

# EFFECTIVE AND APPROXIMATE PILA-WILKIE TYPE COUNTING WITH COMPLEX-ANALYTIC SETS

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ABSTRACT. We establish an effective point-counting statement of Pila-Wilkie type for the zero-set of any computable complex-analytic function over a ball. For our proof, we develop an approximate counting result to cover rational points *close* to an analytic subset of a ball. This second theorem is unique among results of Pila-Wilkie type in that it applies uniformly to any analytic subset of a ball.

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## 1. INTRODUCTION

In [25], Pila and Wilkie give a subpolynomial upper bound, in terms of their heights, on the number of rational points inside the transcendental part of a set definable in an o-minimal expansion of the real field. This point counting theorem applies to a wide range of sets and functions of fundamental interest, and has had striking applications to arithmetic geometry, functional transcendence, and Hodge theory. See [2] for a simplified proof and self contained treatment of the Pila-Wilkie theorem, and [27, 1, 20, 23] for a detailed description of the aforementioned applications.

A salient instance of such an application is the instrumental role played by counting of Pila-Wilkie type in various parts of the Pila-Zannier strategy employed for a full resolution of the André-Oort conjecture [24]. In particular, this proof used that a sharpening of the Pila-Wilkie theorem due to the second author [5] for certain sets defined using leaves of foliations over number fields can be used to give lower bounds for Galois orbits of special points on Shimura varieties [11]. In this paper, we establish various effective Pila-Wilkie type statements and use many ideas from [5] for this purpose.

Although the Pila-Wilkie theorem cannot be made effective in general, various sharp and effective versions have been realized for sets cut out by functions satisfying a certain form of differential equations [10, 6, 3]. The statement in [3] due

to the second author effectivizes various applications of the Pila-Wilkie theorem to arithmetic geometry and Hodge theory.

Our first theorem is an effective Pila-Wilkie theorem for the zero-set of any *computable*<sup>1</sup> complex analytic function over a ball. We work with the height function  $\mathcal{H} : \mathbb{Q} \rightarrow \mathbb{R}$  by  $\mathcal{H}(\frac{a}{b}) := \max(|a|, |b|) \in \mathbb{N}^{\geq 1}$  for coprime  $a, b \in \mathbb{Z}$ ,  $b \neq 0$ . By convention  $\mathcal{H}(0) = 1$ , and for  $a = (a_1, \dots, a_n) \in \mathbb{Q}^n$ , we extend  $\mathcal{H}$  to be given by  $\mathcal{H}(a) := \max\{\mathcal{H}(a_1), \dots, \mathcal{H}(a_n)\}$ . For  $T \in \mathbb{R}^{\geq 2}$  and  $X \subseteq \mathbb{C}^n$ , we set  $X(\mathbb{Q}, T) := \{a \in X \cap \mathbb{Q}^n : \mathcal{H}(a) \leq T\}$ , the (finite) set of rational points of  $X$  of height at most  $T$ .

**Theorem 1.1.** *Let  $X := \{z \in B : g(z) = 0\}$ , where  $B \subseteq \mathbb{C}^n$  is a ball and  $g$  is analytic on  $\bar{B}^{1/2}$ . If the function  $g$  is computable and  $X$  does not contain any semialgebraic curve, then for every  $\epsilon > 0$  we can compute  $c = c(g, \epsilon) > 0$  such that for all  $T$*

$$\#X(\mathbb{Q}, T) \leq cT^\epsilon.$$

For any ball,  $B^{1/2}$  denotes the ball with the same center as  $B$  and twice the radius. Throughout the rest of the paper we fix  $B \subset \mathbb{C}^n$  to denote the ball of radius  $1/2$  around the origin. We prove Theorem 1.1 in Section 6 for this fixed  $B$  which clearly implies the general statement above. Note the  $T^\epsilon$  asymptotic in Theorem 1.1 is optimal due to [22, Example 7.5].

Our proof of Theorem 1.1 relies on certain approximate Pila-Wilkie type counting technology. Let  $X = \{z \in B : g_1(z) = \dots = g_m(z) = 0\}$ , where the  $g_1, \dots, g_m$  are complex-analytic functions on  $\bar{B}^{1/2}$  with  $\|g_i\|_{\bar{B}^{1/2}} \leq 1$ . We call such an  $X$  an *analytic set* and set

$$X(\omega) := \{z \in B : |g_1(z)|, \dots, |g_m(z)| \leq \omega\},$$

which we call the  $\omega$ -*lemniscate* of  $X$ .

We are now ready to state our second main theorem whose form parallels the more intricate block version of the Pila-Wilkie theorem. This statement is unique among Pila-Wilkie type counting results in that the counting parameters, in particular the constant  $c$ , do not depend on the analytic set  $X$ .

**Theorem 1.2.** *For every  $\epsilon$  and  $\nu$ , there are  $\theta, c = \text{poly}_n(\nu, \epsilon^{-1})$ , and  $d = \text{poly}_n(\epsilon^{-1})$  such that for every analytic set  $X$  and all  $T$ ,  $X(T^{-\theta})(\mathbb{Q}, T)$  is contained in at most  $cT^\epsilon$  many  $d$ -blocks in  $X(T^{-\nu})$ .*

A  $d$ -block in  $X(\omega)$  is a holomorphic image of a polydisc inside the set  $X(\omega)$ , where the graph of each co-ordinate of this map is an analytic component of an algebraic variety of degree at most  $d$ . This notion is defined in more detail in Section 5, and the  $\text{poly}_n(\cdot)$  notation is defined below. Although we state and prove this theorem for  $B$  fixed as above, a similar statement for analytic sets in any ball is an immediate consequence.

The fact that Theorem 1.2 gives a  $d$  which depends only on  $\epsilon$  plays a critical role in our proof of Theorem 1.1. To realize this technical feature we prove and employ a Pila-Wilkie statement for algebraic varieties – Theorem 3.6. This theorem is of independent interest and says that for any  $\epsilon$  and algebraic variety  $W$ , the rational points of height at most  $T$  on  $W$  lie on  $O(T^\epsilon)$  many algebraic varieties of  $W$  whose degree depend only on  $\epsilon$  and not on the degree of  $W$ . We also record a variant

<sup>1</sup>see Section 2 for the definition of a computable complex analytic function.

where these rational points can be covered by algebraic varieties whose number and degrees grow polynomially in  $\log(T)$ . We also include a similar logarithmic growth variant for Theorem 1.2 as well– Theorem 5.3. These  $\log(T)$ -variants carry independent interest and are not used for the proof of Theorem 1.1.

**Notations and conventions.** Throughout we work with the conventions that  $k, l, m, N \in \mathbb{N} = \{0, 1, \dots\}$ ,  $\mu, d, e, i, j, n \in \mathbb{N}^{\geq 1} := \{1, 2, \dots\}$ . For  $\alpha, \sigma \in \mathbb{N}^l$ , we set  $|\sigma| := \sigma_1 + \dots + \sigma_l \in \mathbb{N}$ . Also  $t \in \mathbb{R}$ ;  $\beta, \gamma, \delta, \epsilon, \rho, \omega, C \in (0, 1)_{\mathbb{R}} := \{t : 0 < t < 1\}$ ;  $\eta, c, T \in \mathbb{R}^{\geq 2} := \{t : t \geq 2\}$ ;  $\theta, \nu, r, M \in \mathbb{R}^> := \{t : t > 0\}$ . We follow the natural convention that any symbol with an added subscript will denote an object in the same domain as the plain symbol, so  $d_\nu \in \mathbb{N}^{\geq 1}$ ,  $\eta_1 \in \mathbb{R}^{\geq 2}$ , and so on. We will denote Euler’s constant by  $e$ .

For a ball or polydisc  $A \subset \mathbb{C}^n$  with center  $p \in \mathbb{C}^n$ , set  $A^\delta := p + \delta^{-1}(A - p)$ . For a point  $p \in \mathbb{C}^n$  and a ball/polydisc  $A$  centered at the origin,  $A_p$  denotes the ball/polydisc with the same radius/polyradius as  $A$  but centered at  $p$ .

For any domain  $\Omega \subseteq \mathbb{C}^n$  bounded domain, we let  $\mathcal{O}(\Omega)$  denote the set of complex-valued analytic functions on  $\Omega$ . We shall work with complex cells and cellular maps and related technology from [9]. For a complex cell  $\mathcal{C} \subset \mathbb{C}^n$ ,  $\mathcal{O}(\mathcal{C})$  will denote the set of cellular maps with domain  $\mathcal{C}$ , and we use this notation so that no confusion may arise. For a function  $h \in \mathcal{O}(\Omega)$  and a ball or polydisc  $A \subseteq \Omega$ , we set  $\|h\|_A := \max_{z \in \bar{A}} |h(z)|$ .

The notation  $s = \text{poly}_n(t_1, t_2, t_3)$  means that  $s \leq P_n(t_1, t_2, t_3)$ , where  $n \rightarrow P_n$  is a universally fixed mapping with  $P_n$  being polynomials with positive coefficients.

## 2. THE 0-DIMENSIONAL CASE FOR THEOREM 1.1

We begin by defining our notion of computability for complex-valued functions.

**Definition 2.1.** A complex-analytic function  $f : \bar{B}^{1/2} \rightarrow \mathbb{C}$  is computable if

- (1) There is an algorithm, that given any computable  $z \in \bar{B}^{1/2}$  and  $n \in \mathbb{N}^{\geq 1}$ , gives  $q \in \mathbb{Q}$  such that  $|f(z) - q| < 2^{-n}$ .
- (2) there is a computable function  $h_f : \mathbb{N} \rightarrow \mathbb{N}$ , such that  $|z - w| < 1/h_f(n)$  implies  $|f(z) - f(w)| < 1/n$ .

We will use the following implication of [19, Lemma 1, Example 1] to get an upper bound for the size of the finite set  $X = \{z \in D^2 : g(z) = 0\}$ .

**Lemma 2.2.** Let  $f : D \rightarrow \mathbb{C}$  be analytic function on a disc. Then we have that

$$\#\{z \in D^2 : g(z) = 0\} < 5 \cdot \log(\|g\|_D / \|g\|_D^2), \quad \text{and,}$$

$$\#\{z \in D^\eta : g(z) = 0\} < (1/\log(\eta/2)) \cdot \log(\|g\|_D / \|g\|_D^\eta) \quad \text{for } \eta > 2.$$

*Proof of Theorem 1.1, Case  $n = 1$ :* Let  $g : D \rightarrow \mathbb{C}$  be a non-constant computable analytic function, where  $D \subset \mathbb{C}$  is a disc around the origin. By Lemma 2.2 it suffices to compute a number  $\tau$  such that  $\tau > \|g\|_D / \|g\|_{D^2}$ .

Starting with  $k = 1$  and iterating over  $k \in \mathbb{N}^{\geq 1}$ , we obtain a finite set  $S_1$  of computable numbers in  $\partial D$  containing such that any point of  $\partial D$  is at a distance of less than  $1/h_g(k)$  from some point of  $S_1$ . We compute a subset  $S_2$  for  $\partial D^2$  similarly; here  $h_g$  is the computable functions given by Definition 2.1. Using Definition 2.1(1), we get non-negative rational numbers  $m$  and  $M$  such that  $\|g\|_D < M + 2/k$  and

$m - 2/k < \|g\|_{D^2} < m + 2/k$ . Since  $g$  is not identically zero, there is a  $k$  for which  $m - 2/k \geq 0$ , and for this  $k$  we have that

$$\|g\|_D / \|g\|_{D^2} < (M + 2/k)/m - 2/k.$$

The proof is complete.  $\square$

### 3. AN EFFECTIVE PILA-WILKIE TYPE STATEMENT FOR ALGEBRAIC VARIETIES

In this section, we show that the rational points of height at most  $T$  on a algebraic varieties are covered by subvarieties of uniformly bounded degree, and that the number of such subvarieties grows subpolynomially in  $T$ . This essentially follows from the Pila-Wilkie counting strategy and we use the powerful cellular parameterization theorem from [9] to include sharp effectivity features. Theorem 3.5 parallels the block version of the Pila-Wilkie theorem, except that we count rational points on algebraic instead of transcendental objects.

As alluded to in the introduction, we will employ Theorem 3.5 to arrange that  $d$  depends solely on  $\epsilon$  in Theorem 1.2, and will later use this feature in a critical way for the proof of Theorem 1.1. We start with some simple calculations that will be useful also in later sections.

**Some combinatorial notation and basic facts.** In this subsection, suppose  $\mu, e, m \in \mathbb{N}^{\geq 1}$  are positive integers satisfying  $\mu > e(m+1)$ . Set  $L(n, l) := \binom{n+l-1}{l}$ .

Let  $k$  be the largest integer such that  $\sum_{l=0}^k eL(m, l) < \mu$ ; note the condition on  $\mu$  implies that  $k \geq 1$ . We have that  $\sum_{l=0}^k eL(m, l) = eL(m+1, k)$  and also,

$$\frac{k^m}{m!} < L(m+1, k) < \frac{\mu}{e} \leq L(m+1, k+1) < \frac{2k^m(m+1)^m}{m!}. \quad (1)$$

This gives  $k > \frac{1}{m+1} \left(\frac{m! \mu}{2e}\right)^{\frac{1}{m}}$ , and  $k < \left(\frac{m! \mu}{e}\right)^{\frac{1}{m}}$ . Let  $S(m, k) := \sum_{l=0}^k eL(m, l) \cdot l$ . Then  $S(m, k) = eL(m+2, k-1) > \frac{ek}{2(m+1)} L(m+1, k+1)$  using the fact that  $2m(k+1) > m+k+1$ . The lower bounds for  $L(m+1, k+1)$  and  $k$  from (1) imply

$$S(m, k) > \left(\frac{m!}{2e}\right)^{\frac{1}{m}} \frac{\mu^{1+\frac{1}{m}}}{2(m+1)^2}. \quad (2)$$

We set  $E_m := \frac{1}{2(m+1)^2} \left(\frac{m!}{2e}\right)^{\frac{1}{m}}$ , so we have that  $S(m, k) > E_m \mu^{1+\frac{1}{m}}$ .

