EFFECTIVE AND APPROXIMATE PILA-WILKIE TYPE COUNTING WITH COMPLEX-ANALYTIC SETS

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Abstract. We obtain an effective Pila-Wilkie statement for the zero-set of computable complex-analytic functions over a relatively compact domain. For our proof, we develop technology to measure the density of rational points in a relatively compact domain *close* to being a common solution to a family of complex-analytic functions.

This second theorem parallels the block version of the Pila-Wilkie theorem and will ensue further applications. A unique feature of this result is that the different parameter constants associated with counting of Pila-Wilkie type work uniformly for any collection of functions.

1. Introduction

In [17], Pila and Wilkie give a subpolynomial upper bound, in terms of their heights, on the number of rational points inside the transcendental part of a set definable in an o-minimal expansion of the real field. This point counting theorem applies to a wide range of sets and functions of fundamental interest, and has had striking applications to arithmetic geometry, functional transcendence, and Hodge theory. See [2] for a simplified proof and self contained treatment of the Pila-Wilkie theorem and [19, 1, 16, 18] for a detailed description of its applications.

The bound given by the Pila-Wilkie theorem cannot be made effective general, but various effectivity results have been realized for sets cut out by functions satisfying a certain form of differential equations, see [4, 12, 3]. Among the effective results, the statement in [3] applies to the widest collection of sets, namely all sets definable in the Pfaffian closure of the expansion of the reals by Log-Noetherian functions, and this result effectivizes various known applications of the Pila-Wilkie theorem to arithmetic geometry and Hodge theory.

In this paper we establish an effective Pila-Wilkie type statement for sets cut out by any set of *computable* for a complex analytic functions over a compact domain as follows.

Theorem 1.1. *Let $X = \{z \in \Omega : \ g(z) = 0\}$, where $\Omega \subseteq \mathbb{C}^n$ is relatively compact and g is a analytic function on Ω. If the function g is computable and X does not contain any semialgebraic curve, then for every $\epsilon > 0$ there is a computable constant $c = c(X, \epsilon) > 0$ such that for all T*

$$\#X(\mathbb{Q}, T) \ \leqslant \ cT^{\epsilon}.$$

Our proof of Theorem 1.1 relies on 'approximate' Pila-Wilkie type counting technology for complex-analytic sets over compact domains, which constitutes the second main theorem of this paper.

Let $X = \{z \in \Omega : \ g_i(z) = 0, \ i \in I\}$, where $\Omega \subseteq \mathbb{C}^n$ is relatively compact, I is a possibly infinite index set, and the g_i are analytic functions on Ω. We call such an

X a *complex-analytic set*. This second main theorem parallels the more intricate block version of the Pila-Wilkie theorem and is unique in the realm of results of this type that the corresponding constant c does not depend on the set X in question.

Theorem 1.2. *Let ϵ, ν be given. Then there are contants $\theta, c = \mathrm{poly}_n(\nu, \epsilon^{-1})$ and $d = \mathrm{poly}_n(\epsilon^{-1})$ such that for every complex-analytic set X and all T, we have that $X(T^{-\theta})(\mathbb{Q}, T)$ is contained in at most cT^ϵ many d-blocks in $X(T^{-\nu})$.*

Notations and conventions. Throughout we work with the conventions that $k, l, m, N \in \mathbb{N} = \{0, 1 \dots\}$, $\mu, d, e, i, j, n \in \mathbb{N}^{\geqslant 1} := \{1, 2, \dots\}$. For $\alpha, \sigma \in \mathbb{N}^l$, we set $|\sigma| := \sigma_1 + \dots + \sigma_l \in \mathbb{N}$. Also $t \in \mathbb{R}$; $\beta, \gamma, \delta, \epsilon, \rho, \omega, C \in (0,1)_\mathbb{R} := \{t : \ 0 < t < 1\}$; $\eta, c, T \in \mathbb{R}^{\geqslant 2} := \{t : \ t \geqslant 2\}$; $\theta, \nu, r, M \in \mathbb{R}^{>} := \{t : \ t > 0\}$. We follow the natural convention that any symbol with an added subscript will denote an object in the same domain as the plain symbol, so $d_\nu \in \mathbb{N}^{\geqslant 1}, \eta_1 \in \mathbb{R}^{\geqslant 2}$, and so on. We will denote Euler's constant by e.

For a ball or polydisc $A \subset \mathbb{C}^n$ with center $p \in \mathbb{C}^n$, set $A^\delta := p + \delta^{-1}(A - p)$. For a point $p \in \mathbb{C}^n$ and a ball/polydisc A centered at the origin, A_p denotes the ball/polydisc with the same radius/polyradius as A but centered at p.

Throughout the text, $\Omega \subset \mathbb{C}^n$ denotes a relatively compact domain. We let $\mathcal{O}(\Omega)$ denote the set of complex-valued analytic functions on Ω. For a function $h \in \mathcal{O}(\Omega)$ and a ball or polydisc $A \subseteq \Omega$, we set $\|h\|_A := \max_{z \in \bar{A}} |h(z)|$. Throughout $B := B(1/2) \subset \mathbb{C}^n$ denotes the ball of radius $1/2$ around the origin.

2. An effective Pila-Wilkie type statement for algebraic sets

In this section, we record an interesting application of the cellular parameterization theorem from [9] using the Bombieri-Pila determinant method [10]. We will employ this result to arrange that d depends solely on ϵ in Theorem 1.2, and will later use this feature in a critical way for the proof of Theorem 1.1. First some simple calculations that will be useful in the proof below and also in the next section.

Some combinatorial notation and basic facts. In this subsection, suppose $\mu, e, m \in \mathbb{N}^{\geqslant 1}$ are positive integers satisfying $\mu > e(m+1)$. Set $L(n, l) := \binom{n+l-1}{l}$.

Let k be the largest integer such that $\sum_{l=0}^k eL(m, l) < \mu$; note that the condition on μ implies that $k \geqslant 1$. We have that $\sum_{l=0}^k eL(m, l) = eL(m+1, k)$ and also,

$$\frac{k^m}{m!} < L(m+1, k) < \frac{\mu}{e} \leqslant L(m+1, k+1) < \frac{2k^m(m+1)^m}{m!}. \tag{1}$$

This gives $k > \frac{1}{m+1}(\frac{m!\mu}{2e})^{\frac{1}{m}}$, and $k < (\frac{m!\mu}{e})^{\frac{1}{m}}$. Let $S(m, k) := \sum_{l=0}^k eL(m, l) \cdot l$. Then $S(m, k) = emL(m+2, k-1) > \frac{ek}{2(m+1)} L(m+1, k+1)$ using the fact that $2m(k+1) > m + k + 1$. The lower bounds for $L(m+1, k+1)$ and k from (1) imply

$$S(m, k) > \big(\frac{m!}{2e}\big)^{\frac{1}{m}} \frac{\mu^{1+\frac{1}{m}}}{2(m+1)^2}. \tag{2}$$

We set $E_m := \frac{1}{2(m+1)^2}(\frac{m!}{2e})^{\frac{1}{m}}$, and so we have that $S(m, k) > E_m \mu^{1+\frac{1}{m}}$.

Bombieri-Pila determinants with complex cells. We refer the reader to [9] for basic definitions and formulations around complex cells. Let $\mathscr{C} \subseteq \mathbb{C}^n$ be a cell of length n and m be the number of its D-fibers. We work under the umbrella assumption that $m < n$. For $m \geqslant 1$, let $\boldsymbol{\delta} \in \mathbb{R}^n$ be given by $\delta < 1/2$ in the $D-$coordinates and $\delta^{E'_m \mu^{1+1/m}}$ in the A, D_0-coordinates of $\mathscr{C}$, where $E'_m = \frac{1}{4(m+1)^2}$. Applying (2) for $e = 1$ gives $S(m,k) > E'_m \mu^{1+1/m}$. This inequality can be employed at the end of the proof of [9, Lemma 93] to get the following restatement of that result. When $m = 0$, we set $\boldsymbol{\delta} = (\delta^{\mu^2}/16, \dots, \delta^{\mu^2}/16)$.

Proposition 2.1. *Suppose $f_i \in \mathcal{O}(\mathscr{C}^{\boldsymbol{\delta}})$ with $\|f_i\|_{\mathscr{C}^{\boldsymbol{\delta}}} \leqslant M$, and $p_j \in \mathscr{C}$ for $i, j \leqslant \mu$. Then if $m \geqslant 1$,*

$$|\det(\mathbf{f}, \mathbf{p})| \ := \ \det\big(f_i(p_j)\big)_{i,j \leqslant \mu} \ \leqslant M^\mu \mu^{(n+1)^2 \mu} \delta^{E'_m \mu^{1+\frac{1}{m}}};$$

and when $m = 0$,

$$|\det(\mathbf{f}, \mathbf{p})| \ \leqslant M^\mu \mu^{(n+1)^2 \mu} \delta^{\mu^3/16}.$$

The choice of a comparatively smaller E_m allows us to do away with the $\delta^{-O_n(\mu)}$ factor in the upper bound from [9, Lemma 93]. This is a direct consequence of (2), which we leave as an exercise to the reader along with with working out $(n+1)^2\mu$ as an upper bound for the exponent of μ. Our formulation helps us to make certain estimates more explicit and gives a stronger statement in the case of $m = 0$.

Interpolating rational points by polynomials. Let $d \geqslant 2$ and $m < l \leqslant n$. Set $\mu = L(l+1, d)$, and note that $\mu \geqslant l + d > d$, $\frac{d^l}{l!} \leqslant \mu \leqslant 2d^l$. Also if $l \geqslant 2$, then $d(l+1) \leqslant \mu \leqslant d^l \frac{l^k}{k!}$ for all $1 \leqslant k \leqslant l$. In addition to these observations, we will use the fact that $(2d)^{\frac{1}{d}} \leqslant 2$ for all d. Note that this implies that $\mu^{\frac{1}{ld}} \leqslant 2$.

