A REPROOF OF THE PILA-WILKIE THEOREM

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Abstract. we prove the Pila-Wilkie theorem, following the original paper [2], but exploiting cell decomposition more thoroughly to simplify the deduction from its main ingredients.

1. Introduction

First some notation needed to state the theorem. Throughout, $d, k, l, m, n \in \mathbb{N}$, and $\varepsilon, c, K \in \mathbb{R}^{>}$. An o-minimal field is in this note an expansion of a field which is o-minimal as a structure. It is well known that the underlying field of an 0-minimal field must be real closed. We define the *multiplicative height function* $\mathrm{H} : \mathbb{Q} \to \mathbb{R}$ by $\mathrm{H}(\frac{a}{b}) := \max(|a|, |b|) \in \mathbb{N}^{\geqslant 1}$ for coprime $a, b \in \mathbb{Z}$, $b \neq 0$. Thus $\mathrm{H}(0) = \mathrm{H}(1) = \mathrm{H}(-1) = 1$, and for $q \in \mathbb{Q}$ we have $\mathrm{H}(q) \geqslant 2$ if $q \notin \{0, 1, -1\}$, $\mathrm{H}(q) = \mathrm{H}(-q)$, and $\mathrm{H}(q^{-1}) = \mathrm{H}(q)$ for $q \neq 0$. For $n \geqslant 1$ and $a = (a_1, ..., a_n) \in \mathbb{Q}^n$,

$$\mathrm{H}(a) := \max\{\mathrm{H}(a_i) : 1 \leqslant i \leqslant n\} \in \mathbb{N}^{\geqslant 1}.$$

Let $X \subseteq \mathbb{R}^n$, $n \geqslant 1$. We set $X(\mathbb{Q}) = X \cap \mathbb{Q}^n$. Throughout T ranges over real numbers $\geqslant 1$, and we set $X(\mathbb{Q}, T) := \{a \in X(\mathbb{Q}) : \mathrm{H}(a) \leqslant T\}$ be the (finite) set of rational points of X of height $\leqslant T$, and set $N(X, T) := \#X(\mathbb{Q}, T) \in \mathbb{N}$.

The *algebraic part* of X, denoted by X^{alg}, is the union of the connected infinite semialgebraic subsets of X. We also set

$$X^{\mathrm{tr}} := X \setminus X^{\mathrm{alg}} \qquad (\textit{the transcendental part of } X).$$

We can now state the Pila-Wilkie theorem, also called the *Counting Theorem*:

Theorem 1.1. *Let $\widetilde{\mathbb{R}}$ be an o-minimal expansion of the real field and let $X \subseteq \mathbb{R}^n$ be definable in $\widetilde{\mathbb{R}}$, $n \geqslant 1$. Then for all ε there is a c such that for all T,*

$$N(X^{\mathrm{tr}}, T) \leqslant cT^{\varepsilon}.$$

Roughly speaking, it says there are few rational points on the transcendental part of a set definable in an *o*-minimal expansion of the real field: the number of such points grows slower than any power T^{ε} with T bounding their height. To apply the counting theorem one needs to describe Xalg in some useful way. This typically involves Ax-Schanuel type transcendence results.

The proof of Theorem 1.1 depends on two intermediate results. The first of these has nothing to do with o-minimality. To state it we again need to introduce some notation. For $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}^m$ we set $|\alpha| := \alpha_1 + \cdots + \alpha_m$, and given a field F (often $F = \mathbb{R}$) we set $x^{\alpha} := x_1^{\alpha_1} \cdots x_m^{\alpha_n}$ for the usual m-tuple $x = (x_1, \ldots, x_m)$ of coordinate functions on F^m, and likewise $a^{\alpha} := a_1^{\alpha_1} \cdots a_m^{\alpha_m}$ for

$a = (a_1, \dots, a_m) \in F^m$. Let $U \subseteq \mathbb{R}^m$ be open. For a function $f : U \to \mathbb{R}$ of class C^k and $\alpha \in \mathbb{N}^m$, $|\alpha| \leqslant k$, we let

$$f^{(\alpha)} \ := \ \frac{\partial^{|\alpha|}}{\partial x^\alpha} f$$

denote the corresponding partial derivative of order α. We extend the above to C^k-maps $f = (f_1, \dots, f_n) : U \to \mathbb{R}^n$, where

$$f^{(\alpha)} \ := \ (f_1^{(\alpha)}, \dots, f_n^{(\alpha)}) : U \to \mathbb{R}^n$$

for α as before. This includes the case $m = 0$, where $\mathbb{R}^0$ has just one point and any map $f : U \to \mathbb{R}^n$ is of class C^k for all k, with $f^{(\alpha)} = f$ for the unique $\alpha \in \mathbb{N}^0$. For $a = (a_1, \dots, a_n) \in \mathbb{R}^n$ we set $|a| := \max\{|a_1|, \dots, |a_n|\} \in \mathbb{R}$, which by convention is 0 if $n = 0$; this conflicts with our notation $|\alpha|$ for $\alpha \in \mathbb{N}^n$, but in practice no confusion will arise.

