EFFECTIVE AND APPROXIMATE PILA-WILKIE TYPE COUNTING WITH COMPLEX-ANALYTIC SETS

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Abstract. We establish an effective point-counting statement of Pila-Wilkie type for the zero-set of any computable complex-analytic function over a ball. For our proof, we develop an approximate counting result to cover rational points *close* to an analytic subset of a ball. This second theorem is unique among results of Pila-Wilkie type in that it applies uniformly to any analytic subset of a ball.

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1. Introduction

In [17], Pila and Wilkie give a subpolynomial upper bound, in terms of their heights, on the number of rational points inside the transcendental part of a set definable in an o-minimal expansion of the real field. This point counting theorem applies to a wide range of sets and functions of fundamental interest, and has had striking applications to arithmetic geometry, functional transcendence, and Hodge theory. See [2] for a simplified proof and self contained treatment of the Pila-Wilkie theorem, and [20, 1, 16, 19] for a detailed description of the aforementioned applications.

The bound given by the Pila-Wilkie theorem cannot be made effective general, but various effectivity results have been realized for sets cut out by functions satisfying a certain form of differential equations, see [4, 12, 3]. Among these results, the statement in [3] gives the most general effective statement. This result applies to all sets definable in the Pfaffian closure of the expansion of the reals by Log-Noetherian functions, and effectivizes various applications of the Pila-Wilkie theorem to arithmetic geometry and Hodge theory.

In this paper we establish an effective Pila-Wilkie type statement for the zero-set of any *computable*[1] complex analytic function over a ball. We work with the height function $\mathcal{H}:\mathbb{Q}\to\mathbb{R}$ by $\mathcal{H}(\frac{a}{b}) := \max(|a|,\ |b|)\in\mathbb{N}^{\geqslant 1}$ for coprime $a,b\in\mathbb{Z}$,

[1]see Section 2 for the definition of a computable complex analytic function.

$b \neq 0$. Thus $\mathcal{H}(0) = 1$, and for $a = (a_1, \ldots, a_n) \in \mathbb{Q}^n$, we extend $\mathcal{H}$ to be given by $\mathcal{H}(a) := \max\{\mathcal{H}(a_1), \ldots, \mathcal{H}(a_n)\}$. For $T \in \mathbb{R}^{\geqslant 2}$ and $X \subseteq \mathbb{C}^n$, we set $X(\mathbb{Q}, T) := \{a \in X \cap \mathbb{Q}^n : \mathcal{H}(a) \leqslant T\}$, the (finite) set of rational points of X of height at most T.

Theorem 1.1. *Let $X := \{z \in B : g(z) = 0\}$, where $B \subseteq \mathbb{C}^n$ is a ball and g is a analytic function on $B^{1/2}$. If the function g is computable and X does not contain any semialgebraic curve, then for every $\epsilon > 0$ there is a computable constant $c = c(X, \epsilon) > 0$ such that for all T*

$$\#X(\mathbb{Q}, T) \leqslant cT^\epsilon.$$

For any ball, $B^{1/2}$ denotes the ball with the same center as B and twice the radius. Throughout the rest of the paper we fix $B \subset \mathbb{C}^n$ to denote the ball of radius $1/2$ around the origin. We prove Theorem 1.1 in Section 6 for this fixed B which clearly implies the general statement above. Note the asymptotics of T^ϵ in Theorem 1.1 are optimal due to [18, Example 7.5].

Our proof of Theorem 1.1 relies on certain approximate Pila-Wilkie type counting technology. Let $X = \{z \in B : g_1(z) = \ldots = g_m(z) = 0\}$, where the $g_1, \ldots, g_m$ are complex-analytic functions on $B^{1/2}$ with $\|g_i\|_{B^{1/2}} \leqslant 1$. We call such an X an *analytic* set and set

$$X(\omega) := \{z \in B : |g_1(z)|, \ldots, |g_m(z)| \leqslant \omega\},$$

which we call the *ω-lemniscate of X*.

We are now ready to state our second main theorem whose form parallels the more intricate block version of the Pila-Wilkie theorem, and the statement is unique in the realm of results of this type in that the counting parameters, in particular the constant c, does not depend on the analytic set X.

Theorem 1.2. *Let ϵ and ν be given. Then there are $\theta, c = \mathrm{poly}_n(\nu, \epsilon^{-1})$ and $d = \mathrm{poly}_n(\epsilon^{-1})$ such that for every analytic set X and all T, we have that $X(T^{-\theta})(\mathbb{Q}, T)$ is contained in at most cT^ϵ many d-blocks in $X(T^{-\nu})$.*

A d-block in $X(\omega)$ is a holomorphic image of a polydisc inside the set $X(\omega)$, where the graph of each co-ordinate of this map is an analytic component of an algebraic variety of degree at most d. This notion is defined in more detail in Section 5, and the $\mathrm{poly}_n(\cdot)$ notation is defined below. Although we state and prove this theorem for fixed B, a similar statement for analytic sets in any ball is an immediate consequence.

The fact that Theorem 1.2 gives a d which depends only on ϵ plays a critical role in our proof of Theorem 1.1. To realize this technical feature we prove and employ a Pila-Wilkie statement for algebraic sets – Theorem 3.6. This theorem is of independent interest and says that for any ϵ and algebraic set W, the rational points of height at most T on W lie on $O(T^\epsilon)$ many algebraic subsets of W whose degree depend only on ϵ and not on the degree of W. We also record a variant where these rational points can be covered by algebraic subsets whose number and degrees grow polynomially in $\log(T)$. We also include a similar logarithmic growth variant for Theorem 1.2 as well– Theorem 5.3. These $\log(T)$-variants are not used for the proof of Theorem 1.1.

Notations and conventions. Throughout we work with the conventions that $k, l, m, N \in \mathbb{N} = \{0, 1 \ldots\}$, $\mu, d, e, i, j, n \in \mathbb{N}^{\geqslant 1} := \{1, 2, \ldots\}$. For $\alpha, \sigma \in \mathbb{N}^l$, we set

$|\sigma| := \sigma_1 + \ldots + \sigma_l \in \mathbb{N}$. Also $t \in \mathbb{R}$; $\beta, \gamma, \delta, \epsilon, \rho, \omega, C \in (0,1)_{\mathbb{R}} := \{t :\ 0 < t < 1\}$; $\eta, c, T \in \mathbb{R}^{\geqslant 2} := \{t :\ t \geqslant 2\}$; $\theta, \nu, r, M \in \mathbb{R}^{>} := \{t :\ t > 0\}$. We follow the natural convention that any symbol with an added subscript will denote an object in the same domain as the plain symbol, so $d_\nu \in \mathbb{N}^{\geqslant 1}, \eta_1 \in \mathbb{R}^{\geqslant 2}$, and so on. We will denote Euler's constant by e.

For a ball or polydisc $A \subset \mathbb{C}^n$ with center $p \in \mathbb{C}^n$, set $A^\delta := p + \delta^{-1}(A - p)$. For a point $p \in \mathbb{C}^n$ and a ball/polydisc A centered at the origin, A_p denotes the ball/polydisc with the same radius/polyradius as A but centered at p.

For any domain $\Omega \subseteq \mathbb{C}^n$ bounded domain, we let $\mathcal{O}(\Omega)$ denote the set of complex-valued analytic functions on Ω. We shall work with complex cells and cellular maps and related technology from [9]. For a complex cell $\mathscr{C} \subset \mathbb{C}^n$, $\mathcal{O}(\mathscr{C})$ will denote the set of cellular maps with domain $\mathscr{C}$, and we use this notation so that no confusion may arise. For a function $h \in \mathcal{O}(\Omega)$ and a ball or polydisc $A \subseteq \Omega$, we set $\|h\|_A := \max_{z \in \bar{A}} |h(z)|$.

The notation $s = \mathrm{poly}_n(t_1, t_2, t_3)$ means that $s \leqslant P_n(t_1, t_2, t_3)$, where $n \to P_n$ is a universally fixed mapping with P_n being polynomials with positive coefficients.

2. The 0-dimensional case for Theorem 1.1

We begin by defining the notion of computability for complex-valued functions.