Let $\mathbf{f} := (f_1, \dots, f_n)$ be a map, where $f_1, \dots, f_n$ are complex-valued functions, and $\mathbf{p} := \{p_1, \dots, p_\mu\}$ a collection of points. Let J be the set $\boldsymbol{j} = \{j_1, \dots, j_l\}$ with $1 \leqslant j_1, \dots, j_l \leqslant n$. Let $\boldsymbol{j} = \{j_1, \dots, j_l\} \in J$ and $z_{\boldsymbol{j}} := \{z_{j_1}, \dots, z_{j_l}\} \subseteq \{z_1, \dots, z_n\}$ be the set of variables corresponding to $\boldsymbol{j}$. Set

$$\det{}^d_{\boldsymbol{j}}(\mathbf{f}, \mathbf{p}) := \det(\mathbf{f}^\sigma(p_i))_{\sigma \in \Sigma^d_{\boldsymbol{j}}, i \leqslant \mu},$$

where $\Sigma^d_{\boldsymbol{j}} := \{\alpha \in \mathbb{N}^n : \ |\alpha| \leqslant d \text{ and } \alpha_j = 0 \text{ if } j \notin \boldsymbol{j}\}$, and $\mathbf{f}^\sigma := f_1^{\sigma_1} \cdots f_n^{\sigma_n}$. Note $\#\Sigma^d_{\boldsymbol{j}} = \mu$. Let S be a set of points in the common domain of the f_i, $i \leqslant n$. Then by a routine linear algebra argument, $(f_1, \dots, f_n)(S) \subseteq \mathbb{C}^n$ is contained in an hypersurface in $z_{\boldsymbol{j}}$ variables of degree at most d if and only if $\det^d_{\boldsymbol{j}}(\mathbf{f}, \mathbf{p}) = 0$ for any $\mathbf{p} \subset S$ of size μ; we use this observation freely in this subsection. Here, by a *hypersurface in $z_{\boldsymbol{j}}$ variables of degree at most d* we mean the zero set of a non-trivial polynomial in the $z_{\boldsymbol{j}}$ variables of total degree at most d .

We will be interested in counting rational points, and towards that end we define the *height function* $\mathcal{H} : \mathbb{Q} \to \mathbb{R}$ by $\mathcal{H}(\frac{a}{b}) \ := \ \max(|a|, \ |b|) \in \mathbb{N}^{\geqslant 1}$ for coprime $a, b \in \mathbb{Z}$, $b \neq 0$. Thus $\mathcal{H}(0) = 1$, and for $a = (a_1, \dots, a_n) \in \mathbb{Q}^n$, we extend $\mathcal{H}$ to be given by $\mathcal{H}(a) \ := \ \max\{\mathcal{H}(a_1), \dots, \mathcal{H}(a_n)\}$. Recall $T \in \mathbb{R}^{\geqslant 2}$ and let $Y \subseteq \mathbb{C}^n$. We set $Y(\mathbb{Q}, T) := \{a \in Y \cap \mathbb{Q}^n : \ \mathcal{H}(a) \leqslant T\}$ be the (finite) set of rational points of Y of height at most T.

Recall $\mathscr{C} \subseteq \mathbb{C}^n$ denotes a cell of length n and m be the number of its D-fibers, and $n < m$.

Lemma 2.2. *Suppose* $\mathbf{f} = (f_1, \dots, f_n)$ *is a cellular map with domain* $\mathcal{O}(\mathscr{C}^{\boldsymbol{\delta}})$ *and* $\|f_i\|_{\mathscr{C}^\delta} \leqslant 2n$. *Let* $\boldsymbol{j} \in J$ *and* $C := 2^{-8(m+1)^4(n+1)^2}$. *Then for every* ϵ *there is* $d < (\frac{4(m+1)^4}{\epsilon})^{m+1}$ *such that if*

$$\delta \leqslant CT^{-\epsilon},$$

then $\mathbf{f}(\mathscr{C})(\mathbb{Q}, T)$ *is contained in a hypersurface in* $z_{\boldsymbol{j}}$ *variables of degree at most* d.

Proof. Fix and ϵ and set $l = m+1$. Recall $\mu = L(l+1, d)$ and suppose $p_1, \dots, p_\mu \in \mathscr{C}$ are such that $\mathbf{f}(p_1), \dots, \mathbf{f}(p_\mu) \in \mathbf{f}(\mathscr{C})(\mathbb{Q}, T)$.

Case I: $m = 0$. Here $\mu = d+1$, and choose d such that $\frac{1}{\epsilon} < d < \frac{4}{\epsilon}$. Proposition 2.1 and [9, Lemma 95] give that

$$T^{-d(d+1)} \leqslant |\det{}^d_{\boldsymbol{j}}(\mathbf{f}, \mathbf{p})| \leqslant (2n)^{d+1}(d+1)^{(n+1)^2(d+1)}\delta^{(d+1)^3/16}$$

or $\det{}^d_{\boldsymbol{j}}(\mathbf{f}, \mathbf{p}) = 0$. So $\det{}^d_{\boldsymbol{j}}(\mathbf{f}, \mathbf{p}) = 0$ if

$$\delta \leqslant C'T^{-1/d},$$

where $C' := (2n(d+1)^{(n+1)^2})^{-16/(d+1)^2} \geqslant C = 2^{-8(n+1)^2}$, since $(d+1)^{4/(d+1)^2} \leqslant 2$ and $(2n)^{4/(d+1)^2} \leqslant 2^{(n+1)^2}$ for all d, n. Since we chose d such that $1/d < \epsilon$ we have the desired conclusion.

Case II: $m \geqslant 1$. Choose d such that $(\frac{4l^4}{\epsilon})^m < d < (\frac{4l^4}{\epsilon})^l$. By Proposition 2.1 and [9, Lemma 95] we have that

$$T^{-l\mu d} \leqslant |\det{}^d_{\boldsymbol{j}}(\mathbf{f}, \mathbf{p})| \leqslant (2n)^\mu \mu^{(n+1)^2\mu}\delta^{E'_m \mu^{1+\frac{1}{m}}}$$

or $\det{}^d_{\boldsymbol{j}}(\mathbf{f}, \mathbf{p}) = 0$. So $\det{}^d_{\boldsymbol{j}}(\mathbf{f}, \mathbf{p}) = 0$ if

$$\delta \leqslant C'T^{-4l^3 d\mu^{\frac{-1}{m}}},$$

where $C' = (2n)^{-4l^2}\mu^{-4l^2(n+1)^2\mu^{\frac{-1}{m}}}$. Using $\frac{d^l}{l!} \leqslant \mu \leqslant 2d^l$ and $(2d)^{\frac{1}{d}} \leqslant 2$, we see that $\mu^{\mu^{\frac{-1}{m}}} \leqslant 2^{l^2}$, and hence $C = 2^{-8l^4(n+1)^2} \leqslant C'$. We now use again that $\mu \geqslant \frac{d^l}{l!}$ to see that by our choice of d we have that $4l^3 d\mu^{\frac{-1}{m}} \leqslant 4l^4 d^{\frac{-1}{m}} < \epsilon$. So we have shown that $\delta \leqslant CT^{-\epsilon}$ implies that $\mathbf{f}(\mathscr{C})(\mathbb{Q}, T)$ is contained in a hypersurface in $z_{\boldsymbol{j}}$ variables of degree at most d and the proof is complete. ☐

Covering rational points on a variety by subvarieties of smaller degree. Throughout the rest of this section we set $l = m + 1$, and recall J is the set of $\boldsymbol{j} = \{j_1, \dots, j_{m+1}\}$ with $1 \leqslant j_1, \dots, j_{m+1} \leqslant n$. Set $\mathscr{P}_n := D(1)^n$.

An algebraic variety is of the form $Z(I) := \{z \in \mathbb{C}^n : \ h(a) = 0 \text{ for all } h \in I\}$, where I is a non-trivial ideal of $\mathbb{C}[z_1, \dots, z_n]$. The complexity (or degree) of an algebraic variety is as defined in [9, subsubsection 2.2.3]. For the following fact we work with the lexicographical ordering on monomials.

Fact 1. *The dimension of an algebraic variety* $Z(I) \subseteq \boldsymbol{k}^n$ *is the maximal size of a set* $S \subseteq \{z_1, \dots, z_n\}$ *such that no monomial in the variables in* S *is the leading monomial of an element of* I.

Lemma 2.3. *Let* ϵ *be given and* $W \subseteq \mathbb{C}^n$ *be an algebraic variety of complexity* d_0 *with* $\dim(W) = m < n$. *Then for any* $\boldsymbol{j} \in J$, *there are* $c = \mathrm{poly}_n(d_0, \epsilon^{-1}) \geqslant 1$ *and* $d = \mathrm{poly}_n(\epsilon^{-1})$ *such that for all* T, *there is* $N \leqslant cT^\epsilon$ *and hypersurfaces in* $z_{\boldsymbol{j}}$ *variables* $H_1, \dots, H_N$ *of degree at most* d *with* $(W \cap \mathscr{P}_n)(\mathbb{Q}, T) \subseteq \bigcup_{i=1}^N H_i$.

Proof. The case $m = 0$ is immediate and we assume from now on that $m \geqslant 1$. Obtain $C = O_n(1)$ and $d = \mathrm{poly}_n(\epsilon^{-1})$ using the corresponding estimates from Lemma 2.2. Fix $\boldsymbol{j} \in J$, and set $\mu = L(m+2, d)$ and $\delta = CT^{\frac{\epsilon}{2n}}$. The tuple $\boldsymbol{\delta} \in \mathbb{R}^n$ will be defined with regards to cells as in the context of Proposition 2.1 and will depend on the fiber composition of the cell in question.

We employ [9, Theorem 8] to get a real cellular cover of $\mathscr{P}_n \cap W$ of size $\mathrm{poly}_n(d_0)$ admitting $\frac{1}{2}$-extensions. For every cell $\mathscr{C}$ of this covering, apply [9, Lemma 94] to obtain a covering of $\mathscr{C}$ by $\mathrm{poly}_n\big(\mu \log(\frac{1}{\delta})\big) \cdot \delta^{-\dim(\mathscr{C})}$ cells such that each cell in this refinement admits a $\boldsymbol{\delta}$-extension. This gives a covering of $\mathscr{P}_n \cap W$ by $\mathrm{poly}_n(d_0, d, \log T) \cdot T^{\frac{\epsilon}{2}} = \mathrm{poly}_n(d_0, \epsilon^{-1})T^{\epsilon}$ maps; we use here that $d = \mathrm{poly}_n(\epsilon^{-1})$ and that $(\log T)^{O_n(1)} \leqslant O_n(1)\epsilon^{-1}T^{\epsilon/2}$ for all T.

Take a map of this cover, $\mathbf{f} : \mathscr{C}^{\boldsymbol{\delta}} \to \mathscr{P}_n^{1/2} \cap W$. Then our choice of δ implies by Lemma 2.2 that $\mathbf{f}(\mathscr{C})(\mathbb{Q}, T)$ is contained in a hypersurface in variables $z_{\boldsymbol{j}}$ of degree at most d, and the proof is complete. $\square$

Theorem 2.4. *Let ϵ be given and $W \subseteq \mathbb{C}^n$ be an algebraic variety of complexity d_0. Then there are $c = \mathrm{poly}_n(d_0, \epsilon^{-1})$ and $d = \mathrm{poly}_n(\epsilon^{-1})$ such that for all T, there is $N \leqslant cT^{\epsilon}$ and algebraic varieties $W_1, \ldots, W_N \subseteq W$ of complexity at most d with $(W \cap \mathscr{P}_n)(\mathbb{Q}, T) \subseteq \bigcup_{i=1}^{N} W_i$.*

Proof. We can and do assume that $W \subset \mathbb{C}^n$. So $m := \dim(W) < n$, and we proceed by induction on m.

