APPM 4600 — HOMEWORK # 2

1. (a) Show that $(1+x)^n = 1 + nx + o(x)$ as $x \to 0$.
   (b) Show that $x \sin\sqrt{x} = O(x^{3/2})$ as $x \to 0$.
   (c) Show that $e^{-t} = o(\frac{1}{t^2})$ as $t \to \infty$.
   (d) Show that $\int_0^{\varepsilon} e^{-x^2} dx = O(\varepsilon)$ as $\varepsilon \to 0$.

2. Consider solving $\mathbf{Ax} = \mathbf{b}$ where $\mathbf{A} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1+10^{-10} & 1-10^{-10} \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The exact solution is $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and the inverse of $\mathbf{A}$ is $\begin{bmatrix} 1-10^{10} & 10^{10} \\ 1+10^{10} & -10^{10} \end{bmatrix}$. In this problem we will investigate a perturbation in $\mathbf{b}$ of $\begin{bmatrix} \Delta b_1 \\ \Delta b_2 \end{bmatrix}$ and the numerical effects of the condition number.

   (a) Find an exact formula for the change in the solution between the exact problem and the perturbed problem Δx.
   (b) What is the condition number of $\mathbf{A}$?
   (c) Let Δb_1 and Δb_2 be of magnitude 10^{-5}; not necessarily the same value. What is the relative error in the solution? What is the relationship between the relative error, the condition number, and the perturbation. Is the behavior different if the perturbations are the same? Which is more realistic: same value of perturbation or different value of perturbation?

3. Recall the concept of a *relative condition number* $\kappa_f(x)$ for a function $f(x)$. For $\widetilde{x} = x + \delta x$, and $\delta x \to 0$, it gives us an upper bound on the relative error on the output $\widetilde{y} = f(\widetilde{x})$. That is:

$$\frac{|f(x) - f(\widetilde{x})|}{|f(x)|} \leq \kappa_f(x)\frac{|x - \widetilde{x}|}{|x|}$$

   For a differentiable function $f(x)$, there is a formula for the relative condition number:

$$\kappa_f(x) = \left|\frac{xf'(x)}{f(x)}\right|$$

   Let $f(x) = e^x - 1$.

   (a) What is the relative condition number $\kappa_f(x)$? Are there any values of x for which this is ill-conditioned (for which $\kappa_f(x)$ is very large)?
   (b) Consider computing $f(x)$ via the following algorithm:

```
y = math.e^x
return y -1
```

   Is this algorithm stable? Justify your answer

(c) Let x have the value $9.999999995000000 \times 10^{-10}$, in which case the true value for $f(x)$ is equal to 10^{-9} up to 16 decimal places. How many correct digits does the algorithm listed above give you? Is this expected?

(d) Find a polynomial approximation of $f(x)$ that is accurate to 16 digits for $x = 9.999999995000000 \times 10^{-10}$. Hint: use Taylor series, and remember that 16 digits of accuracy is a relative error, not an absolute one.

(e) Verify that your answer from part (d) is correct.

(f) [Optional] How many digits of precision do you have if you do a simpler Taylor series?

(g) [Fact; no work required] Matlab provides `expm1` and Python provides `numpy.expm1` which are special-purpose algorithms to compute $e^x - 1$ for $x \approx 0$. You could compare your Taylor series approximation with `expm1`.

4. *Practicing Python*

(a) Create a vector $\mathbf{t}$ with entries starting at 0 incrementing by $\frac{\pi}{30}$ to π. Then create the vector $\mathbf{y} = \cos(\mathbf{t})$.
Write a code that evaluates the following sum:

$$S = \sum_{k=1}^{N} \mathbf{t}(k)\mathbf{y}(k)$$

Print the statement "the sum is: S" with the numerical value of S.

(b) *Wavy circles.* In one figure, plot the parametric curve

$$x(\theta) = R(1 + \delta r \sin(f\theta + p))\cos(\theta)$$
$$y(\theta) = R(1 + \delta r \sin(f\theta + p))\sin(\theta)$$

for $0 \le \theta \le 2\pi$ and for $R = 1.2$, $\delta r = 0.1$, $f = 15$ and $p = 0$. Make sure to adjust the scale so that the axis have the same scale.
In a second figure, use a *for* loop to plot 10 curves and let with $R = i$, $\delta r = 0.05$, $f = 2 + i$ for the i^{th} curve. Let the value of p be a uniformly distributed random number (look up `random.uniform`) between 0 and 2.