Definition 2.1. *A function $f : B^{1/2} \to \mathbb{C}$ is* computable *if*

(1) for every computable sequence $\{z_i\}_{i \in N}$ of complex tuples in $B^{1/2} \subset \mathbb{C}^n$, the sequence $\{f(z_i)\}_{i \in \mathbb{N}}$ is also computable.
(2) there is a computable function $d_f : \mathbb{N} \to \mathbb{N}$, such that $|z - w| < 1/d(n)$ implies $|f(z) - f(w)| < 1/n$.

We will use the following implication of [15, Lemma 1, Example 1] to get an upper bound for the size of the finite set $X = \{z \in D^2 : g(z) = 0\}$.

Lemma 2.2. *Let $f : D \to \mathbb{C}$ be analytic function on a disc. Then we have that*

$$\#\{z \in D^2 : g(z) = 0\} \ <\ 5 \cdot \log(\|g\|_D / \|g\|_D^2), \quad \textit{and,}$$

$$\#\{z \in D^\eta : g(z) = 0\} \ <\ \big(1/\log(\eta/2)\big) \cdot \log(\|g\|_D / \|g\|_D^\eta) \quad \textit{for } \eta > 2.$$

Proof of Theorem 1.1, Case $n = 1$: Let $g : D \to \mathbb{C}$ be a non-constant computable analytic function, where $D \subset \mathbb{C}$ is a disc around the origin. By Lemma 2.2 it suffices to compute a number τ such that $\tau > \|g\|_D / \|g\|_D^2$. Starting with $k = 1$ and iterating over $k \in \mathbb{N}^{\geqslant 1}$, we compute $1/d_g(k)$-nets S_1 and S_2, for ∂D^2 and ∂D respectively. Here d_g is the computable functions given by Definition 2.1, and we let m and M be the maximum of the absolute values of g at the points of S_1 and S_2 respectively. If m and M are both non-zero then

$$\|g\|_D / \|g\|_D^2 < (M + 1/k)/m,$$

and we are done. This process terminates for some since g has only finitely many zeros in D. □

3. An effective Pila-Wilkie type statement for algebraic sets

In this section, we show that the rational points of height at most T on a algebraic sets are covered by algebraic subsets of uniformly bounded degree, and that the number of such algebraic subsets grows subpolynomially in T. This essentially follows from the Pila-Wilkie counting strategy and we use the powerful cellular parameterization theorem from [9] to include sharp effectivity feaures. Theorem 3.5 parallels the block version of the Pila-Wilkie theorem, except that we count rational points on algebraic instead of transcendental objects.

As alluded to in the introduction, we will employ Theorem 3.5 to arrange that d depends solely on ϵ in Theorem 1.2, and will later use this feature in a critical way for the proof of Theorem 1.1. We start with some simple calculations that will be useful also in later sections.

Some combinatorial notation and basic facts. In this subsection, suppose $\mu, e, m \in \mathbb{N}^{\geqslant 1}$ are positive integers satisfying $\mu > e(m+1)$. Set $L(n,l) := \binom{n+l-1}{l}$.

Let k be the largest integer such that $\sum_{l=0}^{k} eL(m,l) < \mu$; note the condition on μ implies that $k \geqslant 1$. We have that $\sum_{l=0}^{k} eL(m,l) = eL(m+1,k)$ and also,

$$\frac{k^m}{m!} < L(m+1,k) < \frac{\mu}{e} \leqslant L(m+1,k+1) < \frac{2k^m(m+1)^m}{m!}. \tag{1}$$

This gives $k > \frac{1}{m+1}(\frac{m!\mu}{2e})^{\frac{1}{m}}$, and $k < (\frac{m!\mu}{e})^{\frac{1}{m}}$. Let $S(m,k) := \sum_{l=0}^{k} eL(m,l)\cdot l$. Then $S(m,k) = emL(m+2,k-1) > \frac{ek}{2(m+1)}L(m+1,k+1)$ using the fact that $2m(k+1) > m+k+1$. The lower bounds for $L(m+1,k+1)$ and k from (1) imply

$$S(m,k) > \big(\frac{m!}{2e}\big)^{\frac{1}{m}} \frac{\mu^{1+\frac{1}{m}}}{2(m+1)^2}. \tag{2}$$

We set $E_m := \frac{1}{2(m+1)^2}(\frac{m!}{2e})^{\frac{1}{m}}$, so we have that $S(m,k) > E_m \mu^{1+\frac{1}{m}}$.

3.1. **Bombieri-Pila determinants with complex cells.** We refer the reader to [9] for basic definitions and formulations around complex cells. Let $\mathscr{C} \subseteq \mathbb{C}^n$ be a cell of length n and m be the number of its D-fibers. We work under the umbrella assumption that $m < n$. For $m \geqslant 1$, let $\boldsymbol{\delta} \in \mathbb{R}^n$ be given by $\delta < 1/2$ in the $D-$coordinates and $\delta^{E'_m \mu^{1+1/m}}$ in the A, D_0-coordinates of $\mathscr{C}$, where $E'_m = \frac{1}{4(m+1)^2}$. Applying (2) for $e=1$ gives $S(m,k) > E'_m \mu^{1+1/m}$. This inequality can be employed at the end of the proof of [9, Lemma 93] to get the following restatement of that result. When $m=0$, we set $\boldsymbol{\delta} = (\delta^{\mu^2}/16, \dots, \delta^{\mu^2}/16)$.

Proposition 3.1. *Suppose $f_i \in \mathcal{O}(\mathscr{C}^{\boldsymbol{\delta}})$ with $\|f_i\|_{\mathscr{C}^{\boldsymbol{\delta}}} \leqslant M$, and $p_j \in \mathscr{C}$ for $i,j \leqslant \mu$. Then if $m \geqslant 1$,*

$$|\det(\mathbf{f},\mathbf{p})| \ := \ \det\big(f_i(p_j)\big)_{i,j\leqslant\mu} \ \leqslant M^\mu \mu^{(n+1)^2\mu} \delta^{E'_m \mu^{1+\frac{1}{m}}};$$

and when $m=0$,

$$|\det(\mathbf{f},\mathbf{p})| \ \leqslant M^\mu \mu^{(n+1)^2\mu} \delta^{\mu^3/16}.$$

The choice of a comparatively smaller E'_m allows us to do away with the $\delta^{-O_n(\mu)}$ factor in the upper bound from [9, Lemma 93]. This is a direct consequence of (2), which we leave as an exercise to the reader along with with working out $(n+1)^2\mu$ as an upper bound for the exponent of μ. Beyond making the estimates more explicit, our formulation includes the stronger statement for the case of $m = 0$ required for Lemma 3.2.

Interpolating rational points by polynomials. Let $d \geqslant 2$ and $m < l \leqslant n$. Set $\mu = L(l+1, d)$, and note that $\mu \geqslant l + d > d$, $\frac{d^l}{l!} \leqslant \mu \leqslant 2d^l$. Also if $l \geqslant 2$, then $d(l+1) \leqslant \mu \leqslant d^l \frac{l^k}{k!}$ for all $1 \leqslant k \leqslant l$. In addition to these observations, we will use the fact that $(2d)^{\frac{1}{d}} \leqslant 2$ for all d. Note that this implies that $\mu^{\frac{1}{ld}} \leqslant 2$.

Let $\mathbf{f} := (f_1, \ldots, f_n)$ be a map, where $f_1, \ldots, f_n$ are complex-valued functions, and $\mathbf{p} := \{p_1, \ldots, p_\mu\}$ a collection of points. Let J_l be the set of $\boldsymbol{j} = \{j_1, \ldots, j_l\}$ with $1 \leqslant j_1 \leqslant \ldots \leqslant j_l \leqslant n$. For every $\boldsymbol{j} = \{j_1, \ldots, j_l\} \in J_l$ we set $z_{\boldsymbol{j}} := \{z_{j_1}, \ldots, z_{j_l}\} \subseteq \{z_1, \ldots, z_n\}$ to be the set of variables corresponding to $\boldsymbol{j}$. Set

$$\det{}^d_{\boldsymbol{j}}(\mathbf{f}, \mathbf{p}) := \det\big(\mathbf{f}^\sigma(p_i)\big)_{\sigma \in \Sigma^d_{\boldsymbol{j}}, i \leqslant \mu},$$

where $\Sigma^d_{\boldsymbol{j}} := \{\alpha \in \mathbb{N}^n : \ |\alpha| \leqslant d \text{ and } \alpha_j = 0 \text{ if } j \notin \boldsymbol{j}\}$, and $\mathbf{f}^\sigma := f_1^{\sigma_1} \cdots f_n^{\sigma_n}$. Note $\#\Sigma^d_{\boldsymbol{j}} = \mu$.