Claim: There are $c_* = \mathrm{poly}_n(d_0, \epsilon^{-1})$, $d_* = \mathrm{poly}_n(\epsilon^{-1})$ such that for all T, we can cover $(W \cap \mathscr{P}_n)(\mathbb{Q}, T)$ by irreducible algebraic varieties $V_1, \ldots, V_l$ of complexity d_* such that $\dim(V_i) \leqslant m$ for all $i \leqslant l$, where $l \leqslant c_* T^{\frac{\epsilon}{2}}$.

We first assume the claim and complete the proof. Set $N_0 := 0$, and consider some $i \leqslant l$. We have that $V_i \subseteq W$ or $\dim(V_i \cap W) < m$. In the former case, set $N_i = N_{i-1} + 1$, $W_{N_i} := V_i$, $c_i = 2$, and $d_i := d_*$. In the latter case, induction gives for all T, $c_i = \mathrm{poly}_n(d_0, \epsilon^{-1})$, $d_i = \mathrm{poly}_n(\epsilon^{-1})$, $M_i \leqslant c_i T^{\frac{\epsilon}{2}}$, and algebraic varieties $V_{N_{i-1}+1}, \ldots, V_{N_{i-1}+M_i} \subseteq V_i \cap W$ of complexity at most d_i that cover $V_i \cap W(\mathbb{Q}, T)$; set $N_i := N_{i-1} + M_i$ in this case. It remains to observe that $c := c_* \cdot \max\{c_1, \ldots, c_l\}$, $d := \max\{d_1, \ldots, d_l\}$, $N := N_l \leqslant cT^{\epsilon}$, and the algebraic varieties $W_1, \ldots, W_N \subseteq W$ are as desired.

We now return to the proof of claim. Recall J is the set of all $\boldsymbol{j} = \{j_1, \ldots, j_{m+1}\}$ with $1 \leqslant j_1, \ldots, j_{m+1} \leqslant n$ and let $L := \#J = \binom{n}{m+1}$. Fix a T, our proof and the resulting constants will not depend on this choice. For every $\boldsymbol{j} \in J$ we apply Lemma 2.3 to get $c_{\boldsymbol{j}} = \mathrm{poly}_n(d_0, \epsilon^{-1})$, $d_{\boldsymbol{j}} = \mathrm{poly}_n(\epsilon^{-1})$ such that $W(\mathbb{Q}, T)$ is covered by at most $c_{\boldsymbol{j}} T^{\frac{\epsilon}{2L}}$ many hypersurfaces in the $z_{\boldsymbol{j}}$ variables of degree $d_{\boldsymbol{j}}$. Let $c' := \max(c_{\boldsymbol{j}})_{\boldsymbol{j} \in J}$, and $d' := \max(d_{\boldsymbol{j}})_{\boldsymbol{j} \in J}$.

Thus we have that $W(\mathbb{Q}, T)$ is contained in at most $(c')^L T^{\frac{\epsilon}{2}} = \mathrm{poly}_n(c') T^{\frac{\epsilon}{2}}$ sets of the form $\cap_{\boldsymbol{j} \in J} H_{\boldsymbol{j}}$, where $H_{\boldsymbol{j}}$ is a hypersurface of degree $d_{\boldsymbol{j}}$ in the $z_{\boldsymbol{j}}$ variables. For all T, let $V_1, \ldots, V_l$ be the collection of all the irreducible components of all the sets $\cap_{\boldsymbol{i} \in I} H_{\boldsymbol{i}}$ covering $W(\mathbb{Q}, T)$ as in the last sentence. Note that the complexity of V_i is $d_* = \mathrm{poly}_n(d')$ for all $i \leqslant l$, and also that $l \leqslant c_* T^{\frac{\epsilon}{2}}$, where $c_* = \mathrm{poly}_n(d')\mathrm{poly}_n(c') = \mathrm{poly}_n(d_0, \epsilon^{-1})$.

Moreover, $\dim(V_i) \leqslant m$ for all $i \leqslant l$ by construction of the hypersurfaces $(H_{\boldsymbol{j}})_{\boldsymbol{j} \in J}$ and Fact 1. We have shown that c_*, d_*, and $V_1, \ldots, V_l$ are as desired, which finishes the proof of the claim and in turn the completes the proof of the theorem. $\square$

3. Interpolating rational points on lemniscates

Let $A \subset B^{\frac{1}{2}} \subset \mathbb{C}^{m+1}$ be a ball or polydisc. We introduce the *Bernstein index* of g with respect to the pair $A^\eta \subset A$,

$$\mathscr{B}_A^\eta(g) := \|g\|_A / \|g\|_{A^\eta}.$$

So $\mathscr{B}_A^\eta(g) > 1$. We will make use of the the following fact [5, Lemma 17].

Lemma 3.1. *Let $U \subseteq \mathbb{C}$ be a disc and $h \in \mathcal{O}(\bar{U})$. Then there is a disc V concentric with U such that $U^4 \subset V \subset \bar{U}^2$, and*

$$\min_{z \in \partial V} |h(z)| \geqslant \|h\|_U \mathrm{e}^{-\chi \cdot \mathscr{B}_U^2(h)},$$

where $\chi \in \mathbb{R}$ is a constant with $18 < \chi < 19$.

Remark. Throughout χ will denote the constant as in the statement above.

3.1. Bounding determinants.

Set $X := \{z \in B^{\frac{1}{2}} : \ g = 0\}$, and the ω-*lemniscate of* X to be $X(\omega) := \{z \in B : \ |g| \leqslant \omega\}$. Note that X is of pure dimension m.

Let $\Delta := \Delta_h \times \Delta_v \subseteq B^{\frac{1}{2}}$, with polydiscs $\Delta_h \subset \mathbb{C}^m$ and $\Delta_v \subset \mathbb{C}$. We say that Δ is a *Weierstrass polydisc* for X if $X \cap (\bar{\Delta}_h \times \partial \Delta_v) = \emptyset$. The proofs of [8, Fact 5] and [14, II.B Theorem 2] imply that g is given by a unit times a Weierstrass polynomial of degree, say e, in z_{m+1} on Δ; and so we can perform Weierstrass division by g. We shall use this last observation shortly.

Let D be a polydisc such that that $D^\delta \subseteq \Delta$, for some δ to chosen later. We start by developing an analogue of [7, Lemma 9] for $X(\omega)$. For Δ as above, Δ' will denote the polydisc given by $\Delta' := \Delta_h \times \Delta_v^{1/3}$, and set $\mathcal{M} := \mathbb{N}^m \times \{0, 1, \ldots, e-1\}$. For convenience of notation we set

$$\beta_0 := \omega \|g\|_\Delta^{-1} \mathrm{e}^{\chi \cdot \mathscr{B}_\Delta^2(g)} \qquad \text{and} \qquad \beta := \omega \|g\|_\Delta^{-1} \mathrm{e}^{\chi \cdot \mathscr{B}_\Delta^7(g)}.$$

To respect the conventions in force, we assume that ω and g are such that $\beta, \beta_0 < 1$.

Throughout this section, let $f_i \in \mathcal{O}(\Delta')$ with $\|f_i\|_{\Delta'} \leqslant M$, and points $p_j \in D \cap X(\omega)$ for $i, j \leqslant \mu$. For the interpolation result, our primary step would be to obtain a sufficiently small bound for

$$\det(\mathbf{f}, \mathbf{p}) := \det\big(f_i(p_j)\big)_{i,j \leqslant \mu}.$$

We first handle the special case of $m = 0$. So Δ is a disc, $\mathcal{M} = \{0, \ldots, e-1\}$ and $\Delta' = \Delta^{1/3}$.

Proposition 3.2. *Let $m = 0$, $\mu > e$, and $\delta < \frac{1}{4}$. Then*

$$|\det(\mathbf{f}, \mathbf{p})| \ \leqslant \ (8\mu M)^\mu \beta_0^{\mu - e}.$$

Proof. Throughout the proof i ranges over positive natural numbers not greater than μ. We perform Weierstrass division by g and use [8, Proposition 7] to obtain

$$f_i = \ P_i + g \cdot q_i,$$

with P_i being a polynomial of degree at most $e-1$, $q_i \in \mathcal{O}(\Delta')$, and $\|P_i\|_\Delta \leqslant 3M$. In the context of Weierstrass division, also see [14, II.D Theorem 1].

We work towards bounding the function

$$q_i = (f_i - P_i)/g.$$

Lemma 3.1 applied to $U = \Delta$ together with the maximum modulus principle gives

$$\|q_i\|_D \leqslant 4M \|g\|_\Delta^{-1} \mathrm{e}^{\chi \cdot \mathscr{B}_\Delta^2(g)}.$$

In more detail, we use here that $f_i - P_i$ is bounded above by $4M$. Moreover, the assumption that $\delta < \frac{1}{4}$ implies that $D \subset (D^\delta)^4 \subseteq \Delta^4$, and then Lemma 3.1 gives an upper bound for $1/g$ on the boundary of a disc containing D. This implies the above upper bound for q_i on this boundary, and then the fact that $q_i \in \mathcal{O}(\Delta')$ gives that this upper bound holds over the whole disc D, by the maximum modulus principle.

For $j \leqslant \mu$, using $p_j \in X(\omega) \cap D$ and recalling the definition of β_0 we see that

$$|g \cdot q_i(p_j)| \leqslant 4M\beta_0.$$

Expanding $\det(\mathbf{f}, \mathbf{p})$ linearly with respect to each row we obtain 2^μ interpolation determinants. Let us consider one such determinant $\det_I$, so that for each i, every occurrence of f_i is replaced by either P_i or $g \cdot q_i$. Since the degree of P_i is bounded above by $e-1$, if there are more than e rows of $\det_I$ which feature P_i we must have $\det_I = 0$. Hence for non-zero $\det_I$, we can have at least $\mu - e$ many rows featuring gq_i. This gives that $|\det_I| \leqslant \mu! 4^\mu M^\mu \beta_0^{\mu-e}$. Now using $\mu! \leqslant \mu^\mu$ we obtain the desired upper bound on $|\det(\mathbf{f}, \mathbf{p})|$. □

For the $m > 0$ case we will need the following sharper version of [7, Lemma 3].

