{48} & a_{49} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} & a_{57} & a_{58} & a_{59} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} & a_{67} & a_{68} & a_{69} \\ a_{71} & a_{72} & a_{73} & a_{74} & a_{75} & a_{76} & a_{77} & a_{78} & a_{79} \\ a_{81} & a_{82} & a_{83} & a_{84} & a_{85} & a_{86} & a_{87} & a_{88} & a_{89} \\ a_{91} & a_{92} & a_{93} & a_{94} & a_{95} & a_{96} & a_{97} & a_{98} & a_{99} \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \\ T_9 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \\ F_7 \\ F_8 \\ F_9 \end{bmatrix}$$

With elements (e) in the range from $1 \cdots 12$ inclusive. Each element has an element matrix of the form.

$$A^{(e)} = \begin{bmatrix} a_{11}^{(e)} & a_{12}^{(e)} & a_{13}^{(e)} \\ a_{21}^{(e)} & a_{22}^{(e)} & a_{23}^{(e)} \\ a_{31}^{(e)} & a_{32}^{(e)} & a_{33}^{(e)} \end{bmatrix} \qquad F^{(e)} = \begin{bmatrix} F_1^{(e)} \\ F_2^{(e)} \\ F_3^{(e)} \end{bmatrix}$$

Then for every element we add the components of the local basis to the global system. By the method

$$\begin{aligned} A_{node(e,I)node(e,J)} &= A_{node(e,I)node(e,J)} + A_{IJ}^{(e)} \\ F_{node(e,I)} &= F_{node(e,I)} + F_I^{(e)} \\ I, J &= 1,2,3 \quad \text{the vertices of that element.} \end{aligned}$$

Where *node* gives the global index of that vertex of the given element. Using this method we can find the terms for A_{ij} and F_i.

Finding all the values for F_i will be found as the below.

$$\begin{aligned} F_1 &= F_3^{(6)} + F_2^{(7)} + F_1^{(8)} + F_1^{(9)} \\ F_2 &= F_1^{(1)} + F_1^{(2)} + F_1^{(3)} \\ F_3 &= F_3^{(1)} + F_1^{(4)} + F_1^{(5)} \\ F_4 &= F_3^{(3)} + F_2^{(11)} + F_3^{(12)} \\ F_5 &= F_2^{(2)} + F_2^{(10)} + F_2^{(12)} \\ F_6 &= F_3^{(2)} + F_2^{(3)} + F_2^{(6)} + F_2^{(8)} + F_1^{(10)} \\ F_7 &= F_2^{(1)} + F_3^{(3)} + F_2^{(4)} + F_1^{(6)} + F_1^{(7)} \\ F_8 &= F_3^{(4)} + F_2^{(5)} + F_3^{(7)} + F_3^{(9)} + F_3^{(11)} \\ F_9 &= F_3^{(8)} + F_2^{(9)} + F_3^{(10)} + F_1^{(11)} + F_1^{(12)} \end{aligned}$$

I also do this for all A_ij, which will then be used to fill in the matrix. All of the values are found to be

$$\begin{aligned} A_{11} &= A_{33}^{(6)} + A_{22}^{(7)} + A_{11}^{(8)} + A_{11}^{(9)} \\ A_{12} &= 0 \\ A_{13} &= 0 \\ A_{14} &= 0 \\ A_{15} &= 0 \\ A_{16} &= A_{32}^{(6)} + A_{12}^{(8)} \\ A_{17} &= A_{31}^{(6)} + A_{21}^{(7)} \\ A_{18} &= A_{23}^{(7)} + A_{13}^{(9)} \\ A_{19} &= A_{13}^{(8)} + A_{12}^{(9)} \end{aligned}$$

$$\begin{aligned} A_{21} &= 0 \\ A_{22} &= A_{11}^{(1)} + A_{11}^{(2)} + A_{11}^{(3)} \\ A_{23} &= A_{13}^{(1)} \\ A_{24} &= 0 \\ A_{25} &= A_{12}^{(2)} \\ A_{26} &= A_{13}^{(2)} + A_{12}^{(3)} \\ A_{27} &= A_{12}^{(1)} + A_{13}^{(3)} \\ A_{28} &= 0 \\ A_{29} &= 0 \end{aligned}$$

$$\begin{aligned} A_{31} &= 0 \\ A_{32} &= A_{31}^{(1)} \\ A_{33} &= A_{33}^{(1)} + A_{11}^{(4)} + A_{11}^{(5)} \\ A_{34} &= A_{13}^{(5)} \\ A_{35} &= 0 \\ A_{36} &= 0 \\ A_{37} &= A_{32}^{(1)} + A_{12}^{(4)} \\ A_{38} &= A_{13}^{(4)} + A_{12}^{(5)} \\ A_{39} &= 0 \end{aligned}$$

