PHYSICALY BASED RENDERING

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1. Introduction

2. Radiometry & Light Attenuation

2.1. Radiometry Recap. What to measure in a simulation?

Definition 2.1 (Radiant flux)**.** Total amount of energy passing thought surface (measured per second). $\Phi\,[W]$.

This is not sufficient to measure, because it is the energy per surface area, then consider a large radiant flux that could either be

(1) a lot of energy over a small surface.
(2) a little energy over a large surface.

Thus this metric is too ambiguous, and not good enough for us.

Definition 2.2 (Irradiance)**.** Total amount of energy passing through a unit area. $E\left[W/m^2\right]$.

This is still ambiguous, we haven't fixed the angle which the light arrives to the surface. So

(1) a lot of energy in a huge angle
(2) a little energy in a small angle

will measure the same irradiance.

Definition 2.3 (Radiance)**.** Total energy passing through a unit area with unit angle $I\left[W/\left(m^2\cdot sr\right)\right]$[1].

This is the measurement that we will be using.

2.2. The most fundamental question. How much light exits a surface point in a given direction?

The complete answer is given by Maxwell equations! In practice, we don't do this, it would require simulations on the size of wavelengths of light, and that is overly complex.

The real solution is the rendering equation!

2.3. The scalar product.

$$\vec{a}\cdot\vec{b}\equiv||\vec{a}||\cdot||\vec{b}||\cos\theta.$$

For this work we will always assume that the vectors are normalized, and so this becomes

$$\vec{a}\cdot\vec{b}\equiv\cos\theta.$$

2.4. Terminology.

The key components in the terminology are depicted in figure 2.1[2].

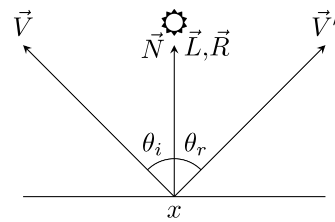

Figure 2.1. Basic terminology

$\vec{V}$	direction towards the viewer (eye,camera)
$\vec{N}$	surface normal
$\vec{L}$	vector point towards the light source
$\vec{R}$	reflected ray direction $\vec{R}=\vec{L}-2\vec{N}\left(\vec{L}\cdot\vec{N}\right)$.
θ_i,θ_r	incident and reflected angles.

2.5. Light Attenuation.

When the sun is directly above as in figure 2.1, then

$$\left(\vec{L}\cdot\vec{N}\right)\Phi=\cos\alpha|_{\alpha\approx0}\Phi=\Phi.$$

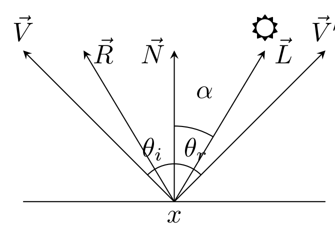

Figure 2.2. Light attenuation with sun at 45°.

When the sun is at an angle as in figure 2.2, then

$$\left(\vec{L}\cdot\vec{N}\right)\Phi=\cos\alpha|_{\alpha\approx45^\circ}\Phi=0.7\Phi.$$

[1] Angle in more dimensions is called solid angle, for which the unit is steradians(sr)

[2] Vectors are always pointing away from the point of intrest x

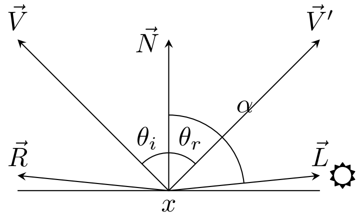

FIGURE 2.3. Light attenuation with sun at 90°.

When the sun is at an angle as in figure 2.3, then

$$\left(\vec{L}\cdot\vec{N}\right)\Phi = \cos\alpha|_{\alpha\approx 90^\circ}\Phi = 0.$$

Thus we can model this light attenuation using this dot product.

3. BRDF models & The Rendering Equation

3.1. Materials. What makes the difference between materials?

Different materials reflect incoming light to different directions and absorb different amounts of it.

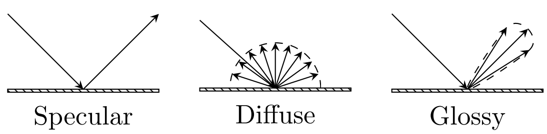

FIGURE 3.4. Specula, diffuse, and spread reflections from a surface

Specular: For one incoming direction there is only one outgoing direction, this is what a mirror would be.