Lemma 3.3. *Let $m > 0$. Any $f \in \mathcal{O}(\bar{\Delta}')$ can be decomposed in the form*

$$f = \sum_{\alpha \in \mathcal{M}} m_\alpha(f) + q \cdot g,$$

with $q \in \mathcal{O}(\bar{\Delta}')$, $\|m_\alpha(f)\|_\Delta \leqslant 3\|f\|_{\Delta'}$ for $\alpha \in \mathcal{M}$, $\|\sum_{\alpha \in \mathcal{M}} m_\alpha(f)\|_\Delta \leqslant 3\|f\|_{\Delta'}$, and

$$\|q\|_{\Delta^{14}} \leqslant 4\|f\|_{\Delta'}\|g\|_\Delta^{-1} \mathrm{e}^{\chi \cdot \mathscr{B}^7_\Delta(g)}.$$

Proof. The displayed decomposition and the claims

$$q \in \mathcal{O}(\bar{\Delta}'), \quad \|m_\alpha(f)\|_\Delta \leqslant 3\|f\|_{\Delta'}, \quad \|\sum_{\alpha \in \mathcal{M}} m_\alpha(f)\|_\Delta \leqslant 3\|f\|_{\Delta'}$$

follow directly from [8, Proposition 7, Theorem 3]. We observe that this implies that $\|q \cdot g\|_\Delta \leqslant 4\|f\|_{\Delta'}$; a fact to be used later in this proof.

Towards proving the last claim, let $p \in \partial\Delta^7$ be such that $|g(p)| = \|g\|_{\Delta^7}$. We now choose and fix new coordinates $\mathbf{w}$ such that $\mathbf{w} = \mathbf{z} - p$; in other words the point p is the origin in the $\mathbf{w}$-coordinates. We set $g_\mathbf{w}$ to the function g in the $\mathbf{w}$-coordinates. Set $\Upsilon := \Delta_p^{7/6}$; note that $\Upsilon \subset \Delta$ and $\Delta^{14} \subset \Upsilon^4$.

Let $B' \subset B^{\frac{1}{2}}$ be a ball with center $\mathbf{p}$ with $\Upsilon \subset B'$. Set $\Lambda := \{\lambda = (\lambda_1, \ldots, \lambda_n) \in \partial B' : \lambda_1 \in \mathbb{R}\}$. For any $\lambda \in \Lambda$, we set $g_\lambda : D(1) \to \mathbb{C}$ to be given by

$$g_\lambda(w) := g_\mathbf{w}(w\lambda),$$

where $D(1)$ is the open unit disc. Set $M_\lambda := \|g_\lambda\|_{D(1)}$, $m_\lambda := \|g_\lambda\|_{D(1)^2}$.

Applying 3.1 to the function g_λ we get a disc V_λ concentric with $D(1)$ with $D(1)^4 \subset V_\lambda \subset \overline{D(1)}^2$ and

$$\min_{w \in \partial V_\lambda} |g_\lambda(w)| \geqslant M_\lambda \mathrm{e}^{-\chi \cdot \ln \frac{M_\lambda}{m_\lambda}}.$$

Now $M_\lambda \leqslant \|g\|_\Delta$, and $m_\lambda \geqslant \|g\|_{\Delta^7}$ since $g_\mathbf{w}(\mathbf{0}) = \|g\|_{\Delta^7}$. Since $\chi > 1$, we have

$$\min_{w \in \partial V_\lambda} |g_\lambda(w)| \geqslant \|g\|_\Delta \mathrm{e}^{-\chi \cdot \mathscr{B}^7_\Delta(g)}.$$

Notice that the lower bound in the above inequality does not depend on λ; we are now ready to finish the proof. Recall that $\|q\cdot g\| \leqslant 4\|f\|_{\Delta'}$. Take any $a\in\Delta^{14}$. Note that $\Delta^{14}\subset\Upsilon^4\subset(B')^4$, and so there are $\lambda_0\in\Lambda$ and $w_0\in D(1)^4\subset V_{\lambda_0}$ such that $a-p=w_0\lambda_0$. So the fact that $q\in\mathcal{O}(\bar{\Delta}')$, and the maximum principle applied to $q(p+w\lambda_0):\bar{V}_{\lambda_0}\to\mathbb{C}$ gives that

$$|q(a)| \leqslant 4\|f\|_{\Delta'}\|g\|_{\Delta}^{-1}\mathrm{e}^{\chi\cdot\mathscr{B}_{\Delta}^{7}(g)}.$$

It only remains to notice that our upper bound does not depend on the choice of $a\in\Delta^{14}$. This gives us the last claim and the proof is complete. $\square$

We are now ready for the $m>0$ version of Proposition 3.2; set $L(n,l):=\binom{n+l-1}{l}$. First some simple calculations that will be useful in the proof below. We work with $m\geqslant 1$, $\mu>e(m+1)$, and let k be the largest integer such that $\sum_{l=0}^{k} eL(m,l)<\mu$. We have that $\sum_{l=0}^{k} eL(m,l)=eL(m+1,k)$ and so by the assumption on μ we have $k\geqslant 1$. Also,

$$\frac{k^m}{m!} < L(m+1,k) < \frac{\mu}{e} \leqslant L(m+1,k+1) < \frac{2k^m(m+1)^m}{m!}.$$

This gives $k>\frac{1}{m+1}(\frac{m!\mu}{2e})^{\frac{1}{m}}$, and $k<(\frac{m!\mu}{e})^{\frac{1}{m}}$. Let $S(m,k):=\sum_{l=0}^{k} eL(m,l)\cdot l$. Then $S(m,k)=emL(m+2,k-1)>\frac{ek}{2(m+1)}L(m+1,k+1)$. Now using our lower bounds for $L(m+1,k+1)$ and k we obtain,

$$S(m,k) > \big(\frac{m!}{2e}\big)^{\frac{1}{m}}\frac{\mu^{1+\frac{1}{m}}}{2(m+1)^2}.$$

We set $E_m:=\frac{1}{2(m+1)^2}(\frac{m!}{2e})^{\frac{1}{m}}$, so that $S(m,k)>E_m\mu^{1+\frac{1}{m}}$. We are now ready to state the result.

Proposition 3.4. *Let $m\geqslant 1$ and $\mu>e(m+1)$. Let $k_0\geqslant(m!\mu/e)^{\frac{1}{m}}$ and assume $\delta<1/14$. Then for $\rho:=\max(\delta,\beta^{1/k_0})$ we have that*

$$|\det(\mathbf{f},\mathbf{p})| \;\leqslant\; (4(1.1)^m\mu^3M)^{\mu}\cdot\rho^{E_m\cdot\mu^{1+\frac{1}{m}}}.$$

Proof. We set

$$k:=\max\Big\{j:\ \sum_{l=0}^{j} eL(m,l)<\mu\Big\}.$$

So by our calculations just before the statement and our assumption on μ we have $k\geqslant 1$, and the upper bound on k gives $k_0>k$. Throughout the proof i ranges over positive natural numbers not greater than μ. Set $\mathcal{M}^{\leqslant k}:=\{\alpha\in\mathcal{M}:\ |\alpha|\leqslant k\}$ and apply Lemma 3.3 to obtain

$$f_i=\sum_{\alpha\in\mathcal{M}^{\leqslant k}} m_\alpha(f_i)+R_k(f_i)+gq_i,$$

where $\|m_\alpha(f)\|_\Delta\leqslant 3M$ for all $\alpha\in\mathcal{M}^{\leqslant k}$, and

$$\|q_i\|_D\leqslant 4M\|g\|_{\Delta}^{-1}\mathrm{e}^{\chi\cdot\mathscr{B}_{\Delta}^{7}(g)}\leqslant 4M\beta/\omega.$$

Note here we used $\delta<\frac{1}{14}$, which gives that $D\subseteq\Delta^{1/\delta}\subset\Delta^{14}$. By our choice of k, the number of terms in each of the expansions above does not exceed $\mu+1$. Moreover, [7, Proposition 8], the fact that $k_0>k$ and our assumptions on the constants β and δ give for every $j\leqslant\mu$ and $\alpha\in\mathcal{M}^{\leqslant k}$ that

$$\|m_\alpha(f_i)\|_D \leqslant C_0\rho^{|\alpha|}, \quad \|R_k(f_i)\|_D \leqslant C_0\rho^k, \qquad |g\cdot q_i(p_j)| \leqslant C_0\rho^k,$$

$$\text{where} \quad C_0 := \frac{4eL(m,k)M}{(1-\delta)^m} < 4(1.1)^m\mu M;$$

the upper bound for C_0 comes by using $L(m,k) < L(m+1,k) < \frac{\mu}{e}$ and $\delta < \frac{1}{14}$.

As in the proof of [7, Lemma 9], expanding $\det(\mathbf{f},\mathbf{p})$ linearly we get a sum of at most $(\mu+1)^\mu$ interpolation determinants $\det_I$, such that $|\det_I| \leqslant \mu! C_0^\mu \rho^S$, where $S = S(m,k) = \sum_{l=0}^k eL(m,l)\cdot l > E_m\mu^{1+1/m}$, the last inequality coming from our calculations just preceding the statement of the proposition. Note that $\mu > e(m+1)$ implies that $\mu \geqslant 3$ and then $\mu!(\mu+1)^\mu \leqslant \mu^{2\mu}$. The proof is complete. □

3.2. **Covering rational points by hypersurfaces.** We now turn to analogues of results in [7, Section 3.3], and follow the general strategy there. Most of this subsection is bookkeeping, and we arrange things in way that allows for a clean statement of Corollary 3.7. This result is the only one that we employ later.

Let $d \geqslant 2$, $m < l \leqslant n$, and $\mu = L(l+1,d)$. So $\mu \geqslant l+d > d$, $\frac{d^l}{l!} \leqslant \mu \leqslant 2d^l$. Also if $l \geqslant 2$, then $d(l+1) \leqslant \mu \leqslant d^l\frac{l^k}{k!}$ for all $1 \leqslant k \leqslant l$. In addition to these observations, we will use the fact that $(2d)^{\frac{1}{d}} \leqslant 2$ for all d. Note that this implies that $\mu^{\frac{1}{ld}} \leqslant 2$.