**3.1. Bombieri-Pila determinants with complex cells.** We refer the reader to [9] for basic definitions and formulations around complex cells. Let  $\mathcal{C} \subseteq \mathbb{C}^n$  be a cell of length  $n$  and  $m$  be the number of its  $D$ -fibers. We work under the umbrella assumption that  $m < n$ . For  $m \geq 1$ , let  $\delta \in \mathbb{R}^n$  be given by  $\delta < 1/2$  in the  $D$ -coordinates and  $\delta^{E'_m \mu^{1+1/m}}$  in the  $A, D_0$ -coordinates of  $\mathcal{C}$ , where  $E'_m = \frac{1}{4(m+1)^2}$ . Applying (2) for  $e = 1$  gives  $S(m, k) > E'_m \mu^{1+1/m}$ . This inequality can be employed at the end of the proof of [9, Lemma 93] to get the following restatement of that result. When  $m = 0$ , we set  $\delta = (\delta^{\mu^2}/16, \dots, \delta^{\mu^2}/16)$ .

**Proposition 3.1.** *Suppose  $f_i \in \mathcal{O}(\mathcal{C}^\delta)$  with  $\|f_i\|_{\mathcal{C}^\delta} \leq M$ , and  $p_j \in \mathcal{C}$  for  $i, j \leq \mu$ . Then if  $m \geq 1$ ,*

$$|\det(\mathbf{f}, \mathbf{p})| := \det(f_i(p_j))_{i,j \leq \mu} \leq M^\mu \mu^{(n+1)^2 \mu} \delta^{E'_m \mu^{1+\frac{1}{m}}};$$

*and when  $m = 0$ ,*

$$|\det(\mathbf{f}, \mathbf{p})| \leq M^\mu \mu^{(n+1)^2 \mu} \delta^{\mu^3/16}.$$

The choice of a comparatively smaller  $E'_m$  allows us to do away with the  $\delta^{-O_n(\mu)}$  factor in the upper bound from [9, Lemma 93]. This is a direct consequence of (2), which we leave as an exercise to the reader along with working out  $(n+1)^2\mu$  as an upper bound for the exponent of  $\mu$ . Beyond making the estimates more explicit, our formulation includes the stronger statement for the case of  $m = 0$  required for Lemma 3.2.

*Interpolating rational points by polynomials.* Let  $d \geq 2$  and  $m < l \leq n$ . Set  $\mu = L(l+1, d)$ , and note that  $\mu \geq l + d > d$ ,  $\frac{d^l}{l!} \leq \mu \leq 2d^l$ . Also if  $l \geq 2$ , then  $d(l+1) \leq \mu \leq d^l \frac{l^k}{k!}$  for all  $1 \leq k \leq l$ . In addition to these observations, we will use the fact that  $(2d)^{\frac{1}{d}} \leq 2$  for all  $d$ . Note that this implies that  $\mu^{\frac{1}{d}} \leq 2$ .

Let  $\mathbf{f} := (f_1, \dots, f_n)$  be a map, where  $f_1, \dots, f_n$  are complex-valued functions, and  $\mathbf{p} := \{p_1, \dots, p_\mu\}$  a collection of points. Let  $J_l$  be the set of  $\mathbf{j} = \{j_1, \dots, j_l\}$  with  $1 \leq j_1 \leq \dots \leq j_l \leq n$ . For every  $\mathbf{j} = \{j_1, \dots, j_l\} \in J_l$  we set  $z_{\mathbf{j}} := \{z_{j_1}, \dots, z_{j_l}\} \subseteq \{z_1, \dots, z_n\}$  to be the set of variables corresponding to  $\mathbf{j}$ . Set

$$\det_{\mathbf{j}}^d(\mathbf{f}, \mathbf{p}) := \det(\mathbf{f}^\sigma(p_i))_{\sigma \in \Sigma_{\mathbf{j}}^d, i \leq \mu},$$

where  $\Sigma_{\mathbf{j}}^d := \{\alpha \in \mathbb{N}^n : |\alpha| \leq d \text{ and } \alpha_j = 0 \text{ if } j \notin \mathbf{j}\}$ , and  $\mathbf{f}^\sigma := f_1^{\sigma_1} \dots f_n^{\sigma_n}$ . Note  $\#\Sigma_{\mathbf{j}}^d = \mu$ .

Let  $S$  be a set of points in the common domain of the  $f_i, i \leq n$ . Then by a routine linear algebra argument,  $(f_1, \dots, f_n)(S) \subseteq \mathbb{C}^n$  is contained in an hypersurface in  $z_{\mathbf{j}}$  variables of degree at most  $d$  if and only if  $\det_{\mathbf{j}}^d(\mathbf{f}, \mathbf{p}) = 0$  for any  $\mathbf{p} \subset S$  of size  $\mu$ ; we use this observation freely in this section and also the next section. By a *hypersurface in  $z_{\mathbf{j}}$  variables of degree at most  $d$*  we mean the zero set of a non-trivial polynomial in the  $z_{\mathbf{j}}$  variables of total degree at most  $d$ . Note that this means that such a hypersurface which has degree at most  $d$  as an algebraic variety in  $\mathbb{C}^n$ . We recall the definition of degree of a variety at the start of the next subsection and note that the equivalence referenced in the last sentence is a well-known fact.

**Lemma 3.2.** *Let  $\mathbf{f} = (f_1, \dots, f_n)$  be a map with  $f_i \in \mathcal{O}(\mathcal{C}^\delta)$  and  $\|f_i\|_{\mathcal{C}^\delta} \leq 2n$ . Set  $C := 2^{-8(m+1)^4(n+1)^2}$ . Then for every  $\epsilon$  there is  $d < (\frac{4(m+1)^4}{\epsilon})^{m+1}$  such that for all  $T$  and  $\mathbf{j} \in J_{m+1}$ , if*

$$\delta \leq CT^{-\epsilon},$$

*then  $\mathbf{f}(\mathcal{C})(\mathbb{Q}, T)$  is contained in a hypersurface in  $z_{\mathbf{j}}$  variables of degree at most  $d$ .*

*Proof.* Fix  $\epsilon$  and set  $l = m+1$ . Recall  $\mu = L(l+1, d)$  and suppose  $p_1, \dots, p_\mu \in \mathcal{C}$  are such that  $\mathbf{f}(p_1), \dots, \mathbf{f}(p_\mu) \in \mathbf{f}(\mathcal{C})(\mathbb{Q}, T)$ .

*Case I:  $m = 0$ .* Here  $\mu = d+1$ , and choose  $d$  such that  $\frac{1}{\epsilon} < d < \frac{4}{\epsilon}$ . Proposition 3.1 and [9, Lemma 95] give that

$$T^{-d(d+1)} \leq |\det_{\mathbf{j}}^d(\mathbf{f}, \mathbf{p})| \leq (2n)^{d+1} (d+1)^{(n+1)^2(d+1)} \delta^{(d+1)^3/16}$$

or  $\det_{\mathbf{j}}^d(\mathbf{f}, \mathbf{p}) = 0$ . So  $\det_{\mathbf{j}}^d(\mathbf{f}, \mathbf{p}) = 0$  if

$$\delta \leq C'T^{-1/d},$$

where  $C' := (2n(d+1)^{(n+1)^2})^{-16/(d+1)^2} \geq C = 2^{-8(n+1)^2}$ , since  $(d+1)^{4/(d+1)^2} \leq 2$  and  $(2n)^{4/(d+1)^2} \leq 2^{(n+1)^2}$  for all  $d, n$ . Since we chose  $d$  such that  $1/d < \epsilon$  we have the desired conclusion.

*Case II:*  $m \geq 1$ . Choose  $d$  such that  $(\frac{4l^4}{\epsilon})^m < d < (\frac{4l^4}{\epsilon})^l$ . By Proposition 3.1 and [9, Lemma 95] we have that

$$T^{-l\mu d} \leq |\det_{\mathbf{j}}^d(\mathbf{f}, \mathbf{p})| \leq (2n)^\mu \mu^{(n+1)^2 \mu} \delta^{E'_m \mu^{1+\frac{1}{m}}}$$

or  $\det_{\mathbf{j}}^d(\mathbf{f}, \mathbf{p}) = 0$ . So  $\det_{\mathbf{j}}^d(\mathbf{f}, \mathbf{p}) = 0$  if

$$\delta \leq C' T^{-4l^3 d \mu^{\frac{-1}{m}}},$$

where  $C' = (2n)^{-4l^2} \mu^{-4l^2(n+1)^2 \mu^{\frac{-1}{m}}}$ . Using  $\frac{d^l}{l!} \leq \mu \leq 2d^l$  and  $(2d)^{\frac{1}{d}} \leq 2$ , we see that  $\mu^{\frac{-1}{m}} \leq 2^{l^2}$ , and hence  $C = 2^{-8l^4(n+1)^2} \leq C'$ . We now use again that  $\mu \geq \frac{d^l}{l!}$  to see that by our choice of  $d$  we have that  $4l^3 d \mu^{\frac{-1}{m}} \leq 4l^4 d^{\frac{-1}{m}} < \epsilon$ . So we have shown that  $\delta \leq CT^{-\epsilon}$  implies that  $\mathbf{f}(\mathcal{C})(\mathbb{Q}, T)$  is contained in a hypersurface in  $z_{\mathbf{j}}$  variables of degree at most  $d$  and the proof is complete.  $\square$

We also record a variant of Lemma 3.2, where in exchange for allowing degree of the interpolating polynomial to grow as a power of  $\log(T)$ , we get a bound for  $\delta$  that works uniformly for all  $T$ .

**Corollary 3.3.** *Let  $\mathbf{f} = (f_1, \dots, f_n)$  be a map with  $f_i \in \mathcal{O}(\mathcal{C}^\delta)$  and  $\|f_i\|_{\mathcal{C}^\delta} \leq 2n$ . Then there is  $C = C(n)$  such that for all  $T$  and  $\mathbf{j} \in J_{m+1}$ , if  $\delta \leq C$  then we have that  $\mathbf{f}(\mathcal{C})(\mathbb{Q}, T)$  is contained in a hypersurface in  $z_{\mathbf{j}}$  variables of degree at most  $\log(T)^{m+1}$ .*

*Proof.* Run the proof of Lemma 3.2 for  $\epsilon = 4(m+1)^4/\log(T)$ .  $\square$

**3.2. Covering rational points on an algebraic variety by subvarieties of bounded degree.** Throughout the rest of this section we set  $l = m+1$ , and recall  $J_{m+1}$  is the set of  $\mathbf{j} = \{j_1, \dots, j_{m+1}\}$  with  $1 \leq j_1, \dots, j_{m+1} \leq n$ .

In this paper, an *algebraic variety* is  $Z(I) := \{z \in \mathbb{C}^n : h(a) = 0 \text{ for all } h \in I\}$ , where  $I$  is a non-trivial ideal of  $\mathbb{C}[z_1, \dots, z_n]$ . Recall that the *degree* of an algebraic variety is the number of intersections (counted with multiplicities) between the variety and a generic affine-linear hyperplane of complementary dimension. This matches the definition in [9, subsection 2.2.3], where the words complexity and degree are used interchangeably to refer to this concept. We will use the following basic fact.

**Fact 3.4.** [13, Chapter 9, Section 5, Corollary 4] *The dimension of an algebraic variety  $Z(I) \subseteq \mathbb{C}^n$  is the maximal size of a set  $S \subseteq \{z_1, \dots, z_n\}$  such that  $I$  contains no nonzero polynomial in only the variables in  $S$ .*

**Lemma 3.5.** *Let  $\epsilon$  be given and  $W \subseteq \mathbb{C}^n$  be an algebraic variety of degree  $d_0$  with  $\dim(W) = m < n$ . Then there are  $c = \text{poly}_n(d_0, \epsilon^{-1}) \geq 1$  and  $d = \text{poly}_n(\epsilon^{-1})$  such that for all  $T$  and  $\mathbf{j} \in J_{m+1}$ , there is  $N \leq cT^\epsilon$  and hypersurfaces in  $z_{\mathbf{j}}$  variables  $H_1, \dots, H_N$  of degree at most  $d$  with  $W(\mathbb{Q}, T) \subseteq \bigcup_{i=1}^N H_i$ .*

*Proof.* Let  $\mathcal{P}_n := D(1)^n$ . It clearly suffices to prove the theorem for  $W \cap sP^n$  in the role of  $W$ . The case  $m = 0$  is immediate and we assume from now on that  $m \geq 1$ . Obtain  $C = O_n(1)$  and  $d = \text{poly}_n(\epsilon^{-1})$  using the corresponding estimates from Lemma 3.2. Fix  $\mathbf{j} \in J_{m+1}$ , and set  $\mu = L(m+2, d)$  and  $\delta = CT^{\frac{\epsilon}{2n}}$ . The tuple  $\delta \in \mathbb{R}^n$  will be defined with regards to cells as in the context of Proposition 3.1 and will hence depend on the fiber composition of the cell in question.

We employ [9, Theorem 8] to get a real cellular cover of  $\mathcal{P}_n \cap W$  of size  $\text{poly}_n(d_0)$  admitting  $\frac{1}{2}$ -extensions. For every cell  $\mathcal{C}$  of this covering, apply [9, Lemma 94] to obtain a covering of  $\mathcal{C}$  by  $\text{poly}_n(\mu \log(\frac{1}{\delta})) \cdot \delta^{-\dim(\mathcal{C})}$  cells such that each cell in this refinement admits a  $\delta$ -extension. This gives a covering of  $\mathcal{P}_n \cap W$  by  $\text{poly}_n(d_0, d, \log T) \cdot T^{\frac{\epsilon}{2}} = \text{poly}_n(d_0, \epsilon^{-1}) T^\epsilon$  maps; we use here that  $d = \text{poly}_n(\epsilon^{-1})$  and that  $(\log T)^{O_n(1)} \leq O_n(1) \epsilon^{-1} T^{\epsilon/2}$  for all  $T$ .

Take a map of this cover,  $\mathbf{f} : \mathcal{C}^\delta \rightarrow \mathcal{P}_n^{1/2} \cap W$ . Then our choice of  $\delta$  implies by Lemma 3.2 that  $\mathbf{f}(\mathcal{C})(\mathbb{Q}, T)$  is contained in a hypersurface in variables  $z_j$  of degree at most  $d$ , and the proof is complete.  $\square$

**Theorem 3.6.** *Let  $\epsilon$  and an algebraic variety  $W \subseteq \mathbb{C}^n$  of degree  $d_0$  be given. Then there are  $c = \text{poly}_n(d_0, \epsilon^{-1})$  and  $d = \text{poly}_n(\epsilon^{-1})$  such that for all  $T$ , there is  $N \leq cT^\epsilon$  and subvarieties  $W_1, \dots, W_N \subseteq W$  of degree at most  $d$  such that  $W(\mathbb{Q}, T) \subseteq \bigcup_{i=1}^N W_i$ .*

*Proof.* We can and do assume that  $W \subset \mathbb{C}^n$ . So  $m := \dim(W) < n$ , and we proceed by induction on  $m$ .

