Stock Price Prediction using Fractional Gradient-Based Long Short Term Memory

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Abstract. Deep Learning is considered one of the most effective strategies used by hedge funds to maximize profits. But Deep Neural Networks (DNN) lack theoretical analysis of memory exploitation. Some traditional time series methods such as Auto-Regressive Integrated Moving Average (ARIMA) and Generalized Auto-Regressive Conditional Heteroskedasticity (GARCH) work only when the entire series is pre-processed or when the whole data is available. Thus, it fails in a live trading system. So, there is a great need to develop techniques that give more accurate stock/index predictions. This study has exploited fractional-order derivatives' memory property in the backpropagation of LSTM for stock predictions. As the history of previous stock prices plays a significant role in deciding the future price, fractional-order derivatives carry the past information along with itself. So, the use of Fractional-order derivatives with neural networks for this time series prediction is meaningful and helpful.

1. Introduction

Stock equity plays a significant role in a nation's economy. The stock market prediction focuses on the extraction of future movement of the stock price. The accurate forecast of share price movement results in great profit avenues. Thus, this makes the investors eager about the direction of market prices. The stock market is highly dynamic, and it is a challenging task to predict the movement, as it depends on numerous factors such as politics, economic growth, etc. There are two conventional ways of predicting the market the one is technical, and the other is fundamental. The technical analysis investigates price change's direction to predict the future stock values, while fundamental analysis depends on analysing the unstructured textual information.

These traditional approaches are now considered inferior due to increased computational power, where larger data sets can be analysed more accurately in a short duration. With these approaches, new artificial intelligence approaches such as machine learning, computational intelligence, deep learning, and others originate. Linear regression, moving average, support vector machine, k-nearest neighbour algorithm [1], ARIMA [2], and LSTM [3] are machine learning algorithms widely used for stock price prediction. Most of these methods are not able to reflect the significant changes in the stock price. Linear models like ARIMA work only for a particular time-series data. It does not work for a different set of data. LSTM works fine for this problem, but the results can still be further improved. Predicting a stock price is a challenging task on account of its history dependency and high volatility.

This study predicts the stock price movement in the future using the integrated approach involving neural network variant LSTM and fractional derivatives. The customer will have a certain number of years for which the money is invested. To predict the portfolio's value, we have to expect each stock's price given the number of years (with/without dividends). Our task is Sequence Learning (Time Series Analysis). From a sequence of Historical Prices, we will train our model to predict the future price.

2. Literature Review

In the recent decade, enormous deep learning-based techniques have been introduced for stock prediction [4]. Before that, Artificial Neural Network (ANN) was used extensively for stock market prediction [5]. Deep learning works significantly better than simple ANN. Several other techniques are carried out to analyse the market, some of which are statistical methods, fractional calculus-based [6,7,8], and multi-criteria decision-making techniques [9,10]. Pritam et al. [9] constructed a portfolio considering the customer's risk preferences using the fuzzy technique for order of preference by similarity to ideal solution. The fuzzy analytic hierarchy process has been used in [10] to select Bombay Stock Exchange Sensitive Index (BSE SENSEX) sectors' dominance for optimal equity. In [6], the stock market has been analysed using fractional differential equations. Fractional Calculus has also been applied to ANNs due to the long-term memory or non-locality of fractional derivatives [11-14]. Wang et al. [11] used the fractional steepest descent algorithm for training three-layered neural networks and proved its monotonicity and convergence. Stochastic gradient descent algorithm has been generalized to Fractional-order Gradient Descent algorithm (FGD), the fractional-order derivative replaces the integer-order derivative, and the update rule becomes