Now define for $k, n \geqslant 1$ and $X \subseteq \mathbb{R}^n$ a *strong k-parametrization of X* to be a C^k-map $f : (0,1)^m \to \mathbb{R}^n$, $m < n$, with image X, such that $|f^{(\alpha)}(a)| \leqslant 1$ for all $\alpha \in \mathbb{N}^m$ with $|\alpha| \leqslant k$ and all $a \in (0,1)^m$. We also define a *hypersurface in $\mathbb{R}^n$ of degree $\leqslant d$* to be the zeroset in $\mathbb{R}^n$ of a nonzero polynomial in $x = (x_1, \dots, x_n)$ over $\mathbb{R}$ of (total) degree $\leqslant d$.

Theorem 1.2. *Let $n \geqslant 1$ be given. Then for any $d \geqslant 1$ there are $k = k(n,\ d) \geqslant 1$, $\varepsilon = \varepsilon(n,\ d)$, and $c = c(n,\ d)$, such that if $X \subseteq \mathbb{R}^n$ has a strong k-parametrization, then for all T at most cT^ε hypersurfaces in $\mathbb{R}^n$ of degree $\leqslant d$ are enough to cover $X(\mathbb{Q}, T)$, with $\varepsilon(n,\ d) \to 0$ as $d \to \infty$.*

Let now R be any o-minimal field, and let $X \subseteq R^n$ be definable. We continue to use the notational conventions set above for $\mathbb{R}$, with U and f now definable in R instead. We can then introduce the notion of a *definable strong k-parametrization of X* as before, with R and the interval $(0,1)_R$ in R instead of $\mathbb{R}$ and the real interval $(0,1)$ and where f is definable. The second intermediate result in the proof of the Pila-Wilkie theorem is about decomposing a definable set in an o-minimal field into finitely many definable subsets that are more manageable:

Theorem 1.3. *Given an o-minimal field R, any definable set $X \subseteq [-1,1]_R^n$ with empty interior is for any $k \geqslant 1$ a finite union of definable subsets, each having a definable strong k-parametrization.*

We use Theorem 1.3 not just when R is an *o*-minimal expansion $\widetilde{\mathbb{R}}$ of the real field, even though Theorem 1.1 is only about definable sets in such expansions. This is because by model theory we obtain from Theorem 1.3 a uniform version of the corresponding result for any o-minimal expansion $\widetilde{\mathbb{R}}$ of the real field. Here 'uniform' means that if instead of a single definable $X \subseteq \mathbb{R}^n$ we have a definable family $(X_b)_{b \in B}$ of such sets, then the decomposition of X_b into definable subsets and their k-parametrizations can also be taken to depend definably on $b \in B$.

2. Proof of the Counting Theorem from the two ingredients

First some elementary facts about $X^{\rm alg}$ and $X^{\rm tr}$ for $X \subseteq \mathbb{R}^n$. The first is obvious.

Lemma 2.1. *If $X = X_1 \cup \cdots \cup X_m$, then $X^{\rm alg} \supseteq X_1^{\rm alg} \cup \cdots \cup X_m^{\rm alg}$, and thus*

$$X^{\rm tr} \ \subseteq \ X_1^{\rm tr} \cup \cdots \cup X_m^{\rm tr}.$$

Note also that if $n \geqslant 1$ and X is open in $\mathbb{R}^n$, then $X^{\mathrm{tr}} = \emptyset$.

Lemma 2.2. *Suppose $S \subseteq \mathbb{R}^n$ is semialgebraic, $X \subseteq S$, and $f : S \to \mathbb{R}^m$ is semialgebraic and injective, and maps X homeomorphically onto $Y = f(X) \subseteq \mathbb{R}^m$. Then $f(X^{\mathrm{alg}}) = Y^{\mathrm{alg}}$ and thus $f(X^{\mathrm{tr}}) = Y^{\mathrm{tr}}$.*

Proof. It is clear that $f(X^{\mathrm{alg}}) \subseteq Y^{\mathrm{alg}}$. Also, for any connected infinite semialgebraic set $C \subseteq Y$, the set $f^{-1}(C) \subseteq S$ is semialgebraic (since C and f are), contained in X (since f is injective), hence connected and infinite, and thus $f^{-1}(C) \subseteq X^{\mathrm{alg}}$. This shows $f^{-1}(Y^{\mathrm{alg}}) \subseteq X^{\mathrm{alg}}$, and thus $f(X^{\mathrm{alg}}) = Y^{\mathrm{alg}}$. $\square$