Let S be a set of points in the common domain of the f_i, $i \leqslant n$. Then by a routine linear algebra argument, $(f_1, \ldots, f_n)(S) \subseteq \mathbb{C}^n$ is contained in an hypersurface in $z_{\boldsymbol{j}}$ variables of degree at most d if and only if $\det^d_{\boldsymbol{j}}(\mathbf{f}, \mathbf{p}) = 0$ for any $\mathbf{p} \subset S$ of size μ; we use this observation freely in this section and also the next section. By a *hypersurface in $z_{\boldsymbol{j}}$ variables of degree at most d* we mean the zero set of a non-trivial polynomial in the $z_{\boldsymbol{j}}$ variables of total degree at most d, and equivalently a hypersurface which has degree at most d as an algebraic variety in $\mathbb{C}^n$. We recall the definition of degree of a variety at the start of the next subsection and note that the equivalence referenced in the last sentence is a well-known fact.

Lemma 3.2. *Let $\mathbf{f} = (f_1, \ldots, f_n)$ be a map with $f_i \in \mathcal{O}(\mathscr{C}^{\boldsymbol{\delta}})$ and $\|f_i\|_{\mathscr{C}^{\boldsymbol{\delta}}} \leqslant 2n$. Set $C := 2^{-8(m+1)^4(n+1)^2}$. Then for every ϵ there is $d < (\frac{4(m+1)^4}{\epsilon})^{m+1}$ such that for all T and $\boldsymbol{j} \in J_{m+1}$, if*

$$\delta \leqslant CT^{-\epsilon},$$

then $\mathbf{f}(\mathscr{C})(\mathbb{Q}, T)$ is contained in a hypersurface in $z_{\boldsymbol{j}}$ variables of degree at most d.

Proof. Fix and ϵ and set $l = m+1$. Recall $\mu = L(l+1, d)$ and suppose $p_1, \ldots, p_\mu \in \mathscr{C}$ are such that $\mathbf{f}(p_1), \ldots, \mathbf{f}(p_\mu) \in \mathbf{f}(\mathscr{C})(\mathbb{Q}, T)$.

Case I: $m = 0$. Here $\mu = d+1$, and choose d such that $\frac{1}{\epsilon} < d < \frac{4}{\epsilon}$. Proposition 3.1 and [9, Lemma 95] give that

$$T^{-d(d+1)} \leqslant |\det{}^d_{\boldsymbol{j}}(\mathbf{f}, \mathbf{p})| \leqslant (2n)^{d+1}(d+1)^{(n+1)^2(d+1)}\delta^{(d+1)^3/16}$$

or $\det^d_{\boldsymbol{j}}(\mathbf{f}, \mathbf{p}) = 0$. So $\det^d_{\boldsymbol{j}}(\mathbf{f}, \mathbf{p}) = 0$ if

$$\delta \leqslant C' T^{-1/d},$$

where $C' := (2n(d+1)^{(n+1)^2})^{-16/(d+1)^2} \geqslant C = 2^{-8(n+1)^2}$, since $(d+1)^{4/(d+1)^2} \leqslant 2$ and $(2n)^{4/(d+1)^2} \leqslant 2^{(n+1)^2}$ for all d, n. Since we chose d such that $1/d < \epsilon$ we have the desired conclusion.

Case II: $m \geqslant 1$. Choose d such that $(\frac{4l^4}{\epsilon})^m < d < (\frac{4l^4}{\epsilon})^l$. By Proposition 3.1 and [9, Lemma 95] we have that

$$T^{-l\mu d} \leqslant |\det{}^d_{\boldsymbol{j}}(\mathbf{f}, \mathbf{p})| \leqslant (2n)^\mu \mu^{(n+1)^2\mu} \delta^{E'_m \mu^{1+\frac{1}{m}}}$$

or $\det^d_{\boldsymbol{j}}(\mathbf{f},\mathbf{p})=0$. So $\det^d_{\boldsymbol{j}}(\mathbf{f},\mathbf{p})=0$ if

$$\delta \leqslant C' T^{-4l^3 d \mu^{\frac{-1}{m}}},$$

where $C' = (2n)^{-4l^2}\mu^{-4l^2(n+1)^2\mu^{\frac{-1}{m}}}$. Using $\frac{d^l}{l!} \leqslant \mu \leqslant 2d^l$ and $(2d)^{\frac{1}{d}} \leqslant 2$, we see that $\mu^{\mu^{\frac{-1}{m}}} \leqslant 2^{l^2}$, and hence $C = 2^{-8l^4(n+1)^2} \leqslant C'$. We now use again that $\mu \geqslant \frac{d^l}{l!}$ to see that by our choice of d we have that $4l^3 d\mu^{\frac{-1}{m}} \leqslant 4l^4 d^{\frac{-1}{m}} < \epsilon$. So we have shown that $\delta \leqslant CT^{-\epsilon}$ implies that $\mathbf{f}(\mathscr{C})(\mathbb{Q},T)$ is contained in a hypersurface in $z_{\boldsymbol{j}}$ variables of degree at most d and the proof is complete. □

We also record a variant of Lemma 3.2, where in exchange for allowing degree of the interpolating polynomial to grow as a power of $\log(T)$, we get a bound for δ that works uniformly for all T.

Corollary 3.3. *Let $\mathbf{f} = (f_1,\dots,f_n)$ be a map with $f_i \in \mathcal{O}(\mathscr{C}^{\boldsymbol{\delta}})$ and $\|f_i\|_{\mathscr{C}^{\boldsymbol{\delta}}} \leqslant 2n$. Then there is $C = C(n)$ such that for all T and $\boldsymbol{j} \in J_{m+1}$, if $\delta \leqslant C$ then we have that $\mathbf{f}(\mathscr{C})(\mathbb{Q},T)$ is contained in a hypersurface in $z_{\boldsymbol{j}}$ variables of degree at most $\log(T)^{m+1}$.*

Proof. Run the proof of Lemma 3.2 for $\epsilon = 4(m+1)^4/\log(T)$. □

3.2. **Covering rational points on an algebraic set by algebraic subsets of bounded degree.** Throughout the rest of this section we set $l = m+1$, and recall J_{m+1} is the set of $\boldsymbol{j} = \{j_1,\dots,j_{m+1}\}$ with $1 \leqslant j_1,\dots,j_{m+1} \leqslant n$. Set $\mathscr{P}_n := D(1)^n$.

In this paper, an *algebraic set* is $Z(I) := \{z \in \mathbb{C}^n : \ h(a) = 0 \text{ for all } h \in I\}$, where I is a non-trivial ideal of $\mathbb{C}[z_1,\dots,z_n]$. So an algebraic set is in particular an algebraic variety, and recall that the *degree* of an algebraic variety is the number of intersections between the variety and a generic affine-linear hyperplane of complementary dimension. This matches the definition in [9, subsubsection 2.2.3], where the words complexity and degree are used interchangeably to refer to this concept. We shall use the following fact.