$$\Delta w = -\eta \frac{\partial^{\nu} E}{\partial w^{\nu}} \tag{1}$$

where $0 < \eta < 1$ is the learning rate, and $\nu > 0$ is the fractional order of differentiation. The fractional derivative was used for the backpropagation algorithm for Feed Forward Neural Networks (FNNs) by Chen et al. [14] in 2013. The simulation results demonstrated that the convergence speed based on fractional-order FNNs was much faster than integer-order FNNs. In 2015, Pu [15] paid attention to the fractional-order gradient method. It was observed that this method might not lead to the actual extreme point. This defect was rectified by Chen [16,17] by using the truncation and short memory principle in the fractional-order gradient method. Khan et al. [13,18] proposed fractional-order backpropagation through a time algorithm for Recurrent Neural Networks (RNNs) and Radial Basis Function neural networks. In [18], FGD is the amalgamation of the conventional and the modified Riemann–Liouville derivative-based fractional gradient descent method. In 2018, Bao et al. [19] proposed a deep fractional-order BP neural network with L_2 regularisation term and the order α can be any positive real number. Caputo's derivative-based fractional gradient method for backpropagation of Convolutional Neural Networks has been introduced, which successfully converges to real extreme point [20].

RNNs are capable of processing sequential data. However, it is challenging to train long term dependencies in these kinds of structures. LSTM is a special kind of networks capable of learning long term dependencies using dedicated circuits. Memory or history plays a significant role in the training of RNNs/LSTMs. Hence, the application of fractional derivatives that inherently incorporate history in computation seems very meaningful and beneficial for training these types of Neural Networks. Due to fractional derivatives' non-locality property, fractional-order LSTM/RNN networks are expected to learn long-term dependencies more easily than the integer-order gradient-based LSTM/RNN. For dealing with the complexity of stock market data, we are using fractional-order LSTM for its rigorous analysis. In the following sections, the concept of LSTM and the basics of fractional derivatives are briefly described.

3. Preliminaries

3.1. Basics of Fractional Calculus

Fractional calculus was originated around three centuries ago. The inventor of calculus, Leibniz, was asked by L'Hopital about derivatives' behaviour when order is non-integer. At that moment, Leibniz replied that "It will lead to some useful consequences in the future." This response caught the attention of other great mathematicians, Lacroix, Riemann, Liouville, Abel, and many others. These mathematicians started working in fractional derivatives and gave many definitions for fractional integrals and derivatives. Abel was the first mathematician to incorporate the use of fractional calculus in solving the tautochrone problem. But still, fractional calculus was considered a field of abstract research involving complicated mathematical calculations. Due to the advent of computational support, fractional calculus started emerging and applied to several areas of Science, Engineering, and Economics [21-31]. Various other definitions of fractional derivatives were introduced, which can be used for solving real-life problems [33]. The most used definitions are as follows:

3.1.1. Riemann-Liouville Fractional Integral Operator.: Let $\alpha > 0$ and g be the function which is piecewise continuous on $G' = (0, \infty)$ and integrable on any finite subinterval of $G = [0, \infty)$. Then for $t > a$ and $t, a \in \mathbb{R}$ the following equation

$$ {}_{a}D_{t}^{-\alpha} g(t) = \frac{1}{\Gamma(\alpha)} \int_{a}^{t} (t-\xi)^{\alpha-1} g(\xi) d\xi \tag{2}$$

represents the Riemann-Liouville fractional integral of function g of order α. This is the most generally used variant of fractional integration. Gamma function is not defined at zero and negative integers. Thus, this definition cannot be generalized for non-integer differentiation. So separate definitions were introduced for fractional differentiation. The three most frequently used definitions of the fractional derivative are:

3.1.2. Riemann-Liouville Fractional Derivative. Suppose that $\alpha > 0,\ t > a$ and $m \in \mathbb{N}$ such that $m - 1 \leq \alpha < m$, then to get α^{th}-derivative of $g(t)$ we first integrate $g(t)$ fractionally up to $m-\alpha$, then differentiate m times, i.e.