**Claim:** There are  $c_* = \text{poly}_n(d_0, \epsilon^{-1})$ ,  $d_* = \text{poly}_n(\epsilon^{-1})$  such that for all  $T$ , we can cover  $W(\mathbb{Q}, T)$  by irreducible algebraic varieties  $V_1, \dots, V_l$  of degree  $d_*$  such that  $\dim(V_i) \leq m$  for all  $i \leq l$ , where  $l \leq c_* T^{\frac{\epsilon}{2}}$ .

We first assume the claim and complete the proof. Set  $N_0 := 0$ , and consider some  $i \leq l$ . We have that  $V_i \subseteq W$  or  $\dim(V_i \cap W) < m$ . In the former case, set  $N_i = N_{i-1} + 1$ ,  $W_{N_i} := V_i$ ,  $c_i = 2$ , and  $d_i := d_*$ . In the latter case, induction gives for all  $T$ ,  $c_i = \text{poly}_n(d_0, \epsilon^{-1})$ ,  $d_i = \text{poly}_n(\epsilon^{-1})$ ,  $M_i \leq c_i T^{\frac{\epsilon}{2}}$ , and algebraic varieties  $V_{N_{i-1}+1}, \dots, V_{N_{i-1}+M_i} \subseteq V_i \cap W$  of degree at most  $d_i$  that cover  $(V_i \cap W)(\mathbb{Q}, T)$ ; set  $N_i := N_{i-1} + M_i$  in this case. It remains to observe that  $c := c_* \cdot \max\{c_1, \dots, c_l\}$ ,  $d := \max\{d_1, \dots, d_l\}$ ,  $N := N_l \leq cT^\epsilon$ , and the algebraic varieties  $W_1, \dots, W_N \subseteq W$  are as desired.

We now return to the proof of claim.  $J_{m+1}$  is the set of all  $\mathbf{j} = \{j_1, \dots, j_{m+1}\}$  with  $1 \leq j_1, \dots, j_{m+1} \leq n$  and let  $L := \#J_{m+1} = \binom{n}{m+1}$ . Fix a  $T$ , our proof and the resulting constants will not depend on this choice. We apply Lemma 3.5 to get for every  $\mathbf{j} \in J_{m+1}$ , constants  $c_j = \text{poly}_n(d_0, \epsilon^{-1})$ ,  $d_j = \text{poly}_n(\epsilon^{-1})$  such that  $W(\mathbb{Q}, T)$  is covered by at most  $c_j T^{\frac{\epsilon}{2L}}$  many hypersurfaces in the  $z_j$  variables of degree  $d_j$ . Let  $c' := \max(c_j)_{j \in J_{m+1}}$ , and  $d' := \max(d_j)_{j \in J_{m+1}}$ .

Thus we have that  $W(\mathbb{Q}, T)$  is contained in at most  $(c')^L T^{\frac{\epsilon}{2}} = \text{poly}_n(c') T^{\frac{\epsilon}{2}}$  sets of the form  $\cap_{j \in J_{m+1}} H_j$ , where  $H_j$  is a hypersurface of degree  $d_j$  in the  $z_j$  variables. For all  $T$ , let  $V_1, \dots, V_l$  be the collection of all the irreducible components of all the sets  $\cap_{i \in I} H_i$  covering  $W(\mathbb{Q}, T)$  as in the last sentence. Note that the degree of  $V_i$  is  $d_* = \text{poly}_n(d')$  for all  $i \leq l$ , and also that  $l \leq c_* T^{\frac{\epsilon}{2}}$ , where  $c_* = \text{poly}_n(d') \text{poly}_n(c') = \text{poly}_n(d_0, \epsilon^{-1})$ .

Moreover,  $\dim(V_i) \leq m$  for all  $i \leq l$  by construction of the hypersurfaces  $(H_j)_{j \in J_{m+1}}$  and Fact 3.4. We have shown that  $c_*$ ,  $d_*$ , and  $V_1, \dots, V_l$  are as desired, which finishes the proof of the claim and the theorem.  $\square$

The following variant of Theorem 3.6 shows that given an algebraic variety  $W$ , we can cover  $W(\mathbb{Q}, T)$  by subvarieties whose number can only grow polynomially in  $\log(T)$  if we allow the bound on the degree to also grow as a polynomial in  $\log(T)$ .

**Theorem 3.7.** *Let  $W \subseteq \mathbb{C}^n$  be an algebraic variety of degree  $d_0$ . Then there are functions  $c(t) = \text{poly}_n(d_0, t)$  and  $d(t) = \text{poly}_n(t)$  such that for all  $T$ , there is*

$N \leq c(\log(T))$  and subvarieties  $W_1, \dots, W_N \subseteq W$  of degree at most  $d(\log(T))$  with  $W(\mathbb{Q}, T) \subseteq \bigcup_{i=1}^N W_i$ .

*Proof.* We obtain this by running the proof of Theorem 3.6 for  $\epsilon = 1/\log(T)$ . Note we already have Corollary 3.3 to play the role of Lemma 3.2.  $\square$

#### 4. INTERPOLATING RATIONAL POINTS ON LEMNISCATES

Throughout this section,  $g \in \mathcal{O}(\bar{B}^{\frac{1}{2}})$  and is non-constant. Let  $A \subseteq B^{\frac{1}{2}} \subset \mathbb{C}^{m+1}$  be a ball or polydisc. We introduce the *Bernstein index* of  $g$  with respect to the pair  $A^\eta \subset A$ ,

$$\mathcal{B}_A^\eta(g) := \log(\|g\|_A / \|g\|_{A^\eta}).$$

So  $\mathcal{B}_A^\eta(g) > 0$  for all non-constant  $g$ ; recall  $\eta \geq 2$  by the convention set forth in the introduction. We will make use of the the following fact [4, Lemma 17].

**Lemma 4.1.** *Let  $U \subseteq \mathbb{C}$  be a disc and  $h \in \mathcal{O}(\bar{U})$ . Then there is a disc  $V$  concentric with  $U$  such that  $U^4 \subset V \subset \bar{U}^2$ , and*

$$\min_{z \in \partial V} |h(z)| \geq \|h\|_U e^{-20\mathcal{B}_U^2(h)}.$$

The exponent 20 can be easily computed with help from [19, Example 1].

**4.1. Bounding Bombieri-Pila determinants.** Set  $X := \{z \in B : g = 0\}$ , and recall the  $\omega$ -lemniscate of  $X$  is the  $X(\omega) = \{z \in B : |g| \leq \omega\}$ . Note that  $X$  is of pure dimension  $n - 1$  by [12, Proposition 2.6], and hence [8, Proposition 7] applies to  $X$ .

Let  $\Delta := \Delta_h \times \Delta_v \subseteq B^{\frac{1}{2}}$ , with polydiscs  $\Delta_h \subset \mathbb{C}^m$  and  $\Delta_v \subset \mathbb{C}$ . We say that  $\Delta$  is a *Weierstrass polydisc* for  $X$  if  $X \cap (\bar{\Delta}_h \times \partial\Delta_v) = \emptyset$ . The proofs of [8, Fact 5] and [18, II.B Theorem 2] imply that  $g$  is given by a unit times a Weierstrass polynomial of degree,  $e = e(g, \Delta) = e(X, \Delta)$ , in  $z_{m+1}$  on  $\Delta$ ; and so we can perform Weierstrass division by  $g$ . We shall use this observation shortly.

Let  $D$  be a polydisc such that  $D^\delta \subseteq \Delta$ , for some  $\delta$  to chosen later. We start by developing an analogue of [7, Lemma 9] for  $X(\omega)$ . For  $\Delta$  as above,  $\Delta'$  will denote the polydisc given by  $\Delta' := \Delta_h \times \Delta_v^{1/3}$ , and set  $\mathcal{M} := \mathbb{N}^m \times \{0, 1, \dots, e-1\}$ . For convenience of notation we set

$$\beta_0 := \omega \|g\|_\Delta^{-1} e^{20\mathcal{B}_\Delta^2(g)} \quad \text{and} \quad \beta := \omega \|g\|_\Delta^{-1} e^{20\mathcal{B}_\Delta^7(g)}.$$

Throughout we assume that  $\omega$  and  $g$  are such that  $\beta, \beta_0 < 1$ .

Let  $f_i \in \mathcal{O}(\bar{\Delta}')$  with  $\|f_i\|_{\Delta'} \leq M$ , and points  $p_j \in D \cap X(\omega)$  for  $i, j \leq \mu$ ; we will work with this setup throughout this section. For the interpolation result, our primary step would be to obtain a sufficiently small bound for

$$\det(\mathbf{f}, \mathbf{p}) := \det(f_i(p_j))_{i,j \leq \mu}.$$

We first handle the special case of  $m = 0$ . So  $\Delta$  is a disc,  $\mathcal{M} = \{0, \dots, e-1\}$  and  $\Delta' = \Delta^{1/3}$ .

**Proposition 4.2.** *Let  $m = 0$ ,  $\mu > e$ , and  $\delta < \frac{1}{4}$ . Then*

$$|\det(\mathbf{f}, \mathbf{p})| \leq (8\mu M)^\mu \beta_0^{\mu-e}.$$



*Proof.* For each  $i \leq \mu$  we perform Weierstrass division by  $g$  to obtain  $P_i$ , a polynomial of degree at most  $e - 1$ , such that

$$f_i(z) = P_i(z) + g(z) \cdot q_i(z)$$

for all  $z \in \Delta$ . Note that  $q_i \in \mathcal{O}(\Delta)$  and employ [8, Proposition 7] to observe that  $\|P_i\|_\Delta \leq 3M$ . In the context of Weierstrass division, also see [18, II.D Theorem 1].

We work towards bounding the function

$$q_i = (f_i - P_i)/g.$$

Lemma 4.1 applied to  $U = \Delta$  together with the maximum modulus principle gives

$$\|q_i\|_D \leq 4M \|g\|_\Delta^{-1} e^{20B_\Delta^2(g)}.$$

In more detail, we use Lemma 4.1 to get a lower bound for  $|g|$  on the boundary of a disc containing  $D$ . Note that  $\delta < \frac{1}{4}$  and  $D^\delta \subseteq \Delta$  implies that  $D \subset (D^\delta)^4 \subseteq \Delta^4$ . Then the fact that  $q_i \in \mathcal{O}(\Delta)$  together with  $\|f_i - P_i\|_\Delta \leq 4M$  gives that the upper bound above holds over the whole disc  $D$ , by the maximum modulus principle.

For  $j \leq \mu$ ,  $p_j \in X(\omega) \cap D$  and the definition of  $\beta_0$  implies that

$$|g(p_j) \cdot q_i(p_j)| \leq 4M\beta_0.$$

Expanding  $\det(\mathbf{f}, \mathbf{p})$  linearly with respect to each row we obtain  $2^\mu$  determinants. Let us consider one such determinant  $\det_I$ , so that for each  $i$ , every occurrence of  $f_i$  is replaced by either  $P_i$  or  $g \cdot q_i$ . Since the degree of  $P_i$  is bounded above by  $e - 1$ , if there are more than  $e$  rows of  $\det_I$  which feature  $P_i$  we must have  $\det_I = 0$ . Hence for non-zero  $\det_I$ , we have at least  $\mu - e$  many rows featuring the  $g \cdot q_i$  terms. This implies that  $|\det_I| \leq \mu! 4^\mu M^\mu \beta_0^{\mu-e}$  and using  $\mu! \leq \mu^\mu$  we obtain the desired upper bound on  $|\det(\mathbf{f}, \mathbf{p})|$ .  $\square$

For the  $m > 0$  case we will need the following sharper version of [7, Lemma 3]. For  $\alpha \in \mathcal{M}$ , we let  $m_\alpha(f)$  denote a monomial of the form  $c_\alpha z^\alpha$ .

**Lemma 4.3.** *Let  $m > 0$ . Any  $f \in \mathcal{O}(\bar{\Delta}')$  can be decomposed in the form*

$$f = \sum_{\alpha \in \mathcal{M}} m_\alpha(f) + q \cdot g,$$

with  $q \in \mathcal{O}(\bar{\Delta})$ ,  $\|m_\alpha(f)\|_\Delta \leq 3\|f\|_{\Delta'}$  for  $\alpha \in \mathcal{M}$ ,  $\|\sum_{\alpha \in \mathcal{M}} m_\alpha(f)\|_\Delta \leq 3\|f\|_{\Delta'}$ , and

$$\|q\|_{\Delta^{14}} \leq 4\|f\|_{\Delta'} \|g\|_\Delta^{-1} e^{20 \cdot B_\Delta^7(g)}.$$

*Proof.* The displayed decomposition and the claims

$$q \in \mathcal{O}(\bar{\Delta}'), \quad \|m_\alpha(f)\|_\Delta \leq 3\|f\|_{\Delta'}, \quad \left\| \sum_{\alpha \in \mathcal{M}} m_\alpha(f) \right\|_\Delta \leq 3\|f\|_{\Delta'}$$

follow directly from [8, Proposition 7]. Note this implies that  $\|q \cdot g\|_\Delta \leq 4\|f\|_{\Delta'}$ .

Towards proving the last claim, let  $p \in \partial\Delta^7$  be such that  $|g(p)| = \|g\|_{\Delta^7}$ . We now choose and fix new coordinates  $\mathbf{w}$  such that  $\mathbf{w} = \mathbf{z} - p$ ; in other words the point  $p$  is the origin in the  $\mathbf{w}$ -coordinates. We set  $g_{\mathbf{w}}$  to the function  $g$  in the  $\mathbf{w}$ -coordinates. Set  $\Upsilon := \Delta_p^{7/6}$ ; note that  $\Upsilon \subset \Delta$  and  $\Delta^{14} \subset \Upsilon^4$ .

