Case Study 1 :- 2D Unsteady Conduction Equation

General eqn :-

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+q_v = \rho \cdot C_p \cdot \frac{\partial T}{\partial t}$$

To get :- 2D, Unsteady, without heat generation.

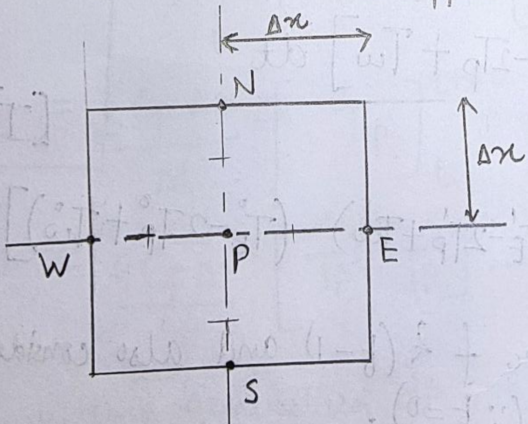
$q_v = 0$

$$\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)=0$$

ρ

$$\therefore \underbrace{\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)}_{①}+\underbrace{\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)}_{②} = \underbrace{\rho C_p \frac{\partial T}{\partial t}}_{③}$$

Figure :- Assuming square plate, $\rightarrow$ Copper



Solving eqn ①, $\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)$

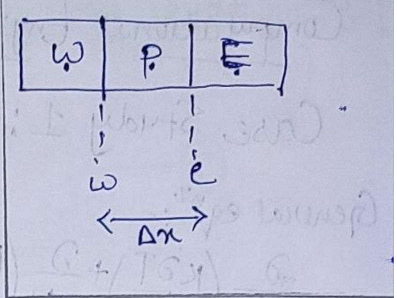
Double integrate it wrt time (dt) and space (dx). $\underbrace{\text{space}}_{2D}$

$$\iint \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) dx\, dt$$

Using limits as;

Time $t = t$, $x = w$

$t = t + \Delta t$, $x = e$



$$\Rightarrow \int_t^{t+\Delta t} \int_w^e \frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right) dx\, dt$$

$$= \int_t^{t+\Delta t} \left[k \frac{\partial T}{\partial x} \right]_w^e dt$$

$$= \int_t^{t+\Delta t} \left\{ (k_e - k_w) \cdot \left[\left(\frac{\partial T}{\partial x}\right)_e - \left(\frac{\partial T}{\partial x}\right)_w \right] \right. dt$$

$$= \int_t^{t+\Delta t} k \cdot \left[\frac{T_E - T_P}{x_e} - \left(\frac{T_P - T_W}{x_w} \right) \right] dt$$

$\Delta x = x_e - x_w$

$[T]_t^{t+\Delta t} = T' - T^{\circ}$

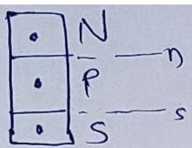
$$= \frac{k}{\Delta x} \int_t^{t+\Delta t} \left[T_E - 2T_P + T_W \right] dt$$

$$= \frac{k}{\Delta x} \left[(T_E' - 2T_P' + T_W') - (T_E^{\circ} - 2T_P^{\circ} + T_W^{\circ}) \right]$$

Using weighing factor f & $(f-1)$ and also considering explicit method. ($\because f = 0$).

$$= \frac{k}{\Delta x} \left[f (T_E' - 2T_P' + T_W') - (f-1)(T_E^{\circ} - 2T_P^{\circ} + T_W^{\circ}) \right]$$

$$= \frac{k}{\Delta x} \left[T_E^{\circ} - 2T_P^{\circ} + T_W^{\circ} \right] \quad (4)$$

for 

$$\left\{\iint \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) dy\, dt\right\}$$ $dy = dx$ here.

we get,

$$\frac{k}{\Delta x}\left[T_N^0 - 2T_P^0 + T_S^0\right] \quad (5)$$

Solving eqn (3), $\displaystyle\int_t^{t+\Delta t}\int_e^w \rho\, C_p \cdot \frac{\partial T}{\partial t}\, dx\, dt$

$$= \rho C_p \cdot \int_t^{t+\Delta t} \frac{\partial T}{\partial t}\, dt \cdot \int_e^w dx$$

$$= \rho C_p \cdot \int_t^{t+\Delta t} \partial T\, dt \cdot (x_w - x_e)$$

$$\int_t^{t+\Delta t} \partial t\, dt \longrightarrow [t]_t^{t+\Delta t}$$

$$= \frac{\rho C_p \Delta x}{\Delta t}\left[T_P\right]_t^{t+\Delta t}$$

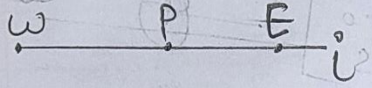
$$= \frac{\rho C_p \Delta x}{\Delta t}\left[T_P^1 - T_P^0\right] \quad (6)$$

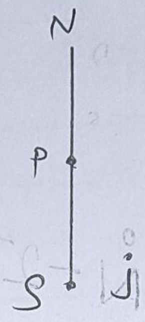
rewriting generalized discritised equation as,