$$ {}_{a}D_{t}^{\alpha} g(t) = \begin{cases} \dfrac{d^m}{dt^m} \left\{ \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_{a}^{t} (t-\xi)^{m-\alpha-1} g(\xi) d\xi \right\} & \text{When } m-1 \leq \alpha < m. \\ \dfrac{d^m g(t)}{dt^m} & \text{When } \alpha = m. \end{cases} \tag{3}$$

where Γ is the Gamma function, is called the Riemann-Liouville fractional derivative of function g of order α. This definition requires only a continuous function for its application, but this definition has some limitations. Firstly, it does not give the differentiation of a constant function to be zero. Secondly, while solving fractional differential equations involving this derivative, Laplace transform is used, and Laplace transform of this derivative involves initial conditions at fractional orders that are not having any physical interpretations. Caputo introduced another definition of derivative for dealing with such problems, as explained below [34]. But it requires the function to be differentiable.

3.1.3. Caputo Fractional Derivative. Suppose that $\alpha > 0,\ t > a$ and $m \in \mathbb{N}$ such that $m - 1 \leq \alpha < m$, then to get α^{th}-derivative of $g(t)$ we first integrate $g(t)$ fractionally up to $m-\alpha$, then differentiate m times, i.e.

$$ {}_{a}^{C}D_{t}^{\alpha}g(t) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)}\displaystyle\int_{a}^{t}(t-\tau)^{n-\alpha-1}g^{(n)}(\xi)d\xi & m-1 \le \alpha < m. \\ \dfrac{d^{m}g(t)}{dt^{m}}, & \alpha = m. \end{cases} \quad (4) $$

is called the Caputo fractional derivative of function g of order α.

3.1.4. Grünwald-Letnikov (G-L) Fractional Derivative. The $\alpha > 0$ order derivative of a function g is expressed as the limit of a sum by

$$ D^{\alpha}g(t) = \lim_{\substack{h\to 0 \\ nh=t-a}} \sum_{r=0}^{n}(-1)^{r}\binom{\alpha}{r}g(t-rh) \quad (5) $$

where $\binom{\alpha}{r} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-r+1)\Gamma(r+1)}$. This discrete version introduced by Grünwald and Letnikov [35,36] does not require the function to be differentiable or continuous.

It is visible in the definition that evaluation of fractional derivative at t considers the value of the function at all past values $t-nh.$ But while evaluating integer-order derivative at t considers the value of the function at t and $t-h$. The main advantage of using fractional derivatives is their memory property. These can consider history dependency and non-local distribution of the system. Another advantage of fractional derivatives is that the order of differentiation acts as an additional degree of freedom. On these grounds, fractional calculus has been exploited in anomalous diffusion [37], viscoelasticity [38], signal and image processing [39], and biology [40].

3.1.5. Fractional Chain Rule. Due to computational complexity, an approximated chain rule has been used, which is given in [41], obtained by using the fractional Taylor's series on a differentiable function and that is summarised as:

$$ g(x+h) = g(x) + \frac{h^{\alpha}}{\Gamma(1+\alpha)}g^{(\alpha)}(x) + \frac{h^{2\alpha}}{\Gamma(2+\alpha)}g^{(2\alpha)}(x) + \frac{h^{3\alpha}}{\Gamma(3+\alpha)}g^{(3\alpha)}(x) + \cdots \quad (6) $$

$$ \Rightarrow \frac{\Delta g(x)}{h^{\alpha}} = \frac{1}{\Gamma(1+\alpha)}g^{(\alpha)}(x) + \frac{h^{\alpha}}{\Gamma(2+\alpha)}g^{(2\alpha)}(x) + \frac{h^{2\alpha}}{\Gamma(3+\alpha)}g^{(3\alpha)}(x) + \cdots \quad (7) $$

Taking limit $h \to 0$, we get

$$ lim_{h\to 0}\frac{\Delta g(x)}{h^{\alpha}} = \frac{1}{\Gamma(1+\alpha)}g^{(\alpha)}(x) \quad (8) $$