Fact 3.4. [11, Chapter 9, Section 5, Corollary 4] *The dimension of an algebraic set $Z(I) \subseteq \mathbb{C}^n$ is the maximal size of a set $S \subseteq \{z_1,\dots,z_n\}$ such that I contains no nonzero polynomial in only the variables in S.*

Lemma 3.5. *Let ϵ be given and $W \subseteq \mathbb{C}^n$ be an algebraic set of degree d_0 with $\dim(W) = m < n$. Then there are $c = \mathrm{poly}_n(d_0,\epsilon^{-1}) \geqslant 1$ and $d = \mathrm{poly}_n(\epsilon^{-1})$ such that for all T and $\boldsymbol{j} \in J_{m+1}$, there is $N \leqslant cT^\epsilon$ and hypersurfaces in $z_{\boldsymbol{j}}$ variables $H_1,\dots,H_N$ of degree at most d with $(W \cap \mathscr{P}_n)(\mathbb{Q},T) \subseteq \bigcup_{i=1}^N H_i$.*

Proof. The case $m = 0$ is immediate and we assume from now on that $m \geqslant 1$. Obtain $C = O_n(1)$ and $d = \mathrm{poly}_n(\epsilon^{-1})$ using the corresponding estimates from Lemma 3.2. Fix $\boldsymbol{j} \in J_{m+1}$, and set $\mu = L(m+2,d)$ and $\delta = CT^{\frac{\epsilon}{2n}}$. The tuple $\boldsymbol{\delta} \in \mathbb{R}^n$ will be defined with regards to cells as in the context of Proposition 3.1 and will hence depend on the fiber composition of the cell in question.

We employ [9, Theorem 8] to get a real cellular cover of $\mathscr{P}_n \cap W$ of size $\mathrm{poly}_n(d_0)$ admitting $\frac{1}{2}$-extensions. For every cell $\mathscr{C}$ of this covering, apply [9, Lemma 94] to obtain a covering of $\mathscr{C}$ by $\mathrm{poly}_n\big(\mu \log(\frac{1}{\delta})\big) \cdot \delta^{-\dim(\mathscr{C})}$ cells such that each cell in this refinement admits a $\boldsymbol{\delta}$-extension. This gives a covering of $\mathscr{P}_n \cap W$ by $\mathrm{poly}_n(d_0,d,\log T) \cdot T^{\frac{\epsilon}{2}} = \mathrm{poly}_n(d_0,\epsilon^{-1})T^\epsilon$ maps; we use here that $d = \mathrm{poly}_n(\epsilon^{-1})$ and that $(\log T)^{O_n(1)} \leqslant O_n(1)\epsilon^{-1}T^{\epsilon/2}$ for all T.

Take a map of this cover, $\mathbf{f} : \mathscr{C}^{\boldsymbol{\delta}} \to \mathscr{P}_n^{1/2} \cap W$. Then our choice of δ implies by Lemma 3.2 that $\mathbf{f}(\mathscr{C})(\mathbb{Q}, T)$ is contained in a hypersurface in variables $z_{\boldsymbol{j}}$ of degree at most d, and the proof is complete. □

Theorem 3.6. *Let ϵ and an algebraic set $W \subseteq \mathbb{C}^n$ of degree d_0 be given. Then there are $c = \mathrm{poly}_n(d_0, \epsilon^{-1})$ and $d = \mathrm{poly}_n(\epsilon^{-1})$ such that for all T, there is $N \leqslant cT^\epsilon$ and algebraic sets $W_1, \dots, W_N \subseteq W$ of degree at most d with $(W \cap \mathscr{P}_n)(\mathbb{Q}, T) \subseteq \bigcup_{i=1}^N W_i$.*

Proof. We can and do assume that $W \subset \mathbb{C}^n$. So $m := \dim(W) < n$, and we proceed by induction on m.

Claim: There are $c_* = \mathrm{poly}_n(d_0, \epsilon^{-1})$, $d_* = \mathrm{poly}_n(\epsilon^{-1})$ such that for all T, we can cover $(W \cap \mathscr{P}_n)(\mathbb{Q}, T)$ by irreducible algebraic sets $V_1, \dots, V_l$ of degree d_* such that $\dim(V_i) \leqslant m$ for all $i \leqslant l$, where $l \leqslant c_* T^{\frac{\epsilon}{2}}$.

We first assume the claim and complete the proof. Set $N_0 := 0$, and consider some $i \leqslant l$. We have that $V_i \subseteq W$ or $\dim(V_i \cap W) < m$. In the former case, set $N_i = N_{i-1} + 1$, $W_{N_i} := V_i$, $c_i = 2$, and $d_i := d_*$. In the latter case, induction gives for all T, $c_i = \mathrm{poly}_n(d_0, \epsilon^{-1})$, $d_i = \mathrm{poly}_n(\epsilon^{-1})$, $M_i \leqslant c_i T^{\frac{\epsilon}{2}}$, and algebraic sets $V_{N_{i-1}+1}, \dots, V_{N_{i-1}+M_i} \subseteq V_i \cap W$ of degree at most d_i that cover $V_i \cap W(\mathbb{Q}, T)$; set $N_i := N_{i-1} + M_i$ in this case. It remains to observe that $c := c_* \cdot \max\{c_1, \dots, c_l\}$, $d := \max\{d_1, \dots, d_l\}$, $N := N_l \leqslant cT^\epsilon$, and the algebraic sets $W_1, \dots, W_N \subseteq W$ are as desired.

We now return to the proof of claim. J_{m+1} is the set of all $\boldsymbol{j} = \{j_1, \dots, j_{m+1}\}$ with $1 \leqslant j_1, \dots, j_{m+1} \leqslant n$ and let $L := \#J_{m+1} = \binom{n}{m+1}$. Fix a T, our proof and the resulting constants will not depend on this choice. We apply Lemma 3.5 to get for every $\boldsymbol{j} \in J_{m+1}$, constants $c_{\boldsymbol{j}} = \mathrm{poly}_n(d_0, \epsilon^{-1})$, $d_{\boldsymbol{j}} = \mathrm{poly}_n(\epsilon^{-1})$ such that $W(\mathbb{Q}, T)$ is covered by at most $c_{\boldsymbol{j}} T^{\frac{\epsilon}{2L}}$ many hypersurfaces in the $z_{\boldsymbol{j}}$ variables of degree $d_{\boldsymbol{j}}$. Let $c' := \max(c_{\boldsymbol{j}})_{\boldsymbol{j} \in J_{m+1}}$, and $d' := \max(d_{\boldsymbol{j}})_{\boldsymbol{j} \in J_{m+1}}$.

Thus we have that $W(\mathbb{Q}, T)$ is contained in at most $(c')^L T^{\frac{\epsilon}{2}} = \mathrm{poly}_n(c') T^{\frac{\epsilon}{2}}$ sets of the form $\cap_{\boldsymbol{j} \in J_{m+1}} H_{\boldsymbol{j}}$, where $H_{\boldsymbol{j}}$ is a hypersurface of degree $d_{\boldsymbol{j}}$ in the $z_{\boldsymbol{j}}$ variables. For all T, let $V_1, \dots, V_l$ be the collection of all the irreducible components of all the sets $\cap_{\boldsymbol{i} \in I} H_{\boldsymbol{i}}$ covering $W(\mathbb{Q}, T)$ as in the last sentence. Note that the degree of V_i is $d_* = \mathrm{poly}_n(d')$ for all $i \leqslant l$, and also that $l \leqslant c_* T^{\frac{\epsilon}{2}}$, where $c_* = \mathrm{poly}_n(d')\mathrm{poly}_n(c') = \mathrm{poly}_n(d_0, \epsilon^{-1})$.

Moreover, $\dim(V_i) \leqslant m$ for all $i \leqslant l$ by construction of the hypersurfaces $(H_{\boldsymbol{j}})_{\boldsymbol{j} \in J_{m+1}}$ and Fact 3.4. We have shown that c_*, d_*, and $V_1, \dots, V_l$ are as desired, which finishes the proof of the claim and the theorem. □

The following variant of Theorem 3.6 shows that given an algebraic set W, we can cover $W(\mathbb{Q}, T)$ by algebraic subsets whose number can only grow polynomially in $\log(T)$ if we allow the bound on the degree to also grow as a polynomial in $\log(T)$.