Let  $B' \subset B^{\frac{1}{2}}$  be a ball with center  $\mathbf{p}$  with  $\Upsilon \subset B'$ . Set  $\Lambda := \{\lambda = (\lambda_1, \dots, \lambda_n) \in \partial B' : \lambda_1 \in \mathbb{R}\}$ . For any  $\lambda \in \Lambda$ , we set  $g_\lambda : D(1) \rightarrow \mathbb{C}$  to be given by

$$g_\lambda(w) := g_{\mathbf{w}}(w\lambda),$$

where  $D(1)$  is the open unit disc. Set  $M_\lambda := \|g_\lambda\|_{D(1)}$ ,  $m_\lambda := \|g_\lambda\|_{D(1)^2}$ .

Applying 4.1 to the function  $g_\lambda$  we get a disc  $V_\lambda$  concentric with  $D(1)$  with  $D(1)^4 \subset V_\lambda \subset \overline{D(1)}^2$  and

$$\min_{w \in \partial V_\lambda} |g_\lambda(w)| \geq M_\lambda e^{-20 \log \frac{M_\lambda}{m_\lambda}}.$$

Now  $M_\lambda \leq \|g\|_\Delta$ , and  $m_\lambda \geq \|g\|_{\Delta^7}$  since  $g_{\mathbf{w}}(\mathbf{0}) = \|g\|_{\Delta^7}$ , which gives

$$\min_{w \in \partial V_\lambda} |g_\lambda(w)| \geq \|g\|_\Delta e^{-20 \mathcal{B}_\Delta^7(g)}.$$

Notice that the lower bound in the above inequality does not depend on  $\lambda$ ; we are now ready to finish the proof. Recall that  $\|q \cdot g\| \leq 4\|f\|_{\Delta^7}$ . Take any  $a \in \Delta^{14}$ . Since  $\Delta^{14} \subset \Upsilon^4 \subset (B')^4$ , there are  $\lambda_0 \in \Lambda$  and  $w_0 \in D(1)^4 \subset V_{\lambda_0}$  such that  $a - p = w_0 \lambda_0$ . So the fact that  $q \in \mathcal{O}(\bar{\Delta})$ , and the maximum principle applied to  $q(p + w \lambda_0) : \bar{V}_{\lambda_0} \rightarrow \mathbb{C}$  gives that

$$|q(a)| \leq 4\|f\|_{\Delta^7} \|g\|_\Delta^{-1} e^{20 \mathcal{B}_\Delta^7(g)}.$$

It only remains to notice that our upper bound does not depend on the choice of  $a \in \Delta^{14}$ . This gives us the last claim and the proof is complete.  $\square$

We are now ready for the  $m > 0$  version of Proposition 4.2.

**Proposition 4.4.** *Let  $m \geq 1$  and  $\mu > e(m+1)$ . Let  $k_0 \geq (m!\mu/e)^{\frac{1}{m}}$  and assume  $\delta < 1/14$ . Then for  $\rho := \max(\delta, \beta^{1/k_0})$  we have that*

$$|\det(\mathbf{f}, \mathbf{p})| \leq (4(1.1)^m \mu^3 M)^\mu \cdot \rho^{E_m \cdot \mu^{1+\frac{1}{m}}}.$$

*Proof.* We set

$$k := \max \left\{ j : \sum_{l=0}^j eL(m, l) < \mu \right\}.$$

By our calculations at the start of Section 3 and our assumption on  $\mu$  we have  $k \geq 1$ , and the upper bound on  $k$  gives  $k_0 > k$ . Throughout the proof  $i$  ranges over positive natural numbers not greater than  $\mu$ . Set  $\mathcal{M}^{\leq k} := \{\alpha \in \mathcal{M} : |\alpha| \leq k\}$  and apply Lemma 4.3 to obtain

$$f_i = \sum_{\alpha \in \mathcal{M}^{\leq k}} m_\alpha(f_i) + R_k(f_i) + gq_i,$$

where  $\|m_\alpha(f)\|_\Delta \leq 3M$  for all  $\alpha \in \mathcal{M}^{\leq k}$ , and

$$\|q_i\|_D \leq 4M \|g\|_\Delta^{-1} e^{20 \mathcal{B}_\Delta^7(g)} \leq 4M\beta/\omega.$$

Note here we used  $D \subseteq \Delta^{1/\delta} \subset \Delta^{14}$  since  $\delta < \frac{1}{14}$ . By our choice of  $k$ , the number of terms in the expansions of  $f_i$  above does not exceed  $\mu + 1$ . Moreover we have by [7, Proposition 8] and  $k_0 > k$  that for every  $j \leq \mu$  and  $\alpha \in \mathcal{M}^{\leq k}$ ,

$$\|m_\alpha(f_i)\|_D \leq C_0 \delta^{|\alpha|}, \quad \|R_k(f_i)\|_D \leq C_0 \delta^k, \quad |g(p_j) \cdot q_i(p_j)| \leq 4M\beta$$

$$\text{where } C_0 := \frac{4eL(m, k)M}{(1-\delta)^m} < 4(1.1)^m \mu M;$$

the upper bound for  $C_0$  comes by using  $L(m, k) < L(m+1, k) < \frac{\mu}{e}$  and  $\delta < \frac{1}{14}$ .

As in the proof of [7, Lemma 9], expanding  $\det(\mathbf{f}, \mathbf{p})$  linearly we get a sum of at most  $(\mu+1)^\mu$  interpolation determinants  $\det_I$ , such that  $|\det_I| \leq \mu! C_0^\mu \rho^S$ , where

$S = S(m, k) = \sum_{l=0}^k eL(m, l) \cdot l > E_m \mu^{1+1/m}$ , the last inequality coming from our calculations from Section 3. Note that  $\mu > e(m+1)$  implies that  $\mu \geq 3$  and then  $\mu!(\mu+1)^\mu \leq \mu^{2\mu}$ . The proof is complete.  $\square$

**4.2. Covering rational points by hypersurfaces.** We now turn to analogues of results in [7, Section 3.3], and follow the general strategy there. We arrange things in way that allows for a clean statement of Corollary 4.7, which will be the only result from this section to be directly used later in the paper.

We borrow much of the setup from Subsection 3.1. In particular  $d \geq 2$ ,  $m < l \leq n$ ,  $\mu = L(l+1, d)$ , and recall that  $J_l$  is the set of all  $\mathbf{j} = \{j_1, \dots, j_l\}$  with  $1 \leq j_1 \leq \dots \leq j_l \leq n$ . Throughout  $X$ ,  $\Delta$ ,  $\Delta'$ , and  $D$  are as in the previous subsection. So  $\dim(X) = m$ ,  $\Delta \subseteq \mathbb{C}^{m+1}$  is a polydisc for  $X$  and  $D^\delta \subseteq \Delta$ . Suppose  $f_i \in \mathcal{O}(\Delta')$  with  $\|f_i\|_{\Delta'} \leq 1$  for all  $i \leq n$ , and set  $Y := \mathbf{f}(X(\omega) \cap D) \subset \mathbb{C}^n$ .

**Lemma 4.5.** *Let  $m = 0$  and  $\delta < \frac{1}{4}$ . For every  $\epsilon$  and  $d$  with  $\frac{3l^2}{\epsilon} \leq \frac{d}{\epsilon} < \frac{4l^2}{\epsilon}$ , we have for all  $T$  and  $\mathbf{j} \in J_l$  that if*

$$\beta_0^{\frac{2l}{\mu d}} \leq \frac{1}{4} T^{-\epsilon},$$

*then  $Y(\mathbb{Q}, T)$  is contained in a hypersurface in  $z_{\mathbf{j}}$  variables of degree at most  $d$ .*

*Proof.* Fix an  $\epsilon$ . Let  $p_1, \dots, p_\mu \in X(\omega) \cap D$  be such that  $\mathbf{f}(p_1), \dots, \mathbf{f}(p_\mu) \in Y(\mathbb{Q}, T)$ , and suppose  $\det_{\mathbf{j}}^d(\mathbf{f}, \mathbf{p}) \neq 0$ . Choose  $d$  satisfying  $e \frac{3l^2}{\epsilon} \leq d < e \frac{4l^2}{\epsilon}$ . Note that this implies that  $\frac{d}{\epsilon} > 3$ , and so  $\frac{2l^2}{\frac{d}{\epsilon}-1} < e \frac{3l^2}{d} \leq \epsilon$ . Then by  $\mu > d > e$ , Proposition 4.2, and [7, Lemma 10] we have that

$$T^{-l\mu d} \leq \det_{\mathbf{j}}^d(\mathbf{f}, \mathbf{p}) \leq (8\mu)^\mu \beta_0^{\mu-e}.$$

So for  $C := (8\mu)^{-\frac{2l}{d(\mu-e)}}$ , any choice of  $\beta_0 \leq (CT^{-\epsilon})^{\frac{\mu d}{2l}}$  gives a contradiction to the displayed inequalities; since  $\frac{2l^2}{\mu-e} < \frac{2l^2}{d-e} = \frac{2l^2}{e(\frac{d}{\epsilon}-1)} \leq \frac{2l^2}{\frac{d}{\epsilon}-1} < \epsilon$ . This also gives  $\frac{2l}{d(\mu-e)} < \frac{\epsilon}{ld} < \frac{1}{ld}$ ; and to make our choice of  $C$  independent of  $d$ , we use  $\mu^{\frac{1}{ld}} \leq 2$ , and then reset  $C := 1/4$ ; we use here that  $ld \geq 3$ . Now the contradiction in hand implies that  $\det_{\mathbf{j}}^d(\mathbf{f}, \mathbf{p}) = 0$ , and this yields an interpolating hypersurface in  $z_{\mathbf{j}}$  variables of degree at most  $d$  as claimed.  $\square$

**Lemma 4.6.** *Let  $m \geq 1$ ,  $\delta < \frac{1}{14}$ , and  $k_0 := ld^{l/m}$ . There exists  $C = C(l, m) < 1$  such that for every  $\epsilon$  there is  $d$  with  $\frac{d}{\epsilon} < (\frac{4l^3 l^{\frac{l-1}{m}}}{\epsilon})^l$  such that for all  $T$  and  $\mathbf{j} \in J_l$  if*

$$\rho := \max(\delta, \beta^{\frac{1}{k_0}}) \leq CT^{-\epsilon},$$

*then  $Y(\mathbb{Q}, T)$  is contained in a hypersurface in  $z_{\mathbf{j}}$  variables of degree at most  $d$ .*

*Proof.* Fix an  $\epsilon$  and let  $d$  be such that  $d > e(\frac{4l^3 l^{\frac{l-1}{m}}}{\epsilon})^m$ . Clearly we can, and do choose  $d$  so that  $d < e(\frac{4l^3 l^{\frac{l-1}{m}}}{\epsilon})^l$ . We have that  $m \geq 1$ , so that  $l \geq 2$ . As noted before, this means that  $\mu \geq d(l+1)$  which implies  $\mu > e(m+1)$  since  $d > e$ . Also recall that  $\mu \leq d^{\frac{l^m}{m!}}$ , and so  $k_0 = ld^{l/m} \geq (m!\mu)^{1/m}$  satisfies the assumption on  $k_0$  in the statement of Proposition 4.4.

Let  $p_1, \dots, p_\mu \in X(\omega) \cap D$  be such that  $\mathbf{f}(p_1), \dots, \mathbf{f}(p_\mu) \in Y(\mathbb{Q}, T)$ , and suppose that  $\det_{\mathbf{j}}^d(\mathbf{f}, \mathbf{p}) \neq 0$ . Then Proposition 4.4 and [7, Lemma 10] give

$$T^{-l\mu d} \leq \det_{\mathbf{j}}^d(\mathbf{f}, \mathbf{p}) \leq (4(1.1)^m \mu^3)^\mu \cdot \rho^{E_m \cdot \mu^{1+1/m}},$$

where  $E_m = \frac{1}{2(m+1)^2} \left(\frac{m!}{2e}\right)^{\frac{1}{m}} > \frac{1}{4l^2 e^{\frac{1}{m}}}$ . So we get a contradiction if

$$\rho < CT^{-4l^3 e^{\frac{1}{m}} d \mu^{-\frac{1}{m}}},$$

for  $C := (4(1.1)^m \mu^3)^{-4l^2 e^{\frac{1}{m}} \mu^{-\frac{1}{m}}}$ . This choice of  $C$  depends on  $d$  and in turn  $\epsilon$ , and also  $e$ , but in the statement we promised a  $C$  that depends on just  $m$  and  $M$ . Using  $d \geq e$ ,  $l! \leq l^{l-1}$ , and recalling that  $\frac{d^l}{l!} \leq \mu$  we observe

$$(4(1.1)^m \mu^3)^{-\frac{4l^2 e^{\frac{1}{m}}}{\mu^{\frac{1}{m}}}} \geq (4(1.1)^l \mu^3)^{-\frac{4l^2 l^{\frac{l-1}{d}}}{d}}.$$

Now we use  $\mu \leq 2d^l$ ,  $(2d)^{\frac{1}{d}} \leq 2$ , and also  $32^{\frac{1}{3l}} (1.1)^{1/3} < 32^{1/6} (1.1) < 2$  to see

$$(4(1.1)^l \mu^3)^{-\frac{4l^2 l^{\frac{l-1}{d}}}{d}} > 2^{-12l^3 l^{\frac{l-1}{m}}}.$$

So we change reset  $C$  to  $C := 2^{-12l^3 l^{\frac{l-1}{m}}}$ . We use again that  $\frac{d^l}{l!} \leq \mu$ ,  $l! \leq l^{l-1}$ , to see that our choice of  $d$  gives

$$4l^3 e^{\frac{1}{m}} d \mu^{-\frac{1}{m}} < 4l^3 l^{\frac{l-1}{m}} (e/d)^{\frac{1}{m}} < \epsilon.$$

So any  $\rho \leq CT^{-\epsilon}$  gives the desired contradiction, and the corollary follows.  $\square$

We now record a consequence that we will apply later, here we choose  $l = m + 1$  which is the number of  $z_j$  variables. Also included is a viable upper bound on  $\beta$  that is uniform in  $m$ . Let  $v_m : \mathbb{N} \rightarrow \{0, 1\}$  be such that  $v_m = 0 \iff m = 0$ .

