GROTHENDIECK RINGS OF ORDERED SUBGROUPS OF $\mathbb{Q}$

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ABSTRACT. Let G be a proper subgroup of $\mathbb{Q}$ and S_G be the set of primes p such that $\frac{a}{p^n} \in G$ for all $a \in G, n \in \mathbb{N}$. We show that the model-theoretic Grothendieck ring of the ordered abelian group $(G; +, <)$ is a quotient of $(\mathbb{Z}/q\mathbb{Z})[T]/(T + T^2)$, where q is the largest odd integer that divides $p - 1$ for all $p \notin S_G$.

This implies that the Grothendieck ring of $(G; +, <)$ is trivial in various salient cases, for example when S_G is finite, or when S_G does not contain some prime of the form $2^n + 1$, $n \in \mathbb{N}$.

1. Introduction

The model-theoretic notion of a Grothendieck ring was introduced in [12, 8] and captures interesting combinatorial and geometric properties of sets definable in a first-order structure. In this paper, we focus on computing the Grothendieck rings of subgroups of $\mathbb{Q}$ in the language of ordered abelian groups. The edge cases are known: the Grothendieck ring of $(\mathbb{Z}; +, <)$ is trivial [12], and for the o-minimal structure $(\mathbb{Q}; +, <)$ this ring is isomorphic to $\mathbb{Z}[T]/(T^2 + T)$ [11].

Grothendieck ring of a structure. For an $\mathcal{L}$-structure $\mathcal{M}$, we denote by $\mathrm{Def}(\mathcal{M})$ the family of definable (with parameters) subsets of $\mathcal{M}$. The *Grothendieck group* of $\mathcal{M}$ is the abelian group generated by symbols $[X]$, $X \in \mathrm{Def}(\mathcal{M})$, with the following relations:

- $[X] = [Y]$ when X and Y are definably isomorphic,
- $[X \cup Y] = [X] + [Y]$ when $X \cap Y = \emptyset$.

Extending this group by multiplication defined by $[X][Y] = [X \times Y]$ we obtain the *Grothendieck ring* of $\mathcal{M}$; denoted by $K_0(\mathcal{M})$. We will refer to $[X]$ as the *value* of the definable set X in $K_0(\mathcal{M})$.

In this paper, p always denotes a prime integer and G is a proper subgroup of $\mathbb{Q}$. We let S_G denote the set of p such that $\frac{a}{p^n} \in G$ for all $a \in G, n \in \mathbb{N}$. Note that since G is a proper subgroup of $\mathbb{Q}$, S_G is a proper subset of the set of integer primes, and we say that p is *fully inverted in* G if $p \in S_G$.

Theorem 1.1. *The Grothendieck ring of the ordered abelian group $(G; +, <)$ is a quotient of $(\mathbb{Z}/q\mathbb{Z})[T]/(T + T^2)$, where q is the largest odd integer which divides $p - 1$ for all $p \notin S_G$.*

We prove this theorem in Section 3 although the triviality of the Grothendieck ring for various cases of S is already given in Cor 2.6.

2020 *Mathematics Subject Classification.* Primary 03C07, 03C64, 06F20.

Key words and phrases. Archimedean ordered abelian groups, Grothendieck ring.

[†]Bhardwaj was partially funded by KU Leuven IF C16/23/010.

Grothendieck rings often play a crucial role in motivic integration [10, 6, 14], and the structures of ordered abelian groups are especially relevant in this context. Apart from determining the Grothendieck ring, it is also an important theme to study definable sets up to definable bijection, sometimes even up to some special definable bijections, depending on the context. This leads to some finer classification results, see for example [1, 2, 5], but we do not address this finer challenge in this paper.

There has been much work on Grothendieck rings of various structures in the language of rings. It follows easily that the Grothendieck ring of real closed fields is given by $\mathbb{Z}$ [12], and in contrast the same paper shows that the Grothendieck ring of the complex field embeds a polynomial ring over $\mathbb{Z}$ generated by continuum many indeterminates. Cluckers and Haskell showed in [7] that the field of p-adic numbers is trivial. Moreover, the Grothendieck ring of formal Laurent series over p-adic numbers and formal Laurent series over local fields of strictly positive characteristic are also trivial by work of Cluckers [3]. See also [13] for another exact computation of the Grothendieck ring of a first-order structure.