Theorem 3.7. *Let $W \subseteq \mathbb{C}^n$ be an algebraic set of degree d_0. Then there are $c = \mathrm{poly}_n\big(d_0, \log(T)\big)$ and $d = \mathrm{poly}_n\big(\log(T)\big)$ such that for all T, there is $N \leqslant c$ and algebraic sets $W_1, \dots, W_N \subseteq W$ of degree at most d with $(W \cap \mathscr{P}_n)(\mathbb{Q}, T) \subseteq \bigcup_{i=1}^N W_i$.*

Proof. We obtain this by running the proof of Theorem 3.6 for $\epsilon = 1/\log(T)$. Note we already have Corollary 3.3 to play the role of Lemma 3.2. □

4. Interpolating rational points on lemniscates

Throughout this section, $g \in \mathcal{O}(B^{\frac{1}{2}})$ and is non-constant. Let $A \subset B^{\frac{1}{2}} \subset \mathbb{C}^{m+1}$ be a ball or polydisc. We introduce the *Bernstein index* of g with respect to the pair $A^\eta \subset A$,

$$\mathcal{B}_A^\eta(g) := \log\big(\|g\|_A / \|g\|_{A^\eta}\big).$$

So $\mathcal{B}_A^\eta(g) > 0$ for all non-constant g; recall $\eta \geqslant 2$ by the convention set forth in the introduction. We will make use of the the following fact [5, Lemma 17].

Lemma 4.1. *Let $U \subseteq \mathbb{C}$ be a disc and $h \in \mathcal{O}(\bar{U})$. Then there is a disc V concentric with U such that $U^4 \subset V \subset \bar{U}^2$, and*

$$\min_{z \in \partial V} |h(z)| \geqslant \|h\|_U \mathrm{e}^{-20\mathcal{B}_U^2(h)}.$$

The exponent 20 can be easily computed with help from [15, Example 1].

4.1. **Bounding Bombieri-Pila determinants.** Set $X := \{z \in B^{:}\ g = 0\}$, and recall the ω-lemniscate of X is the $X(\omega) = \{z \in B :\ |g| \leqslant \omega\}$. Note that X is of pure dimension $n-1$ by [10, Proposition 2.6], and hence [8, Proposition 7] applies to X.

Let $\Delta := \Delta_h \times \Delta_v \subseteq B^{\frac{1}{2}}$, with polydiscs $\Delta_h \subset \mathbb{C}^m$ and $\Delta_v \subset \mathbb{C}$. We say that Δ is a *Weierstrass polydisc* for X if $X \cap (\bar{\Delta}_h \times \partial\Delta_v) = \emptyset$. The proofs of [8, Fact 5] and [14, II.B Theorem 2] imply that g is given by a unit times a Weierstrass polynomial of degree, $e = e(g, \Delta) = e(X, \Delta)$, in z_{m+1} on Δ; and so we can perform Weierstrass division by g. We shall use this observation shortly.

Let D be a polydisc such that that $D^\delta \subseteq \Delta$, for some δ to chosen later. We start by developing an analogue of [7, Lemma 9] for $X(\omega)$. For Δ as above, Δ' will denote the polydisc given by $\Delta' := \Delta_h \times \Delta_v^{1/3}$, and set $\mathcal{M} := \mathbb{N}^m \times \{0, 1, \ldots, e-1\}$. For convenience of notation we set

$$\beta_0 := \omega\|g\|_\Delta^{-1}\mathrm{e}^{20\mathcal{B}_\Delta^2(g)} \qquad \text{and} \qquad \beta := \omega\|g\|_\Delta^{-1}\mathrm{e}^{20\mathcal{B}_\Delta^7(g)}.$$

Throughout we assume that ω and g are such that $\beta, \beta_0 < 1$.

Let $f_i \in \mathcal{O}(\bar{\Delta}')$ with $\|f_i\|_{\Delta'} \leqslant M$, and points $p_j \in D \cap X(\omega)$ for $i, j \leqslant \mu$; we will work with this setup throughout this section. For the interpolation result, our primary step would be to obtain a sufficiently small bound for

$$\det(\mathbf{f}, \mathbf{p}) := \det\big(f_i(p_j)\big)_{i,j \leqslant \mu}.$$

We first handle the special case of $m = 0$. So Δ is a disc, $\mathcal{M} = \{0, \ldots, e-1\}$ and $\Delta' = \Delta^{1/3}$.

Proposition 4.2. *Let $m = 0$, $\mu > e$, and $\delta < \frac{1}{4}$. Then*

$$|\det(\mathbf{f}, \mathbf{p})| \ \leqslant\ (8\mu M)^\mu \beta_0^{\mu - e}.$$

Proof. For each $i \leqslant \mu$ we perform Weierstrass division by g to obtain P_i, a polynomial of degree at most $e-1$, such that

$$f_i(z) = \ P_i(z) + g(z) \cdot q_i(z)$$

for all $z \in \Delta$. Note that $q_i \in \mathcal{O}(\Delta)$ and employ [8, Proposition 7] to observe that $\|P_i\|_\Delta \leqslant 3M$. In the context of Weierstrass division, also see [14, II.D Theorem 1].

We work towards bounding the function

$$q_i = (f_i - P_i)/g.$$

Lemma 4.1 applied to $U = \Delta$ together with the maximum modulus principle gives

$$\|q_i\|_D \leqslant 4M\|g\|_\Delta^{-1} \mathrm{e}^{20\mathcal{B}^2_\Delta(g)}.$$

In more detail, we use Lemma 4.1 to get a lower bound for $|g|$ on the boundary of a disc containing D. Note that $\delta < \frac{1}{4}$ and $D^\delta \subseteq \Delta$ implies that $D \subset (D^\delta)^4 \subseteq \Delta^4$. Then the fact that $q_i \in \mathcal{O}(\Delta)$ together with $\|f_i - P_i\|_\Delta \leqslant 4M$ gives that the upper bound above holds over the whole disc D, by the maximum modulus principle.

For $j \leqslant \mu$, $p_j \in X(\omega) \cap D$ and the definition of β_0 implies that

$$|g(p_j) \cdot q_i(p_j)| \leqslant 4M\beta_0.$$

Expanding $\det(\mathbf{f}, \mathbf{p})$ linearly with respect to each row we obtain 2^μ determinants. Let us consider one such determinant $\det_I$, so that for each i, every occurrence of f_i is replaced by either P_i or $g \cdot q_i$. Since the degree of P_i is bounded above by $e-1$, if there are more than e rows of $\det_I$ which feature P_i we must have $\det_I = 0$. Hence for non-zero $\det_I$, we have at least $\mu - e$ many rows featuring the $g \cdot q_i$ terms. This implies that $|\det_I| \leqslant \mu! 4^\mu M^\mu \beta_0^{\mu - e}$ and using $\mu! \leqslant \mu^\mu$ we obtain the desired upper bound on $|\det(\mathbf{f}, \mathbf{p})|$. ☐

For the $m > 0$ case we will need the following sharper version of [7, Lemma 3]. For $\alpha \in \mathcal{M}$, we let $m_\alpha(f)$ denote a monomial of the form $c_\alpha z^\alpha$.

Lemma 4.3. *Let $m > 0$. Any $f \in \mathcal{O}(\bar{\Delta}')$ can be decomposed in the form*

$$f = \sum_{\alpha \in \mathcal{M}} m_\alpha(f) + q \cdot g,$$

with $q \in \mathcal{O}(\bar{\Delta})$, $\|m_\alpha(f)\|_\Delta \leqslant 3\|f\|_{\Delta'}$ for $\alpha \in \mathcal{M}$, $\|\sum_{\alpha \in \mathcal{M}} m_\alpha(f)\|_\Delta \leqslant 3\|f\|_{\Delta'}$, and

$$\|q\|_{\Delta^{14}} \leqslant 4\|f\|_{\Delta'}\|g\|_\Delta^{-1} \mathrm{e}^{20 \cdot \mathcal{B}^7_\Delta(g)}.$$

Proof. The displayed decomposition and the claims

$$q \in \mathcal{O}(\bar{\Delta}'), \quad \|m_\alpha(f)\|_\Delta \leqslant 3\|f\|_{\Delta'}, \quad \|\sum_{\alpha \in \mathcal{M}} m_\alpha(f)\|_\Delta \leqslant 3\|f\|_{\Delta'}$$

follow directly from [8, Proposition 7]. Note this implies that $\|q \cdot g\|_\Delta \leqslant 4\|f\|_{\Delta'}$

Towards proving the last claim, let $p \in \partial\Delta^7$ be such that $|g(p)| = \|g\|_{\Delta^7}$. We now choose and fix new coordinates $\mathbf{w}$ such that $\mathbf{w} = \mathbf{z} - p$; in other words the point p is the origin in the $\mathbf{w}$-coordinates. We set $g_\mathbf{w}$ to the function g in the $\mathbf{w}$-coordinates. Set $\Upsilon := \Delta_p^{7/6}$; note that $\Upsilon \subset \Delta$ and $\Delta^{14} \subset \Upsilon^4$.