**Corollary 4.7.** *Let  $\delta < \frac{1}{4}$  and  $C = 2^{-12(m+1)^4}$ . For every  $\epsilon$  there exists  $d$  satisfying  $\frac{d}{e} < \left(\frac{4(m+1)^4}{\epsilon}\right)^{m+1}$  such that for all  $T$  and  $\mathbf{j} \in J_{m+1}$  if*

$$\beta \leq (CT^{-\epsilon})^{ld^2} \quad \text{and} \quad v_m \delta \leq CT^{-\epsilon},$$

*then  $Y(\mathbb{Q}, T)$  is contained in a hypersurface in  $z_j$  variables of degree at most  $d$ .*

*Proof.* This follows easily from the previous two lemmas. Fix an  $\epsilon$  and  $T$ , and suppose  $\beta$  and  $\delta$  satisfy the displayed inequalities. For  $m = 0$  since  $\beta_0 < \beta$ , we have  $\beta_0^{\frac{2}{\mu^d}} < \beta^{\frac{1}{d^2}} < CT^{-\epsilon} < \frac{1}{4}T^{-\epsilon}$ . Now a direct application of Corollary 4.5 finishes this case. For  $m \geq 1$ , the bound we assume on  $\beta$  implies that  $\beta^{\frac{1}{k_0}} \leq CT^{-\epsilon}$ , and so the assumption on  $\delta$  gives  $\rho := \max(\delta, \beta^{\frac{1}{k_0}}) \leq CT^{-\epsilon}$ , and we are done by a direct application of Corollary 4.6.  $\square$

**Corollary 4.8.** *Let  $\delta < \frac{1}{4}$ . Then there is  $C = C(n)$  such that for all  $T$  and  $\mathbf{j} \in J_{m+1}$  if*

$$\beta \leq C^{\log(T)^{2(m+1)}} \quad \text{and} \quad v_m \delta \leq CT^{-\epsilon},$$

*then  $Y(\mathbb{Q}, T)$  is contained in a hypersurface in  $z_j$  variables of degree at most  $\log(T)^{m+1}$ .*

*Proof.* Run the proof of Corollary 4.7 for  $\epsilon = 4e(m+1)^4 / \log(T)$ .  $\square$

## 5. COUNTING APPROXIMATE RATIONAL SOLUTIONS OF ANALYTIC FAMILIES

Throughout this section, we fix  $X = \{z \in B : g_1(z) = \dots = g_m(z) = 0\}$ , with  $g_i \in \mathcal{O}(\bar{B}^{1/2})$  and  $\|g_i\|_{B^{1/2}} \leq 1$ . Recall that  $B = B(\frac{1}{2}) \subset \mathbb{C}^n$  is the ball of radius  $1/2$  around the origin.

**Lemma 5.1.** *Suppose  $g \in \mathcal{O}(B)$  such that  $|g(0)| \geq \omega_0$ ,  $\|g\|_B \leq 1$  and let  $\eta_0 \in \mathbb{R}^{\geq 2}$ . There is a unitary transformation  $\mathbf{u}$  and a polydisc  $\Delta$  for  $Z := \{z \in B : g \circ \mathbf{u}(z) = 0\}$  such that  $B^{2\eta_0\eta_1} \subset \Delta \subset B^{2\eta_0}$  for  $\eta_1 = \text{poly}_n(-\log(\omega_0)/\log(\eta_0))$ . Moreover, we have that  $e(Z, \Delta) < -\log(\omega_0)/\log(\eta_0)$ .*

*Proof.* Fix  $\eta_0 \in \mathbb{R}^{\geq 2}$ . Since the analytic set  $\{z \in B : g(z) = 0\}$  is of pure dimension by [12, Proposition 2.6], [5, Corollary 18] gives an affine unitary map  $\mathbf{u}$  and a polydisc  $\Delta$  for  $Z := \{z \in B : g \circ \mathbf{u}(z) = 0\}$  such that  $B^{2\eta_0\eta_1} \subset \Delta \subset B^{2\eta_0}$  for  $\eta_1 = \text{poly}_n(\text{vol}(Z \cap B^{2\eta_0}))$ . We bound the volume of  $Z$  in  $B^{2\eta_0}$  by use of an analogue of Crofton's formula for complex analytic sets. More precisely, we employ [12, Proposition 14.6.3] which gives that

$$\text{vol}(Z \cap B^{2\eta_0}) = O_n(1) \int_{G(1,n)} \#(Z \cap B^{2\eta_0} \cap L) dL,$$

where  $G(1, n)$  denotes the space of all 1-dimensional linear subspaces of  $\mathbb{C}^n$  with standard measure.

For all but finite complex lines  $L$  passing through the origin, the intersection  $Z \cap L$  is finite, and for such a line we apply Lemma 2.2 to get that

$$\#(Z \cap B^{2\eta_0} \cap L) < (1/\log(\eta_0)) \cdot \log(\|g \circ \mathbf{u}\|_{B \cap L} / \|g \circ \mathbf{u}\|_{B^{2\eta_0} \cap L}).$$

Then for all but finite  $L \in G(1, n)$ , it follows that  $\#(Z \cap B^{2\eta_0} \cap L) < -\log(\omega_0)/\log(\eta_0)$  by using  $\|g\|_B \leq 1$  and  $|g(0)| \geq \omega_0$ , which implies in particular that  $e(Z, \Delta) < -\log(\omega_0)/\log(\eta_0)$ . This gives that  $\text{vol}(Z \cap B^{2\eta_0}) \leq -O_n(1) \log(\omega_0)/\log(\eta_0)$  and the proof is finished.  $\square$

We borrow the definition of the notion of an algebraic map and its complexity from [9, subsubsection 2.2.3]. A map  $\phi = (\phi_1, \dots, \phi_n) : D(1)^l \rightarrow \mathbb{C}^n$  is *algebraic of complexity  $d$*  if for each  $i = 1, \dots, n$ , the graph of  $\phi_i$  is an analytic component of  $(D(1)^l \times \mathbb{C}) \cap W_i$ , where  $W_i \cap D(1)^l \times \mathbb{C}$  is an algebraic variety of degree at most  $d$ . A collection of maps has complexity  $d$  if all its members have complexity  $d$ .

For a algebraic map  $\phi : D(1)^l \rightarrow B^{1/2}$  of complexity  $d$ , we say that  $\phi(D(\frac{1}{2})^l)$  is a  *$d$ -block in  $X(\omega)$*  if  $\phi(D(\frac{1}{2})^l) \subseteq X(\omega)$ . We sometimes just call such an object a  *$d$ -block* when the ambient set  $X(\omega)$  is clear from context. In particular any point or polydisc inside  $X(\omega)$  are 1-blocks. We are ready to state and prove the following more precise version of Theorem 1.2.

**Theorem 5.2.** *Let  $\epsilon, \nu$  be given. Suppose  $\theta$  is such that*

$$\theta \geq 20\nu + 2^{8(n+1)} n^{10n+7} \epsilon^{-2(n+1)} \cdot \nu^2.$$

*Then there are  $c = \text{poly}_n(\nu, \epsilon^{-1})$  and  $d = \text{poly}_n(\epsilon^{-1})$  such that for all  $T$ , we have that  $X(T^{-\theta})(\mathbb{Q}, T)$  is contained in at most  $cT^\epsilon$  many  $d$ -blocks in  $X(T^{-\nu})$ .*

*Proof.* Let  $\epsilon, \nu$  and  $\theta$  be given as in the statement above. All blocks in the proof are blocks in  $X(T^{-\nu})$ . We show the consequence of the theorem locally for  $X \cap A$ , where  $A$  is a block. This will suffice as we control the size of  $A$  so that it is not

too small; this will be made clear soon. We demonstrate our method around the origin.

Fix some  $T$ , this choice will not effect our process. For  $\eta_0 = T^{\epsilon/4n}$  and  $\omega_0 = T^{-\nu}$ , we get  $\eta_1 = \text{poly}_n(4n\nu/\epsilon)$  as given by Lemma 5.1. Set  $\eta = 8\eta_0\eta_1 C^{-1}T^{\epsilon/4n}$ , where  $C = C(n) = 2^{-12n^4}$  comes from Corollary 4.7. Suppose  $\|g_i\|_{B^\eta} \leq T^{-\nu}$  for all  $i = 1, \dots, m$ . then there is a polydisc  $D$  with  $B^{n\eta} \subset D \subset B^\eta$ , and we have  $D \subseteq X(T^{-\nu})$ , so  $D$  is a 1-block.

Otherwise  $\|g_i\|_{B^\eta} > T^{-\nu}$  for some  $i \in \{1, \dots, m\}$ ; set  $g = g_i$ . Let  $p$  be a point on  $\bar{B}^\eta$  where the maximum of  $g$  is attained, so that  $|g(p)| > T^{-\nu}$ . We apply Lemma 5.1 for  $p$  in the role of the origin and  $B_p^2$  in the role of  $B$  to obtain a unitary transformation  $\mathbf{u}$  and a Weierstrass polydisc  $\Delta$  for  $Z := \{z \in B : g \circ \mathbf{u}(z) = 0\}$  such that  $B_p^{4\eta_0\eta_1} \subset \mathbf{u}(\Delta) \subset B_p^{4\eta_0}$ . Set  $D := \Delta^{C^{-1}T^{4\epsilon/n}}$  and note that  $B^\eta \subset D$ . We also say that  $\Delta$  is a polydisc for  $X$ .

We apply Corollary 4.7 for  $\omega = T^{-\theta}$  and  $Y = \mathbf{u}(Z(T^{-\theta}) \cap D)$ . So  $m = n - 1$  and Lemma 5.1 gives that  $e(X, \Delta) < \frac{4n\nu}{\epsilon}$ . Recall  $\beta = T^{-\theta} \|g \circ \mathbf{u}\|_{\Delta}^{-1} e^{20B_{\Delta}^7(g)}$ , and with  $\frac{\epsilon}{4n}$  in the role of  $\epsilon$  we obtain  $d' < \left(\frac{16n^5}{\epsilon}\right)^n$ , such that for

$$\delta := CT^{-\frac{\epsilon}{4n}} \quad \text{and} \quad d_\nu := \frac{4n\nu}{\epsilon} \cdot d',$$

if  $\beta \leq (CT^{-\epsilon})^{nd_\nu^2}$ , then  $Y(\mathbb{Q}, T)$  is contained in an hypersurface in  $\mathbb{C}^n$  of degree  $d_\nu$ .

At this stage we have shown, irrespective of whether  $\|g\|_{B^\eta} < T^{-\nu}$  or not, that  $B$  can be covered by at most  $(2 \cdot n\eta)^n = c'T^{\frac{\epsilon}{2}}$  unitary images of polydiscs for  $X$  with  $c' = \text{poly}_n(\epsilon^{-1}, \nu)$ . If we show, in a manner that does not depend on working around the origin, the choice of  $T$  and the polydisc  $D$ , that there are  $c_1 = \text{poly}_n(\epsilon^{-1}, \nu)$ ,  $d_1 = \text{poly}_n(\epsilon^{-1})$  such that  $Y(T^{-\theta})(\mathbb{Q}, T)$  is contained in at most  $c_1 T^{\frac{\epsilon}{2}}$   $d_1$ -blocks in  $X(T^{-\nu})$ , then  $\mathbf{u}(Z)(T^{-\theta})(\mathbb{Q}, T)$  will be contained in at most  $c_1 c' T^\epsilon$  many  $d_1$ -blocks, and the proof will be complete.

We have that  $D \subseteq \Delta^{14}$ , since  $\delta < C \leq 2^{-12}$ , and working in the  $\Delta$ -coordinates gives that  $\|g \circ \mathbf{u}\|_{\Delta^7} > T^{-\nu}$ . Our assumption on  $\theta$  implies that

$$\beta \leq T^{-\theta} \|g \circ \mathbf{u}\|_{\Delta^7}^{-20} \leq T^{-2^{8(n+1)} n^{10n+7} \epsilon^{-2(n+1)} \nu^2} < T^{-(\epsilon n + 12n^5) \left(\frac{4n}{\epsilon} \left(\frac{16n^5}{\epsilon}\right)^n \nu\right)^2}.$$

Note that  $d_\nu^2 < \left(\frac{4n}{\epsilon} \left(\frac{(2n)^5}{\epsilon}\right)^n \nu\right)^2$  and  $C = 2^{-12n^4} \geq T^{-12n^4}$ . So we have that  $\beta < (CT^{-\epsilon})^{nd_\nu^2}$ , and as noted before this gives that  $Y(T^{-\theta})(\mathbb{Q}, T)$  is contained in an hypersurface, say  $H_\nu$ , of degree at most  $d_\nu = \text{poly}_n(\epsilon^{-1}, \nu)$ .

The degree of  $H_\nu$ , i.e.  $d_\nu$ , depends on  $\theta$ , which going forward will be an obstruction to our aim of covering  $Y(T^{-\theta})(\mathbb{Q}, T)$  with blocks in  $X(T^{-\nu})$  of degree depending on just  $\epsilon$  and  $n$ . To remedy this, we apply Theorem 3.6 to obtain  $c_2 = \text{poly}_n(\epsilon^{-1}, \nu)$ ,  $d_2 = \text{poly}_n(\epsilon^{-1})$ , and cover  $H_\nu(\mathbb{Q}, T)$  by  $c_2 T^{\frac{\epsilon}{8}}$  many algebraic subvarieties of  $H_\nu$  of degree bounded by  $d_2$ .

Let  $W$  be an algebraic variety from this cover, so the degree of  $W$  is bounded by  $d_2$ . Using that  $\mathbf{u}(D)$  is the image of a polydisc by unitary transformation, we employ [9, Theorem 8] to get a real cellular cover of  $\mathbf{u}(D) \cap W$  of size and complexity  $d_3 = \text{poly}_n(d_2)$  admitting  $\frac{1}{2}$ -extensions. Then apply [9, Lemma 94] to further refine each of the cells so that each cell  $\mathcal{C}$  in the refinement admits a  $\delta_*$ -extension. Here  $\delta_*$  is given by  $\delta_* = C_* T^{\frac{\epsilon}{8n}}$  in the  $D$ -coordinates and  $\delta_*^{E'_m \mu^{1+1/m}}$  in the  $A, D_0$ -coordinates of  $\mathcal{C}$ , where  $\mu = L(n+1, d_*)$ , and  $C_* = 2^{-4n^4(n+1)^2}$ ,  $d_* = \text{poly}_n(\epsilon^{-1})$  are from Lemma 3.2; this choice will be recalled later. Note we

are in the setting of Lemma 3.2 since the dimension  $D \cap W$ , and hence the number of  $D$ -fibers of  $\mathcal{C}$ , is at most  $n - 1$ . We have arrived now at a covering of algebraic complexity  $d_4 = \text{poly}_n(d_3) = \text{poly}_n(\epsilon^{-1})$  for  $D \cap W$  by  $\text{poly}_n(\epsilon^{-1}, \log T) \cdot T^{\frac{\epsilon}{8}} < \text{poly}_n(\epsilon^{-1}) \cdot T^{\frac{\epsilon}{4}}$  such cells.

