Homework 2

Neel Sanghvi

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Problem 1

Part a: Show that $(1+x)^n = 1 + nx + o(x)$ **as** $x \to 0$.

Solution

1. First, given f and g are real valued functions then $f(x) = o(g(x))$ when $\lim_{x\to a} \frac{f(x)}{g(x)} = 0$.

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k = 1 + nx + \binom{n}{2} x^2 + \binom{n}{3} x^3 + ... = 1 + nx + f(x)$$

2. Let $g(x) = x$ so

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \binom{n}{2} x + \binom{n}{3} x^2 + ... = 0$$

3. This means that $f(x) = o(g(x)) = o(x)$. So,

$$(1+x)^n = 1 + nx + o(x) \tag{1}$$

Part b: Show that $x \sin\sqrt{x} = O(x^{3/2})$ **as** $x \to 0$.

Solution

1. First, $f(x) = O(g(x))$ when $\big|f(x)\big| \leq C\big|g(x)\big|$.

$$f(x) = x\sin\sqrt{x} \tag{2}$$

$$\lim_{x\to 0} \frac{f(x)}{x^{3/2}} = \lim_{x\to 0} \frac{x\sin\sqrt{x}}{x^{3/2}} = \lim_{x\to 0} \frac{\sin\sqrt{x}}{\sqrt{x}} = 1$$

Note that in the case of $\lim_{x\to 0} \frac{\sin\sqrt{x}}{\sqrt{x}} = 1$, we can use L'Hopital's rule here to show that $f(x) \simeq g(x)$ around 0. This shows that $x\sin\sqrt{x} = O(x^{3/2})$ as $x \to 0$

Part c: Show that $e^{-t} = o\left(\frac{1}{t^2}\right)$ **as** $t \to \infty$

Solution

1. First, given f and g are real valued functions then $f(x) = o(g(x))$ when $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$.

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{t\to\infty} \frac{e^-t}{\frac{1}{t^2}} = \lim_{t\to\infty} \frac{t^2}{e^t} = \frac{\infty}{\infty} \overset{\text{H}}{=} \lim_{t\to\infty} \frac{2t}{-e^{-t}} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{2}{e^{-t}} = 0$$

Part d: Show that $\int_0^\epsilon e^{-x^2}\,dx = O\left(\epsilon\right)$ as $\epsilon \to 0$

Solution

1. First, given f and g are real valued functions then $f(x) = o(g(x))$ when $\lim_{x\to a} \frac{f(x)}{g(x)} = 0$.

$$\lim_{\epsilon\to 0} \frac{f(\epsilon)}{g(\epsilon)} = \lim_{x\to 0} \frac{\int_0^\epsilon e^{-x^2} dx}{\epsilon} = \frac{0}{0} \overset{\text{H}}{=} \lim_{\epsilon\to 0} \frac{e^{-\epsilon^2}}{1} = 1 \tag{3}$$

   This shows that $\int_0^\epsilon e^{-x^2}\,dx = O\left(\epsilon\right)$

Problem 2

Part a: Find an exact formula for the change in the solution between the exact problem and the perturbed problem Δx

Solution

1. It is given that $\boldsymbol{Ax} = \boldsymbol{b}$ and $\boldsymbol{x} = \boldsymbol{A^{-1}b}$.
2. Now when b is perturbed then some $\boldsymbol{\tilde{x} = A^{-1}(b + \Delta b)}$
3. The perturbed problem $\Delta x = |\tilde{x} - x|$

$$\Delta x = |\tilde{x} - x| = A^{-1}b + A^{-1}\Delta b - A^{-1}b = A^{-1}\Delta b = \begin{bmatrix} \Delta b_1 + 10^{10}(\Delta b_2 - \Delta b_1) \\ \Delta b_2 + 10^{10}(\Delta b_1 - \Delta b_2) \end{bmatrix}$$

Part b: What is the condition number of A

Solution

The condition number $\kappa = \|\mathbf{A}\| \left\|\mathbf{A^{-1}}\right\| = 2 \times 10^{10}$.

Part c

Solution

1. The relative error in the solution is defined by $\frac{\|\Delta x\|}{\|x\|}$
2. Let $\Delta b_1 = 1 \times 10^-5$ and $\Delta b_2 = 2 \times 10^-5$

$$\Delta x = \begin{bmatrix} \Delta b_1 + 10^{10}(\Delta b_2 - \Delta b_1) \\ \Delta b_2 + 10^{10}(\Delta b_1 - \Delta b_2) \end{bmatrix} = \begin{bmatrix} 1 \times 10^-5 + 1 \times 10^5 \\ 2 \times 10^-5 - 1 \times 10^5 \end{bmatrix} \approx \begin{bmatrix} 1 \times 10^5 \\ -1 \times 10^5 \end{bmatrix}$$

$$\frac{\|\Delta x\|}{\|x\|} = \frac{\sqrt{2} * 10^5}{\sqrt{2}} = 10^5 \tag{4}$$

3. The relation between the condition number, relative error and perturbation is that $\Delta x \simeq \Delta b \times \kappa$.
4. It can be seen that even though the perturbation in **b** is very small, the change in the solution is very large. This is why it is the problem is ill-conditioned. Even if the pertubations are the same, Δx is still going to be on the same order of magnitude.