Let $\mathbf{f} := (f_1,\dots,f_n)$ be a collection of complex-valued functions, and $\mathbf{p} := \{p_1,\dots,p_\mu\}$ a collection of points. Let J be the set $\boldsymbol{j} = \{j_1,\dots,j_l\}$ with $1 \leqslant j_1,\dots,j_l \leqslant n$. For the rest of the section, we fix a $\boldsymbol{j} = \{j_1,\dots,j_l\} \in J$, and let $z_{\boldsymbol{j}} := \{z_{j_1},\dots,z_{j_l}\} \subseteq \{z_1,\dots,z_n\}$ be the set of variables corresponding to $\boldsymbol{j}$. Set

$$\det{}^d_{\boldsymbol{j}}(\mathbf{f},\mathbf{p}) := \det(\mathbf{f}^\sigma(p_i))_{\sigma\in\Sigma^d_{\boldsymbol{j}}, i\leqslant\mu},$$

where $\Sigma^d_{\boldsymbol{j}} := \{\alpha \in \mathbb{N}^n : \ |\alpha| \leqslant d \text{ and } \alpha_j = 0 \text{ if } j \notin \boldsymbol{j}\}$, and $\mathbf{f}^\sigma := f_1^{\sigma_1}\cdots f_n^{\sigma_n}$. Note $\#\Sigma^d_{\boldsymbol{j}} = \mu$. Let S be a set of points in the common domain of the f_i, $i \leqslant n$. Then by a routine linear algebra argument, $(f_1,\dots,f_n)(S) \subseteq \mathbb{C}^n$ is contained in an hypersurface of degree d at most d in the $z_{\boldsymbol{j}}$ variables if and only if $\det^d_{\boldsymbol{j}}(\mathbf{f},\mathbf{p}) = 0$ for any $\mathbf{p} \subset S$ of size μ; we use this observation freely in this subsection. Here, by a *hypersurface of degree at most d in the $z_{\boldsymbol{j}}$ variables* we mean the zero set of a non-trivial polynomial of degree at most d in the $z_{\boldsymbol{j}}$ variables.

We will be interested in counting rational points, and towards that end we define the *height function* $\mathcal{H} : \mathbb{Q} \to \mathbb{R}$ by $\mathcal{H}(\frac{a}{b}) := \max(|a|,\ |b|) \in \mathbb{N}^{\geqslant 1}$ for coprime $a, b \in \mathbb{Z}$, $b \neq 0$. Thus $\mathcal{H}(0) = 1$, and for $a = (a_1,\dots,a_n) \in \mathbb{Q}^n$, we extend $\mathcal{H}$ to be given by $\mathcal{H}(a) := \max\{\mathcal{H}(a_1),\dots,\mathcal{H}(a_n)\}$. Recall $T \in \mathbb{R}^{\geqslant 2}$ and let $Y \subseteq \mathbb{C}^n$. We set $Y(\mathbb{Q},T) := \{a \in Y \cap \mathbb{Q}^n : \ \mathcal{H}(a) \leqslant T\}$ be the (finite) set of rational points of Y of height at most T.

Throughout this subsection X, Δ, Δ', and D are as before. So $\dim(X) = m$, $\Delta \subseteq \mathbb{C}^{m+1}$ is a polydisc for X and $D^\delta \subseteq \Delta$. Suppose $f_i \in \mathcal{O}(\Delta')$ with $\|f_i\|_{\Delta'} \leqslant 1$ for all $i \leqslant n$, and set $Y := \mathbf{f}(X(\omega)\cap D) \subset \mathbb{C}^n$.

Lemma 3.5. *Let $m = 0$ and $\delta < \frac{1}{4}$. For every ϵ and d with $\frac{3l^2}{\epsilon} \leqslant \frac{d}{e} < \frac{4l^2}{\epsilon}$, if we have that*

$$\beta_0^{\frac{2l}{\mu d}} \leqslant \frac{1}{4}T^{-\epsilon},$$

then $Y(\mathbb{Q},T)$ is contained in a hypersurface of degree at most d in the $z_{\boldsymbol{j}}$ variables.

Proof. Fix an ϵ. Let $p_1,\dots,p_\mu \in X(\omega)\cap D$ be such that $\mathbf{f}(p_1),\dots,\mathbf{f}(p_\mu) \in Y(\mathbb{Q},T)$, and suppose $\det^d_{\boldsymbol{j}}(\mathbf{f},\mathbf{p}) \neq 0$. Choose d satisfying $e\frac{3l^2}{\epsilon} \leqslant d < e\frac{4l^2}{\epsilon}$. Note that this implies that $\frac{d}{3e} > 1$, and so $\frac{2l^2}{\frac{d}{e}-1} < e\frac{3l^2}{d} < \epsilon$. Then $\mu > d > e$ and Proposition 3.2 and [7, Lemma 10] give

$$T^{-l\mu d} \;\leqslant\; \det{}^d_{\boldsymbol{j}}(\mathbf{f},\mathbf{p}) \;\leqslant\; (8\mu)^\mu \beta_0^{\mu-e}.$$

So for $C := (8\mu)^{-\frac{2l}{d(\mu-e)}}$, any choice of $\beta_0 \leqslant (CT^{-\epsilon})^{\frac{\mu d}{2l}}$ gives a contradiction to the displayed inequalities; since $\frac{2l^2}{\mu-e} < \frac{2l^2}{d-e} = \frac{2l^2}{e(\frac{d}{e}-1)} \leqslant \frac{2l^2}{\frac{d}{e}-1} < \epsilon$. This also gives $\frac{2l}{d(\mu-e)} < \frac{\epsilon}{ld} < \frac{1}{ld}$; and to make our choice of C independent of d, we use $\mu^{\frac{1}{ld}} \leqslant 2$, and then reset $C := 1/4$; we use here that $ld \geqslant 3$. Now the contradiction in hand implies that $\det^d_{\boldsymbol{j}}(f,\mathbf{p}) = 0$, and this yields an interpolating hypersurface of degree d in the $z_{\boldsymbol{j}}$ variables as claimed. $\square$

Lemma 3.6. *Let $m \geqslant 1$, $\delta < \frac{1}{14}$, and $k_0 := ld^{l/m}$. There exists $C = C(l,m) < 1$ such that for every ϵ there is d with $\frac{d}{e} < (\frac{4l^3 l^{\frac{l-1}{m}}}{\epsilon})^l$ such that if*

$$\rho := \max(\delta, \beta^{\frac{1}{k_0}}) \leqslant CT^{-\epsilon},$$

then $Y(\mathbb{Q},T)$ is contained in a hypersurface of degree at most d in the $z_{\boldsymbol{j}}$ variables.

Proof. Fix an ϵ. A choice that will be justified later, we take d such that $d > e(\frac{4l^3 l^{\frac{l-1}{m}}}{\epsilon})^m$. Clearly we can, and do choose d so that $d < e(\frac{4l^3 l^{\frac{l-1}{m}}}{\epsilon})^l$. We have that $m \geqslant 1$, so that $l \geqslant 2$. As noted before, this means that $\mu \geqslant d(l+1)$. So since $d > e$, and $\mu > e(m+1)$, which is a assumption for Proposition 3.4. Also recall that $\mu \leqslant d^l \frac{l^m}{m!}$, and so $k_0 = ld^{l/m} \geqslant (m!\mu)^{1/m}$ satisfies the assumption on k_0 in the statement of Proposition 3.4.

Let $p_1,\dots,p_\mu \in X(\omega)\cap D$ be such that $\mathbf{f}(p_1),\dots,\mathbf{f}(p_\mu) \in Y(\mathbb{Q},T)$, and suppose that $\det^d_{\boldsymbol{j}}(\mathbf{f},\mathbf{p}) \neq 0$. Then Proposition 3.4 and [7, Lemma 10] give

$$T^{-l\mu d} \;\leqslant\; \det{}^d_{\boldsymbol{j}}(\mathbf{f},\mathbf{p}) \;\leqslant\; \big(4(1.1)^m\mu^3\big)^\mu \cdot \rho^{E_m\cdot\mu^{1+1/m}},$$

where $E_m = \frac{1}{2(m+1)^2}(\frac{m!}{2e})^{\frac{1}{m}} > \frac{1}{4l^2 e^{\frac{1}{m}}}$. So we get a contradiction if

$$\rho < CT^{-4l^3 e^{\frac{1}{m}} d\mu^{-\frac{1}{m}}},$$

for $C := \big(4(1.1)^m\mu^3\big)^{-4l^2 e^{\frac{1}{m}}\mu^{-\frac{1}{m}}}$. This choice of C depends on d and in turn ϵ, and also e, but in the statement we promised a C that depends on just m and M. Using $d \geqslant e$, $l! \leqslant l^{l-1}$, and recalling that $\frac{d^l}{l!} \leqslant \mu$ we observe

$$\big(4(1.1)^m\mu^3\big)^{\frac{-4l^2 e^{\frac{1}{m}}}{\mu^{\frac{1}{m}}}} \geqslant \big(4(1.1)^l\mu^3\big)^{\frac{-4l^2 l^{\frac{l-1}{m}}}{d}}.$$

Now we use $\mu \leqslant 2d^l$, $(2d)^{\frac{1}{d}} \leqslant 2$, and also $32^{\frac{1}{3l}}(1.1)^{1/3} < 32^{1/6}(1.1) < 2$ to see

$$\big(4(1.1)^l\mu^3\big)^{\frac{-4l^2 l^{\frac{l-1}{m}}}{d}} > 2^{-12l^3 l^{\frac{l-1}{m}}}.$$

So we change reset C to $C := 2^{-12l^3 l^{\frac{l-1}{m}}}$. We use again that $\frac{d^l}{l!} \leqslant \mu$, $l! \leqslant l^{l-1}$, to see that our choice of d gives

$$4l^3 e^{\frac{1}{m}} d\mu^{-\frac{1}{m}} < 4l^3 l^{\frac{l-1}{m}} (e/d)^{\frac{1}{m}} < \epsilon.$$

So any $\rho \leqslant CT^{-\epsilon}$ gives the desired contradiction, and the corollary follows. □

We now record a consequence that we will apply, here we choose $l = m+1$ so that $\#z_{\mathbf{j}} = l = m+1$. Also included is a viable upper bound on β that is uniform in m. Let $\upsilon_m : \mathbb{N} \to \{0,1\}$ be such that $\upsilon_m = 0 \iff m = 0$.

Corollary 3.7. *Let $\delta < \frac{1}{14}$ and $C = 2^{-12l^4}$. For every ϵ there exists d satisfying $\frac{d}{e} < \left(\frac{4l^4}{\epsilon}\right)^l$ such that if*

$$\beta \leqslant (CT^{-\epsilon})^{ld^2} \quad \text{and} \quad \upsilon_m \delta \leqslant CT^{-\epsilon},$$

then $Y(\mathbb{Q},T)$ is contained in a hypersurface of degree at most d in the $z_{\mathbf{j}}$ variables.