Pick a component of this cover, say  $\mathbf{f} : \mathcal{C}^{\delta_*} \rightarrow \mathbf{u}(D)^{\frac{1}{2}} \cap W$ , so complexity of  $\mathbf{f}$  is at most  $d_4$ . It suffices to prove that: there is  $c_3 = \text{poly}_n(\theta, \epsilon^{-1})$  and  $d_5 = \text{poly}_n(\epsilon^{-1})$  such that  $(Y(T^{-\theta}) \cap f(\mathcal{C}))(\mathbb{Q}, T)$  is contained in at most  $c_3 T^{\frac{\epsilon}{8}} d_5$ -blocks.

We proceed by induction on  $k = \dim(\mathcal{C}) \leq n - 1$ . The base case  $k = 0$  is trivial, and also implies our claim in the case of  $n = 1$ . Now let  $n \geq 2$ , and our inductive assumption is that we have the desired claim if  $k \leq n - 2$ . Suppose  $k = n - 1$  and let  $m_*$  be the number of  $D$ -fibers of  $\mathcal{C}$ ; so  $m_* \leq n - 1$ . Our supposition implies that  $\dim(W) = n - 1$ , and let  $I_W$  be an ideal of  $\mathbb{C}[z_1, \dots, z_n]$  with  $Z(I_W) = W$ . By Fact 3.4 we have a variable, say  $z_W \in \{z_1, \dots, z_n\}$ , such that every element of  $I_W$  features  $z_W$ .

If  $m_* < n - 1$ , then our choice of  $C_*$  and  $\delta_*$  allows us to apply Lemma 3.2 with  $l = m_* + 1 < n$ , and choice of some  $\mathbf{j} \subseteq \{z_1, \dots, z_n\} \setminus \{z_W\}$ . This gives a hypersurface, say  $H_*$ , of degree at most  $d_*$  in the  $z_{\mathbf{j}}$  variables such that  $\mathbf{f}(\mathcal{C})(\mathbb{Q}, T)$  is contained in  $H_*$ . Note this gives by Fact 3.4 that  $\dim(W \cap H_*) \leq n - 2$ , and that the degree of  $W \cap H_*$  is bounded by  $d_6 = \text{poly}_n(d_2, d_*) = \text{poly}_n(\epsilon^{-1})$  by Bezout's theorem, see [9, subsubsection 2.2.3]. Now using that the complexity of  $\mathbf{f}$  is bounded by  $d_4$ , we obtain a cellular covering of  $\mathbf{u}(D) \cap W \cap H_*$  following a similar process and entailing similar properties as for the case of  $\mathbf{u}(D) \cap W$  earlier. The size and complexity of this covering can clearly be controlled as per our requirements, and since cells in this covering are of dimension  $\leq n - 2$ , we finish this case by invoking the inductive assumption.

Now suppose that  $m_* = n - 1 = k$ ; then no fiber of  $\mathcal{C}$  is of  $A$  or  $D_0$  type. By renormalizing  $\mathcal{C}$  we get a polydisc  $\Delta_* \subset \mathbb{C}^{n-1}$ , and an algebraic map  $\mathbf{h} : \Delta_*^{\delta_*} \rightarrow \mathbf{u}(D)^{\frac{1}{2}} \cap W$  of complexity at most  $d_7 = \text{poly}_n(d_4)$  such that  $\mathbf{f}(\mathcal{C}) \subseteq \mathbf{h}(\Delta_*)$ . If  $\mathbf{h}(\Delta_*) \subseteq X(T^{-\nu})$ , then  $\mathbf{h}(\Delta_*)$  is a  $d_7$ -block, since  $\delta_* < \frac{1}{2}$ . Otherwise consider the analytic set  $X_* := \mathbf{h}^{-1}(X) = \{z \in \Delta_* : g_i(\mathbf{h}(z)) = 0, i = 1, \dots, m\}$ , and note that  $\dim(X_*) \leq m_* - 1 < n - 1$ . We use  $\delta_* < 2^{-n^2}$ , to get a ball  $B_*$  around the origin with  $\Delta_* \subset B_* \subset B_*^{\frac{1}{2}} \subset \Delta_*^{\delta_*}$ . Now we are in the earlier situation with  $B_*^{\frac{1}{2}}, X_*$  in the role of the  $B^{\frac{1}{2}}, X$  respectively. Since  $\dim(X_*) \leq m - 1$ , we use the same process as before, but now apply Corollary 4.4 with  $l = \dim(X_*) + 1 \leq n - 1$ , to cover  $B_*$  by  $\text{poly}_n(\theta, \epsilon^{-1})$  many polydiscs, such that for every polydisc  $D_*$  in this collection,  $\mathbf{h}(X_* \cap D_*)(\mathbb{Q}, T)$  is contained in an hypersurface of degree at most  $d_\nu$ , say  $H_*$ , in  $z_{\mathbf{j}}$  variables, where  $\mathbf{j} := \{z_1, \dots, z_n\} \setminus \{z_W\}$ . Note this gives by Fact 3.4 that  $\dim(W \cap H_*) \leq n - 2$ , and that the degree of  $W \cap H_*$  is bounded by  $d_8 = \text{poly}_n(d_2, d_\nu)$  by Bezout's theorem, see [9, subsubsection 2.2.3]. So as we follow the rest of the process for  $B_*^{\frac{1}{2}}, \mathbf{h}, X_*$ , using that the complexity of  $\mathbf{h}$  is bounded by  $d_7 = \text{poly}_n(\epsilon^{-1})$ , we end up with a cellular cover of the appropriate size and complexity, but now the dimension of the cells in the covering is at most  $n - 2$ . So we invoke induction again, and the proof is complete.  $\square$

**Theorem 5.3.** *For every  $\nu$  there are functions  $\theta(t) = \text{poly}_n(\nu, t)$ ,  $c(t) = \text{poly}_n(\nu, t)$ , and  $d(t) = \text{poly}_n(t)$  such that for every analytic set  $X$  and all  $T$ , we have that  $X(T^{-\theta(\log(T))})(\mathbb{Q}, T)$  is contained in at most  $c(\log(T))$  many  $d(\log(T))$ -blocks in  $X(T^{-\nu})$ .*

## 6. EFFECTIVE PILA-WILKIE COUNTING

Throughout this section, we fix  $X = \{z \in B : g(z) = 0\}$ ,  $g$  non-constant complex analytic on  $\bar{B}^{1/2} \subset \mathbb{C}^n$  with  $n \geq 2$  and  $\|g\|_{B^{1/2}} \leq 1$ , and assume that  $X$  does not contain any semialgebraic set of positive dimension. To establish Theorem 1.1, it clearly suffices to prove the statement for such an  $X$ ; recall that we already proved the  $n = 1$  case of Theorem 1.1 in Section 2.

**6.1. Chow co-ordinates.** We recall some details about the Chow variety from [16, Chapter 4]. Let  $Y$  be an irreducible variety of dimension 1 and degree  $d$  in the projective space  $\mathbb{P}^n$ , and let  $R_Y(f_1, f_2)$  denote the *resultant* of  $Y$ . The resultant is an irreducible polynomial of degree  $d$  in 2 indeterminate linear forms, which vanishes whenever the projective linear subspace  $\{z \in \mathbb{P}^n : f_1(z) = f_2(z) = 0\}$  of  $\mathbb{P}^n$  intersects  $Y$ . The coefficients of  $R_Y$  give the *Chow coordinates* of  $Y$ , and we can extend this definition to assign chow coordinates to any 1-dimensional effective cycle in  $\mathbb{P}^n$  of degree  $d$ . The chow coordinates of all 1-dimensional effective cycles in  $\mathbb{P}^n$  of degree  $d$  forms a closed projective variety, and we denote it by  $\mathcal{C}_{d,n}$ . The variety  $\mathcal{C}_{d,n}$  is explicitly computable by [17].

Since  $\mathcal{C}_{d,n}$  is a closed projective variety, we have that  $\mathcal{C}_{d,n}$  is compact in the topology induced by the Fubini-study metric. We denote this metric by  $\text{dist}_{\mathcal{C}_{d,n}}(\cdot, \cdot)$  and topological notions referred in the context of  $\mathcal{C}_{d,n}$  are with respect to this metric.

In this section, a *curve* is a (possibly reducible) affine algebraic variety in  $\mathbb{C}^n$  of pure dimension 1. We work with the canonical embedding  $\mathbb{C}^n \rightarrow \mathbb{P}^n$  given by  $(z_1, \dots, z_n) \rightarrow (z_1, \dots, z_n, 1)$  and each curve  $W$  in  $\mathbb{C}^n$  of degree  $d$  corresponds to a effective 1-dimensional algebraic cycle in  $\mathbb{P}^n$  of degree  $d$ , and we let  $c_W$  denote the chow coordinates of this algebraic cycle. Conversely, to each chow coordinate  $c \in \mathcal{C}_{d,n} \setminus (0, \dots, 0, 1)$ , we associate a curve  $W_c$  in  $\mathbb{C}^n$  whose degree is at most  $d$ . We set

$$\mathcal{C}_{d,n}^* := \{W_c : c \in \mathcal{C}_{d,n} \setminus (0, \dots, 0, 1) \text{ and } W_c \cap \bar{B}^{2/3} \neq \emptyset\}.$$

**Lemma 6.1.** *We can compute constants  $C = C(d, n)$  and  $\delta = \delta(d, n)$  such that the following holds. For  $W_1, W_2 \in \mathcal{C}_{d,n}^*$ ,*

$$\text{dist}_{\mathcal{C}_{d,n}}(c_{W_1}, c_{W_2}) < \epsilon \implies \text{dist}_H(W_1, W_2) < C\epsilon^\delta.$$

Here  $\text{dist}_H(W_1, W_2)$  is the Hausdorff distance between  $W_1 \cap \bar{B}^{1/2}$  and  $W_2 \cap \bar{B}^{1/2}$ .

*Proof.* We recall that the resultant  $R_Y$  of a degree  $d$  curve  $Y$  gives a set  $G_Y$  of polynomials of degree  $d$  whose zero set is the curve [14, Subsection 3.3]. Suppose we have two curves  $W_1, W_2 \in \mathcal{C}_{d,n}^*$  with  $\text{dist}_{\mathcal{C}_{d,n}}(c_{W_1}, c_{W_2}) < \epsilon$ . Then this process of obtaining  $G_Y$  from  $R_Y$  gives constants  $C_1 = C_1(d, n)$  and  $\delta_1 = \delta_1(d, n)$  such that for every  $P \in G_{W_1}$  there is  $Q \in G_{W_2}$  with  $|P(a) - Q(a)| \leq C_1\epsilon^{\delta_1}$  for all  $a \in \bar{B}^{1/2}$ .

An effective version of the Łojasiewicz inequality [26, Theorem 7] gives constants  $C_2 = C_2(D, n)$  and  $\delta_2 = \delta_2(D, n)$  such that for all curves  $Y$  of degrees at most  $d$ , we have for any  $a \in \bar{B}^{1/2}$  that

$$\text{dist}(a, Y \cap \bar{B}^{1/2}) \leq C_2 \max_{P \in G_Y} |P(a)|^{\delta_2}.$$

Combining the last two paragraphs we get for any point  $a \in W_1 \cap \bar{B}^{1/2}$ , that  $\text{dist}(a, W_2 \cap \bar{B}^{1/2}) \leq C_2 C_1^{\delta_1} \epsilon^{\delta_1 \delta_2}$ . The same bound holds for the distance of any point on  $W_2 \cap \bar{B}^{1/2}$  and  $W_1 \cap \bar{B}^{1/2}$ , and the proof is complete since the constants we produced are independent of  $W_1$  and  $W_2$ .  $\square$



**Corollary 6.2.** *The space  $\mathcal{C}_{d,n}^*$  is compact with respect to the  $\text{dist}_H(\cdot, \cdot)$  metric.*

**6.2. Preliminaries with polydiscs.** Throughout this section, we only consider polydiscs of the type  $\Delta = \Delta_z \times \Delta_w \subset \mathbb{C}^n$  with  $\dim \Delta_z = 1$  and  $\dim \Delta_w = n - 1$ , and  $\pi_z : \Delta \rightarrow \Delta_z$  will denote the usual projection. Recall that  $\Delta$  is a polydisc for a curve  $W$  if  $W \cap (\bar{\Delta}_z \times \partial \Delta_w) = \emptyset$  (in this definition, we identify  $W$  with its unitary image in the co-ordinates of  $\Delta$ ). Also, we say that  $\Delta$  is a  $\eta$ -gap polydisc for  $W$  if  $(W + B^\eta) \cap (\bar{\Delta}_z \times \partial \Delta_w) = \emptyset$ .

We say that  $\Delta_1 = \Delta_{z_1} \times \Delta_{w_1}, \dots, \Delta_n = \Delta_{z_n} \times \Delta_{w_n}$  are *orthogonal polydiscs* if under a unitary change of co-ordinates, we have that  $z_1$  is the first co-ordinate in  $\mathbb{C}^n$ ,  $z_2$  is the second co-ordinate in  $\mathbb{C}^n$  and so on. We shall employ the following easy fact later.