Diffuse: For one incoming direction on there are many possible outcomes.

Glossy/Spread: This is a mixture of specular and diffuse.

3.2. **BRDF.** We will create a probability density function that takes

(1) an incoming light direction
(2) a point on the surface

as parameters, and outputs the probability of a given outward direction. We define this function as

Definition 3.1 (Bidirectional reflectance distribution function(BRDF))**.**

$$f_r\left(\vec{\omega}, x, \vec{\omega}'\right).$$

What about materials that don't reflect all incoming light, some things that could transmit the light. There are some examples in figure 3.5.

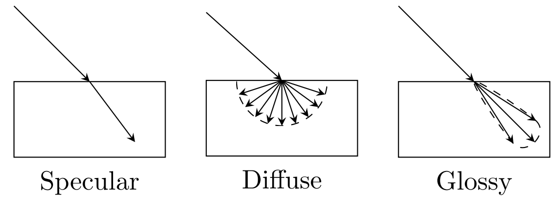

FIGURE 3.5. Transmitted models of light

Things are transparent because atom are mostly empty. Things are opaque, because the electrons are absobing the photons. The electrons then jump to an outer orbit. Glass is transparent because the orbits of the electrons are too far apart, so that if an electron absorbs a photon it does not get enough energy to jump to the higher orbit. Note that if the light is of a higher energy (not in the visible spectrum) then it could be opaque. For instance glass is opaque to UV-light.

3.3. **BTDF.** For light that is also transmited, we have the Bidirectional transmittance distribution function.

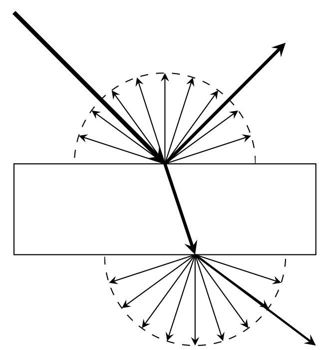

FIGURE 3.6. Light that is both reflected and transmitted

3.4. **BSDF.** To generalize the BRDF and BTDF, we use the BSDF (S is for scattering).

3.5. **BRDF Properties.**

(1) Helmholtz-reciprocity: The direction of the ray of light can be reversed

$$\forall\vec{\omega},\vec{\omega}': \quad f_r\left(\vec{\omega}, x, \vec{\omega}'\right) = f_r\left(\vec{\omega}', x, \vec{\omega}\right)$$

(2) Positivity: It is an impossibility for an exit direction to have a negative probability.

$$\forall\vec{\omega},\vec{\omega}': \quad f_r\left(\vec{\omega}, x, \vec{\omega}'\right) \geq 0$$

(3) Energy conservation: An object may reflect or absorb incoming light, but no more can com out than the incoming amount.

$$\int_\Omega f_r\left(\vec{\omega}, x, \vec{\omega}'\right)\cos\theta d\vec{\omega}' \leq 1$$

This means that we add up all the incoming energy from all possible directions, taking the light attenuation into account. If it equals 1, all light is is reflected, if it is less than 1 some of the light is absorbed. More than 1 is impossible.

3.6. **The Rendering Equation.** An object can emit light itself. It also receives light form different directions, which it will either reflect or absorb, therefore

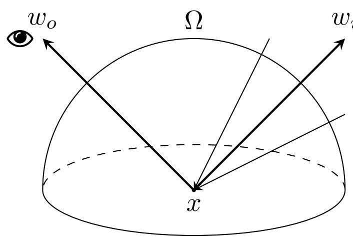

FIGURE 3.7. Rendering Equation Representation

$$L_o(x,\vec{\omega}) = \underbrace{L_e(x,\vec{\omega})}_{\text{emitted}} + \underbrace{\int_\Omega L_i(x,\vec{\omega})\, f_r(\vec{\omega},x,\vec{\omega}')\cos\theta d\vec{\omega}'}_{\text{reflected incoming light}}$$

Where

$L_o(x,\vec{\omega})$: exiting radiance in point x towards direction $\vec{\omega}$.
$L_e(x,\vec{\omega})$: emitted radiance in point x towards direction $\vec{\omega}$.
$\int_\Omega \cdots d\vec{\omega}'$: Sum of the incoming radiance from all $\vec{\omega}'$ directions[3].
$L_i(x,\vec{\omega}')$: Incoming radiance from the direction $\vec{\omega}'$ to point x.
$f_r(\vec{\omega},x,\vec{\omega}')$: BRDF.
$\cos\theta$: light attenuation.