$$A_{41} = 0$$
$$A_{42} = 0$$
$$A_{43} = A^{(5)}_{31}$$
$$A_{44} = A^{(5)}_{33} + A^{(11)}_{22} + A^{(12)}_{33}$$
$$A_{45} = A^{(12)}_{32}$$
$$A_{46} = 0$$
$$A_{47} = 0$$
$$A_{48} = A^{(5)}_{32} + A^{(11)}_{23}$$
$$A_{49} = A^{(11)}_{21} + A^{(12)}_{31}$$

$$A_{51} = 0$$
$$A_{52} = A^{(2)}_{21}$$
$$A_{53} = 0$$
$$A_{54} = A^{(12)}_{23}$$
$$A_{55} = A^{(2)}_{22} + A^{(10)}_{22} + A^{(12)}_{22}$$
$$A_{56} = A^{(2)}_{23} + A^{(10)}_{21}$$
$$A_{57} = 0$$
$$A_{58} = 0$$
$$A_{59} = A^{(10)}_{23} + A^{(12)}_{21}$$

$$A_{61} = A^{(6)}_{23} + A^{(8)}_{21}$$
$$A_{62} = A^{(2)}_{31} + A^{(3)}_{21}$$
$$A_{63} = 0$$
$$A_{64} = 0$$
$$A_{65} = A^{(2)}_{23} + A^{(10)}_{21}$$
$$A_{66} = A^{(2)}_{33} + A^{(3)}_{22} + A^{(6)}_{22} + A^{(8)}_{22} + A^{(10)}_{11}$$
$$A_{67} = A^{(3)}_{23} + A^{(6)}_{21}$$
$$A_{68} = 0$$
$$A_{69} = A^{(8)}_{23} + A^{(10)}_{13}$$

$$A_{71} = A^{(6)}_{13} + A^{(7)}_{12}$$
$$A_{72} = A^{(1)}_{12} + A^{(3)}_{13}$$
$$A_{73} = A^{(1)}_{23} + A^{(4)}_{21}$$
$$A_{74} = 0$$
$$A_{75} = 0$$
$$A_{76} = A^{(3)}_{32} + A^{(6)}_{12}$$
$$A_{77} = A^{(1)}_{22} + A^{(3)}_{33} + A^{(4)}_{22} + A^{(6)}_{11} + A^{(7)}_{11}$$
$$A_{78} = A^{(4)}_{23} + A^{(7)}_{13}$$
$$A_{79} = 0$$

$$A_{81} = A^{(7)}_{32} + A^{(9)}_{31}$$
$$A_{82} = 0$$
$$A_{83} = A^{(4)}_{31} + A^{(5)}_{21}$$
$$A_{84} = A^{(5)}_{23} + A^{(11)}_{32}$$
$$A_{85} = 0$$
$$A_{86} = 0$$
$$A_{87} = A^{(4)}_{32} + A^{(7)}_{31}$$
$$A_{88} = A^{(4)}_{33} + A^{(5)}_{22} + A^{(7)}_{33} + A^{(9)}_{33} + A^{(11)}_{33}$$
$$A_{89} = A^{(9)}_{32} + A^{(11)}_{31}$$

$$A_{91} = A^{(8)}_{31} + A^{(9)}_{21}$$
$$A_{92} = 0$$
$$A_{93} = 0$$
$$A_{94} = A^{(11)}_{12} + A^{(12)}_{13}$$
$$A_{95} = A^{(10)}_{32} + A^{(12)}_{12}$$
$$A_{96} = A^{(8)}_{32} + A^{(10)}_{31}$$
$$A_{97} = 0$$
$$A_{98} = A^{(9)}_{23} + A^{(11)}_{13}$$
$$A_{99} = A^{(8)}_{33} + A^{(9)}_{22} + A^{(10)}_{33} + A^{(11)}_{11} + A^{(12)}_{11}$$