Acknowledgements. Our foremost thanks goes to Pierre Touchard for suggesting this research direction for the second author's Masters dissertation topic. We also express our gratitude to Raf Cluckers for his support and guidance throughout this project. In addition, we are grateful to Pierre Touchard, Mathias Stout, and Floris Vermeulen for helpful discussions; indeed we attribute the idea for Lemma 2.5 to these discussions.

2. Grothendieck ring values of unary definable sets

Let G be a proper subgroup of $\mathbb{Q}$. As defined in the introduction, S_G is the set fully inverted primes in G. For every positive integer n, we let n_{S_G} denote the positive integer that divides n, such that only primes in S_G divide n_{S_G}, and no prime in S_G divides n/n_{S_G}. Also for $n/n_{S_G} = p_{i_1}^{e_1} \cdots p_{i_m}^{e_m}$, we set $n_{K_G} := p_{i_1}^{k_{i_1}} \cdots p_{i_m}^{k_{i_m}}$, the product of the prime factors of n/n_{S_G} raised to the power up to which they are invertible in G. This means that if a is the smallest positive integer in G, then $a/n_{K_G} \in G$, and if p divides n_{K_G}, then $a/(p \cdot n_K) \notin G$.

For any $a, b \in \mathbb{Z}$ with $\gcd(a, b) = 1$, it is easy to see that the Grothendieck ring of G and $\frac{a}{b}G$ are the same. Thus it suffices to only consider subgroups of $\mathbb{Q}$ which contain $\mathbb{Z}$. Furthermore, we only need to consider the case where G is dense in $\mathbb{Q}$ with respect to the order topology. Since otherwise there is $a/b \in \mathbb{Q}$ with $\gcd(a, b) = 1$ such that $\frac{a}{b}G = \mathbb{Z}$ and the Grothendieck ring of $(G; +, <)$ is trivial. Hence we assume going forward that

$$\mathbb{Z} \subset G \subset \mathbb{Q} \qquad \text{and} \qquad G \text{ is dense in } \mathbb{Q}.$$

2.1. **Values of intervals.** In the rest of this section, we compute certain values realized by sets of the form $(a, b) \cap G$, where $a, b \in \mathbb{Q}$ with $a < b$. We call such sets $\mathbb{Q}$-*intervals* in G and when the context is clear, we abuse notation and use (a, b) to denote the $\mathbb{Q}$-interval $(a, b) \cap G$.

These sets are clearly definable in $(G; +, <)$, and we show in Proposition 3.1 that the Grothendieck ring of $(G; +, <)$ is in fact generated by the values of $\mathbb{Q}$-intervals. Hence one major focus of this paper is to study these values, and we adapt the results of [11, Claim 7] and employ various number theoretic tricks for this purpose.

Lemma 2.1. *The following hold for bounded $\mathbb{Q}$-intervals in G.*

(i) $[(a,b)] = -1$ *if* $a, b \in G$.
(ii) $[(a,b)] = -\frac{1}{2}$ *for some* $a \in G$ *and* $b \in \mathbb{Q}\backslash G$.

Proof. We first prove (i). For $a, b \in M$ we have that

$$[(a,\infty)] \;=\; [(a,b)] + [\{b\}] + [(b,\infty)].$$

Since $b - a \in M$, the map $x \to x + b - a$ gives a definable isomorphism from (a,∞) to (b,∞). It follows that $[(a,b)] = -1$ as desired, since the value of a singleton is 1.

For (ii), note that not all primes are fully inverted, and take p to be the smallest prime not in S_G. Suppose $k \in \mathbb{N}$ such that $1/p^k \in G$ and $1/p^{k+1} \notin G$. We get that

$$\begin{aligned}
[(0,\frac{1}{p^k})] &= [(0,\frac{1}{p^{k+1}})] + [(\frac{1}{p^{k+1}},\frac{1}{p^k})] \\
&= [(0,\frac{1}{p^{k+1}})] + [(0,\frac{p-1}{p^{k+1}})] \\
&= 2 \cdot [(0,\frac{1}{p^{k+1}})]
\end{aligned}$$

Here we use that $x \to \frac{1}{p^k} - x$ and $x \to x/(p-1)$ are definable isomorphisms. Note by the choice of p, all prime factors of $p-1$ are in S_G. By (i), we know that $[(0,\frac{1}{p^k})] = -1$ and hence $[(0,\frac{1}{p^{k+1}})] = -\frac{1}{2}$. □

Lemma 2.2. *Set* $X := [(0,+\infty)]$. *The following hold for unbounded $\mathbb{Q}$-intervals in G.*

(i) $[(a,+\infty)] = [(-\infty,a)] = X$ *for any* $a \in G$.
(ii) $[(-\infty,+\infty)] = 2X + 1$.
(iii) $[(c,+\infty)] = [(-\infty,c)] = X + \frac{1}{2}$ *for some* $c \in \mathbb{Q}\backslash G$.