Difficulties

(1) The exitant radiance of a point x depends on the incoming radiance of every other point, which also depend on x. Thus every point depends on every other point, and so would be infinitely recursive.
(2) It is absolutely hopeless to compute this integral in closed form.
(3) The integral is also infinite-dimensional. We need to compute an infinite number of bounces of light.
(4) It is often singular.

As of now, we have insufficient knowledge to even give an approximate solution to this equation.

The solution is to compute the entirety of the light path, then move the last bounce point, and compute the contribution of each individual light source to the sensor. This is not what we will be using. But here it is

$$\begin{aligned} L_o(p'\to p) = L_e(p'\to p) \\ + \int_A L_i(p^n\to p')\, f_r(p^n\to p'\to p) \\ \cdot G(p^n\leftrightarrow p')\, dA(p^n) \end{aligned}$$

$L_o(p'\to p)$: outgoing radiance from point p' to p.
$L_e(p'\to p)$: emitted radiance from p' to p.
$\int_\Omega \cdots dA(p^n)'$: sum for all possible p^n surface points.
$L_i(p^n\to p')$: incoming radiance form point p^n to p'.
$f_r(p^n\to p'\to p)$: probability of $p^n\to p'\to p$ transfer.
$G(p^n\leftrightarrow p')$: geometry term.

4. Diffuse, Specular, and Ambient Shading

4.1. **BRDF types.** Illumination model for *ambient* BRDF(simplified):

$$I = k_a I_a$$

k_a: ambient coefficient of an object.
I_a: ambient intensity of the scene/light source.

Illumination model for *diffuse* BRDF(simplified);

$$I = k_d\left(\vec{L}\cdot\vec{N}\right)$$

k_d: diffuse coefficient of an object[4]
$\vec{L}$: vector point to the light source
$\vec{N}$: surface normal

Illumination model for *specular* BRDF(simplified):

$$I = k_s\left(\vec{V}\cdot\vec{R}\right)^n$$

k_s: specular coefficient of an object
$\vec{V}$: vector pointing to the viewer/camera
$\vec{R}$: reflected direction of the light ray
$(\cdot)^n$: shininess factor.

The view vector is in the formula, therefor it should look differently if we turn out heads.

The sum of these three type of illumination model is called Phong shading, and is a descriptive representation of the shading.

[3] Ω is only the hemisphere, because the $\cos\theta$ from below would be negative, and we don't need to deal with this light

[4] The diffuse and other coefficients may take different values for different wavelengths. An object may absorb most of the green light, but may reflect all incoming red light in the meantime. The proper term would be $k_{d,\lambda}$.

4.2. The Illumination Equation. This is the simplification of the rendering equation

$$I = k_a I_a + I_i \left(k_d \left(\vec{L} \cdot \vec{N} \right) + k_s \left(\vec{V} \cdot \vec{R} \right)^n \right)$$

- Only accounts for the direct illumination of the light sources.
- The indirect illumination is neglected, the ambient term helps to reinject some of the lost energy into the system.
- This is indeed a very crude approximation of physically reality.

This could be summed for all light sources, we kept this notation for simplicity.

Something is missing, what is missing?

4.3. The Algorithm.

(1) Construction of the camera/eye rays
(2) Intersection with the scene object
(3) Shading
(4) Reflection/refraction directions
(5) Recursion

The ray doesn't stop at the first intersection, we have to trace it further as it will either get reflected or refracted.