Thus assembling this into the global matrix we find $[A]$ to be

$$\begin{bmatrix}
A^{(6)}_{33} + A^{(7)}_{22} + A^{(8)}_{11} + A^{(9)}_{11} & 0 & 0 & 0 & 0 & A^{(6)}_{32} + A^{(8)}_{12} & A^{(6)}_{31} + A^{(7)}_{21} & A^{(7)}_{23} + A^{(9)}_{13} & A^{(8)}_{13} + A^{(9)}_{12} \\
0 & A^{(1)}_{11} + A^{(2)}_{11} + A^{(3)}_{11} & A^{(1)}_{13} & 0 & A^{(2)}_{12} & A^{(2)}_{13} + A^{(3)}_{12} & A^{(1)}_{12} + A^{(3)}_{13} & 0 & 0 \\
0 & A^{(1)}_{31} & A^{(1)}_{33} + A^{(4)}_{11} + A^{(5)}_{11} & A^{(5)}_{13} & 0 & 0 & A^{(1)}_{32} + A^{(4)}_{12} & A^{(4)}_{13} + A^{(5)}_{12} & 0 \\
0 & 0 & A^{(5)}_{31} & A^{(5)}_{33} + A^{(11)}_{22} + A^{(12)}_{33} & A^{(12)}_{32} & 0 & 0 & A^{(5)}_{32} + A^{(11)}_{23} & A^{(11)}_{21} + A^{(12)}_{31} \\
0 & A^{(2)}_{21} & 0 & A^{(12)}_{23} & A^{(2)}_{22} + A^{(10)}_{22} + A^{(12)}_{22} & A^{(2)}_{23} + A^{(10)}_{21} & 0 & 0 & A^{(10)}_{23} + A^{(12)}_{21} \\
A^{(6)}_{23} + A^{(8)}_{21} & A^{(2)}_{31} + A^{(3)}_{21} & 0 & 0 & A^{(2)}_{23} + A^{(10)}_{21} & A^{(2)}_{33} + A^{(3)}_{22} + A^{(6)}_{22} + A^{(8)}_{22} + A^{(10)}_{11} & A^{(3)}_{23} + A^{(6)}_{21} & 0 & A^{(8)}_{23} + A^{(10)}_{13} \\
A^{(6)}_{13} + A^{(7)}_{12} & A^{(1)}_{12} + A^{(3)}_{13} & A^{(1)}_{23} + A^{(4)}_{21} & 0 & 0 & A^{(3)}_{32} + A^{(6)}_{12} & A^{(1)}_{22} + A^{(3)}_{33} + A^{(4)}_{22} + A^{(6)}_{11} + A^{(7)}_{11} & A^{(4)}_{23} + A^{(7)}_{13} & 0 \\
A^{(7)}_{32} + A^{(9)}_{31} & 0 & A^{(4)}_{31} + A^{(5)}_{21} & A^{(5)}_{23} + A^{(11)}_{32} & 0 & 0 & A^{(4)}_{32} + A^{(7)}_{31} & A^{(4)}_{33} + A^{(5)}_{22} + A^{(7)}_{33} + A^{(9)}_{33} + A^{(11)}_{33} & A^{(9)}_{32} + A^{(11)}_{31} \\
A^{(8)}_{31} + A^{(9)}_{21} & 0 & 0 & A^{(11)}_{12} + A^{(12)}_{13} & A^{(10)}_{32} + A^{(12)}_{12} & A^{(8)}_{32} + A^{(10)}_{31} & 0 & A^{(9)}_{23} + A^{(11)}_{13} & A^{(8)}_{33} + A^{(9)}_{22} + A^{(10)}_{33} + A^{(11)}_{11} + A^{(12)}_{11}
\end{bmatrix}$$

with the coresponding forcing matrix $[F]$ as

$$\begin{bmatrix}
F^{(6)}_3 + F^{(7)}_2 + F^{(8)}_1 + F^{(9)}_1 \\
F^{(1)}_1 + F^{(2)}_1 + F^{(3)}_1 \\
F^{(1)}_3 + F^{(4)}_1 + F^{(5)}_1 \\
F^{(3)}_3 + F^{(11)}_2 + F^{(12)}_3 \\
F^{(2)}_2 + F^{(10)}_2 + F^{(12)}_2 \\
F^{(2)}_3 + F^{(3)}_2 + F^{(6)}_2 + F^{(8)}_2 + F^{(10)}_1 \\
F^{(1)}_2 + F^{(3)}_3 + F^{(4)}_2 + F^{(6)}_1 + F^{(7)}_1 \\
F^{(4)}_3 + F^{(5)}_2 + F^{(7)}_3 + F^{(9)}_3 + F^{(11)}_3 \\
F^{(8)}_3 + F^{(9)}_2 + F^{(10)}_3 + F^{(11)}_1 + F^{(12)}_1
\end{bmatrix}$$