In order to apply Theorem 1.3 we need to reduce to the case $X \subseteq [-1,1]^n$. This is done as follows. For $I \subseteq \{1,\dots,n\}$, set

$$X_I \;:=\; \{a \in X : \; |a_i| > 1 \text{ for all } i \in I, \; |a_i| \leqslant 1 \text{ for all } i \notin I\}$$

and define the semialgebraic map $f_I : \mathbb{R}^n_I \to \mathbb{R}^n$ by $f_I(a) = b$ where $b_i := a_i^{-1}$ for $i \in I$ and $b_i := a_i$ for $i \notin I$; so for $I = \emptyset$ we have $\mathbb{R}^n_I = [-1,1]^n$ and f_I is the inclusion map into $\mathbb{R}^n$. Note that for $a \in \mathbb{Q}^n$ we have $f_I(a) \in \mathbb{Q}^n$ and $\mathrm{H}(a) = \mathrm{H}\big(f_I(a)\big)$. Moreover, X is the disjoint union of the sets X_I, and for $Y_I = f_I(X_I)$ we have $Y_I \subseteq [-1,1]^n$, $Y_I^{\mathrm{tr}} = f_I(X_I^{\mathrm{tr}})$ by Lemma 2.2, so $N(Y_I^{\mathrm{tr}}, T) = N(X_I^{\mathrm{tr}}, T)$ for all T.

The sketch below actually proves the Counting Theorem, modulo a uniformity assumption that arises at the end of the sketch. This motivates a stronger "definable family" version of the theorem, which we then prove as in the sketch.

Sketch of the proof of Theorem 1.1 from Theorems 1.2 and 1.3. Let $X \subseteq \mathbb{R}^n$ be definable in the o-minimal expansion $\widetilde{\mathbb{R}}$ of the real field, $n \geqslant 1$. We need to show that there are 'few' rational points on X outside X^{alg}. We proceed by induction on n. By Lemma 2.1 and the remark following it we can remove the interior of X in $\mathbb{R}^n$ from X and arrange that X has empty interior. As indicated just before this sketch we also arrange $X \subseteq [-1,1]^n$.

Let ε be given, and take $d \geqslant 1$ so large that $\varepsilon(n,d) \leqslant \varepsilon/2$ in Theorem 1.2, and take $k = k(n,d)$. By Theorem 1.3 for $\widetilde{\mathbb{R}}$, $X = X_1 \cup \dots \cup X_M$, $M \in \mathbb{N}$, where each $X_i \subseteq \mathbb{R}^n$ is definable in $\widetilde{\mathbb{R}}$ and admits a strong k-parametrization.

By Theorem 1.2, $X(\mathbb{Q},T) \subseteq \bigcup_{i=1}^M \bigcup_{j=1}^J H_{ij}$, where the H_{ij} are hypersurfaces in $\mathbb{R}^n$ of degree $\leqslant d$, and $J \in \mathbb{N}$, $J \leqslant cT^{\varepsilon/2}$, $c = c(n,d)$ as in that theorem.

Let H be any hypersurface in $\mathbb{R}^n$ of degree $\leqslant d$. We aim for an upper bound on $N\big((X \cap H)^{\mathrm{tr}}, T\big)$ of the form $c_1 T^{\varepsilon/2}$ with $c_1 \in \mathbb{R}^{>}$ independent of H and T. (If we achieve this, then applying this to the hypersurfaces H_{ij} we obtain

$$N(X^{\mathrm{tr}}, T) \;\leqslant\; MJc_1T^{\varepsilon/2} \;\leqslant\; M \cdot cT^{\varepsilon/2} \cdot c_1 T^{\varepsilon/2} \;=\; Mcc_1 \cdot T^{\varepsilon},$$

and we are done.) Take semialgebraic cells $C_1, \dots, C_m$ in $\mathbb{R}^n$ such that

$$H \;=\; C_1 \cup \dots \cup C_m$$

and let $C = C_\mu$ be one of those cells. Then by [1, (III, 2.7)] we have a semialgebraic homeomorphism $p = p_C : C \to p(C) = p(C_\mu)$ onto an open cell $p(C_\mu)$ in $\mathbb{R}^{n_\mu}$ with $n_\mu < n$, and so p maps $X \cap C_\mu$ homeomorphically onto its image $Y_\mu \subseteq p(C_\mu) \subseteq \mathbb{R}^{n_\mu}$. For $a \in C_\mu \cap \mathbb{Q}^n$ we have $\mathrm{H}\big(p(a)\big) \leqslant \mathrm{H}(a)$, since p is given by omitting some coordinates. The hypersurfaces of degree $\leqslant d$ in $\mathbb{R}^n$ belong to a single semialgebraic family, hence by [1, (III, 3.6)] we can (and do) take here $m \leqslant m(d,n)$, with $m(d,n) \in \mathbb{N}$ depending only on d,n. Now

$$(X \cap H)^{\mathrm{tr}} \subseteq (X \cap C_1)^{\mathrm{tr}} \cup \cdots \cup (X \cap C_m)^{\mathrm{tr}}.$$