5. The Fresnel Equation and Schlick's Approximation

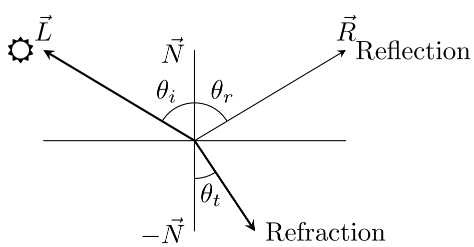

Figure 5.8. A light ray with both reflection and refraction

5.1. The Fresnel equation (simplified). Schlick's approximation

$$R(\theta) = R_0 + (1 - R_0)(1 - \cos\theta)^5$$

$R(\theta)$**:** probability of reflection when the incident ray angle is θ.

R_0**:** probability of reflection on normal incidence($\theta = 0$).

$$R_0 = \left(\frac{n_1 - n_2}{n_1 + n_2} \right)^2$$

For an air/vacuum-medium interaction, where $n_2 = 1$, then the interactions are given by

$$R_0 = \left(\frac{n_1 - 1}{n_1 + 1} \right)^2.$$

This gives the probability of reflection. If the ray is not reflected, it will be refracted according to the law of energy conservation. Therefore the probability of transmission is given by

$$T(\theta) = 1 - R(\theta).$$

For instance

$$R(0°) = R_0$$
$$R(90°) = R_0 + (1 - R_0) = 1$$

We can think of this function as an interpolation between these two states.

For Air-glass interaction $n_{\text{glass}} = 1.5$

$$R(\theta) = \left(\frac{0.5}{2.5} \right)^2 + \left(1 - \left(\frac{0.5}{2.5} \right)^2 \right) (1 - \cos\theta)^5$$

We should expect very high probability of refraction when the light ray comes from the direction of the normal (theta is zero). High probability of refraction means that the reflection probability is low, the function value should be low at zero.

$$R(0) < 0.1$$

At around 60 degrees, refraction rays seem to be brighter, so the function value should be low there. 60 degrees in radians is about 1.

$$R(1) < 0.2$$

Always reflect the rays when they come from almost parallel to the surface.

$$R(1.7) \approx 1$$

Lets check if the equations match our expectations.

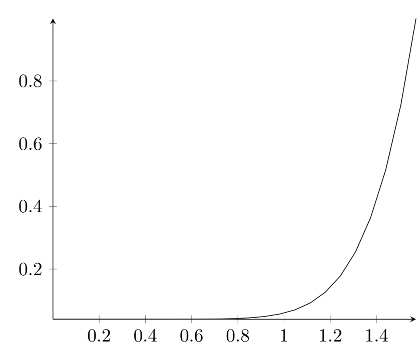

Figure 5.9. Plot of Schlick's Approximation for air-glass interface.

This matches what we expect... But if we continue extrapolating, it seems to be going above a probability of 1.

Lets do another experiment, but with a vacuum to vacuum interface.

$$R(\theta) = (1 - \cos\theta)^5$$

We expect that there will be no reflection, so $R(\theta) = 0$.

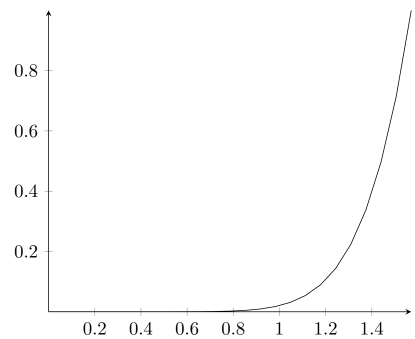

FIGURE 5.10. Schlick's approximation for vacuum-vacuum interface.

This is not what we want, $R(\theta) \neq 0$.

The Schlick-Approximation is used to efficiently calculate vacuum-medium type of interactions.

The original Fresnel equation:

$$R_s(\theta) = \left| \frac{n_1 \cos\theta - n_2 \sqrt{1 - \left(\frac{n_1}{n_2} \sin\theta\right)^2}}{n_1 \cos\theta + n_2 \sqrt{1 - \left(\frac{n_1}{n_2} \sin\theta\right)^2}} \right|^2$$

What does this say about the vacuum-vacuum interaction? $n_1 = n_2 = 1$.

$$R_s(\theta) = 0$$

This is what we expect it to be. So the complete equation provides the full explanation and solution.

6. Snell's Law and Total Internal Reflection