Now we apply the boundary conditions the matrix becomes

$$\begin{bmatrix}
A^{(6)}_{33} + A^{(7)}_{22} + A^{(8)}_{11} + A^{(9)}_{11} & 0 & 0 & 0 & 0 & A^{(6)}_{32} + A^{(8)}_{12} & A^{(6)}_{31} + A^{(7)}_{21} & A^{(7)}_{23} + A^{(9)}_{13} & A^{(8)}_{13} + A^{(9)}_{12} \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
A^{(6)}_{23} + A^{(8)}_{21} & A^{(2)}_{31} + A^{(3)}_{21} & 0 & 0 & A^{(2)}_{23} + A^{(10)}_{21} & A^{(2)}_{33} + A^{(3)}_{22} + A^{(6)}_{22} + A^{(8)}_{22} + A^{(10)}_{11} & A^{(3)}_{23} + A^{(6)}_{21} & 0 & A^{(8)}_{23} + A^{(10)}_{13} \\
A^{(6)}_{13} + A^{(7)}_{12} & A^{(1)}_{12} + A^{(3)}_{13} & A^{(1)}_{23} + A^{(4)}_{21} & 0 & 0 & A^{(3)}_{32} + A^{(6)}_{12} & A^{(1)}_{22} + A^{(3)}_{33} + A^{(4)}_{22} + A^{(6)}_{11} + A^{(7)}_{11} & A^{(4)}_{23} + A^{(7)}_{13} & 0 \\
A^{(7)}_{32} + A^{(9)}_{31} & 0 & A^{(4)}_{31} + A^{(5)}_{21} & A^{(5)}_{23} + A^{(11)}_{32} & 0 & 0 & A^{(4)}_{32} + A^{(7)}_{31} & A^{(4)}_{33} + A^{(5)}_{22} + A^{(7)}_{33} + A^{(9)}_{33} + A^{(11)}_{33} & A^{(9)}_{32} + A^{(11)}_{31} \\
A^{(8)}_{31} + A^{(9)}_{21} & 0 & 0 & A^{(11)}_{12} + A^{(12)}_{13} & A^{(10)}_{32} + A^{(12)}_{12} & A^{(8)}_{32} + A^{(10)}_{31} & 0 & A^{(9)}_{23} + A^{(11)}_{13} & A^{(8)}_{33} + A^{(9)}_{22} + A^{(10)}_{33} + A^{(11)}_{11} + A^{(12)}_{11}
\end{bmatrix}$$

with the forcing matrix of

$$\begin{bmatrix} F_3^{(6)} + F_2^{(7)} + F_1^{(8)} + F_1^{(9)} \\ B_1 \\ B_2 \\ B_3 \\ B_4 \\ F_3^{(2)} + F_2^{(3)} + F_2^{(6)} + F_2^{(8)} + F_1^{(10)} \\ F_2^{(1)} + F_3^{(3)} + F_2^{(4)} + F_1^{(6)} + F_1^{(7)} \\ F_3^{(4)} + F_2^{(5)} + F_3^{(7)} + F_3^{(9)} + F_3^{(11)} \\ F_3^{(8)} + F_2^{(9)} + F_3^{(10)} + F_1^{(11)} + F_1^{(12)} \end{bmatrix}$$

Plutting all of this into the system format, it becomes possible to solve the system for the values of $[T]$.

$$\begin{bmatrix} A_{33}^{(6)} + A_{22}^{(7)} + A_{11}^{(8)} + A_{11}^{(9)} & 0 & 0 & 0 & 0 & A_{32}^{(6)} + A_{12}^{(8)} & A_{31}^{(6)} + A_{21}^{(7)} & A_{23}^{(7)} + A_{13}^{(9)} & A_{13}^{(8)} + A_{12}^{(9)} \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ A_{23}^{(6)} + A_{21}^{(8)} & A_{31}^{(2)} + A_{21}^{(3)} & 0 & 0 & A_{23}^{(2)} + A_{21}^{(10)} & A_{33}^{(2)} + A_{22}^{(3)} + A_{22}^{(6)} + A_{22}^{(8)} + A_{11}^{(10)} & A_{23}^{(3)} + A_{21}^{(6)} & 0 & A_{23}^{(8)} + A_{13}^{(10)} \\ A_{13}^{(6)} + A_{12}^{(7)} & A_{12}^{(1)} + A_{13}^{(3)} & A_{23}^{(1)} + A_{21}^{(4)} & 0 & 0 & A_{32}^{(3)} + A_{12}^{(6)} & A_{22}^{(1)} + A_{33}^{(3)} + A_{22}^{(4)} + A_{11}^{(6)} + A_{11}^{(7)} & A_{23}^{(4)} + A_{13}^{(7)} & 0 \\ A_{32}^{(7)} + A_{31}^{(9)} & 0 & A_{31}^{(4)} + A_{21}^{(5)} & A_{23}^{(5)} + A_{32}^{(11)} & 0 & 0 & A_{32}^{(4)} + A_{31}^{(7)} & A_{33}^{(4)} + A_{22}^{(5)} + A_{33}^{(7)} + A_{33}^{(9)} + A_{33}^{(11)} & A_{32}^{(9)} + A_{31}^{(11)} \\ A_{31}^{(8)} + A_{21}^{(9)} & 0 & 0 & A_{12}^{(11)} + A_{13}^{(12)} & A_{32}^{(10)} + A_{12}^{(12)} & A_{32}^{(8)} + A_{31}^{(10)} & 0 & A_{23}^{(9)} + A_{13}^{(11)} & A_{33}^{(8)} + A_{22}^{(9)} + A_{33}^{(10)} + A_{11}^{(11)} + A_{11}^{(12)} \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \\ T_9 \end{bmatrix} = \begin{bmatrix} F_3^{(6)} + F_2^{(7)} + F_1^{(8)} + F_1^{(9)} \\ B_1 \\ B_2 \\ B_3 \\ B_4 \\ F_3^{(2)} + F_2^{(3)} + F_2^{(6)} + F_2^{(8)} + F_1^{(10)} \\ F_2^{(1)} + F_3^{(3)} + F_2^{(4)} + F_1^{(6)} + F_1^{(7)} \\ F_3^{(4)} + F_2^{(5)} + F_3^{(7)} + F_3^{(9)} + F_3^{(11)} \\ F_3^{(8)} + F_2^{(9)} + F_3^{(10)} + F_1^{(11)} + F_1^{(12)} \end{bmatrix}$$