Proof. This follows through easily from the previous two lemmas. Fix an ϵ and T, and suppose β and δ satisfy the displayed inequalities. For $m = 0$ since $\beta_0 < \beta$, we have $\beta_0^{\frac{2}{\mu d}} < \beta^{\frac{1}{d^2}} < CT^{-\epsilon} < \frac{1}{4}T^{-\epsilon}$. Now a direct application of Corollary 3.5 finishes this case. For $m \geqslant 1$, the bound we assume on β implies that $\beta^{\frac{1}{k_0}} \leqslant CT^{-\epsilon}$, and so the assumption on δ gives $\rho := \max(\delta, \beta^{\frac{1}{k_0}}) \leqslant CT^{-\epsilon}$, and then an application of Corollary 3.6 finishes the proof. □

4. Counting rational points near an analytic set

Let $B := B(\frac{1}{2}) \subset \mathbb{C}^n$ be the ball of radius $\frac{1}{2}$ around the origin. Let $g \in \mathcal{O}(B^{\frac{1}{2}})$ such that g is non-constant and $\|g\|_{B^{\frac{1}{2}}} \leqslant 1$, and set $X := \{z \in B^{\frac{1}{2}} : \ g = 0\}$, $X(\omega) := \{z \in B : \ |g| \leqslant \omega\}$, so we are in the setting of the previous section.

Here the notion of an algebraic map and its complexity is as defined in [9, subsubsection 2.2.3]. Let $\phi : D(1)^l \to \mathbb{C}^n$ be an algebraic map of complexity at most d such that $\phi(D(\frac{1}{2})^l) \subseteq X(\omega')$. Then we call such a set $\phi(D(\frac{1}{2})^l)$ a *d-block in $X(\omega')$*. We sometimes just call such an object a d-block when the ambient set $X(\omega')$ is clear from context. In particular any polydisc inside $X(\omega')$ and also a point inside the set $X(\omega')$ are 1-blocks.

Theorem 4.1. *Let ϵ, ν be given. Suppose θ be such that*

$$\chi \cdot \nu + 2^{8(n+1)} n^{10n+7} \epsilon^{-2(n+1)} \cdot \nu^2 = \theta.$$

Then there are $c = \mathrm{poly}_n(\epsilon^{-1}, \nu)$ and $d = \mathrm{poly}_n(\epsilon^{-1})$ such that for all T, we have that $X(T^{-\theta})(\mathbb{Q},T)$ is contained in at most cT^ϵ many d-blocks in $X(T^{-\nu})$.

Proof. Let ϵ, θ, ν, and $X \subset \mathbb{C}^n$ be given as in the statement above. We show the consequence of the theorem locally for $X \cap A$, where A is a block. This will suffice as we control the size of A so that it is not too small; this will be made clear soon. We demonstrate our method around the origin. All blocks in the proof are blocks in $X(T^{-\nu})$.

Fix some T, this choice will not effect our process. Let $\eta := 2 \cdot \eta_2 \cdot C^{-1} \cdot 4T^{\frac{\epsilon}{2n}}$, where η_2 will be chosen later, and $C = C(n) = 2^{-12n^4}$ comes from Corollary 3.7. If $\|g\|_{B^\eta} < T^{-\nu}$, then there is a polydisc D with $B^{n\eta} \subset D \subset B^\eta$, and we have $(X \cap \Delta)(T^{-\theta}) \subseteq D \subseteq X(T^{-\nu})$, so D is a 1-block.

Otherwise $\|g\|_{B^\eta} \geqslant T^{-\nu}$. Let p be a point on $\bar{B}^\eta$ where the maximum of g is attained, so that $|g(p)| \geqslant T^{-\nu}$. Let $\eta_1 := 4T^{\frac{\epsilon}{4n}}$, so $B_p^2 \subset B^{\frac{1}{2}}$, and apply [6, Corollary 18] to obtain a Weierstrass polydisc Δ for $\mathbf{u}^{-1}(X)$, centered around p; where $\mathbf{u} : B^{\frac{1}{2}} \to B^{\frac{1}{2}}$ is a map that implements the requisite linear unitary change of co-ordinates. Moreover, [6, Corollary 18] gives η_2 such that $B_p^{\eta_1\eta_2} \subset \mathbf{u}(\Delta) \subset \mathbf{u}(\Delta') \subset B_p^{\eta_1}$ with $\eta_2 = \mathrm{poly}_n\big(\mathrm{vol}(X \cap B_p^{\eta_1})\big)$. This is the η_2 we use in the definition of η; the next paragraph shows that $\eta_2 = \mathrm{poly}_n(\epsilon^{-1}, \theta)$ and that the choice does not depend on T.

The fact $\|g\|_{B_p^{\eta_1}} > T^{-\nu}$ gives an upper for $\mathrm{vol}(X \cap B_p^{\eta_1})$ as follows. Let L be a complex line passing through p. In the notation of [15], $\varrho < T^{-\frac{\epsilon}{4n}}$, and so $\gamma(B_p^2 \cap L, \bar{B}_p^{\eta_1} \cap L) < \frac{4n}{\epsilon \ln(T)}$. Now applying [15, Lemma 1] around p gives $|X \cap B_p^{\eta_1} \cap L| < \frac{4n\nu}{\epsilon}$; in particular we have that $e(X, \Delta) < \frac{4n\nu}{\epsilon}$. Then an analogue of Crofton's formula [11, Proposition 14.6.3] implies that $\mathrm{vol}(X \cap B_p^{\eta_1}) = O_n(1)4n\nu\epsilon^{-1}$ so that this gives $\eta_2 = \mathrm{poly}_n(\epsilon^{-1}, \theta)$ as claimed before. For a similar application of this analogue of Crofton's formula, see [6, subsection 5.3]. Note we have $\eta > \eta \cdot CT^{-\frac{\epsilon}{4n}} > \eta_1\eta_2$, which implies $\mathbf{0} \in B^{\eta CT^{-\frac{\epsilon}{4n}}} \subset B_p^{\eta_1\eta_2} \subset \mathbf{u}(\Delta)$ since $p \in \bar{B}^\eta$.

Recall $n = m+1$ and $e(X, \Delta) < \frac{4n\nu}{\epsilon}$, so we employ Corollary 3.7 with $\frac{\epsilon}{4n}$ in the role of ϵ to obtain $d' < \big(\frac{16n^5}{\epsilon}\big)^n$, such that for

$$\delta := CT^{-\frac{\epsilon}{4n}} \quad \text{and} \quad d_\nu := \frac{4n\nu}{\epsilon} \cdot d',$$

if $\beta \leqslant (CT^{-\epsilon})^{nd_\nu^2}$, then $\big(X(T^{-\theta}) \cap \mathbf{u}(\Delta)^{\frac{1}{\delta}}\big)(\mathbb{Q}, T)$ is contained in an hypersurface in $\mathbb{C}^n$ of degree d_ν. Set $D := \mathbf{u}(\Delta)^{\frac{1}{\delta}}$, and note that $\mathbf{0} \in B^\eta \subset D$.

At this stage we have shown, irrespective of whether $\|g\|_{B^\eta} < T^{-\nu}$ or not, that B can be covered by at most $(2 \cdot n\eta)^n = c'T^{\frac{\epsilon}{2}}$ unitary images of polydiscs, where $c' = \mathrm{poly}_n(\epsilon^{-1}, \theta)$. If we show, in a manner that does not depend on working around the origin and the choice of T, that there are $c_1 = \mathrm{poly}_n(\epsilon^{-1}, \theta)$, $d_1 = \mathrm{poly}_n(\epsilon^{-1})$ such that $(X \cap D)(T^{-\theta})(\mathbb{Q}, T)$ is contained in at most $c_1 T^{\frac{\epsilon}{2}}$ d_1-blocks in $X(T^{-\nu})$, then $X(T^{-\theta})(\mathbb{Q}, T)$ will be contained in at most $c_1c'T^\epsilon$ many d_1-blocks, and the proof will be complete. This task is trivial for the case of $\|g\|_D < T^{-\nu}$, so we can and do assume for the rest of the proof that $\|g\|_D \geqslant T^{-\nu}$ and focus on $X \cap D$.

We have that $D \subseteq \mathbf{u}(\Delta)^{14}$, since $\delta < C \leqslant 2^{-12}$, and working in the Δ-coordinates this gives $\|g\|_{\Delta^7} > T^{-\nu}$. Our assumption on ν implies that implies that

$$\beta \leqslant T^{-\theta}\|g\|_{\Delta^7}^{-\chi} \leqslant T^{2^{8(n+1)}n^{10n+7}\epsilon^{-2(n+1)}\nu^2} < T^{-(\epsilon n + 12n^5)\left(\frac{4n}{\epsilon}\left(\frac{16n^5}{\epsilon}\right)^n\nu\right)^2}.$$

Note that $d_\nu^2 < \big(\frac{4n}{\epsilon}\big(\frac{(2n)^5}{\epsilon}\big)^n\nu\big)^2$ and $C = 2^{-12n^4} \geqslant T^{-12n^4}$. So we have that $\beta < (CT^{-\epsilon})^{nd_\nu^2}$, and as noted before this gives that $(X \cap D)(T^{-\theta})(\mathbb{Q}, T)$ is contained in an hypersurface, say H_ν, of degree at most $d_\nu = \mathrm{poly}_n(\epsilon^{-1}, \theta)$.

The degree of H_ν, i.e. d_ν, depends on θ, which going forward will be an obstruction to our aim of covering $X(T^{-\theta})(\mathbb{Q}, T)$ with blocks of degree depending on just n and ϵ. To remedy this, we apply Theorem 2.4 to obtain $c_2 = \mathrm{poly}_n(\theta, \epsilon^{-1})$, $d_2 = \mathrm{poly}_n(\epsilon^{-1})$, and cover $H_\nu(\mathbb{Q}, T)$ by $c_2 T^{\frac{\epsilon}{8}}$ many algebraic varieties of complexity bounded by d_2, all contained in H_ν.

Let W be an algebraic variety from this cover, so the complexity of W is bounded by d_2. Using that D is the image of a polydisc by an algebraic map of complexity $d_3 = O_n(1) = \mathrm{poly}_n(\epsilon^{-1})$, we employ [9, Theorem 8] to get a real cellular cover of $D \cap W$ of size and complexity $d_4 = \mathrm{poly}_n(d_2, d_3)$ admitting $\frac{1}{2}$-extensions. Then apply [9, Lemma 94] to further refine each of the cells so that each cell $\mathscr{C}$ in the refinement admits a $\boldsymbol{\delta_*}$-extension. Here $\boldsymbol{\delta_*}$ is given by $\delta_* = C_*T^{\frac{\epsilon}{8n}}$ in the $D-$coordinates and $\delta_*^{E'_m\mu^{1+1/m}}$ in the A, D_0-coordinates of $\mathscr{C}$, where $\mu = L(n +$

$1, d_*)$, and $C_* = 2^{-4n^4(n+1)^2}$, $d_* = \text{poly}_n(\epsilon^{-1})$ are from Lemma 2.2; this choice will be recalled later. Note we are in the setting of Lemma 2.2 since the dimension $D \cap W$, and hence the number of D-fibers of $\mathscr{C}$, is at most $n-1$. We have arrived now at a covering of algebraic complexity $d_5 = \text{poly}_n(d_4) = \text{poly}_n(\epsilon^{-1})$ for $D \cap W$ by $\text{poly}_n(\epsilon^{-1}, \log T) \cdot T^{\frac{\epsilon}{8}} < \text{poly}_n(\epsilon^{-1}) \cdot T^{\frac{\epsilon}{4}}$ such cells.