Also, $lim_{h\to 0}\frac{\Delta^{\alpha}g(x)}{h^{\alpha}} = g^{(\alpha)}(x) \Rightarrow \Delta^{\alpha}g(x) \approx \Gamma(1+\alpha)\Delta g(x) \quad 0<\alpha<1.$

From the above equation, we can also say $d^{\alpha}g(x) \approx \Gamma(1+\alpha)dg(x),\ 0<\alpha<1$. Using this result,

$$ \frac{d^{\alpha}g\big(u(x)\big)}{dx^{\alpha}} = \frac{d^{\alpha}g\big(u(x)\big)}{du^{\alpha}}\frac{du^{\alpha}}{dx^{\alpha}} = \frac{\Gamma(1+\alpha)dg\big(u(x)\big)}{\Gamma(1+\alpha)du}u_{x}^{\alpha}(x) = g_{u}'(u)u_{x}^{\alpha}(x) \quad (9) $$

Hence,

$$D_x^{\alpha} f\big(u(x)\big) = D_u^{1} f(u)\, D_x^{\alpha} u(x)\,. \tag{10}$$

3.2. Long Short Term Memory (LSTM)
After reviewing the state-of-the-art literature, we noticed the various kinds of neural networks that could be used for this prediction. As for this prediction, we are using time series data. For such data, either Deep Neural Networks (DNN) or Recurrent Neural Networks (RNN) are used. But DNN struggles when there are abrupt changes in the data. Also, DNN does not have any memory. RNNs cover the entire sequence starting from the beginning and then help predict the next term. These networks have loops in them, allowing the information to persist. But as the gap increases, Vanilla RNNs fail to connect the information (resulting in less accuracy in predictions). Also, vanilla RNN sometimes suffers from the problem of vanishing gradient. Hence, we have chosen the LSTM variant for price prediction, which was introduced for solving this problem of the gradient.

4. Proposed Approach
The algorithm employed for predicting the stock price involves the following steps: **1.)** Data Exploration **2.)** Data Pre-processing **3.)** Forward Propagation **4.)** Fractional Gradient Descent Algorithm for training. The approach involves coding the LSTM from scratch.

4.1. Data Exploration and Processing
In this study, we have used the data of Google.com for a decade, January 2007 - June 2017. These data points are indexed in time order. The goal of the study is to predict the closing price for any given date after training. The data was obtained from the Google finance application programming interface, which is shown in Table 1.

Table 1: Google.com data sample

Date	Open	High	Low	Close	Adjusted close	value
2015-11-25	107.510002	107.660004	107.250000	107.470001	101.497200	1820300
2015-11-27	107.589996	107.760002	107.220001	107.629997	101.648300	552400
2015-11-30	107.779999	107.849998	107.110001	107.169998	101.213867	3618100
2015-12-01	107.589996	108.209999	107.370003	108.180000	102.167740	2443600
2015-12-02	107.099998	108.269997	106.879997	107.050003	101.100533	2937200
2015-12-03	107.290001	107.480003	105.059998	105.449997	99.589470	3345600
2015-12-04	107.809998	107.540001	105.620003	107.389999	101.421646	4520000
2015-12-07	107.230003	107.269997	106.059998	106.550003	100.628342	3000500
2015-12-08	107.940002	106.400002	105.269997	105.910004	100.023895	3149600

4.2. Neural Network Architecture and Training
LSTM is used for obtaining better predictions as these are capable of learning long-term dependencies. We have optimized the model by tuning the following parameters. The model consists of three layers of nodes, and each layer contains 64 nodes. The training and testing data used are 80% and 20%, respectively. The validation sets are kept 0:5% of the training data. Random values initialize the network parameters, and the batch size is 512. We train 20 epochs for our fractional-order LSTM.