Proof. The assertions in (i), (ii) are immediate, and then (iii) follows by a direct application of Lemma 2.1(ii). □

Lemma 2.3. *In the Grothendieck ring of* $(G;+,<)$ *we have that* $X^2 + X = 0$.

Proof. For $I := (0,\infty) \cap G$, we set

$$(f,g)_I := \{(x,y) \in I \times G : \; f(x) < y < g(x)\},$$

where f and g are definable functions in $(G;+,<)$, or $f, g \in \{-\infty,+\infty\}$, with $f(x) < g(x)$ for all $x \in I$. We split I^2 into three disjoint parts and obtain

$$[I^2] = [(0,\mathrm{Id}_I)_I] + [\Gamma(\mathrm{Id}_I)] + [(\mathrm{Id}_I,\infty)_I], \tag{1}$$

where Id_I is the identity map on I and $\Gamma(\mathrm{Id}_I)$ denotes its graph. We work with three definable isomorphisms given as follows.

$$\begin{gathered}
f : (0,\mathrm{Id}_I)_I \to (\mathrm{Id}_I,\infty)_I : (x,y) \to (y,x) \\
g : I^2 \to (\mathrm{Id}_I,\infty)_I : (x,y) \to (x,x+y) \\
h : \Gamma(\mathrm{Id}_I) \to I : (x,x) \to x
\end{gathered}$$

Hence we may rewrite (1) as

$$[I^2] = [I^2] + [I] + [I^2],$$

and it follows that $X^2 + X = 0$ as claimed. □

2.2. **Torsion from non-fully inverted primes.** We show that the primes not in S_G induce a certain precise torsion in the Grothendieck ring of $(G;+,<)$. We shall need the following simple fact.

Fact 2.4. *Suppose $n \geqslant 2$ such that $n_{S_G} = 1$. Then for all $y \in G$, n divides exactly one of $y, y + \frac{1}{n_{K_G}}, \dots, y + \frac{n-1}{n_{K_G}}$ in G.*

Proof. Let $n \geqslant 2$ such that $n_{S_G} = 1$ and $y = \frac{a}{b} \in G$, with a and b coprime, be given. Note that n divides n_{K_G} and we can write $b = u \cdot v$ with $a_j \geqslant 0$ such that u divides n_{K_G} and $\gcd(v, n) = 1$. We observe that

$$\frac{a}{b} + \frac{i}{n_{K_G}} \;=\; \frac{a \cdot n_{K_G}/u + vi}{n_{K_G} v}.$$

Since $\gcd(v, n) = 1$, there is unique $i \in \{0, \dots n-1\}$ such that

$$a \cdot n_{K_G}/u + vi \;=\; 0 \mod n,$$

and the proof is complete. □

Lemma 2.5. *Consider a prime number p such that $p \notin S_G$ and let $k \geqslant 0$ be such that $1/p^k \in G$ and $1/p^{k+1} \notin G$. Then in the Grothendieck ring of $(G;+,<)$ we have that $q = 0$ where q is the largest odd factor of $p-1$.*

Proof. The idea is to partition $(-\infty, +\infty) \cap G$ into equivalence classes modulo p. For $i = 0, \dots, p-1$, set $N_i \coloneqq \{x \in G : \; p \mid (x + \frac{i}{p^k})\}$. Then by Lemma 2.4 we have

$$2X + 1 \;=\; [(-\infty, +\infty)] \;=\; \sum_{i=0}^{p-1} [N_i].$$

For $i = 0, \dots, p-1$ we have a definable bijection $N_i \to G$ given by $x \to \frac{x + \frac{i}{p^k}}{p}$. This implies that

$$2X + 1 \;=\; p \cdot (2X + 1)$$

and hence $(p-1)(2X+1) = 0$. Multiplying both sides by $2X+1$ and employing Lemma 2.3 we get that $p - 1 = 0$. Since 2 is invertible in the Grothendieck ring of $(G;+,<)$ by Lemma 2.1(ii), the desired conclusion follows. □

Corollary 2.6. *The Grothendieck ring of the ordered abelian group $(G;+,<)$ is trivial if no odd prime divides $p-1$ for every prime $p \notin S_G$.*