Let $B' \subset B^{\frac{1}{2}}$ be a ball with center $\mathbf{p}$ with $\Upsilon \subset B'$. Set $\Lambda := \{\lambda = (\lambda_1, \ldots, \lambda_n) \in \partial B' : \ \lambda_1 \in \mathbb{R}\}$. For any $\lambda \in \Lambda$, we set $g_\lambda : D(1) \to \mathbb{C}$ to be given by

$$g_\lambda(w) := g_\mathbf{w}(w\lambda),$$

where $D(1)$ is the open unit disc. Set $M_\lambda := \|g_\lambda\|_{D(1)}$, $m_\lambda := \|g_\lambda\|_{D(1)^2}$.

Applying 4.1 to the function g_λ we get a disc V_λ concentric with $D(1)$ with $D(1)^4 \subset V_\lambda \subset \overline{D(1)}^2$ and

$$\min_{w \in \partial V_\lambda} |g_\lambda(w)| \geqslant M_\lambda \mathrm{e}^{-20 \log \frac{M_\lambda}{m_\lambda}}.$$

Now $M_\lambda \leqslant \|g\|_\Delta$, and $m_\lambda \geqslant \|g\|_{\Delta^7}$ since $g_\mathbf{w}(\mathbf{0}) = \|g\|_{\Delta^7}$, which gives

$$\min_{w \in \partial V_\lambda} |g_\lambda(w)| \geqslant \|g\|_\Delta \mathrm{e}^{-20\mathcal{B}^7_\Delta(g)}.$$

Notice that the lower bound in the above inequality does not depend on λ; we are now ready to finish the proof. Recall that $\|q \cdot g\| \leqslant 4\|f\|_{\Delta'}$. Take any $a \in \Delta^{14}$. Since $\Delta^{14} \subset \Upsilon^4 \subset (B')^4$, there are $\lambda_0 \in \Lambda$ and $w_0 \in D(1)^4 \subset V_{\lambda_0}$ such that $a - p = w_0\lambda_0$. So the fact that $q \in \mathcal{O}(\bar{\Delta})$, and the maximum principle applied to $q(p + w\lambda_0) : \bar{V}_{\lambda_0} \to \mathbb{C}$ gives that

$$|q(a)| \leqslant 4\|f\|_{\Delta'}\|g\|_{\Delta}^{-1}\mathrm{e}^{20\mathcal{B}_{\Delta}^{7}(g)}.$$

It only remains to notice that our upper bound does not depend on the choice of $a \in \Delta^{14}$. This gives us the last claim and the proof is complete. □

We are now ready for the $m > 0$ version of Proposition 4.2.

Proposition 4.4. *Let $m \geqslant 1$ and $\mu > e(m+1)$. Let $k_0 \geqslant (m!\mu/e)^{\frac{1}{m}}$ and assume $\delta < 1/14$. Then for $\rho := \max(\delta, \beta^{1/k_0})$ we have that*

$$|\det(\mathbf{f}, \mathbf{p})| \;\leqslant\; (4(1.1)^m\mu^3 M)^{\mu} \cdot \rho^{E_m \cdot \mu^{1+\frac{1}{m}}}.$$

Proof. We set

$$k := \max\left\{ j : \; \sum_{l=0}^{j} eL(m,l) < \mu \right\}.$$

By our calculations at the start of Section 3 and our assumption on μ we have $k \geqslant 1$, and the upper bound on k gives $k_0 > k$. Throughout the proof i ranges over positive natural numbers not greater than μ. Set $\mathcal{M}^{\leqslant k} := \{\alpha \in \mathcal{M} : \; |\alpha| \leqslant k\}$ and apply Lemma 4.3 to obtain

$$f_i = \sum_{\alpha \in \mathcal{M}^{\leqslant k}} m_\alpha(f_i) + R_k(f_i) + gq_i,$$

where $\|m_\alpha(f)\|_\Delta \leqslant 3M$ for all $\alpha \in \mathcal{M}^{\leqslant k}$, and

$$\|q_i\|_D \leqslant 4M\|g\|_{\Delta}^{-1}\mathrm{e}^{20\mathcal{B}_{\Delta}^{7}(g)} \leqslant 4M\beta/\omega.$$

Note here we used $D \subseteq \Delta^{1/\delta} \subset \Delta^{14}$ since $\delta < \frac{1}{14}$. By our choice of k, the number of terms in the expansions of f_i above does not exceed $\mu + 1$. Moreover we have by [7, Proposition 8] and $k_0 > k$ that for every $j \leqslant \mu$ and $\alpha \in \mathcal{M}^{\leqslant k}$,

$$\|m_\alpha(f_i)\|_D \leqslant C_0\delta^{|\alpha|}, \quad \|R_k(f_i)\|_D \leqslant C_0\delta^k, \qquad |g(p_j) \cdot q_i(p_j)| \leqslant 4M\beta$$

$$\text{where} \quad C_0 := \frac{4eL(m,k)M}{(1-\delta)^m} < 4(1.1)^m\mu M;$$

the upper bound for C_0 comes by using $L(m,k) < L(m+1,k) < \frac{\mu}{e}$ and $\delta < \frac{1}{14}$.

As in the proof of [7, Lemma 9], expanding $\det(\mathbf{f}, \mathbf{p})$ linearly we get a sum of at most $(\mu+1)^\mu$ interpolation determinants $\det_I$, such that $|\det_I| \leqslant \mu! C_0^\mu \rho^S$, where $S = S(m,k) = \sum_{l=0}^{k} eL(m,l) \cdot l > E_m\mu^{1+1/m}$, the last inequality coming from our calculations from Section 3. Note that $\mu > e(m+1)$ implies that $\mu \geqslant 3$ and then $\mu!(\mu+1)^\mu \leqslant \mu^{2\mu}$. The proof is complete. □

4.2. **Covering rational points by hypersurfaces.** We now turn to analogues of results in [7, Section 3.3], and follow the general strategy there. We arrange things in way that allows for a clean statement of Corollary 4.7, which will be the only result from this section to be directly used later in the paper.

We borrow much of the setup from Subsection 3.1. In particular $d \geqslant 2$, $m < l \leqslant n$, $\mu = L(l+1, d)$, and recall that J_l is the set of all $\boldsymbol{j} = \{j_1, \dots, j_l\}$ with $1 \leqslant j_1 \leqslant \ldots \leqslant j_l \leqslant n$. Throughout X, Δ, Δ', and D are as in the previous subsection. So $\dim(X) = m$, $\Delta \subseteq \mathbb{C}^{m+1}$ is a polydisc for X and $D^\delta \subseteq \Delta$. Suppose $f_i \in \mathcal{O}(\Delta')$ with $\|f_i\|_{\Delta'} \leqslant 1$ for all $i \leqslant n$, and set $Y := \mathbf{f}(X(\omega) \cap D) \subset \mathbb{C}^n$.

Lemma 4.5. *Let $m = 0$ and $\delta < \frac{1}{4}$. For every ϵ and d with $\frac{3l^2}{\epsilon} \leqslant \frac{d}{e} < \frac{4l^2}{\epsilon}$, we have for all T and $\boldsymbol{j} \in J_l$ that if*

$$\beta_0^{\frac{2l}{\mu d}} \leqslant \frac{1}{4} T^{-\epsilon},$$

then $Y(\mathbb{Q}, T)$ is contained in a hypersurface in $z_{\boldsymbol{j}}$ variables of degree at most d.