Where

$$\begin{aligned} A_{ij}^{(e)} &= \rho c_p \int_{E_e} \varphi_i^{(e)} \vec{u} \cdot \nabla \varphi_j^{(e)} d\vec{x} + k \int_{E_e} \nabla \varphi_i^{(e)} \cdot \nabla \varphi_j^{(e)} d\vec{x} \\ F_i^{(e)} &= \int_{E_e} \varphi_i^{(e)} f d\vec{x} \end{aligned}$$

Then we will use the barycentric coordinate system, to convert these integrals to be

$$\begin{aligned} A_{ij}^{(e)} &= \rho c_p \int_0^1 \int_0^1 \varphi_i^{(e)} \vec{u} \cdot \nabla \varphi_j^{(e)} d\lambda_1 d\lambda_2 + k \int_0^1 \int_0^1 \nabla \varphi_i^{(e)} \cdot \nabla \varphi_j^{(e)} d\lambda_1 d\lambda_2 \\ F_i^{(e)} &= \int_0^1 \int_0^1 \varphi_i^{(e)} f d\lambda_1 d\lambda_2 \end{aligned}$$

Now I will expand the ∇ and apply the dot product, to find the expression for $A_{ij}^{(3)}$, I will still do this using the normal basis, as the barycentric system is just a simple manipulation of coordinates.

$$\begin{aligned} A_{ij}^{(e)} &= \rho c_p \int_{E_e} \varphi_i^{(e)} \vec{u} \cdot \nabla \varphi_j^{(e)} d\vec{x} + k \int_{E_e} \nabla \varphi_i^{(e)} \cdot \nabla \varphi_j^{(e)} d\vec{x} \\ &= \rho c_p \int_{E_e} \varphi_i^{(e)} \vec{u} \cdot \left\langle \frac{\partial \varphi_j^{(e)}}{\partial x}, \frac{\partial \varphi_j^{(e)}}{\partial y} \right\rangle d\vec{x} + k \int_{E_e} \left\langle \frac{\partial \varphi_i^{(e)}}{\partial x}, \frac{\partial \varphi_i^{(e)}}{\partial y} \right\rangle \cdot \left\langle \frac{\partial \varphi_j^{(e)}}{\partial x}, \frac{\partial \varphi_j^{(e)}}{\partial y} \right\rangle d\vec{x} \\ &= \rho c_p \int_{E_e} u^1 \varphi_i^{(e)} \frac{\partial \varphi_j^{(e)}}{\partial x} + u^2 \varphi_i^{(e)} \frac{\partial \varphi_j^{(e)}}{\partial y} d\vec{x} + k \int_{E_e} \frac{\partial \varphi_i^{(e)}}{\partial x} \frac{\partial \varphi_j^{(e)}}{\partial x} + \frac{\partial \varphi_i^{(e)}}{\partial y} \frac{\partial \varphi_j^{(e)}}{\partial y} d\vec{x} \end{aligned}$$

Now converting this into the barycentric coordinate system, using the notation that

$$\begin{aligned} \bar{x} &= \lambda_1 X_1 + \lambda_2 X_2 + (1 - \lambda_1 - \lambda_2) X_3 \\ \bar{y} &= \lambda_1 Y_1 + \lambda_2 Y_2 + (1 - \lambda_1 - \lambda_2) Y_3 \end{aligned}$$

where X_i and Y_i are the corresponding x and y coordinates of the i^{th} vertex of that triangular element. Using this notation we can rewrite the expressions for $A_{ij}^{(e)}$ and $F_i^{(e)}$ to be.