Corollary 2.7. *The Grothendieck ring of M is trivial in the following cases:*

- *S contains only finitely many primes.*
- *S does not contain some prime of form $2^n + 1$, $n \in \mathbb{N}$. In other words, S does not contain 2 or a Fermat prime.*

Proof. The second assertion is immediate. For the first assertion, note that for any odd integer $q > 1$, $qm + 2$ contains infinitely many primes p by the Dirichlet prime number theorem and for any such prime p, we have that such that $q \nmid (p-1)$. □

2.3. **Values of intervals revisited.** Our next result shows that the ring generated by the Grothendieck ring values of unary definable sets in $(G;+,<)$ is a quotient of $(\mathbb{Z}/q\mathbb{Z})[T]/(T+T^2)$, where q is the largest odd integer which divides $p-1$ for all $p \notin S_G$.

Lemma 2.8. *Suppose there is an odd prime q which divides $p-1$ for all primes $p \notin S_G$. Then the following holds for $\mathbb{Q}$-intervals in G.*

(i) $[(a,b)] = -\frac{1}{2}$ *if $a \in G$ and $b \in \mathbb{Q}\backslash G$ or vice versa.*
(ii) $[(a,b)] = 0$ *if $a, b \in \mathbb{Q}\backslash G$.*
(iii) $[(a,+\infty)] = [(-\infty,a)] = X + \frac{1}{2}$ *if $a \in \mathbb{Q}\backslash G$.*

Proof. We begin with the proof of (i). It suffices to consider the case of $a \in M$ and $b \in \mathbb{Q} \setminus M$.

By translating suitably we may assume that $a = 0$ and $b = m/n \notin \mathbb{Z}_S$ such that $b < 1$ and $m, n \in \mathbb{N}$ with $\gcd(m,n) = 1$. Since multiplying and dividing by elements of S_G are definable isomorphisms in $(G;+,<)$, we may assume without loss of generality that all prime factors of m and n are not in S_G. For $b < 1/2$, we have that

$$\begin{aligned}[(0,1)] &= [(0,\frac{m}{n})] + [(\frac{m}{n},\frac{n-m}{n})] + [(\frac{n-m}{n},1)] \\ &= 2[(0,\frac{m}{n})] + [(\frac{m}{n},\frac{n-m}{n})].\end{aligned}$$

We will show that $[(\frac{m}{n},\frac{n-m}{n})] = 0$. It then follows that $[(0,\frac{m}{n})] = -1/2$ as desired for $\frac{m}{n} < 1/2$. From this, we also obtain the case $b > 1/2$ as follows

$$[(0,\frac{m}{n})] = [(0,\frac{n-m}{n})] + [(\frac{m-n}{n},\frac{m}{n})] = [(0,\frac{n-m}{n})] = -1/2.$$

Note that our assumptions imply that $2 \in S_G$ and hence $\frac{m}{n} \neq \frac{n-m}{n}$. Suppose $\frac{m}{n} < 1/2$. Since G is dense in $\mathbb{Q}$, we have $c \in G$ such that $\frac{m}{n} < c < \frac{n-m}{n}$. Using definable isomorphisms we get that

$$\begin{aligned}[(\frac{m}{n},\frac{n-m}{n})] &= [(\frac{m}{n},c)] + [\{c\}] + [(c,\frac{n-m}{n})] \\ &= [(\frac{m}{n}-c,0)] + [(0,\frac{n-m}{n}-c)] + 1\end{aligned} \tag{2}$$

Next we aim to find $Q \in S_G$ such that n divides $Q+1$ in the integers. Since we can then translate the interval in the first summand of equation (2) by the constant $T = c + Q(1-c) - \frac{(Q+1)m}{n} \in G$ and multiply the second summand of equation (2) by Q respectively to get our desired claim as follows.

$$\begin{aligned}[(\frac{m}{n},\frac{n-m}{n})] &= [(Q(1-c) - Q\frac{m}{n}, T)] + [(0, Q\big(\frac{n-m}{n}-c\big)] + 1 \\ &= [Q\big(\frac{n-m}{n}-c\big), T)] + [(0, Q\big(\frac{n-m}{n}-c\big)] + 1 \\ &= [(0,T)] + 1 \\ &= 0\end{aligned}$$

Here we used Lemma 2.1(i). By our assumption, we have an odd prime q which divides $p-1$ for all $p \notin S_G$, and we solve for prime Q satisfying

$$Q \equiv -1 \mod n \qquad \text{and} \qquad Q \equiv 2 \mod q.$$

Note that $q \in S_G$ and hence we have that q is coprime to n as well. Then by the Chinese remainder theorem and Dirichlet's theorem on primes in arithmetic progressions we have that there is a prime Q satisfying the displayed divisibility conditions. Note such a Q necessarily belongs in S_G since q does not divide $Q-1$. This finishes the proof for (i).