4.3. Backward Propagation
To learn the optimal parameters, we have used fractional derivatives to differentiate loss function concerning each of the parameters, and the update rule becomes

$$\Delta w = -\eta \frac{\partial^{\alpha} E}{\partial w^{\alpha}} \tag{11}$$

where α is the fractional order of differentiation and η is the learning rate. The error function is a composite function. Thus, the above-mentioned fractional chain rule is employed. Hence

$$D_w^{\alpha} E\big(J(w)\big) = D_J^{1} E(J) D_w^{\alpha} J(w) \tag{12}$$

The backpropagation code for evaluation of G-L derivative of order 'alpha' is as follows, where we have run the process for $\alpha = 0.9$.

```
GL_previous = f_values[1]
for index in range(2, num_points);
    GL_current = GL_previous*(num_points-alpha-index-1)/(num_points-index) + f_values[index]
    GL_previous=GL_current
return GL_current*(num_points/(domain_end-domain_start))**alpha
```

4.4. Results and Analysis

This section presents the prediction results of the proposed algorithm. The results have been obtained from the test data of Google.com for the past decade. The model also had a validation split, while training phase, to avoid overfitting. Firstly, we have analysed the different order of differentiation 'α' as shown in the graph in Figure 3. We have observed the predicted price for $0 < \alpha < 1$. In Figure 1, the green, blue, orange, and green plots depict the predicted prices at order $1.2, 0.9, 0.7,$ and 0.5, respectively. The error is obtained least at $\alpha = 0.9$. For estimating the error, we have used the root mean squared error.

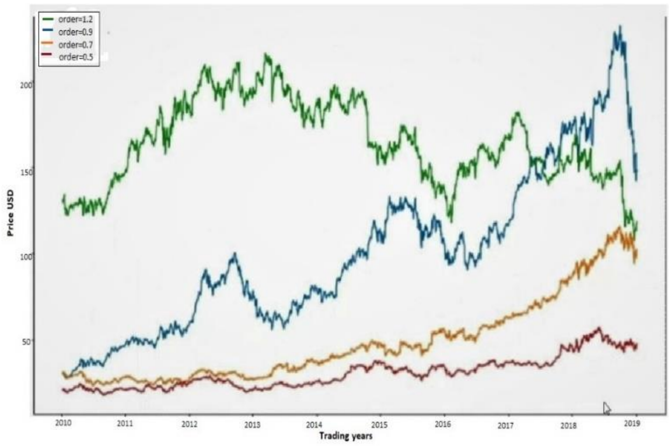

Figure 1: Comparison of prediction for the different order of differentiation

Then we have analysed the results of LSTM obtained at order 0.9 and the adjusted close value of the stock. The graph in Figure 2 depicts that the fractional-order LSTM has successfully predicted the stock's future price trend, i.e., upward, or downward. But the proposed model fails to predict the price with the required accuracy. In Figure 2, the blue and green plots depict the predicted and adjusted close prices, respectively. The proposed LSTM (with fractional derivative) has an error of 0.030% in the training dataset and 0.107% for the test dataset at order 0.9. It is evident from the graph as well that the direction of price movements is well predicted.

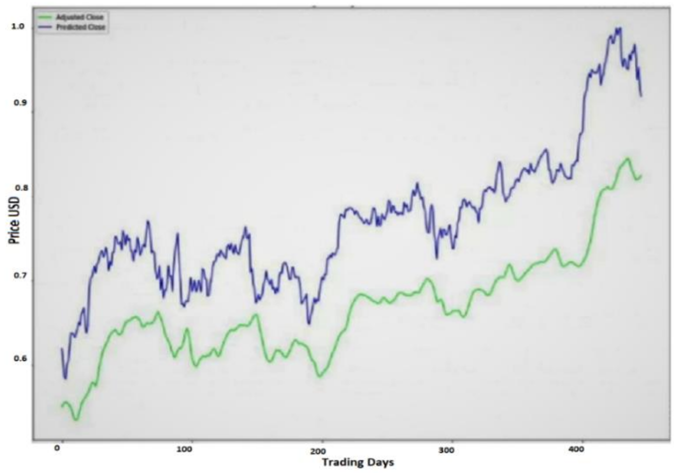

Figure 2: Comparison of predicted result at order 0.9 with the adjusted close stock value