Problem 3

Part a: What is the relative condition number $\kappa_f(x)$ for $f(x) = e^x - 1$

Solution

$$\kappa_f(x) = \left|\frac{xf'(x)}{f(x)}\right| = \left|\frac{xe^x}{e^x - 1}\right|$$

$\kappa_f(x)$ is ill-conditioned when $x \to 0$ as the denominator approaches 0. Moreover, the condition number is very large when **x** is a large number.

Part b

Solution

The algorithm is not stable because x is really close to 0 which would blow up the condition number.

Part c

Solution

Using $x = 9.999999995000000 \times 10^{-10}$ in the algorithm that directly tries to compute $y = e^x - 1$ outputs $y = 2.202546568247173 \times 10^4$ using MATLAB. This is clearly false. It is because so close to 0, the condition number is so large that the smallest perturbation in the input results in a large change in x.

Part d

Solution

The desired accuracy was found using a 2nd order taylor expansion of $e^x - 1 = 1 \times 10^{-9}$ with an accuracy up-to 16 digits. The error using this method was found using the Remainder theorem, the error is $1.666666665833334 \times 10^{-28}$

Part e

Solution

The difference between the answer using the Taylor polynomial and the function `expm1` was so small it could not be resolved by `MATLAB`

Problem 4

Part a

Solution

The complete code is uploaded to the GitHub repository (https://github.com/NeelSanghvi/APPM-4600/). The final ans. is $Sum = -20.050835454935093$

Part b

Solution

The complete code is uploaded to the GitHub repository (https://github.com/NeelSanghvi/APPM-4600) it is in the same file as Problem 4a. The plots are as seen below

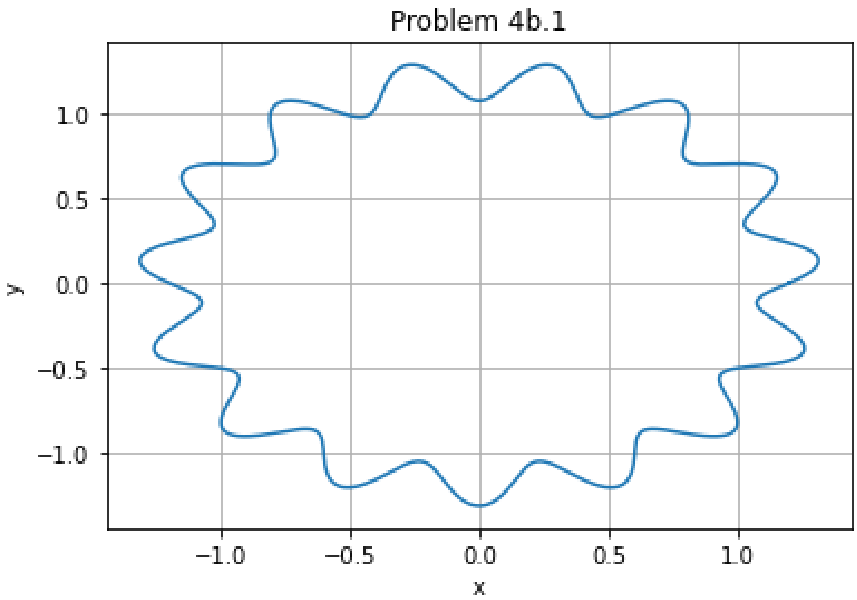

Figure 1: The 1st plot for problem 4b

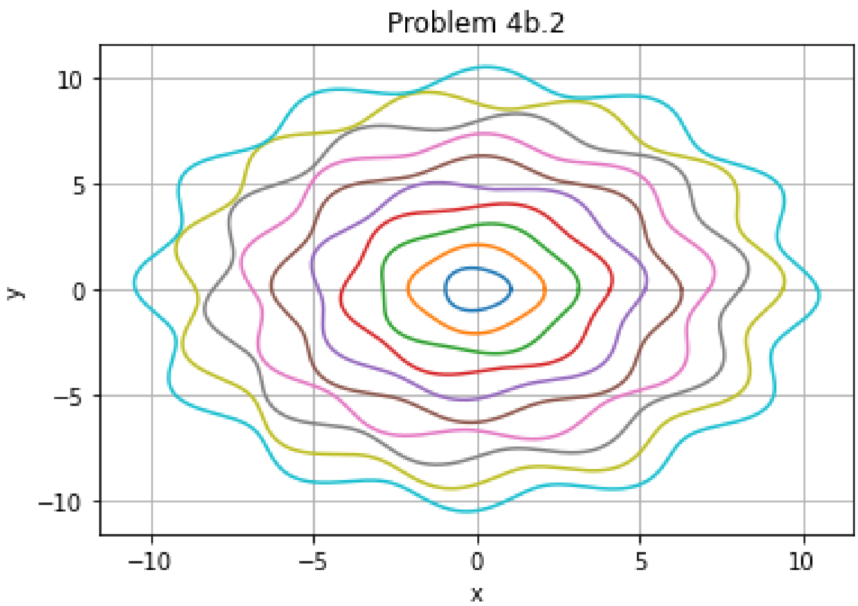

Figure 2: The 2nd plot for problem 4b