$$\begin{aligned} A_{ij}^{(e)} &= \rho c_p \int_0^1 \int_o^1 u^1 \varphi_i^{(e)}(\bar{x}, \bar{y}) \frac{\partial \varphi_j^{(e)}}{\partial x}(\bar{x}, \bar{y}) + u^2 \varphi_i^{(e)}(\bar{x}, \bar{y}) \frac{\partial \varphi_j^{(e)}}{\partial y}(\bar{x}, \bar{y}) d\lambda_1 d\lambda_2 \\ &\quad + k \int_0^1 \int_0^1 \frac{\partial \varphi_i^{(e)}}{\partial x}(\bar{x}, \bar{y}) \frac{\partial \varphi_j^{(e)}}{\partial x}(\bar{x}, \bar{y}) + \frac{\partial \varphi_i^{(e)}}{\partial y}(\bar{x}, \bar{y}) \frac{\partial \varphi_j^{(e)}}{\partial y}(\bar{x}, \bar{y}) d\lambda_1 d\lambda_2 \\ F_i^{(e)} &= \int_0^1 \int_0^1 \varphi_i^{(e)}(\bar{x}, \bar{y}) f(\bar{x}, \bar{y}) d\lambda_1 d\lambda_2 \end{aligned}$$

Using the included python code, we are able to compute the decimal values for these elements, these values are presented in the table below. Note that for the calculations we make some assumptions

$$
\begin{aligned}
\rho &= 1\\
c_p &= 1\\
k &= 1\\
\vec{u} &= \langle 1, 1\rangle\\
f &= (x-0.5)^2 + (y-0.5)^2
\end{aligned}
$$

I want to note that this is 336 integrals and 432 partial derivatives :(

$$
A = \begin{bmatrix}
32. & 0. & 0. & 0. & 0. & -9.33333333 & -6.66666667 & -6.66666667 & -9.33333333\\
0. & 3.38888889 & 1.25 & 0. & 0.25 & -2.44444444 & -2.44444444 & 0. & 0.\\
0. & 0.91666667 & 4.83333333 & 0.91666667 & 0. & 0. & -3.33333333 & -3.33333333 & 0.\\
0. & 0. & 1.25 & 3.38888889 & 0.25 & 0. & 0. & -2.44444444 & -2.44444444\\
0. & 0.58333333 & 0. & 0.58333333 & 1.94444444 & -1.55555556 & 0. & 0. & -1.55555556\\
-6.66666667 & -2.61111111 & 0. & 0. & -3.38888889 & 15.33333333 & -0.44444444 & 0. & -2.22222222\\
-9.33333333 & -2.27777778 & -1.5 & 0. & 0. & -3.11111111 & 17.55555556 & -1.33333333 & 0.\\
-9.33333333 & 0. & -1.5 & -2.27777778 & 0. & 0. & -1.33333333 & 17.55555556 & -3.11111111\\
-6.66666667 & 0. & 0. & -2.61111111 & -3.38888889 & -2.22222222 & 0. & -0.44444444 & 15.33333333
\end{bmatrix}
$$

$$
F = \begin{bmatrix}
0.73333333\\ 1.43333333\\ 0.26666667\\ 1.43333333\\ 2.6\\ 2.175\\ 0.675\\ 0.675\\ 2.175
\end{bmatrix}
$$

Placing this into the system of equations we find

$$
\begin{bmatrix}
32. & 0. & 0. & 0. & 0. & -9.33333333 & -6.66666667 & -6.66666667 & -9.33333333\\
0. & 3.38888889 & 1.25 & 0. & 0.25 & -2.44444444 & -2.44444444 & 0. & 0.\\
0. & 0.91666667 & 4.83333333 & 0.91666667 & 0. & 0. & -3.33333333 & -3.33333333 & 0.\\
0. & 0. & 1.25 & 3.38888889 & 0.25 & 0. & 0. & -2.44444444 & -2.44444444\\
0. & 0.58333333 & 0. & 0.58333333 & 1.94444444 & -1.55555556 & 0. & 0. & -1.55555556\\
-6.66666667 & -2.61111111 & 0. & 0. & -3.38888889 & 15.33333333 & -0.44444444 & 0. & -2.22222222\\
-9.33333333 & -2.27777778 & -1.5 & 0. & 0. & -3.11111111 & 17.55555556 & -1.33333333 & 0.\\
-9.33333333 & 0. & -1.5 & -2.27777778 & 0. & 0. & -1.33333333 & 17.55555556 & -3.11111111\\
-6.66666667 & 0. & 0. & -2.61111111 & -3.38888889 & -2.22222222 & 0. & -0.44444444 & 15.33333333
\end{bmatrix}
\begin{bmatrix} T_1\\ T_2\\ T_3\\ T_4\\ T_5\\ T_6\\ T_7\\ T_8\\ T_9 \end{bmatrix}
=
\begin{bmatrix}
0.73333333\\ 1.43333333\\ 0.26666667\\ 1.43333333\\ 2.6\\ 2.175\\ 0.675\\ 0.675\\ 2.175
\end{bmatrix}
$$

