The Mathematical Theoy of Finite Element Methods

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1 Basic Concepts

The finite element method provides a method for generating algorithms for approximating the solutions of differential equations.

1.1 Weak Formulation of Boundary Value Problem

Considering the two-point boundary value problem

$$-\frac{d^2u}{dx^2} = f \quad (0,1)$$
$$u(0) = 0 \quad u'(1) = 0$$

Taking some function v, and finding the inner product with f, we find

$$(f, v) = \int_0^1 u'(x)v'(x)dx = a(u, v)$$

We define a vector space V as follows

$$V \equiv \left\{ v \in L^2(0,1) : \quad a(v,v) < \infty \text{ and } v(0) = 0 \right\}$$

Using this we can characterize the solution to the differential equation as any function $u \in V$ that satisifes the boundary condition and that satisfies

$$a(u, v) = (f, v) \quad \forall v \in V$$

This is called the *variational* or *weak* formulation of the differential equation. It is variational because v is allowed to vary arbitrarily. It is called weak, because there are other ways in which to interpret the equation with less restrictives assumptions on f.

1.2 Ritz-Galrkin Approximation

Let $S \subset V$ be any (finite dimensional) subspace. Then when we replace V in the previous declaration we find

$$u_S \in S \quad a(u_S, v) = (f, v) \quad \forall v \in S$$

With this approximation, it can be shown that the solution u_S must *exist* and be *unique*.

1.3 Error Estimates

Observing the fundamental *orthogonality* between u and u_S implies

$$a(u - u_S, w) = 0 \quad \forall w \in S$$

Now we define the energy norm as

$$\|v\|_E = \sqrt{a(v, v)}$$

Using this norm, the Schwarz' inequality relates the energy norm and inner-product

$$|a(v, w)| \leq \|v\|_E \, \|w\|_E \quad \forall v, w \in V$$

1.4 Piecewise Polynomial Spaces

1.5 Relationship to Difference Methods

1.6 Computer Implementation of FInite Element Methods

1.7 Local Estimates

1.8 Adaptive Approximation

1.9 Weighted Norm Estimates

2 Sobolev Spaces

This chapter develpps a function space that is used in the variational formulation of differential equations.