For (ii), we start with $c, d \in \mathbb{Q}\backslash G$, and take $a, b \in G$ with $a < c < d < b$. Then we have that

$$-1 = [(a,b)] = [(a,c)] + [(c,d)] + [(d,b)] = -\frac{1}{2} + [(c,d)] - \frac{1}{2}.$$

It follows that $[(c,d)] = 0$.

Assertion (iii) follows by a direct application of (i) and Lemma 2.2(i). □

3. Grothendieck ring of $(G;+,<)$ is generated by values of unary definable sets

It is well-known that the ordered group of integers admits quantifier elimination in the Presburger language – $\big(0,1,+,<,(P_n)_{n\in\mathbb{N}^{\geqslant 1}}\big)$, where for every $n \in \mathbb{N}^{\geqslant 1}$, P_n is unary set given by $a \in P_n$ if and only if $\exists x(a = nx)$. In fact any ordered abelian group that does not have a convex definable subgroup also admits quantifier elimination in this language [4]; in this general setting 1 is defined to be the minimal positive element if such an element exists, and $1 = 0$ otherwise.

So we have that the structure $\big(G;0,+,<,(P_n)_{n\in\mathbb{N}^{\geqslant 1}}\big)$ has quantifier elimination, and we use this fact to transfer certain aspects of cell decomposition from $(\mathbb{Q};+,<)$ to $(G;+,<)$. See [9, Chapter 3] for details about o-minimal cells and o-minimal cell decomposition.

Consider a definable subset $A \subset G^n$. Recall that Fact 2.4 gives that $\neg P_n(x)$ is equivalent to $\bigvee_{i=1}^{n-1} P_n(x + \frac{i}{n_{K_G}})$ for all $n \in \mathbb{N}^{\geqslant 2}$ with $n_{S_G} = 1$. Divisibility conditions for primes in S_G can be dismissed since every element in G is infinitely many times divisible by these primes, therefore the cases we discussed cover all possible kinds of divisibility conditions.

Using this together with quantifier elimination and usual arguments with the Presburger language we obtain $L \in \mathbb{N}^{\geqslant 1}$, which is coprime to all primes in S_G, such that A is a disjoint union of sets defined by formulas $\phi^A_{(l_1,\ldots,l_n)}$ of the following form.

$$\bigvee \Big(\big(\bigwedge_i (\sum_j m_{i_j} x_j = g_i)\big) \wedge \big(\bigwedge_j P_L(x_j - l_j)\big) \wedge \big(\bigwedge_i (c_i < \sum_j e_{i_j} x_j < d_i)\big)\Big),$$

where $l_1, \ldots, l_n \in \{0, \ldots, \frac{L-1}{L_{K_G}}\}$ and $m_{i_j}, e_{i_j} \in \mathbb{Z}$ and $g_i, c_i, d_i \in G$.

Clearly the map $x_j \to (Lz_j + l_j)$ gives a definable isomorphism from the set defined by $\phi^A_{(l_1,\ldots,l_n)}$, to the set defined by the following formula

$$\bigvee \Big(\big(\bigwedge_i (\sum_j m_{i_j}(Lz_j + l_j) = g_i)\big) \wedge \big(\bigwedge_i (c_i < \sum_j e_{i_j}(Lz_j + i_j) < d_i)\big)\Big).$$

This formula has no divisibility conditions and hence the set it defines in $\mathbb{Q}^n$ can be partitioned into cells. This implies that in the Grothendieck ring of $(G;+,<)$ we have that

$$[A] = \sum_{(l_1,\ldots,l_n)} A_{(l_1,\ldots,l_n)} = \sum_i [C_i \cap G^n],$$

where C_i are cells in $(\mathbb{Q};+,<)$.

Quasi-cells. For C a cell in $\mathbb{Q}^n$, we say that $C \cap G^n$ is a *quasi-cell*. Clearly the quasi-cells are definable in G. Our argument above shows that the Grothendieck ring of $(G;+,<)$ is generated by the values of quasi-cells. As advertised earlier, something much stronger is true.