Since the $n_\mu < n$ we can (and do) assume inductively that for all T,

$$N(Y_\mu^{\mathrm{tr}}, T) \leqslant B_\mu T^{\varepsilon/2}, \quad \mu = 1, \dots, m$$

with $B_\mu \in \mathbb{R}^>$ independent of T. Hence for all T,

$$N\big((X \cap C_\mu)^{\mathrm{tr}}\big), T) \leqslant B_\mu T^{\varepsilon/2}, \quad i = 1, \dots, m$$

by Lemma 2.2 applied to the maps $p = p_{C_\mu}$, and thus

$$N\big((X \cap H)^{\mathrm{tr}}, T\big) \leqslant (B_1 + \cdots + B_m) T^{\varepsilon/2}.$$

Assume $B_1, \dots, B_m$ can be taken to depend only on X, ε, not on $H, Y_1, \dots, Y_m$. Then $c_1 := B_1 + \cdots + B_m$ is a positive real number as we were aiming for.

The above sketch is a proof, modulo the assumption at the end. The hypersurfaces H in the sketch belong fortunately to a single semialgebraic family, a fact we already used, and so the sets Y_μ as H varies belong to a single definable family, depending on X. The inductive hypothesis should accordingly include this uniformity, and so the full proof should be carried out not just for one set X, but uniformly for all sets from a definable family, with constants depending only on the family. This is why we need Theorem 1.3 not just for $\widetilde{\mathbb{R}}$ but also for its elementary extensions, though in the above sketch we only used it for $\widetilde{\mathbb{R}}$. (As to the M introduced at the beginning of the sketch, Theorem 1.3 also provides an M that works for all members of the family.) That is the lesson to take away from this sketch.

Remarks on definable families. Given $E \subseteq \mathbb{R}^m$ and $X \subseteq E \times \mathbb{R}^n$ we consider E, X as describing the family $\big(X(s)\big)_{s \in E}$ of sets $X(s) \subseteq \mathbb{R}^n$; these sets $X(s)$ are called the members of the family. If E and X are definable, we call this a *definable family*, and then its members are definable subsets of $\mathbb{R}^n$. (In case $\widetilde{\mathbb{R}}$ is the ordered field of real numbers, we also write *semialgebraic family* instead of *definable family.*)

Given $E \subseteq \mathbb{R}^m$ and $X \subseteq \mathbb{R}^n$ we often divide the resulting family into subfamilies given by a covering $E = E_1 \cup \cdots \cup E_N$, where E_ν is typically the set of $s \in E$ for which $X(s)$ satisfies a certain condition e_ν. Then $X = X_1 \cup \cdots \cup X_N$ with $X_\nu := X \cap E_\nu \times \mathbb{R}^n$, so that $X_\nu(s)$ satisfies e_ν for all $s \in E_\nu$.

For the next lemma (a routine consequence of [1, III, Section 3]) we recall from [1, III, Section 2] that for $\boldsymbol{i} = (i_1, \dots, i_n) \in \{0,1\}^n$ we have a projection map $p_{\boldsymbol{i}} : \mathbb{R}^n \to \mathbb{R}^{i_1 + \cdots + i_n}$ that maps every $\boldsymbol{i}$-cell homeomorphically onto its image.

Lemma 2.3. *Let $d, n \geqslant 1$ and set $D := \binom{d+n}{n}$, the dimension of the $\mathbb{R}$-linear space of polynomials over $\mathbb{R}$ in n variables and of degree $\leqslant d$. Then there are semialgebraic sets $\mathcal{H}, \mathcal{C}_1, \dots, \mathcal{C}_m \subseteq (\mathbb{R}^D \setminus \{0\}) \times \mathbb{R}^n$ such that*

$$\{\mathcal{H}(t) : t \in \mathbb{R}^D \setminus \{0\}\} = \text{ set of hypersurfaces in } \mathbb{R}^n \text{ of degree} \leqslant d,$$

$\mathcal{H}(t) = \mathcal{C}_1(t) \cup \cdots \cup \mathcal{C}_m(t)$ for all $t \in \mathbb{R}^D \setminus \{0\}$, and for each $\mu \in \{1, \dots, m\}$ there is an $\boldsymbol{i} = (i_1, \dots, i_n) \in \{0,1\}^n$, $\boldsymbol{i} \neq (1, \dots, 1)$, with the property that every $\mathcal{C}_\mu(t)$ with $t \in \mathbb{R}^D \setminus \{0\}$ is a semialgebraic $\boldsymbol{i}$-cell in $\mathbb{R}^n$ or empty.