**Fact 6.3.** *Let  $\Delta_1 = \Delta_{z_1} \times \Delta_{w_1}, \dots, \Delta_n = \Delta_{z_n} \times \Delta_{w_n} \subset B^{1/2}$  be orthogonal polydiscs. Suppose for a connected set  $K \subset B^{1/2}$  that  $\text{diam}(\pi_{z_j}(K)) \leq t$  around the center of  $\Delta_{z_j}$  for every  $j = 1, \dots, n$ . Then we have that  $\text{diam}(K) \leq \sqrt{n}t$ .*

**Lemma 6.4.** *There is  $\eta = \text{poly}_n(d) \geq d$  such that for every curve  $W \in \mathcal{C}_{d,n}^*$  and point  $p \in \bar{B}$ , we can obtain orthogonal  $\eta$ -gap polydiscs  $\Delta_1, \dots, \Delta_n \subset B_p^2$  for  $W$ , each centered around  $p$  and containing  $B_p^\eta$ .*

*Proof.* Let  $W \in \mathcal{C}_{d,n}^*$ , and assume without loss of generality that  $p$  is the origin. Fix  $X := W \cap B^2$ . We obtain the desired polydiscs by adapting the strategy for the proof of [5, Corollary 18]. First, we recall the analogue of Crofton's formula [12, Proposition 14.6.3] to get that

$$\text{vol}(X) = O_n(1) \int_{G(n-1,n)} \#(X \cap L) dL,$$

where  $G(n-1, n)$  denotes the space of all  $n-1$ -dimensional linear subspaces of  $\mathbb{C}^n$  with standard measure. Since  $W$  is of degree  $d$ , we get that  $\text{vol}(X) = \text{poly}_n(d)$ .

Next we suitably modify the proof of [5, Proposition 17]. Set  $X' = S^1 \cdot X$  and let  $N(X, \epsilon)$  denote the size of the smallest  $\epsilon$ -net in  $X'$ . By [5, Lemmas 15, 16] we have that

$$N(X', \epsilon) = O_n(\text{vol}(X')\epsilon^{-2 \cdot 1 - 1}) = O_n(\text{poly}_n(d)\epsilon^{-3}).$$

Let  $S$  be a  $4\epsilon$ -separated set in  $B^2$  with  $\#S = O_n(1)\epsilon^{-2n}$  such that  $S$  is closed under  $n$  pairwise orthonormal rotations. Since  $n \geq 2$ , there is an epsilon such that  $\#S > nN(X', \epsilon)$ . For such an epsilon, the proof of [5, Proposition 17] gives balls  $B_1, \dots, B_n \subset B$  of radius  $\epsilon$  with  $1/\epsilon = \text{poly}_n(d)$ , each disjoint from  $X'$ , such that the set of vectors from the origin to the centers of  $B_1, \dots, B_n$  form an orthogonal basis of  $\mathbb{C}^n$ .

Now we follow the proof of [5, Corollary 18], and in turn [7, Theorem 7], to obtain  $\eta = \text{poly}_n(d)$  and orthogonal  $\eta$ -gap polydiscs  $\Delta_1, \dots, \Delta_n \subset B$  for  $W$ , each centered around the origin and containing  $B^\eta$ .  $\square$

**6.3. Distance of a curve from  $X$ .** Given a curve  $W \in \mathcal{C}_{d,n}^*$ , we set  $\mathcal{N}_k^W$  to be a  $\frac{1}{k}$ -net of  $W \cap \bar{B}^{1/2}$  for each  $k \in \mathbb{N}$ . This means that the set of balls of radius  $1/k$  around points of  $\mathcal{N}_k^W$  covers  $W \cap \bar{B}^{1/2}$ , and the balls of radius  $1/2k$  around these points are pairwise disjoint.

Let  $\eta$  be as given by Lemma 6.4, and set  $a_k^W$  to be the  $m^{\text{th}}$ -smallest value in the set  $\{|g(a)| : a \in \mathcal{N}_k^W\}$ , where  $m$  is the largest integer smaller than  $\frac{k^2}{(100\eta)^2}$ ; if  $m = 0$

we set  $a_k^W = 1$ . For a ball  $A \subset \mathbb{C}^n$  of radius  $R$  the analogue of Crofton's formula [12, Proposition 14.6.3, Section 15.1] gives that

$$\text{vol}(W \cap A) \leq d\pi R^2, \quad (3)$$

where  $\text{vol}$  is  $\text{vol}_1$  from [12] and  $d$  is the degree of  $W$ . If the center of  $A$  belongs to  $W$ , then we also have that

$$\text{vol}(W \cap A) \geq \pi R^2. \quad (4)$$

Since  $\mathcal{N}_k^W$  is a  $1/k$ -net of  $W \cap \bar{B}^{1/2}$  and  $W \cap \bar{B}^{2/3} \neq \emptyset$ , we have that

$$\frac{\pi}{16} \leq \text{vol}(W \cap B^{1/2}) \leq \frac{d\pi \mathcal{N}_k^W}{k^2},$$

and hence  $\mathcal{N}_k^W \geq k^2/16d$ . So  $\#\mathcal{N}_k^W > k^2/(100\eta)^2$ , and the definition of  $a_k^W$  makes sense. We work with the following measure of the distance of a curve from  $X$ :

$$\text{dist}(X, W) := \liminf_{k \rightarrow \infty} a_k^W.$$

Since  $g$  is analytic,  $\text{dist}(X, W)$  does not depend on the choice of the nets  $\mathcal{N}_k^W$ .

**Lemma 6.5.** *For any  $W \in \mathcal{C}_{d,n}^*$  we have that  $\text{dist}(X, W) > 0$ .*

*Proof.* Let a curve  $W \in \mathcal{C}_{d,n}^*$  be given. The set  $W_g := \{z \in B^{1/2} : g(z) = 0\} \cap W$  is a bounded semianalytic set and thus has only finitely many connected components [21]. This means that  $W_g$  is finite since  $X$  does not contain any semialgebraic curve. Let  $M$  be such that  $\#W_g < M$ .

Let  $B_1, \dots, B_M$  be pairwise disjoint balls in  $\mathbb{C}^n$  of radius at most  $1/200M\eta d$  centered around points of  $W_g$ . Since  $(W \cap \bar{B}^{1/2}) \setminus \bigcup_{i=1}^M B_i$  is a compact set, there is a  $\tau > 0$  such that  $|g(z)| \geq \tau$  for all  $z \in (W \cap \bar{B}^{1/2}) \setminus \bigcup_{i=1}^M B_i$ .

Let  $S_k$  be number of points of  $\mathcal{N}_k^W$  in  $\bigcup_{i=1}^M B_i$ . Then for  $k > 100M\eta d$ , we have that  $\{z \in W : \exists a \in S_k \text{ and } |a - z| < 1/2k\} \subseteq \bigcup_{i=1}^M B_i^{1/2}$ . Recall for any ball  $A$ ,  $A^{1/2}$  is the ball with the same center and twice the radius of  $A$ . Then applying (3) and (4) we get that

$$\frac{S_k \pi}{4k^2} < \sum_{i=1}^M \text{vol}(W \cap B_i^{1/2}) \leq \frac{d\pi}{(200\eta d)^2}.$$

This gives that  $S_k < k^2/(100\eta)^2$ , and hence  $a_k^W \geq \tau$  for all  $k > 100M\eta d$ . This immediately implies that  $\text{dist}(X, W) \geq \tau > 0$ , and the proof is complete.  $\square$

In fact the compactness  $\mathcal{C}_{d,n}^*$  gives something much stronger.

**Lemma 6.6.** *For all  $d$ , there is  $\tau_0 = \tau_0(d, X) > 0$  such that for any  $W \in \mathcal{C}_{d,n}^*$  we have that*

$$\text{dist}(X, W) > \tau_0.$$

*Proof.* Towards a contradiction, suppose there is a sequence  $W_1, W_2, \dots \in \mathcal{C}_{d,n}^*$  with  $\text{dist}(X, W_m) < 1/m$ ,  $m \in \mathbb{N}^{\geq 1}$ . Then by Lemma 6.2 we have that there is a subsequence  $W_{k_1}, W_{k_2}, \dots$  which converges to some curve  $W_0$  in  $\mathcal{C}_{d,n}^*$ . Then using the uniform continuity of  $g$  over  $\bar{B}^{1/2}$  one easily checks that  $\text{dist}(X, W_0) = 0$ , and we have arrived at a contradiction to Lemma 6.5.  $\square$

#### 6.4. Finishing the proof of Theorem 1.1.

**Resultants of  $g$ .** Let  $\Delta = \Delta_z \times \Delta_w$  be a polydisc for a curve  $W$ . The *analytic resultant* of  $g$  with respect to  $W$ ,  $\Delta$  is the function  $g_{W,\Delta} : \Delta_z \rightarrow \mathbb{C}$ , given by

$$g_{W,\Delta}(z) := \prod_{(z,w) \in W \cap \Delta} g(z,w).$$

Note  $g_W$  is analytic over  $\Delta_{z,W}$  by [8, Fact 5]. If  $W \cap \Delta = \emptyset$ , we set  $g_{W,\Delta}(z) := 1$  for all  $z \in \Delta_z$ .

**Lemma 6.7.** *There is a  $\tau = \tau(d, X) > 0$  such that for every curve  $W$  in  $\mathbb{C}^n$  of degree  $d$ , an  $\eta$ -gap polydisc  $\Delta = \Delta_z \times \Delta_w \subseteq \bar{B}^{2/3}$  for  $W$ , and  $p \in \Delta_z^2$ , we have that*

$$\max_{z \in (\Delta_z^4)_p} |g_{W,\Delta}(z)| > 2^{-\tau}.$$

Moreover, if  $g$  is computable then we can compute a  $\tau$  satisfying the above.

*Proof.* We start with proving the first assertion. Lemma 6.6 gives  $\tau_0 > 0$  such that for all curves in  $\mathbb{C}^n$  of degree  $d$  with  $W \cap \bar{B}^{2/3} \neq \emptyset$ , we have that  $\text{dist}(X, W) > \tau_0$ . Fix such a curve  $Y$  and an  $\eta$ -gap polydisc  $\Delta = \Delta_z \times \Delta_w \subseteq \bar{B}^{2/3}$  for  $Y$ . It suffices to consider the case of  $Y \cap \Delta \neq \emptyset$ , and we assume this going further in the proof.

Using that  $g$  is continuous and  $\bar{B}^{1/2}$  is compact, we get  $k$  such that for all  $w, z \in \bar{B}^{1/2}$ ,  $|w - z| < 1/k$  implies that  $|g(z) - g(w)| < \tau_0/4$ . Let  $\mathcal{N}_k^Y$  be a  $1/k$ -net of  $Y \cap \bar{B}^{1/2}$ . Since  $d(X, Y) > \tau_0$ , it follows that  $a_k^Y > 3\tau_0/4$ , which means the  $m^{\text{th}}$ -smallest value in the set  $\{|g(a)| : a \in \mathcal{N}_k^Y\}$  is greater than  $3\tau_0/4$ , where  $m$  is largest integer smaller than  $\frac{k^2}{(100\eta)^2}$ .

Let  $S_k := \{z \in Y \cap \bar{B}^{1/2} : |g(z)| \leq \frac{\tau_0}{2}\}$ . So we have that  $S_k$  is contained in at most  $\frac{k^2}{(100\eta)^2}$  balls of radius  $1/k$ . Hence the area of the set  $\pi_z(\Delta \cap S_k)$  is at most  $\frac{k^2}{(100\eta)^2} \cdot \frac{\pi}{k^2} \leq \frac{\pi}{(100\eta)^2}$ . We have that the disc  $(\Delta_z^4)_p$  has radius at least  $1/8\eta$ , and hence there is  $a \in (\Delta_z^4)_p \setminus \pi_z(\Delta \cap S_k)$ . This gives that all  $d$  fibers (counted with multiplicity)  $(a, b) \in Y \cap \Delta$  belong in  $(Y \cap \Delta) \setminus S_k$ , and hence  $g_{Y,\Delta}(a) > (\tau_0/2)^d$ . So the choice of  $\tau = -\ln_2(\tau_0/2)^d$  finishes the proof of the first assertion.

For the second assertion, we suppose that  $g$  is computable and obtain the function  $h_g$  as given by Definition 2.1. For  $k \in \mathbb{N}^{\geq 1}$ , we compute  $l_k \in \mathbb{N}^{\geq 1}$  to be the largest integer such that  $h_g(l_k) \leq k$ . Note  $l_k \rightarrow \infty$  as  $k \rightarrow \infty$  and for any  $W \in \mathcal{C}_{d,n}^*$ ,  $|\text{dist}(X, W) - a_k^W| < 2/l_k$ . We iterate on  $k \in \mathbb{N}^{\geq 1}$ , starting with  $k = 1$ . Using the fact that  $\mathcal{C}_{d,n}$  is compact, we explicitly obtain  $Y_1, \dots, Y_{m_k} \in \mathcal{C}_{d,n}^*$  such that for all  $Y' \in \mathcal{C}_{d,n}^*$  there is a  $\mu \in \{1, \dots, m_k\}$  with  $\text{dist}_H(Y', Y_\mu) < 1/2k$ .

Fix a  $\mu \in \{1, \dots, m_k\}$ . We employ [15] to compute a set of points  $U_\mu$  such that  $\text{dist}_H(Y_\mu, U_\mu) < 1/2k$ . Let  $m$  be the largest integer smaller than  $\frac{k^2}{(100\eta)^2}$ . Let  $U_\mu^*$  be the collection of subsets of  $U_\mu$  of size  $m$  such that the points in the subset are pairwise at a distance of at least  $1/k$  from each other. We set

$$b_V := \min_{a \in U_\mu, \text{dist}_H(a, V) \geq \frac{3}{2k}} |q_a|, \quad \text{and} \quad b_k^{Y_\mu} := \max_{V \in U_\mu^*} b_V,$$

where  $q_a \in \mathbb{Q}$  is such that  $|g(a) - q_a| < 1/l_k$  as given by Definition 2.1. For a  $1/k$ -net  $\mathcal{N}_k^{Y_\mu}$  of  $Y \cap \bar{B}^{1/2}$ , there is  $V_0 \in U_\mu^*$  such that for every point  $p$  in the set  $\{a \in \mathcal{N}_k^{Y_\mu} : |g(a)| \leq a_k^{Y_\mu}\}$  there is a unique point in  $V_0$  at a distance less than  $1/2k$  to  $p$ . Hence for all  $p \in U_\mu$  such that  $\text{dist}_H(p, V_0) \geq 3/2k$ , we have that  $|g(p)| > a_k^{Y_\mu} - 1/l_k$ . Hence we have that  $b_k^{Y_\mu} > a_k^{Y_\mu} - 2/l_k$  for all  $\mu$ , and we compute

$\tau_k := \min_{\mu} b_k^{Y_{\mu}}$ . We stop the iteration when we reach a  $k$  such that  $\tau_k - 1/l_k > 0$ . Since we know by Lemma 6.6 that there is  $\tau_0$  such that  $\inf_{W \in \mathcal{C}_{d,n}^*} a_k^W > \tau_0 - 2/l_k$ , and hence  $\tau_k - 1/l_k > \tau_0 - 5/l_k$ , our iterative process always terminates.