5. Conclusion

In this work, a novel fractional backpropagation learning algorithm for the LSTMs is proposed and is applied in portfolio management. The proposed method deployed the fractional gradient descent algorithm involving Caputo's derivative in the learning of LSTM. It is observed that fractional derivatives incorporated in the neural network help predict stock price direction. The method is not precise in indicating the exact Stock Price. Still, it successfully suggests whether the investor should remain invested in the stock for the long-term depending upon stock price movements' direction. The accuracy of fractional-order backpropagation can be studied upon in the future.

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Conflicts of Interest

The authors declare that they have no conflicts of interest to report regarding the present study.

References

[1] Alkhatib K, Najadat H, Hmeidi I and Shatnawi M K A 2013 Stock price prediction using k-nearest neighbor (kNN) algorithm *Int. J. of Business, Humanities and Technology* **3(3)** 32-44

[2] Zhang G P 2003 Time series forecasting using a hybrid ARIMA and neural network model *Neurocomputing* **50** 59-175

[3] Greff K, Srivastava R K, Koutník J, Steunebrink B R and Schmidhuber J 2016 LSTM: A search space Odyssey *IEEE Trans. Neural Netw. Learn Syst.* ***28*(10)** 2222-32

[4] Jiang W 2020 Applications of deep learning in stock market prediction: recent progress *Preprint arXiv:2003.01859*

[5] Vui C S, Soon G K, On C K, Alfred R and Anthony P 2013 A review of stock market prediction with Artificial neural network *Int. Conf. on control system, computing and engineering* (Penang: IEEE) pp 477-82

[6] Wang H 2019 Research on application of fractional calculus in signal real-time analysis and processing

in stock financial market *Chaos, Solitons Fractals* ***128*** 92-97

[7] Cartea A and del-Castillo-Negrete D 2007 Fractional diffusion models of option prices in markets with jumps. *Physica A* ***374*****(2)** 749-63

[8] Pritam K S 2020 *Prospects of fractional calculus in sustainable development goals (Ph.D. Thesis)* BITS Pilani, India

[9] Pritam K S, Mathur T and Agarwal S 2019 An efficient portfolio management for trading under uncertain environment. *Int. J. Supply Chain Manag.* ***8*****(2)** 277

[10] Pritam K S, Mathur T and Agarwal S 2020 Hierarchy of sectors in BSE SENSEX for optimal equity investments using fuzzy AHP. *Intelligent Communication, Control and Devices* (Singapore: Springer) pp 393-404

[11] Wang J, Wen Y, Gou Y, Ye Z and Chen H 2017 Fractional-order gradient descent learning of BP neural networks with Caputo derivative *Neural Netw.* ***89*** 19-30

[12] Wang H, Yu Y, Wen G, Zhang S and Yu J 2015 Global stability analysis of fractional-order Hopfield neural networks with time delay *Neurocomputing* ***154*** 15-23

[13] Khan S, Ahmad J, Naseem I and Moinuddin M 2018 A novel fractional gradient-based learning algorithm for recurrent neural networks *Circuits, Syst. Signal Process.* ***37*****(2)** 593-612

[14] X Chen 2013 *Application of fractional calculus in back propagation neural networks (Ph.D. Thesis)* Nanjing Forestry University, Nanjing, Jiangsu

[15] Pu Y F, Zhou J L, Zhang Y, Zhang N, Huang G and Siarry P 2013 Fractional extreme value adaptive training method: fractional steepest descent approach *IEEE Trans. Neural Netw. Learn Syst.* ***26*****(4)** 653-62

[16] Chen Y, Gao Q, Wei Y and Wang Y 2017 Study on fractional order gradient methods. *Appl. Math. Comput.* ***314*** 310-321

[17] Chen Y, Wei Y, Wang Y and Chen Y 2018 Fractional order gradient methods for a general class of convex functions *Annual American Control Conf.* (Milwaukee: IEEE) pp 3763-67