Reapplying the boundary conditions we find

$$
\begin{bmatrix}
32. & 0. & 0. & 0. & 0. & -9.33333333 & -6.66666667 & -6.66666667 & -9.33333333\\
0. & 1. & 0. & 0. & 0. & 0. & 0. & 0. & 0.\\
0. & 0. & 1. & 0. & 0. & 0. & 0. & 0. & 0.\\
0. & 0. & 0. & 1. & 0. & 0. & 0. & 0. & 0.\\
0. & 0. & 0. & 0. & 1. & 0. & 0. & 0. & 0.\\
-6.66666667 & -2.61111111 & 0. & 0. & -3.38888889 & 15.33333333 & -0.44444444 & 0. & -2.22222222\\
-9.33333333 & -2.27777778 & -1.5 & 0. & 0. & -3.11111111 & 17.55555556 & -1.33333333 & 0.\\
-9.33333333 & 0. & -1.5 & -2.27777778 & 0. & 0. & -1.33333333 & 17.55555556 & -3.11111111\\
-6.66666667 & 0. & 0. & -2.61111111 & -3.38888889 & -2.22222222 & 0. & -0.44444444 & 15.33333333
\end{bmatrix}
\begin{bmatrix} T_1\\ T_2\\ T_3\\ T_4\\ T_5\\ T_6\\ T_7\\ T_8\\ T_9 \end{bmatrix}
=
\begin{bmatrix}
0.73333333\\ B_1\\ B_2\\ B_3\\ B_4\\ 2.175\\ 0.675\\ 0.675\\ 2.175
\end{bmatrix}
$$

Now it becomes possible to solve the system of equations, Again using the python code, we are able to solve this system when we are given the values of B_i.

Let us consider two cases. First when $B_i = 0$, and the second when

$$
\begin{aligned}
B_1 &= 2.5\\
B_2 &= 0.5\\
B_3 &= 2.5\\
B_4 &= 4.5
\end{aligned}
$$

This will be the situation where it should exactly align with the forcing equation.

Using these values for the boundary conditions it is possible to solve for the solution to the system of equations. For this by hand example, I utilized numeric pythons linear algebra solver. I trust that there implementation is accurate, and should provide the correct results, and I was able to verify this by multiplying $[A]$ with the solved for $[T]$, and it resulted in the expected $[F]$ matrix. The values are shown below

For case 1 we find

$$[T] = \begin{bmatrix} 3.72581184e-01 \\ -9.54088910e-17 \\ -2.96059473e-16 \\ -1.03705316e-17 \\ -4.50659102e-16 \\ 3.66396848e-01 \\ 3.26239255e-01 \\ 3.26239255e-01 \\ 3.66396848e-01 \end{bmatrix}$$

And for case 2 we find

$$[T] = \begin{bmatrix} 3.51962034 \\ 2.5 \\ 0.5 \\ 2.5 \\ 4.5 \\ 3.7242765 \\ 3.17810172 \\ 3.17810172 \\ 3.7242765 \end{bmatrix}$$

Using these values I was able to plot the approximation generated by the function. Note that this plotting is done first through my C++ code, which generates a table of data, which is then rendered by matplotlib. These figures are shown here.