Pick a component of this cover, say $\mathbf{f} : \mathscr{C}^{\boldsymbol{\delta_*}} \to D^{\frac{1}{2}} \cap W$, so complexity of $\mathbf{f}$ is at most d_5. It suffices to prove that: there is $c_3 = \text{poly}_n(\theta, \epsilon^{-1})$ and $d_6 = \text{poly}_n(\epsilon^{-1})$ such that $\big(X(T^{-\theta}) \cap f(\mathscr{C})\big)(\mathbb{Q}, T)$ is contained in at most $c_3 T^{\frac{\epsilon}{8}}$ d_6-blocks.

We proceed by induction on $k = \dim(\mathscr{C}) \leqslant n-1$. The base case $k = 0$ is trivial, and implies our claim in the case of $n = 1$. Now let $n \geqslant 2$, and our inductive assumption is that we have the desired claim if $k \leqslant n-2$. Suppose $k = n-1$ and let m_* be the number of D-fibers of $\mathscr{C}$; so $m_* \leqslant n-1$. Our supposition implies that $\dim(W) = n-1$, and let I_W be an ideal of $\mathbb{C}[z_1, \dots, z_n]$ with $Z(I_W) = W$. By Fact 1 we have a variable, say $z_W \in \{z_1, \dots, z_n\}$, such that the leading monomial of every element of I_W features z_W.

If $m_* < n-1$, then our choice of C_* and δ_* allows us to apply Lemma 2.2 with $l = m_* + 1 < n$, and choice of some $\boldsymbol{j} \subseteq \{z_1, \dots, z_n\} \setminus \{z_W\}$. This gives a hypersurface, say H_*, of degree at most d_* in the $z_{\boldsymbol{j}}$ variables such that $\mathbf{f}(\mathscr{C})(\mathbb{Q}, T)$ is contained in H_*. Note this gives by Fact 1 that $\dim(W \cap H_*) \leqslant n-2$, and that the complexity of $W \cap H_*$ is bounded by $d_7 = \text{poly}_n(d_2, d_*) = \text{poly}_n(\epsilon^{-1})$ by Bezout's theorem, see [9, subsubsection 2.2.3]. Now using that the complexity of $\mathbf{f}$ is bounded by d_5, we obtain a cellular covering of $D \cap W \cap H_*$ following a similar process and entailing similar properties as for the case of $D \cap W$ earlier. The size and complexity of this covering can clearly be controlled as per our requirements, and since cells in this covering are of dimension $\leqslant n-2$, we finish this case by invoking the inductive assumption.

Now suppose that $m_* = n-1 = k$; then no fiber of $\mathscr{C}$ is of A or D_0 type. By renormalizing $\mathscr{C}$ we get a polydisc $\Delta_* \subset \mathbb{C}^{n-1}$, and an algebraic map $\mathbf{h} : \Delta_*^{\delta_*} \to D^{\frac{1}{2}} \cap W$ of complexity at most $d_8 = \text{poly}_n(d_5)$ such that $\mathbf{f}(\mathscr{C}) \subseteq \mathbf{h}(\Delta_*)$. If $\mathbf{h}(\Delta_*) \subseteq X(T^{-\nu})$, then $\mathbf{h}(\Delta_*)$ is a d_8-block, since $\delta^* < \frac{1}{2}$. Otherwise consider the analytic set $X_* := \mathbf{h}^{-1}(X) = \{z \in \Delta_* :\ g(\mathbf{h}(z)) = 0\}$, and note that $\dim(X_*) = m_* - 1 < n-1$. We use $\delta_* < 2^{-n^2}$, to get a ball B_* around the origin with $\Delta_* \subset B_* \subset B_*^{\frac{1}{2}} \subset \Delta_*^{\delta}$. Now we are in the earlier situation with $B_*^{\frac{1}{2}}, \mathbf{h}, X_*$ in the role of the $B^{\frac{1}{2}}, \mathbf{u}, \mathbf{u}^{-1}(X)$ respectively. Since $\dim(X_*) = m - 1$, we use the same process as before, but now apply Corollary 3.4 with $l = (m-1)+1 = n-1$, to cover B_* by $\text{poly}_n(\theta, \epsilon^{-1})$ many polydiscs, such that for every polydisc D_* in this collection, $\mathbf{h}(X_* \cap D_*)(\mathbb{Q}, T)$ is contained in an hypersurface of degree at most d_ν, say H_*, in $z_{\boldsymbol{j}}$ variables, where $\boldsymbol{j} := \{z_1, \dots, z_n\} \setminus \{z_W\}$. Note this gives by Fact 1 that $\dim(W \cap H_*) \leqslant n-2$, and that the complexity of $W \cap H_*$ is bounded by $d_9 = \text{poly}_n(d_2, d_\nu)$ by Bezout's theorem, see [9, subsubsection 2.2.3]. So as we follow the rest of the process for $B_*^{\frac{1}{2}}, \mathbf{h}, X_*$, using that the complexity of $\mathbf{h}$ is bounded by $d_8 = \text{poly}_n(\epsilon^{-1})$, we end up with a cellular cover of the appropriate size and complexity, but now the dimension of the cells in the covering is at most $n-2$. So we invoke induction again, and the proof is complete. $\square$

5. Proof of Theorem 1.1

Throughout we work with the conventions that $k, l, m, N \in \mathbb{N} = \{0, 1 \dots\}$, $i, j, n \in \mathbb{N}^{\geqslant 1} := \{1, 2, \dots\}$. Also $t \in \mathbb{R}$; $\delta, \epsilon \in (0,1)_{\mathbb{R}} := \{t : \ 0 < t < 1\}$; $c, T \in \mathbb{R}^{\geqslant 2} := \{t : \ t \geqslant 2\}$; $\theta, \nu, \tau, r \in \mathbb{R}^{>} := \{t : \ t > 0\}$.
Fix $X := \{z \in B : \ g(z) = 0\}$ and g non-zero complex analytic, and suppose we have that $\|g\|_{B^{1/2}} \leqslant 1$.
As before, fix $X := \{z \in B : \ g(z) = 0\}$, g non-constant complex analytic, and suppose we have that $\|g\|_{B^{1/2}} \leqslant 1$.

Throughout this section we assume that X does not contain any semialgebraic set of positive dimension. A *curve* is an algebraic variety of pure dimension 1, and $\mathcal{C}_{d,n}$ denotes the chow space of curves in $\mathbb{C}^n$ of degree d. The space $\mathcal{C}_{d,n}$ is compact with respect to the chow metric [13, Chapter 4, Theorem 1.1]; we denote this metric by $\operatorname{dist}_{\mathcal{C}_{d,n}}(\cdot,\cdot)$ and topological notions referred in the context of $\mathcal{C}_{d,n}$ are with respect to this metric. For a curve W of degree at most d, we abuse notation and let W denote also for its chow coordinates in $\mathcal{C}_{d,n}$.

For a polydisc $\Delta = \Delta_z \times \Delta_w \subset \mathbb{C}^n$, with $\dim \Delta_z = 1$ and $\dim \Delta_w = n-1$, we define the *direction* of Δ to be the slope of any line which is normal to Δ_w. Also $\pi_z : \Delta \to \Delta_z$ denotes the usual projection. Recall that Δ is a polydisc for a curve W if $W \cap (\bar{\Delta}_z \times \partial\Delta_w) = \emptyset$.

Lemma 5.1. *For every curve $W \in C_{d,n}$ and point $p \in \bar{B}$ there are polydiscs $\Delta_1, \dots, \Delta_n \subset B^{1/2}$ for W, each centered around p, with a pairwise orthogonal set of directions.*

Lemma 5.2. *Given d and n, there is a computable function $f_\delta : (0,1) \to (0,1)$ with $f_\delta(t) \to 0$ as $t \to 0$ such that the following holds. Given a curve $W \in \mathcal{C}_{d,n}$ and a polydisc Δ for W, there is $\delta_{W,\Delta}$ such that for all $\delta_1 \leqslant \delta_{W,\Delta}$, $\operatorname{dist}_{\mathcal{C}_{d,n}}(W, W') < \delta_1$ implies that*

(1) Δ is a polydisc for W', and
(2) for each $(a,b) \in W \cap \Delta$, there is $(a,b') \in W' \cap \Delta$ with $|b - b'| < f_\delta(\delta_1)$.

Resultants. Let $\Delta = \Delta_z \times \Delta_w$ be a polydisc for a curve W. The *analytic resultant* of g with respect to W, Δ is the function $g_W : \Delta_z \to \mathbb{C}$, given by

$$g_{W,\Delta}(z) \ := \prod_{(z,w) \in W \cap \Delta} g(z,w).$$

Note g_W is analytic over $\Delta_{z;W}$ by [5, Fact 13].

Lemma 5.3. *There is a covering by open balls $\mathcal{U} = \{U_1, \dots, U_m\}$ of $\mathcal{C}_{d,n}$ such that for each $l = 1, \dots, m$ there are polydiscs $\Delta_{l,1} = \Delta_{z;l,1} \times \Delta_{w;l,1}, \dots, \Delta_{l,n} = \Delta_{z;l,n} \times \Delta_{w;l,n} \subset B^{1/2}$ with pairwise orthogonal directions with the following properties.*

(1) For every $l \in \{1, \dots, m\}$, $\Delta_{l,1}, \dots, \Delta_{l,n}$ are polydiscs for each $W \in U_l$.
(2) Let $W \in \mathcal{C}_{d,n}$ and $p \in \bar{B}$. Then there is a $l \in \{1, \dots, m\}$ with $W \in U_l$ such that $p \in \Delta^4_{z;l,j} \times \Delta_{w;l,j}$ for all $j = 1, \dots, n$.
(3) There is $\tau = \tau(d, X) > 0$ such that for any $W \in \mathcal{C}_{d,n}$ we have that

$$\operatorname{dist}^{\mathcal{U}}(X, W) := \max_{l : W \in U_l} \min_j \max_{0 < a \leqslant 1/4} \min_{z \in \partial\Delta^{1/a}_{z;l,j}} |g_{W,\Delta_{l,j}}(z)| \ \geqslant \ 2^{-\tau}.$$

Proof. Fix a curve $W \in \mathcal{C}_{d,n}$, and a point $p \in \bar{B}$. Apply Lemma 5.1 for W and p to obtain polydiscs $\Delta_1 = \Delta_{z;1} \times \Delta_{w;1}, \dots, \Delta_n = \Delta_{z;n} \times \Delta_{w;n} \subset B^{1/2}$ for W, each centered around p, with a pairwise orthogonal set of directions. Then Lemma 5.2 gives δ_1, δ_2 such that for all $W' \in \mathcal{C}_{d,n}$, $\mathrm{dist}_{\mathcal{C}_{d,n}}(W, W') < \delta_1$ implies that

(1) $\Delta_1, \ldots, \Delta_n$ are polydiscs for W', and
(2) for all $j \in \{1, \ldots, n\}$ and each $(a, b) \in W \cap \Delta_j$, there is $(a, b') \in W' \cap \Delta_j$ such that $|b - b'| < \delta_2$.