We finish the proof by showing that for  $\tau := \tau_k - 1/l_k > 0$ , we have that

$$\max_{z \in \Delta_z^2} |g_{Y,\Delta}(z)| > \tau^d.$$

Recall that  $Y$  is a curve in  $\mathbb{C}^n$  of degree  $d$  and  $\Delta = \Delta_z \times \Delta_w \subset \bar{B}^{2/3}$  is some  $\eta$ -gap polydisc for  $Y$  with  $Y \cap \Delta \neq \emptyset$ . By the choice of the collection  $(Y_{\mu})_{\mu \leq m_k}$ , there is  $Y_{\mu}$  such that  $\text{dist}_H(Y, Y_{\mu}) < 1/2k$ . We have that  $b_k^{Y_{\mu}} \geq \tau_k$  and hence the set  $S_k^{\mu} := \{z \in Y_{\mu} \cap \bar{B}^{1/2} : |g(x)| \leq \tau\}$  is contained in at most  $\frac{k^2}{(100\eta)^2}$  balls of radius  $3/2k$ . It follows that  $S_k' := \{z \in Y \cap \bar{B}^{1/2} : |g(x)| \leq \tau\}$  is contained in at most  $\frac{k^2}{(100\eta)^2}$  balls of radius  $2k$ . Hence the area of the set  $\pi_z(\Delta \cap S_k')$  is at most  $\frac{k^2}{(100\eta)^2} \cdot \frac{\pi}{(2k)^2} \leq \frac{\pi}{(50\eta)^2}$ . We have that the disc  $(\Delta_z^4)_p \subset \mathbb{C}$  has radius at least  $1/8\eta$ , and hence there is  $a \in (\Delta_z^4)_p \setminus \pi_z(\Delta \cap S_k')$ . As in the proof of the first assertion, this gives that all  $d$  fibers (counted with multiplicity)  $(a, b) \in Y \cap \Delta$  belong in  $(Y \cap \Delta) \setminus S_k'$ , and hence  $g_{Y,\Delta}(a) > \tau^d$ . The proof is complete.  $\square$

This parameter  $\tau(d, X)$  given by Lemma 6.7 serves as a relative Bernstein index measure for  $X$ . We arrange that  $\tau(d, X)$  is a non-increasing function of  $d$ .

**Lemma 6.8.** *Let  $W$  be a curve of degree at most  $d$ . Then for any polydisc  $\Delta = \Delta_z \times \Delta_w \subseteq \bar{B}^{2/3}$  for  $W$  and  $p \in \Delta_z^2$ , we have for all  $r \leq 1/2$  that*

$$\max_{z \in (\Delta_z^{2/r})_p} |g_{W,\Delta}(z)| > r^{\tau}.$$

*Proof.* Let  $W$  and  $p \in W \cap B$  be given. We employ Lemma 6.7 to obtain  $\tau$  such that  $\|g_{W,\Delta}\|_{(\Delta_z^4)_p} > 2^{-\tau}$ . Since  $\|g\|_{B^{1/2}} \leq 1$ , we have that  $\|g_{W,\Delta}\|_{(\Delta_z^2)_p} \leq 1$ , and our desired conclusion follows by a direct application of the Hadamard three circle theorem.  $\square$

Recall that  $D(a, r) \subset \mathbb{C}$  denotes the open disc of radius  $r$  around the  $a \in \mathbb{C}$ . We shall use the following fact which follows from the classical Bloch-Landau theorem.

**Lemma 6.9.** *Let  $f : D(0, 1) \rightarrow D(0, 1)$  be an analytic function. For all  $\epsilon < 1/4$ , if  $f(D(0, 1/2))$  does not contain a ball of radius  $\epsilon$ , then  $\text{diam}(f(D(0, 1/2))) < 10\sqrt{\epsilon}$ .*

*Proof.* Let  $f : D(0, 1) \rightarrow D(0, 1)$  be an analytic function such that  $f(D(0, 1/2))$  does not contain a ball of radius  $\epsilon$ . The Cauchy integral formula gives us that  $|f'(z)| \leq 2$  and  $|f''(z)| \leq 8$  for all  $z \in D(0, 1/2)$ . We claim that  $|f'(z)| < 6\sqrt{\epsilon}$  for all  $z \in D(0, 1/2 - \sqrt{\epsilon})$ . The desired conclusion that  $\text{diam}(f(D(0, 1/2))) < 10\sqrt{\epsilon}$  immediately follows from this claim.

Suppose towards a contradiction that there is  $z_0 \in D(1/2 - \sqrt{\epsilon})$  such that  $|f'(z_0)| \geq 6\sqrt{\epsilon}$ . Then for  $w \in \partial D(z_0, \sqrt{\epsilon})$  and  $a \in D(0, \sqrt{\epsilon}/6)$ , we have that

$$|f(w) - f(z_0) - f'(z_0)a - f'(z_0)(w - z_0 - a)| \leq \max_{w \in D(1/2)} |f''(w)|\epsilon/2 \leq 4\epsilon.$$

We combine this with

$$|f'(z_0)(w - z_0 - a)| \geq 5\epsilon$$

and use Rouché's theorem to get that  $f(D(z_0, \sqrt{\epsilon}))$  contains a disc around  $f(z_0)$  of radius  $|f'(z_0)| \cdot \sqrt{\epsilon}/6 \geq \epsilon$ . We have arrived at the desired contradiction and the proof is complete.  $\square$

**Theorem 6.10.** *Let  $X$  and  $\epsilon$  be given. Then there are constants  $d = \text{poly}_n(\epsilon^{-1})$  and  $c = \text{poly}_n(\epsilon^{-1}, \tau(d, X))$  such that for all  $T$  we have that*

$$\#X(\mathbb{Q}, T) \leq cT^\epsilon.$$

*Proof.* For  $X$  and  $\epsilon$ , fix  $d = \text{poly}_n(\epsilon^{-1})$  as given by Theorem 1.2 and let  $\tau = \tau(d, X)$ . Applying Theorem 1.2 for  $\nu = 6\tau$ , we get  $c = \text{poly}_n(\tau, \epsilon^{-1})$  such that for all  $T$ ,  $X(\mathbb{Q}, T)$  is covered by  $cT^\epsilon$  many  $d$ -blocks in  $X(T^{-6\tau})$ . Note here we used that the value of  $d$  given by Theorem 1.2 does not depend on  $\nu$ .

Let  $T \geq 10n$  and suppose  $K$  is a  $d$ -block in  $X(T^{-6\tau})$ . We shall show that  $\text{diam}(K) < T^{-2}$ , which implies that  $\#K(\mathbb{Q}, T) \leq 1$ , and the proof will be complete. It suffices to only consider the case where  $K = \phi(D(0, 1/2))$  for some analytic and algebraic function  $\phi$ . By the definition of a  $d$ -block, there is a curve  $W$  of degree at most  $d$  with  $K \subseteq W$ , and fix a point  $p \in K$ . We apply Lemma 6.4 to get orthogonal polydiscs  $\Delta_1 = \Delta_{z;1} \times \Delta_{w;1}, \dots, \Delta_n = \Delta_{z;n} \times \Delta_{w;n} \subset B^{2/3}$  for  $W$  centered at  $p$ .

Let  $j \in \{1, \dots, n\}$ . By Lemma 6.8 we have that  $\max_{z \in (\Delta_{z;j}^{2/r})_q} |g_{W,\Delta}(z)| > r^\tau$  for all  $r \leq 1/2$  and  $q \in \Delta_{z;j}^2$ . Combining this with  $\|g\|_{B^{1/2}} \leq 1$  and  $\|g\|_K \leq T^{-6\tau}$ , we get that  $\pi_{z;j}(K)$  does not contain a ball of radius  $T^{-6}$  around any  $q \in \Delta_{z;j}^2 \cap \pi_{z;j}(K)$ . Here  $p$  was an arbitrary point in  $K$ , and the same argument holds around any point in  $K$ . This gives that  $\pi_{z;j}(K)$  does not contain a ball of radius  $\sqrt{n}T^{-6}$  and hence  $\text{diam}(\pi_{z;j}(K)) < 10\sqrt{n}T^{-3}$  by Lemma 6.9. This conclusion holds for all  $j \in \{1, \dots, n\}$  and Fact 6.3 implies that  $\text{diam}(K) < 10nT^{-3} \leq T^{-2}$  as desired.  $\square$

*Proof of Theorem 1.1, Case  $n \geq 2$ :* Lemma 6.7 implies that if  $g$  is computable then  $\tau(d, X)$  is also computable for every  $d$ . Hence we are done by Theorem 6.10.  $\square$

## REFERENCES

1. B. Bakker, Y. Brunebarbe, and J. Tsimerman, *o-minimal GAGA and a conjecture of Griffiths*, Invent. Math. **232** (2023), no. 1, 163–228. MR 4557401
2. N. Bhardwaj and L. van den Dries, *On the Pila–Wilkie theorem*, Expositiones Mathematicae **40** (2022), no. 3, 495–542.
3. G. Binyamini, *Log-Noetherian functions*, arXiv e-prints (2024), arxiv.org/abs/2405.16963.
4. ———, *Density of algebraic points on Noetherian varieties*, Geom. Funct. Anal. **29** (2019), no. 1, 72–118. MR 3925105
5. ———, *Point counting for foliations over number fields*, Forum Math. Pi **10** (2022), Paper No. e6, 39. MR 4390301
6. G. Binyamini, G. O. Jones, H. Schmidt, and M.E.M. Thomas, *An effective Pila–Wilkie theorem for sets definable using Pfaffian functions, with some diophantine applications*, arXiv e-prints (2023), arxiv.org/abs/2301.09883.
7. G. Binyamini and D. Novikov, *The Pila–Wilkie theorem for subanalytic families: a complex analytic approach*, Compos. Math. **153** (2017), no. 10, 2171–2194. MR 3705287
8. ———, *Wilkie’s conjecture for restricted elementary functions*, Ann. of Math. (2) **186** (2017), no. 1, 237–275. MR 3665004
9. ———, *Complex cellular structures*, Ann. of Math. (2) **190** (2019), no. 1, 145–248. MR 3990603
10. G. Binyamini, D. Novikov, and B. Zak, *Wilkie’s conjecture for Pfaffian structures*, Ann. of Math. (2) **199** (2024), no. 2, 795–821. MR 4713023
11. G. Binyamini, H. Schmidt, and A. Yafaev, *Lower bounds for Galois orbits of special points on Shimura varieties: a point-counting approach*, Math. Ann. **385** (2023), no. 1-2, 961–973. MR 4542738
12. E. M. Chirka, *Complex analytic sets*, Mathematics and its Applications (Soviet Series), vol. 46, Kluwer Academic Publishers Group, Dordrecht, 1989, Translated from the Russian by R. A. M. Hoksbergen. MR 1111477

13. D. A. Cox, J. Little, and D. O'Shea, *Ideals, varieties, and algorithms*, fourth ed., Undergraduate Texts in Mathematics, Springer, Cham, 2015, An introduction to computational algebraic geometry and commutative algebra. MR 3330490
14. John Dalbec and Bernd Sturmfels, *Introduction to Chow forms*, Invariant methods in discrete and computational geometry (Curaçao, 1994), Kluwer Acad. Publ., Dordrecht, 1995, pp. 37–58. MR 1368005
15. Dufresne E., P. B. Edwards, H. A. Harrington, and J. D. Hauenstein, *Sampling real algebraic varieties for topological data analysis*, arXiv e-prints (2024), [arxiv.org/abs/1802.07716](https://arxiv.org/abs/1802.07716).
16. I. M. Gelfand, M. M. Kapranov, and A. V. Zelevinsky, *Discriminants, resultants, and multidimensional determinants*, Mathematics: Theory & Applications, Birkhäuser Boston, Inc., Boston, MA, 1994. MR 1264417
17. Mark L. Green and Ian Morrison, *The equations defining Chow varieties*, Duke Math. J. **53** (1986), no. 3, 733–747. MR 860668
18. R. C. Gunning and H. Rossi, *Analytic functions of several complex variables*, AMS Chelsea Publishing, Providence, RI, 2009, Reprint of the 1965 original. MR 2568219
19. Y. Il'yashenko and S. Yakovenko, *Counting real zeros of analytic functions satisfying linear ordinary differential equations*, J. Differential Equations **126** (1996), no. 1, 87–105. MR 1382058
20. G. O. Jones and A. J. Wilkie (eds.), *O-minimality and diophantine geometry*, London Mathematical Society Lecture Note Series, vol. 421, Cambridge University Press, Cambridge, 2015, Lecture notes from the LMS-EPSRC course held at the University of Manchester, Manchester, July 2013. MR 3410216
21. S. Łojasiewicz, *Sur les ensembles semi-analytiques*, Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 2, Gauthier-Villars Éditeur, Paris, 1971, pp. 237–241. MR 425152
22. J. Pila, *Integer points on the dilation of a subanalytic surface*, Q. J. Math. **55** (2004), no. 2, 207–223. MR 2068319
23. ———, *Point-counting and the Zilber-Pink conjecture*, Cambridge Tracts in Mathematics, vol. 228, Cambridge University Press, Cambridge, 2022. MR 4420059
24. J. Pila, A.N. Shankar, J. Tsimerman, H. Esnault, and M. Groechenig, *Canonical heights on Shimura varieties and the André-Oort conjecture*, arXiv e-prints (2024), [arxiv.org/abs/2109.08788](https://arxiv.org/abs/2109.08788).
25. J. Pila and A. J. Wilkie, *The rational points of a definable set*, Duke Math. J. **133** (2006), no. 3, 591–616. MR 2228464
26. Pablo Solernó, *Effective Łojasiewicz inequalities in semialgebraic geometry*, Appl. Algebra Engrg. Comm. Comput. **2** (1991), no. 1, 2–14. MR 1209239
27. U. Zannier, *Some problems of unlikely intersections in arithmetic and geometry*, Annals of Mathematics Studies, vol. 181, Princeton University Press, Princeton, NJ, 2012, With appendixes by David Masser. MR 2918151

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