A family version of the counting theorem. We fix an o-minimal expansion $\widetilde{\mathbb{R}}$ of the real field, *definable* is with respect to $\widetilde{\mathbb{R}}$ and so are *cells*, unless specified otherwise. Let $E \subseteq \mathbb{R}^m$ and $X \subseteq E \times \mathbb{R}^n$ be definable, $n \geqslant 1$.

Theorem 2.4. *For all ε there exists a $c = c(X, \varepsilon)$ such that for all $s \in E$ and all T we have $N(X(s)^{\mathrm{tr}}, T) \leqslant cT^{\varepsilon}$.*

Proof. We proceed by induction on n. As in the sketch we reduce to the case where $X(s)$ is for every $s \in E$ a subset of $[-1,1]^n$ with empty interior. Let ε be given, and take $d \geqslant 1$ so large that $\varepsilon(n,d) \leqslant \varepsilon/2$ in Theorem 1.2, and set $k = k(n,d)$. So for every $Z \subseteq \mathbb{R}^n$ with a strong k-parameterization we can cover $Z(\mathbb{Q}, T)$ with at most $CT^{\varepsilon/2}$ hypersurfaces of degree $\leqslant d$ where $C = C(n,d)$ is as in Theorem 1.2. From Theorem 1.3 we obtain definable sets $X_1, \dots, X_M \subseteq E \times \mathbb{R}^n$, $M \in \mathbb{N}$, such that for all $s \in E$, $X(s) = X_1(s) \cup \dots \cup X_M(s)$ and each $X_i(s)$ has a definable strong k-parametrization. Let $s \in E$, and let H be a hypersurface of degree $\leqslant d$. As in the sketch we see that through our choice of k and d it is enough to show that

$$N((X(s) \cap H)^{\mathrm{tr}}, T) \ \leqslant\ c_1 T^{\varepsilon/2}, \text{ for all } T,$$

where c_1 depends only on X, ε, not on s, H, T.

With the present values of d and n, set $D := \binom{d+n}{n}$, $F := \mathbb{R}^D \setminus \{0\}$, and let $\mathcal{H}, \mathcal{C}_1, \dots, \mathcal{C}_m \subseteq F \times \mathbb{R}^n$ be as in Lemma 2.3. For $\mu = 1, \dots, m$, take $\boldsymbol{i}^\mu = (i_1^\mu, \dots, i_n^\mu) \in \{0,1\}^n$, not equal to $(1, \dots, 1)$, such that for all $t \in F$ the subset $C_\mu(t)$ of $\mathbb{R}^n$ is a semialgebraic $\boldsymbol{i}^\mu$-cell or empty, so

$$p_{\boldsymbol{i}^\mu} : \mathbb{R}^n \to \mathbb{R}^{n_\mu}, \quad n_\mu := i_1^\mu + \dots + i_n^\mu < n,$$

maps $C_\mu(t)$ homeomorphically onto its image. Then we have for $\mu = 1, \dots, m$ a definable set $Y_\mu \subseteq (E \times F) \times \mathbb{R}^{n_\mu}$ such that for all $(s,t) \in E \times F$,

$$Y_\mu(s,t) \ = \ p_{\boldsymbol{i}^\mu}\big(X(s) \cap C_\mu(t)\big).$$

Since all $n_\mu < n$ we can assume inductively that for all $(s,t) \in E \times F$ and all T,

$$N(Y_\mu(s,t)^{\mathrm{tr}}, T) \ \leqslant\ B_\mu T^{\varepsilon/2}, \quad \mu = 1, \dots, m$$

with $B_\mu = B_\mu(Y_\mu, \varepsilon) \in \mathbb{R}^{>}$ independent of s, t, T. Hence as in the sketch

$$N\big((X(s) \cap \mathcal{H}(t))^{\mathrm{tr}}, T\big) \ \leqslant\ (B_1 + \dots + B_m) T^{\varepsilon/2}.$$

Then $c_1 := B_1 + \dots + B_m$ is a positive real number as we were aiming for. $\square$

Next a variant where we remove from the sets $X(s)$ only a definable part $V(s)$ of $X(s)^{\mathrm{alg}}$ instead of all of it. The example preceding the statement of Theorem 1.1 shows that this variant is strictly stronger than Theorem 2.4.