Proposition 3.1. *The Grothendieck ring of* $(G;+,<)$ *is generated by the values of unary quasi-cells.*

Proof. It suffices to show that the value of all quasi-cells can be expressed as a linear combination of values of unary quasi-cells. Given a quasi-cell $C \cap G^n$, we proceed by induction on n. The base-cases 1 is immediate.

Take a quasi-cell $C' = C \cap G^{n+1} \subseteq M^{n+1}$, where C is a cell in $\mathbb{Q}^{n+1}$. We first consider the case where C is given as $\Gamma(f)$ where f is a definable continuous function on a cell $A \subseteq \mathbb{Q}^n$. Let $A' = A \cap G^n$ and we have that

$$C' \;=\; \{(x,y) \in A' \times G : \; \sum_{i=1}^{n} a_i x_i + b = my\},$$

for some $a_i, b, m \in \mathbb{Z}$. The function $y = \frac{\sum_{i=1}^n a_i x_i + b}{m}$ is only definable on the elements of A divisible by m. Let $m_T := m/m_{S_G}$ and observe that an element of G is divisible by m if and only if it is divisible by m_T. The projection map onto the first n coordinates π gives a definable isomorphism from C to $\pi(C) = A' \cap P_{m_T}(\sum_{i=1}^n a_i x_i + b)$ and we are done by induction.

The other case it that

$$C \;=\; (f,g)_A \;:=\; \{(x,t) : \; x \in A \text{ and } f(x) < t < g(x)\},$$

where A' is a cell in $\mathbb{Q}^n$, and

$$f, g \in \{\text{continuous functions definable in } (\mathbb{Q};+,<) \text{ with domain } A\} \cup \{-\infty, \infty\},$$

with $f(x) < g(x)$ for all $x \in A$. Set $A' = A \cap G^n$ as before.

If $f = -\infty$ and $g = +\infty$, then $C' = A' \times G$ and $[C'] = [A'] \cdot [G]$. We now proceed to the non-trivial case of $f \neq -\infty$ and $g = +\infty$. We have that

$$C' = \{(x,y) \in A' \times G : \sum_{i=1}^{n} a_i x_i + b < my\},$$

for some $a_i, b, m \in \mathbb{Z}$. Let $m_T := m/m_{S_G}$ as before and observe that

$$[C'] \;=\; \sum_{j=0}^{m_T-1} [\{(x,y) \in A' \times G : \; \sum_{i=1}^{n} a_i x_i + b < my\} \cap P_{m_T}(\sum_{i=1}^{n} a_i x_i + b + \frac{j}{m_K})] \quad (3)$$

For each $j \in \{0, \ldots, m_T - 1\}$, we set $A'_j := A' \cap P_{m_T}(\sum_{i=1}^n a_i x_i + b + \frac{j}{m_K})$ and $C'_j := \{(x,y) \in A'_j \times G : \; \sum_{i=1}^n a_i x_i + b < my\}$, and observe that

$$A'_j \times (\frac{-j}{m \cdot m_K}, +\infty) \to C'_j \;:\; (x,t) \to (x, \frac{a_i x_i + b + \frac{j}{m_K}}{m} + t)$$

is a definable bijection in $(G;+,<)$. Hence we have that

$$[C'] \;=\; \sum_{j=0}^{m_T-1} [C'_j] \;=\; \sum_{j=0}^{m_T-1} [A'_j] \cdot [(\frac{-j}{m \cdot m_K}, +\infty)],$$

and we are done by induction. The case of $f = -\infty$ and $g \neq \infty$ is similar.

Finally, suppose $f \neq -\infty$ and $g \neq \infty$. There is a cell $A \subseteq \mathbb{Q}^n$ such that

$$[C'] = [(-\infty, g)_A \cap G^{n+1}] - [\Gamma(f) \cap G^{n+1}] - [(-\infty, f)_A \cap G^{n+1}],$$

and we employ the the previous cases and induction to conclude the proof. $\square$

Proof of Theorem 1.1. If there is no odd prime which divides $p-1$ for all $p \notin S$, then the Grothendieck ring of $(M; +, <)$ is trivial by Theorem 2.6. In the remaining case, by proposition 3.1, we only need to consider the values realized by $\mathbb{Q}$-intervals. Then the desired result follows immediately by Lemmas 2.1, 2.2, 2.3, 2.5, and 2.8.

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