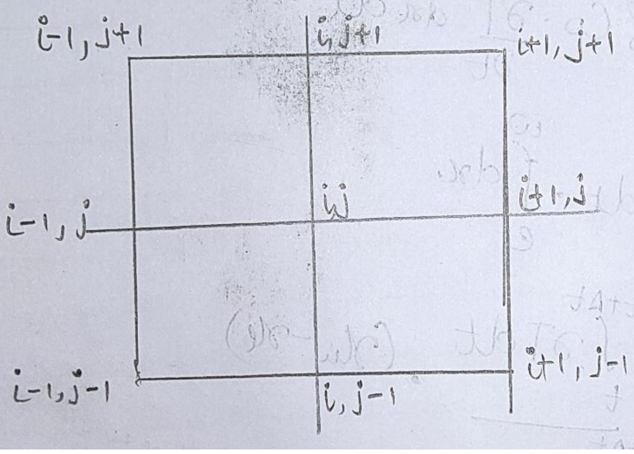
$$\frac{k}{\Delta x}\left[T_E^0 + T_W^0 - 2T_P^0\right] + \frac{k}{\Delta x}\left[T_N^0 + T_S^0 - 2T_P^0\right] = \frac{\rho \cdot C_p \cdot \Delta x}{\Delta t}\left[T_P^1 - T_P^0\right]$$

$$T_P^1 - T_P^0 = \frac{\alpha \Delta t}{(\Delta x)^2}\left[T_E^0 + T_W^0 + T_N^0 + T_S^0 - 4T_P^0\right]$$

$$\Rightarrow \boxed{T_P^1 = T_P^0 + \frac{\alpha \cdot \Delta t}{(\Delta x)^2}\left[T_E^0 + T_W^0 + T_N^0 + T_S^0 - 4T_P^0\right]}$$


W
P
E
i


N
P
S
j


i-1, j+1
i, j+1
i+1, j+1
i-1, j
i, j
i+1, j
i-1, j-1
i, j-1
i+1, j-1

MATLAB Code: -

```
%CASE STUDY on 2-D Unsteady Conduction in Square Copper Plate.
clear all
clc

%% Defining the Variables
L = input('L = ') ; %Length of Plate
n = input('n = '); %Nodes
dx = input('dx = ');%L/(n-1); %Grid spacing in x    %Assuming that our
plate is square.
x = linspace(0,L,n); %Grid points in x
y = linspace(0,L,n); %Grid points in y
alpha = 113/1000000; %Thermal Diffusivity
dt = input('dt = '); %Time Step

%% Maximum Time to be Iterated
T_max = input('T_max = ');

%% Initial Temperature Matrix Values
T = zeros(n,n);

%% Boundary Conditions
T(:,1) = input(' Left Side Node Temp Value = ');
T(:,n) = input(' Right Side Node Temp Value = ');
T(1,:) = input(' Upper Side Node Temp Value = ');
T(n,:) = input(' Bottom Side Node Temp Value = ');

%% Y;X;Time Looping
for t = 0:T_max
  for i = 2:n-1
      for j = 2:n-1
    T_w = T(i-1,j);
    T_p = T(i,j);
    T_e = T(i+1,j);
    T_n = T(i,j-1);
    T_s = T(i,j+1);
    T(i,j) = T_p+(alpha*dt/(dx^2))*(T_e-4*T_p+T_w+T_s+T_n);
      end
  end
end

%% Plot
%plot(x,T)
%surf(x,y,T)
%contour(x,y,T)
contourf(T,1000,'edgecolor','none')
colormap jet
colorbar
xlabel('Distance in X')
ylabel('Distance in X')
zlabel('Temperature')
title('Temperature Profile of a Square Plate')
```

Output [Command Window]: -

```
L = 1
n = 21
dx = 0.05
dt = 1
T_max = 500
 Left Side Node Temp Value = 800
 Right Side Node Temp Value = 100
 Upper Side Node Temp Value = 100
 Bottom Side Node Temp Value = 100
>>
```

Results: -

I compared my Code with *Korosh Agha Mohammad Ghasemi Chemical Engineering at Shiraz University.* Though it was the comparison among Numerical Methods, I was confident enough to know my code is providing correct results. Also, my code is short compared to that of online MATLAB Code.

I used explicit method to discretise my "*Two-Dimensional Heat Conduction equation without Heat Generation*" so the results are showing lower accuracy. Instead of explicit method one can also use Crank Nicolson Method or Implicit Method, etc.

And the results are provided below:

Graph: -

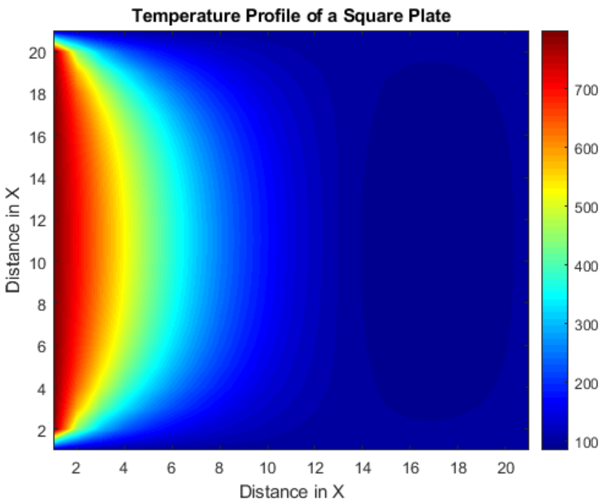


Figure 1 Student Plot

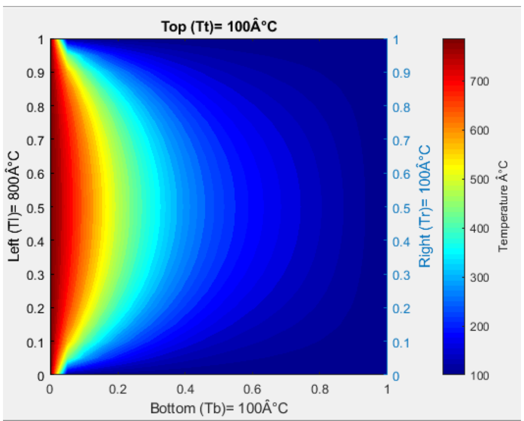


Figure 2 ONLINE MATLAB Coded Plot