Theorem 2.5. *Let ε be given. Then there is a definable set $V = V(X, \varepsilon) \subseteq X$ and a $c = c(X, \varepsilon)$ such that for all $s \in E$ and T,*

$$V(s) \subseteq X(s)^{\mathrm{alg}} \quad \textit{and} \quad N(X(s) \setminus V(s), T) \leqslant cT^{\varepsilon}.$$

Proof. By induction on n. We follow closely the proof of Theorem 2.4 and first reduce to the case that $X(s) \subseteq [-1,1]^n$ for all $s \in E$. Next we remove from each $X(s)$ its interior: the set $V_0 \subseteq E \times \mathbb{R}^n$ given by $V_0(s) =$ interior of $X(s)$, $s \in E$, is definable and V_0 will be part of V. Replacing X by $X \setminus V_0$ we arrange that $X(s) \subseteq [-1,1]^n$ has empty interior, for all $s \in E$. We next take $d, k = k(n,d)$, and $X_1, \dots, X_M \subseteq X$ as before. It now suffices to find a definable $V \subseteq X$ and a constant $c_1 \in \mathbb{R}^{>}$ such that for all $s \in E$, every hypersurface of degree $\leqslant d$ in $\mathbb{R}^n$, and all T we have

$$V(s) \subseteq X(s)^{\text{alg}}, \qquad N\big((X(s) \cap H) \setminus V(s), T\big) \;\leqslant\; c_1 T^{\varepsilon/2}.$$

The proof of Theorem 2.4 gives semialgebraic sets $\mathcal{H}, \mathcal{C}_1, \ldots, \mathcal{C}_m \subseteq F \times \mathbb{R}^n$, and the definable sets $Y_\mu \subseteq E \times F \times \mathbb{R}^{n_\mu}$, $\mu = 1, \ldots, m$. Since $n_\mu < n$, the inductive assumption gives a definable set $W_\mu \subseteq Y_\mu$ and a number $B_\mu = B_\mu(Y_\mu, \varepsilon) \in \mathbb{R}^{>}$ such that for all $s \in E$, $t \in F$ and T we have

$$W_\mu(s,t) \subseteq Y_\mu(s,t)^{\text{alg}} \quad \text{and} \quad N(Y_\mu(s,t) \setminus W_\mu(s,t), T) \leqslant B_\mu T^{\varepsilon}.$$

It is now routine to check that the definable set $V \subseteq X$ with

$$V(s) = \bigcup_{\mu=1}^{m} \bigcup_{t \in F} \mathcal{C}_\mu(t) \cap p_{\boldsymbol{i}^\mu}^{-1}\big(V_\mu(s,t)\big)$$

for all $s \in E$ has the desired property. □

A block family version of the Pila-Wilkie Theorem. We now establish a more refined version of the Counting Theorem which first appeared in [3]. Let $n \geqslant 1$, $X \subseteq \mathbb{R}^n$, and $k \leqslant n$. A *C^1-point in X of dimension k* is a point $a \in X$ such that $X \cap U$ is a k-dimensional embedded C^1-submanifold of $\mathbb{R}^n$, for some open neighborhood U of a in $\mathbb{R}^n$ (in which case $X \cap V$ is an embedded k-dimensional C^1-submanifold of $\mathbb{R}^n$ for all open $V \subseteq U$ with $a \in V$). Note that the set of C^1-points in X of dimension k, call it $\text{reg}_k(X)$, is an embedded C^1-submanifold of $\mathbb{R}^n$ of dimension k, in fact, it is the largest (with respect to inclusion) embedded C^1-submanifold of $\mathbb{R}^n$ of dimension k that is an open subset of X. (We consider the empty subset of $\mathbb{R}^n$ as an embedded C^1-submanifold of $\mathbb{R}^n$ of dimension k, for every $k \leqslant n$.)

What we define next is perhaps best described as a "definable C^1-piece of a semialgebraic set", but this is such a mouthful that we prefer the shorter term "block": a *block of dimension k in $\mathbb{R}^n$* is a definable set $B \subseteq \mathbb{R}^n$ such that for some semialgebraic set $A \subseteq \mathbb{R}^n$ we have $B = \text{reg}_k B \subseteq \text{reg}_k A$. For such B we have $\dim B = k$ if $B \neq \emptyset$, but we do allow $B = \emptyset$. Thus the blocks of dimension 0 in $\mathbb{R}^n$ are exactly the finite subsets of $\mathbb{R}^n$. In view of Lemma 2.6 below, a definable set $B \subseteq \mathbb{R}^n$ is a block of dimension k in $\mathbb{R}^n$ iff B is an open subset of a semialgebraic embedded C^1-submanifold A of $\mathbb{R}^n$ with $\dim A = k$. Thus if B is a block of dimension k in $\mathbb{R}^n$, then $B \times \{0\}$ is a block of dimension k in $\mathbb{R}^{n+1}$.