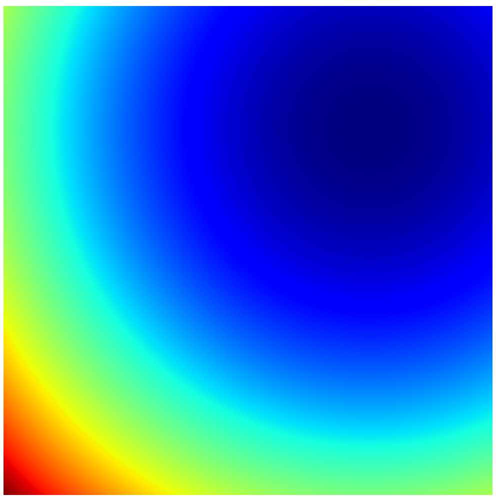

Figure 2. Forcing function

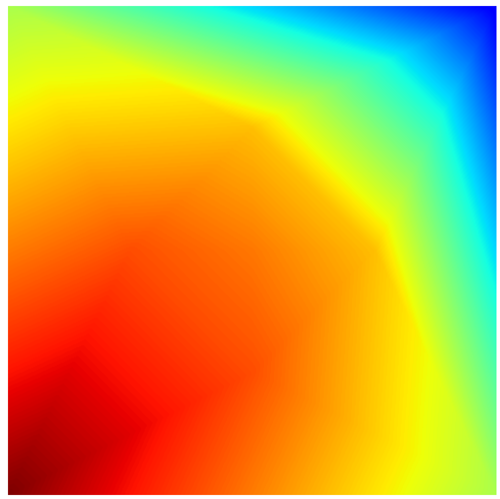

Figure 3. CASE2 approximation with specified boundary conditions

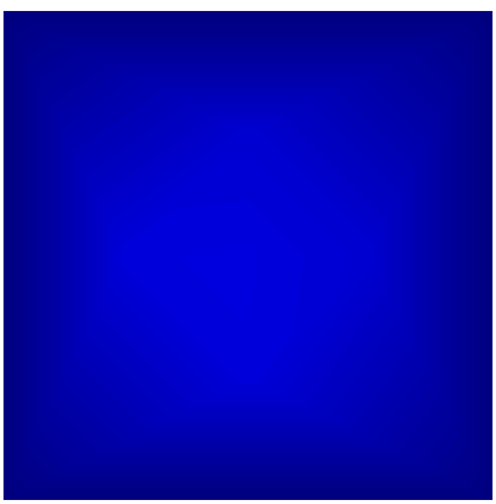

Figure 4. CASE1 approximation with boundary conditions of 0

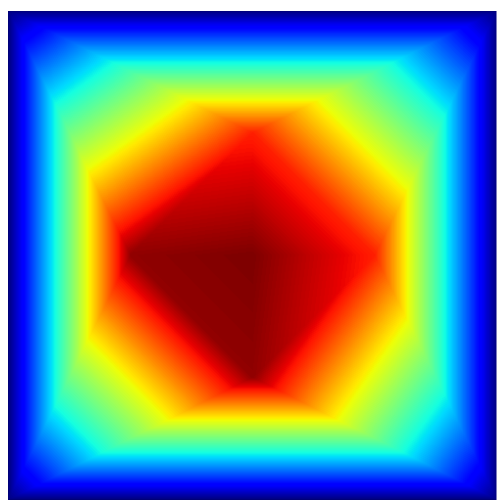

Figure 5. CASE1 approximation with boundary conditions of 0, but with automatic color range.

I just want to note that CASE2 looks weird. I'm hopeful that this is just due the grain size of the mesh, but I also think that a better solution would be if it were rotated by 180°, then inverse the colors. I am fearful that it looks strange because of the C++ plotting that I have setup, so I need to do some more checking of that process. But CASE1 looks pretty good, Its clear that the part in the upper right is indeed lower, and the bottom left is the maximum, but then everything stays relatively close to zero, because of the boundaries.

I've also added a plot of my proposed "better" approximation for case 2.

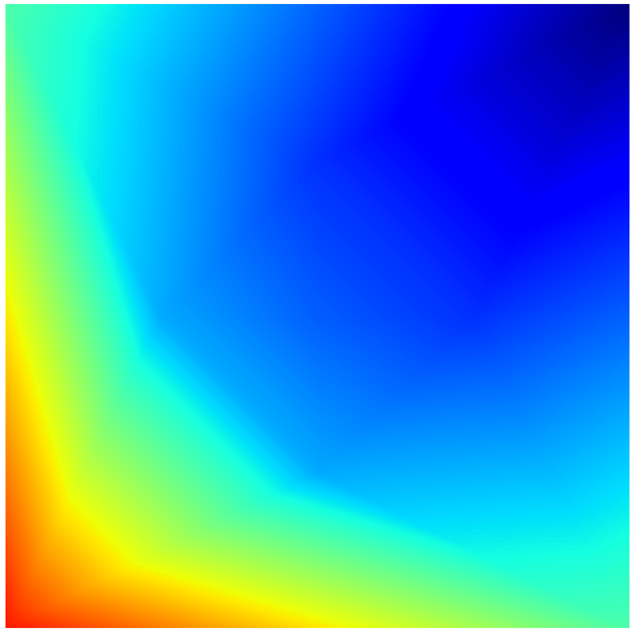

FIGURE 6. A possible better approximation, which is not what is found by the computer???

This was just an initial pass by hand, I have not rigorously verified that all of my algorithms work as intended. So I still need to do more testing for my integration and derivatives algorithms, to make sure that they are not the source of the issue.

One thing is very clear though. THIS IS NOT A PROCESS MEANT FOR HUMANS. It's terrible to do all of this by hand, It is very much meant for computers, but it was good to do it by hand, as some of the matrix assembly make a lot more sense.