Proof. Fix an ϵ. Let $p_1, \dots, p_\mu \in X(\omega) \cap D$ be such that $\mathbf{f}(p_1), \dots, \mathbf{f}(p_\mu) \in Y(\mathbb{Q}, T)$, and suppose $\det_{\boldsymbol{j}}^d(\mathbf{f}, \mathbf{p}) \neq 0$. Choose d satisfying $e\frac{3l^2}{\epsilon} \leqslant d < e\frac{4l^2}{\epsilon}$. Note that this implies that $\frac{d}{e} > 3$, and so $\frac{2l^2}{\frac{d}{e}-1} < e\frac{3l^2}{d} \leqslant \epsilon$. Then by $\mu > d > e$, Proposition 4.2, and [7, Lemma 10] we have that

$$T^{-l\mu d} \;\leqslant\; \det_{\boldsymbol{j}}^d(\mathbf{f}, \mathbf{p}) \;\leqslant\; (8\mu)^\mu \beta_0^{\mu - e}.$$

So for $C := (8\mu)^{-\frac{2l}{d(\mu-e)}}$, any choice of $\beta_0 \leqslant (CT^{-\epsilon})^{\frac{\mu d}{2l}}$ gives a contradiction to the displayed inequalities; since $\frac{2l^2}{\mu - e} < \frac{2l^2}{d-e} = \frac{2l^2}{e(\frac{d}{e}-1)} \leqslant \frac{2l^2}{\frac{d}{e}-1} < \epsilon$. This also gives $\frac{2l}{d(\mu-e)} < \frac{\epsilon}{ld} < \frac{1}{ld}$; and to make our choice of C independent of d, we use $\mu^{\frac{1}{ld}} \leqslant 2$, and then reset $C := 1/4$; we use here that $ld \geqslant 3$. Now the contradiction in hand implies that $\det_{\boldsymbol{j}}^d(f, \mathbf{p}) = 0$, and this yields an interpolating hypersurface in $z_{\boldsymbol{j}}$ variables of degree at most d as claimed. □

Lemma 4.6. *Let $m \geqslant 1$, $\delta < \frac{1}{14}$, and $k_0 := ld^{l/m}$. There exists $C = C(l, m) < 1$ such that for every ϵ there is d with $\frac{d}{e} < (\frac{4l^3 l^{\frac{l-1}{m}}}{\epsilon})^l$ such that for all T and $\boldsymbol{j} \in J_l$ if*

$$\rho := \max(\delta, \beta^{\frac{1}{k_0}}) \leqslant CT^{-\epsilon},$$

then $Y(\mathbb{Q}, T)$ is contained in a hypersurface in $z_{\boldsymbol{j}}$ variables of degree at most d.

Proof. Fix an ϵ and let d be such that $d > e(\frac{4l^3 l^{\frac{l-1}{m}}}{\epsilon})^m$. Clearly we can, and do choose d so that $d < e(\frac{4l^3 l^{\frac{l-1}{m}}}{\epsilon})^l$. We have that $m \geqslant 1$, so that $l \geqslant 2$. As noted before, this means that $\mu \geqslant d(l+1)$ which implies $\mu > e(m+1)$ since $d > e$. Also recall that $\mu \leqslant d^l \frac{l^m}{m!}$, and so $k_0 = ld^{l/m} \geqslant (m!\mu)^{1/m}$ satisfies the assumption on k_0 in the statement of Proposition 4.4.

Let $p_1, \dots, p_\mu \in X(\omega) \cap D$ be such that $\mathbf{f}(p_1), \dots, \mathbf{f}(p_\mu) \in Y(\mathbb{Q}, T)$, and suppose that $\det_{\boldsymbol{j}}^d(\mathbf{f}, \mathbf{p}) \neq 0$. Then Proposition 4.4 and [7, Lemma 10] give

$$T^{-l\mu d} \;\leqslant\; \det_{\boldsymbol{j}}^d(\mathbf{f}, \mathbf{p}) \;\leqslant\; \left(4(1.1)^m \mu^3\right)^\mu \cdot \rho^{E_m \cdot \mu^{1+1/m}},$$

where $E_m = \frac{1}{2(m+1)^2}(\frac{m!}{2e})^{\frac{1}{m}} > \frac{1}{4l^2 e^{\frac{1}{m}}}$. So we get a contradiction if

$$\rho < CT^{-4l^3 e^{\frac{1}{m}} d\mu^{-\frac{1}{m}}},$$

for $C := \big(4(1.1)^m\mu^3\big)^{-4l^2 e^{\frac{1}{m}}\mu^{-\frac{1}{m}}}$. This choice of C depends on d and in turn ϵ, and also e, but in the statement we promised a C that depends on just m and M. Using $d \geqslant e$, $l! \leqslant l^{l-1}$, and recalling that $\frac{d^l}{l!} \leqslant \mu$ we observe

$$\big(4(1.1)^m\mu^3\big)^{\frac{-4l^2 e^{\frac{1}{m}}}{\mu^{\frac{1}{m}}}} \geqslant \big(4(1.1)^l\mu^3\big)^{\frac{-4l^2 l^{\frac{l-1}{m}}}{d}}.$$

Now we use $\mu \leqslant 2d^l$, $(2d)^{\frac{1}{d}} \leqslant 2$, and also $32^{\frac{1}{3l}}(1.1)^{1/3} < 32^{1/6}(1.1) < 2$ to see

$$\big(4(1.1)^l\mu^3\big)^{\frac{-4l^2 l^{\frac{l-1}{m}}}{d}} > 2^{-12l^3 l^{\frac{l-1}{m}}}.$$

So we change reset C to $C := 2^{-12l^3 l^{\frac{l-1}{m}}}$. We use again that $\frac{d^l}{l!} \leqslant \mu$, $l! \leqslant l^{l-1}$, to see that our choice of d gives

$$4l^3 e^{\frac{1}{m}} d\mu^{-\frac{1}{m}} < 4l^3 l^{\frac{l-1}{m}}(e/d)^{\frac{1}{m}} < \epsilon.$$

So any $\rho \leqslant CT^{-\epsilon}$ gives the desired contradiction, and the corollary follows. □

We now record a consequence that we will apply later, here we choose $l = m+1$ which is the number of $z_{\boldsymbol{j}}$ variables. Also included is a viable upper bound on β that is uniform in m. Let $\upsilon_m : \mathbb{N} \to \{0,1\}$ be such that $\upsilon_m = 0 \iff m = 0$.

Corollary 4.7. *Let $\delta < \frac{1}{4}$ and $C = 2^{-12(m+1)^4}$. For every ϵ there exists d satisfying $\frac{d}{e} < \big(\frac{4(m+1)^4}{\epsilon}\big)^{m+1}$ such that for all T and $\boldsymbol{j} \in J_{m+1}$ if*

$$\beta \leqslant (CT^{-\epsilon})^{ld^2} \quad \text{and} \quad \upsilon_m\delta \leqslant CT^{-\epsilon},$$

then $Y(\mathbb{Q},T)$ is contained in a hypersurface in $z_{\boldsymbol{j}}$ variables of degree at most d.

Proof. This follows easily from the previous two lemmas. Fix an ϵ and T, and suppose β and δ satisfy the displayed inequalities. For $m = 0$ since $\beta_0 < \beta$, we have $\beta_0^{\frac{2}{\mu d}} < \beta^{\frac{1}{d^2}} < CT^{-\epsilon} < \frac{1}{4}T^{-\epsilon}$. Now a direct application of Corollary 4.5 finishes this case. For $m \geqslant 1$, the bound we assume on β implies that $\beta^{\frac{1}{k_0}} \leqslant CT^{-\epsilon}$, and so the assumption on δ gives $\rho := \max(\delta, \beta^{\frac{1}{k_0}}) \leqslant CT^{-\epsilon}$, and we are done by a direct application of Corollary 4.6. □

Corollary 4.8. *Let $\delta < \frac{1}{4}$. Then there is $C = C(n)$ such that for all T and $\boldsymbol{j} \in J_{m+1}$ if*

$$\beta \leqslant C^{\log(T)^{2(m+1)}} \quad \text{and} \quad \upsilon_m\delta \leqslant CT^{-\epsilon},$$

then $Y(\mathbb{Q},T)$ is contained in a hypersurface in $z_{\boldsymbol{j}}$ variables of degree at most $\log(T)^{m+1}$.