Lemma 2.6. *Let X, Y be embedded C^1-submanifolds of $\mathbb{R}^n$ of the same dimension with $X \subseteq Y$. Then X is open in Y.*

Proof. Excercise. □

A *block family in $\mathbb{R}^n$ of dimension k* be a definable set $W \subseteq E \times \mathbb{R}^n$, with definable $E \subseteq \mathbb{R}^m$, all whose sections $W(s)$ are blocks of dimension k in $\mathbb{R}^n$.

Lemma 2.7. *Let $X \subseteq \mathbb{R}^n$ be definable, $B \subseteq X$, and $k \geqslant 1$.*

(i) *if B is a block of dimension k in $\mathbb{R}^n$, then $B \subseteq X^{\text{alg}}$;*
(ii) *if W is a block family in $\mathbb{R}^n$ of dimension k, then the union of the sections of W that are contained in X is contained in X^{alg}.*

Proof. Let B be a block of dimension $k \geqslant 1$ in $\mathbb{R}^n$ and take a semialgebraic set $A \subseteq \mathbb{R}^n$ such that $\operatorname{reg}_k B \subseteq \operatorname{reg}_k A$. Let $a \in B$ and take a ball D in $\mathbb{R}^n$ centered at a and an $\boldsymbol{i} = (i_1, \dots, i_k)$ with $1 \leqslant i_1 < \cdots < i_k \leqslant n$ such that $\pi_{\boldsymbol{i}}$ maps $D \cap A$ bijectively onto an open subset U of $\mathbb{R}^k$ with inverse $\phi : U \to \mathbb{R}^n$ of class C^1. By Lemma 2.6 we can take D so small that $D \cap A = D \cap B$. Now U and ϕ are semialgebraic, so $\phi(U)$ is semialgebraic, and thus $a \in \phi(U) \subseteq X^{\mathrm{alg}}$. This proves (i), and (ii) is an immediate consequence. $\square$

Theorem 2.8. *Let $E \subseteq \mathbb{R}^m$ and $X \subseteq E \times \mathbb{R}^n$ be definable, and let ε be given. Then there are $I = I(X, \varepsilon) \in \mathbb{N}$, block families $V_i \subseteq (E \times F_i) \times \mathbb{R}^n$ in $\mathbb{R}^n$ of dimension $v_i \leqslant n$ with definable $F_i \subseteq \mathbb{R}^{\mu_i}$, for $i = 1, \dots, I$, and $c = c(X, \varepsilon)$, such that*

- (i) *$V_i(s, g) \subseteq X(s)$ for all i and $(s, g) \in E \times F_i$;*
- (ii) *for all T and all $s \in E$, $X(s)(\mathbb{Q}, T)$ is covered by at most cT^{ε} blocks $V_i(s, g)$, $g \in F_i$;*
- (iii) *the definable set $V \subseteq E \times \mathbb{R}^n \subseteq \mathbb{R}^{m+n}$ given by*

$$V := \{(s, a) \in E \times \mathbb{R}^n : \ a \in V_i(s, g) \text{ for some } i \text{ with } v_i \geqslant 1 \text{ and some } g \in F_i \ \}$$

has the property that for all T and all $s \in E$ we have

$$V(s) \ \subseteq \ X(s)^{\mathrm{alg}}, \qquad N(X(s) \setminus V(s), T) \ \leqslant \ cT^{\varepsilon}.$$

(Note that if $I = 0$, then $V = \emptyset$.)

Proof. As in the proof of Theorem 2.5, we proceed by induction on n, and follow the proof of Theorem 2.4 closely. We reduce to the case that $X(s) \subseteq [-1, 1]^n$ for all $s \in E$. Also $V_1 \subseteq E \times \mathbb{R}^n$ given by $V_1(s) =$ interior of $X(s)$, is a block family of dimension n, if it is non-empty. V_1 is part of the collection of block families required in the statement, and since $F_1 = \mathbb{R}^0$ we may prove the statement for $X \setminus V_1$ in place of X. Hence, we can assume that $X(s) \subseteq [-1, 1]^n$ is of empty interior for all $s \in E$. Next, we obtain d, k, and M as before. Then, we have $C = C(X, \varepsilon) \in \mathbb{R}^{>}$ with the property that for any $s \in E$, $X(s)(Q, T)$ can be covered by at most $CT^{\varepsilon/2}$ many hypersurfaces in $\mathbb{R}^n$ of degree $\leqslant d$. Therefore it suffices to to find $I, (F_i)_{i \in I}, (V_i)_{i \in I}$, and c which instead of (ii), satisfy