Let $B_\delta^{\mathcal{C}_{d,n}}(W)$ be the set of $W' \in \mathcal{C}_{d,n}$ satisfying $\mathrm{dist}_{\mathcal{C}_{d,n}}(W, W') < \delta$ and set $\delta_1' := \delta_1/2$. By the compactness of $\bar{B}$ and $\mathcal{C}_{d,n}$ it suffices to show there is a $\tau_W > 0$ such that for all curves $W' \in B_{\delta_1'}^{\mathcal{C}_{d,n}}(W)$

$$\mathrm{dist}^W(X, W') := \min_j \max_{0<a\leqslant 1/4} \min_{z\in\partial\Delta_{z;j}^{1/a}} |g_{W',\Delta_j}(z)| > \tau_W.$$

Clearly $\mathrm{dist}^W(X, W') > 0$ for all $W' \in B_{\delta_1}^{\mathcal{C}_{d,n}}(W)$ since X is assumed to not contain any semialgebraic set. Towards a contradiction suppose there is a sequence of curves $W_1, W_2, \ldots$ in $B_{\delta_1'}^{\mathcal{C}_{d,n}}(W)$ such that $\mathrm{dist}^W(X, W_m) < 1/m$ for all $m \in \mathbb{N}^{\geqslant 1}$. Using again that $\mathcal{C}_{d,n}$ is compact we get a subsequence $W_{k_1}, W_{k_2}, \ldots$ which converges in $\mathcal{C}_{d,n}$ to some $W_0 \in \bar{B}_{\delta_1'}^{\mathcal{C}_{d,n}}(W) \subset B_{\delta_1}^{\mathcal{C}_{d,n}}(W)$. Then applying the Lemma 5.2 for W_0 and the polydiscs $\Delta_{W,1}, \ldots, \Delta_{W,n}$, we get that $\mathrm{dist}^W(X, W_0) = 0$ using the continuity of g and that $f_\delta(t) \to 0$ as $t \to 0$. We have arrived at a contradiction and the proof is complete. ☐

We arrange that $\tau(d, X)$ is a non-increasing function of d. This function $\tau(d, X)$ serves as a relative Bernstein index measure for X.

Lemma 5.4. *Let W be a curve of degree at most d and let p be a point in $W \cap B$. Then there are polydiscs $\Delta_1 = \Delta_{z;1} \times \Delta_{w;1}, \ldots, \Delta_n = \Delta_{z;n} \times \Delta_{w;n}$ for W, with a pairwise orthogonal set of directions, such that $\pi_{z;j}(p)$ is the center of $\Delta_{z;j}$ and $\|g_{W,\Delta_j}\|_{\Delta_{z;j}^{1/r}} \geqslant r^\tau$ for all $r \leqslant 1/2$ and $j = 1, \ldots, n$.*

Proof. Let W and p be given as in the statement of the lemma. Apply Lemma 5.3 to get a covering $\mathcal{U}$ of $\mathcal{C}_{d,n}$, polydiscs $\Delta_1', \ldots, \Delta_n'$ for W, and $\tau > 0$ such that $\mathrm{dist}^{\mathcal{U}}(X, W) \geqslant 2^{-\tau}$ and $p \in \Delta_{z;j}'^4 \times \Delta_{w;n}'$ for all $j = 1, \ldots, n$. For each j, set $\Delta_{z;j} := (\Delta_{z;j}'^2)_p$ and $\Delta_j := \Delta_{z;j} \times \Delta_{w;j}' \subset \Delta_j$, and hence $g_{W,\Delta_j}(z)$ is an analytic on the open disc $\Delta_{z;j}$. Fix $j \in \{1, \ldots, n\}$. The fact that $\pi_{z;j}(p) \in \Delta_{z;j}'^4$ implies that the center of $\Delta_{z;j}'$ belongs in $\Delta_{z;j}^2$, and hence

$$\mathrm{dist}^{\mathcal{U}}(X, W) \geqslant 2^{-\tau} \quad \Longrightarrow \quad \|g_{W,\Delta_j}\|_{\Delta_{z;j}^2} \geqslant 2^{-\tau}.$$

Since $\|g\|_{B^{1/2}} \leqslant 1$, we have that $\|g_{W,\Delta_j}\|_{\Delta_{z;j}} \leqslant 1$, and our desired conclusion follows by a direct application of the Hadamard three circle theorem. ☐

Theorem 5.5. *Let X and ϵ be given. Then there are constants $d = \mathrm{poly}_n(\epsilon^{-1})$ and $c = \mathrm{poly}_n\big(\epsilon^{-1}, \tau(d, X)\big)$ such that for all T we have that*

$$\#X(\mathbb{Q}, T) \leqslant cT^\epsilon.$$

Proof. Fix X and ϵ. Also fix $d = \mathrm{poly}_n(\epsilon^{-1})$ as given by Theorem 1.2 and let $\tau = \tau(d, X)$. Applying Theorem 1.2 for $\nu = 3\tau$, we get $c = \mathrm{poly}_n(\tau, \epsilon^{-1})$ such that for all T, $X(\mathbb{Q}, T)$ is covered by cT^ϵ many d-blocks in $X(T^{-3\tau})$. Note here we used that the value of d given by Theorem 1.2 does not depend on ν.

Let K be a d-block in $X(T^{-3\tau})$. We shall show that $\mathrm{diam}(K) < T^{-2}$ for all large enough T, hence it suffices to only consider the case of $\dim K = 1$. Take a curve W of degree at most d with $K \subseteq W$, and fix a point $p \in K$. Apply Lemma 5.4 to obtain polydiscs $\Delta_1 = \Delta_{z;1} \times \Delta_{w;1}, \ldots, \Delta_n = \Delta_{z;n} \times \Delta_{w;n}$ for W, with a pairwise orthogonal set of directions, such that $\pi_{z;j}(p)$ is the center of $\Delta_{z;j}$ and $\|g_{W,\Delta_j}\|_{\Delta_{z,j}^{1/r}} \geqslant r^\tau$ for all $r \leqslant 1/2$ and $j = 1, \ldots, n$.

Then $\|g\|_K \leqslant T^{-3\tau}$ implies that the $\pi_{z;j}\big(\mathrm{diam}(K)\big) \leqslant 2T^{-3}$ for all $j = 1, \ldots, n$. Since the $\pi_{z;j}$ comprise an orthogonal set of directions, we get that $\mathrm{diam}(K) < T^{-2}$ and hence $\#K(\mathbb{Q}, T) \leqslant 1$, for all sufficiently large T. This completes the proof. □

Proof of Theorem 1.2. Fix d. By Theorem 5.5 it suffices to show that if g is computable then so is $\tau(d, X)$. By the compactness of $\bar{B}$, it will suffice to work around a fixed point $p \in \bar{B}$. We run an iterative process and show that it must terminate for some $k \in \mathbb{N}$ and yield a computable upper bound for τ.

Apply Lemma 5.3 to get

(i) a set of open balls $\mathcal{U} = \{U_1, \ldots, U_m\}$ covering $\mathcal{C}_{d,n}$.

(ii) a set of n polydiscs $\Delta_{l,1} = \Delta_{z;l,1} \times \Delta_{w;l,1}, \ldots, \Delta_{l,n} = \Delta_{z;l,n} \times \Delta_{w;l,n}$ with pairwise orthogonal directions for every $l \in \{1, \ldots, m\}$, such that they are polydiscs for each $W \in U_l$ and also $p \in \Delta^4_{z;l,j} \times \Delta_{w;l,j}$ for all $j = 1, \ldots, n$.

(iii) a $\tau = \tau(d, X) > 0$ such that for any $W \in \mathcal{C}_{d,n}$ we have $\mathrm{dist}^{\mathcal{U}}(X, W) \geqslant 2^{-\tau}$.

For each $l \in \{1, \ldots, m\}$, apply Lemma 5.2 for $Y \in U_l$ and the polydiscs $\Delta_{l,1}, \ldots, \Delta_{l,n}$ to obtain δ_Y such that for all $k \in \mathbb{N}^{\geqslant 1}$, $\mathrm{dist}_{\mathcal{C}_{d,n}}(Y, Y') < \delta_Y/k$ implies that

(1) $\Delta_{l,1}, \ldots, \Delta_{l,n}$ are polydiscs for Y', and

(2) for all $j \in \{1, \ldots, n\}$ and each $(a, b) \in Y \cap \Delta_j$, there is $(a, b') \in Y' \cap \Delta_j$ such that $|b - b'| < f_\delta(\delta_Y/k)$.

Using the compactness of $\mathcal{C}_{d,n}$ we get for each $k \in \mathbb{N}^{\geqslant 1}$, $Y_1, \ldots, Y_{m_k} \in \mathcal{C}_{d,n}$ such that for all $Y' \in \mathcal{C}_{d,n}$ there is $\mu \in \{1, \ldots, m_k\}$ with $\mathrm{dist}_{\mathcal{C}_{d,n}}(Y', Y_\mu) < \delta_{Y_l}/k$. Fix a $k \in \mathbb{N}^{\geqslant 1}$. For $\mu \in \{1, \ldots, m_k\}$, $Y_\mu \in U_l$ for some $l \in \{1, \ldots, m\}$, and for $j \in \{1, \ldots, n\}$ and positive integers $i \leqslant k/4$, we sample the value of $g_{Y_\mu, \Delta_{l,j}}$ at a set of k equally spaced points on $\partial\Delta^{k/i}_{z;l,j}$ and compute $\tau_{i,j,k,l}$ as the minimum of these values. Set

$$\tau_k \ := \min_{1 \leqslant l \leqslant m} \min_{j \leqslant n} \max_{i \leqslant k/4} \tau_{i,j,k,l}.$$

Then using property (2) above and the continuity of g we compute δ_k so that $\mathrm{dist}(X, Y') \geqslant \tau_k - \delta_k$ for all $Y' \in \mathcal{C}_{d,n}$. We can compute δ_k since g is computable and f_δ in Lemma 5.2 is a computable function. Lemma 5.2 also implies that $\delta_k \to 0$ as $k \to \infty$ and since we have $\tau > 0$ satisfying (iii) above, there is k for which $\tau_k - \delta_k > 0$ and the proof is complete. □

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