[18] Khan S, Naseem I, Malik M A, Togneri R and Bennamoun M 2018 A fractional gradient descent-based rbf neural network *Circuits, Syst. Signal Process.* ***37*****(12)** 5311-32

[19] Bao C, Pu Y and Zhang Y 2018 Fractional-order deep backpropagation neural network *Comput. Intell. Neurosci.*, **2018** 1-10

[20] Sheng D, Wei Y, Chen Y and Wang Y 2020 Convolutional neural networks with fractional order gradient method *Neurocomputing* ***408*** 42-50

[21] Heymans N and Bauwens J C 1994 Fractal rheological models and fractional differential equations for viscoelastic behavior *Rheologica acta* ***33*****(3)** 210-29

[22] Song L, Xu S and Yang J 2010 Dynamical models of happiness with fractional order *Commun. Nonlinear Sci. Numer. Simul.* ***15*****(3)** 616-628

[23] Metzler R and Klafter J 2000 The random walk's guide to anomalous diffusion: a fractional dynamics approach *Physics reports* 339 pp 1-77

[24] Goyal S P and Mathur T 2006 On generalized fractional diffusion equation *South East Asian J. Math. Math.Sc* **4(2)** 53-61

[25] Mathur T 2004 On generalized fractional diffusion equation-II *J. Raj. Acad. Phy. Sci.* **3** 183-190

[26] Henry B I and Wearne S L 2002 Existence of turing instabilities in a two-species fractional reaction-diffusion system. *SIAM J. Appl. Math.* ***62*****(3)** 870-87

[27] Cottone G, Di Paola M and Santoro R 2010 A novel exact representation of stationary colored Gaussian processes (fractional differential approach) *J. Phys. A: Math. Theor.* ***43*****(8)** 085002

[28] Sugimoto N 1991 Burgers equation with a fractional derivative: hereditary effects on nonlinear acoustic waves *J. Fluid Mech.* ***225*** 631-53

[29] Engheia N 1997 On the role of fractional calculus in electromagnetic theory *IEEE Antennas and Propagation Magazine* **39(4)** 35-46

[30] Mainardi, F 1996 Fractional relaxation-oscillation and fractional diffusion-wave phenomena *Chaos, Solitons Fractals* ***7*****(9)** 1461-77

[31] Ichise M, Nagayanagi Y and Kojima T 1971 An analog simulation of non-integer order transfer functions for analysis of electrode processes *J. Electroanal. Chem. Interf. Electrochem.* ***33*****(2)** 253-65

[32] Podlubny I 1998 *Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications* (San Diego:Elsevier)
[33] Caputo M 1967 Linear models of dissipation whose Q is almost frequency independent—II *Geophys. J.Int.* **13(5)** 529-539
[34] Grunwald A K 1867 Uber" begrente" Derivationen und deren Anwedung. *Zangew Math und Phys* **12** 441-480
[35] Letnikov A V 1868 Theory of differentiation with an arbtraly indicator *Matem Sbornik* **3** 1-68
[36] Chen W, Sun H, Zhang X and Korošak D 2010 Anomalous diffusion modeling by fractal and fractional derivatives. *Comput. Math. with Appl.* **59(5)** 1754-58
[37] Torvik P J and Bagley R L 1984 On the appearance of the fractional derivative in the behavior of real materials. *J. Appl. Mech*. **51(2)** 294-98
[38] Yang Q, Chen D, Zhao T and Chen Y 2016 Fractional calculus in image processing: a review *Fract. Calc. Appl. Anal.* **19(5)** 1222-49
[39] Magin R L 2010 Fractional calculus models of complex dynamics in biological tissues *Comput. Math. with Appl.* **59(5)** 1586-93
[40] Jumarie G 2013 On the derivative chain-rules in fractional calculus via fractional difference and their application to systems modelling *Open Phys. J.* **11(6)** 617-633