- (ii)* for all T, all $s \in E$, and all hypersurfaces H of degree $\leqslant d$, $(X(s) \cap H)(\mathbb{Q}, T)$ is covered by at most $\frac{c}{C} T^{\varepsilon/2}$ blocks $V_i(s, g)$, $g \in F_i$;

As in the proof of 2.5, we make use of the semialgebraic sets $\mathcal{H}, \mathcal{C}_1, \dots, \mathcal{C}_m \subseteq F \times \mathbb{R}^n$, and the definable sets $Y_\mu \subseteq E \times F \times \mathbb{R}^{n_\mu}$, $\mu = 1, \dots, m$. Since $n_\mu < n$, by the induction assumption we have $J_\mu = J_\mu(Y_\mu, \varepsilon) \in \mathbb{N}$, block families $W_{\mu,j} \subseteq \big((E \times F) \times G_{\mu,j}\big) \times \mathbb{R}^{n_\mu}$ of dimension $w_{\mu,j} \leqslant n_\mu$ with definable $G_{\mu,j} \subseteq \mathbb{R}^{\mu,\nu_j}$, for $j = 1, \dots, J_\mu$, and $B_\mu = B_\mu(Y_\mu, \varepsilon) \in \mathbb{R}^{>}$, such that

- (i)$'$ $W_{\mu,j}(s, t, g) \subseteq Y_\mu(s, t)$ for all j and $(s, t, g) \in (E \times F) \times G_{\mu,j}$;
- (ii)$'$ for all T and all $(s, t) \in E \times F$, $Y_\mu(s, t)(\mathbb{Q}, T)$ is covered by at most $B_\mu T^{\varepsilon/2}$ blocks $W_{\mu,j}(s, t, g)$, $g \in G_{\mu,j}$;
- (iii)$'$ the definable set $W_\mu \subseteq (E \times F) \times \mathbb{R}^{n_\mu}$ given by

$$W_\mu := \{(s, t, a) \in (E \times F) \times \mathbb{R}^n : \ a \in W_{\mu,j}(s, t, g) \text{ for } j \text{ with } w_{\mu,j} \geqslant 1,\ g \in G_{\mu,j} \ \}$$

has the property that for all T and all $(s, t) \in E \times F$ we have

$$W_\mu(s, t) \ \subseteq \ Y_\mu(s, t)^{\mathrm{alg}}, \qquad N(Y_\mu(s, t) \setminus W_\mu(s, t), T) \ \leqslant \ B_\mu T^{\varepsilon/2}.$$

Let $I = J_1 + \ldots + J_\mu$, and F_i be the collection definable sets $F \times G_{\mu,j}$, $\mu = 1, \ldots, m$, $j = 1, \ldots, J_\mu$. For $F_i = F \times G_{\mu,j}$, set $V_i \subseteq (E \times F_i) \times \mathbb{R}^n$ to be the definable set given by

$$V_i(s,(t,g)) = \mathcal{C}_\mu(t) \cap p_{\boldsymbol{i}^\mu}^{-1}\big(W_{\mu,j}(s,t,g)\big).$$

The V_i are all block families of dimension $w_{\mu,j} < n$, since $p_{\boldsymbol{i}^\mu}^{-1}$ are all semialgebraic homeomorphisms. It is now straightforward to check that $I, (F_i)_{i\in I}, (V_i)_{i\in I}$, and $c = C(B_1 + \ldots + B_\mu)$ are as desired. □

We can impose on the block families in Theorem 2.8 that all their sections are connected (including the possibility of empty sections). To see this, let $E \subseteq \mathbb{R}^m$ be definable and $V \subseteq E \times \mathbb{R}^n$ a block family of dimension k in $\mathbb{R}^n$. Set

$$L \ := \ \max_{s \in E} \#\{\text{connected components of } V(s)\}.$$

Then Definable Selection and the lexicographic ordering on $\mathbb{R}^n$ give block families $V_1, \ldots, V_L \subseteq E \times \mathbb{R}^n$ of dimension k in $\mathbb{R}^n$ such that for every $s \in E$ the sections $V_1(s), \ldots, V_L(s)$ are connected, open in $V(s)$, pairwise disjoint, and

$$V(s) \ = \ \bigcup_{\lambda=1}^{L} V_\lambda(s).$$

Splitting up each of the block families in Theorem 2.8 in this way and changing the constant c we achieve the connectedness property that we mentioned, without changing the set V of the theorem.

References

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