Proof. Run the proof of Corollary 4.7 for $\epsilon = 4e(m+1)^4/\log(T)$. □

5. Counting approximate rational solutions of analytic families

Throughout this section, we fix $X = \{z \in B : g_1(z) = \ldots = g_m(z) = 0\}$, with $g_i \in \mathcal{O}(B^{1/2})$ and $\|g_i\|_{B^{1/2}} \leqslant 1$. Recall that $B = B(\frac{1}{2}) \subset \mathbb{C}^n$ is the ball of radius $1/2$ around the origin.

Lemma 5.1. *Suppose $g \in \mathcal{O}(B)$ such that $|g(0)| \geqslant \omega_0$, $\|g\|_B \leqslant 1$ and let $\eta_0 \in \mathbb{R}^{\geqslant 2}$. There is a unitary transformation $\mathbf{u}$ and a polydisc Δ for $Z := \{z \in B : g\circ\mathbf{u}(z) = 0\}$ such that $B^{2\eta_0\eta_1} \subset \Delta \subset B^{2\eta_0}$ for $\eta_1 = \operatorname{poly}_n\big(-\log(\omega_0)/\log(\eta_0)\big)$. Moreover, we have that $e(Z,\Delta) < -\log(\omega_0)/\log(\eta_0)$.*

Proof. Fix $\eta_0 \in \mathbb{R}^{\geqslant 2}$. Since the analytic set $\{z \in B : \ g(z) = 0\}$ is of pure dimension by [10, Proposition 2.6], [6, Corollary 18] gives an affine unitary map $\mathbf{u}$ and a polydisc Δ for $Z := \{z \in B : g \circ \mathbf{u}(z) = 0\}$ such that $B^{2\eta_0\eta_1} \subset \Delta \subset B^{2\eta_0}$ for $\eta_1 = \mathrm{poly}_n\big(\mathrm{vol}(Z \cap B^{2\eta_0})\big)$. We bound the volume of Z in $B^{2\eta_0}$ by use of an analogue of Crofton's formula for complex analytic sets. More precisely, we employ [10, Proposition 14.6.3] which gives that

$$\mathrm{vol}(Z \cap B^{2\eta_0}) \ = \ O_n(1) \int_{G(1,n)} \#(Z \cap B^{2\eta_0} \cap L)\ dL,$$

where $G(1, n)$ denotes the space of all 1-dimensional linear subspaces of $\mathbb{C}^n$ with standard measure.

For all but finite complex lines L passing through the origin, the intersection $Z \cap L$ is finite, and for such a line we apply Lemma 2.2 to get that

$$\#(Z \cap B^{2\eta_0} \cap L) \ < \ \big(1/\log(\eta_0)\big) \cdot \log(\|g \circ \mathbf{u}\|_{B\cap L} / \|g \circ \mathbf{u}\|_{B^{2\eta_0}\cap L}).$$

Then for all but finite $L \in G(1, n)$, it follows that $\#(Z \cap B^{2\eta_0} \cap L) < -\log(\omega_0)/\log(\eta_0)$ by using $\|g\|_B \leqslant 1$ and $|g(0)| \geqslant \omega_0$, which implies in particular that $e(Z, \Delta) < -\log(\omega_0)/\log(\eta_0)$. This gives that $\mathrm{vol}(Z \cap B^{2\eta_0}) \leqslant -O_n(1) \log(\omega_0)/\log(\eta_0)$ and the proof is finished. □

We borrow the definition of the notion of an algebraic map and its complexity from [9, subsubsection 2.2.3]. A map $\phi = (\phi_1, \ldots, \phi_n) : D(1)^l \to \mathbb{C}^n$ is *algebraic of complexity* d if for each $i = 1, \ldots, n$ the graph of ϕ_i is an analytic component of $(D(1)^l \times \mathbb{C}) \cap W_i$, where $W_i \cap D(1)^l \times \mathbb{C}$ is an algebraic variety of degree at most d. A collection of maps has complexity d if all its members have complexity d.

For a algebraic map $\phi : D(1)^l \to \mathbb{C}^n$ of complexity d, we say that $\phi(D(\frac{1}{2})^l)$ is a *d-block in $X(\omega)$* if $\phi(D(\frac{1}{2})^l) \subseteq X(\omega)$. We sometimes just call such an object a d-block when the ambient set $X(\omega)$ is clear from context. In particular any point or polydisc inside $X(\omega)$ are 1-blocks. We are ready to state and prove the following more precise version of Theorem 1.2.

Theorem 5.2. *Let ϵ, ν be given. Suppose θ is such that*

$$\theta \quad \geqslant \quad 20\nu + 2^{8(n+1)} n^{10n+7} \epsilon^{-2(n+1)} \cdot \nu^2.$$

Then there are $c = \mathrm{poly}_n(\nu, \epsilon^{-1})$ and $d = \mathrm{poly}_n(\epsilon^{-1})$ such that for all T, we have that $X(T^{-\theta})(\mathbb{Q}, T)$ is contained in at most cT^ϵ many d-blocks in $X(T^{-\nu})$.

Proof. Let ϵ, ν and θ be given as in the statement above. All blocks in the proof are blocks in $X(T^{-\nu})$. We show the consequence of the theorem locally for $X \cap A$, where A is a block. This will suffice as we control the size of A so that it is not too small; this will be made clear soon. We demonstrate our method around the origin.

Fix some T, this choice will not effect our process. For $\eta_0 = T^{\epsilon/4n}$ and $\omega_0 = T^{-\nu}$, we get $\eta_1 = \mathrm{poly}_n(4n\nu/\epsilon)$ as given by Lemma 5.1. Set $\eta = 8\eta_0\eta_1 C^{-1} T^{\epsilon/4n}$, where $C = C(n) = 2^{-12n^4}$ comes from Corollary 4.7. Suppose $\|g_i\|_{B^\eta} \leqslant T^{-\nu}$ for all $i = 1, \ldots, m$. then there is a polydisc D with $B^{n\eta} \subset D \subset B^\eta$, and we have $D \subseteq X(T^{-\nu})$, so D is a 1-block.

Otherwise $\|g_i\|_{B^\eta} > T^{-\nu}$ for some $i \in \{1, \ldots, m\}$; set $g = g_i$. Let p be a point on $\bar{B}^\eta$ where the maximum of g is attained, so that $|g(p)| > T^{-\nu}$. We apply Lemma 5.1 for p in the role of the origin and B_p^2 in the role of B to obtain a unitary transformation $\mathbf{u}$ and a Weierstrass polydisc Δ for $Z := \{z \in B : \ g \circ \mathbf{u}(z) = 0\}$ such that $B_p^{4\eta_0\eta_1} \subset \mathbf{u}(\Delta) \subset B_p^{4\eta_0}$. Set $D := \Delta^{C^{-1}T^{4\epsilon/n}}$ and note that $B^\eta \subset D$.

We apply Corollary 4.7 for $\omega = T^{-\theta}$ and $Y = \mathbf{u}\big(Z(T^{-\theta}) \cap D\big)$. So $m = n-1$ and Lemma 5.1 gives that $e(X, \Delta) < \frac{4n\nu}{\epsilon}$. Recall $\beta = T^{-\theta}\|g \circ \mathbf{u}\|_\Delta^{-1} \mathrm{e}^{20\mathcal{B}_\Delta^7(g)}$, and with $\frac{\epsilon}{4n}$ in the role of ϵ we obtain $d' < \left(\frac{16n^5}{\epsilon}\right)^n$, such that for

$$\delta := CT^{-\frac{\epsilon}{4n}} \quad \text{and} \quad d_\nu := \frac{4n\nu}{\epsilon} \cdot d',$$

if $\beta \leqslant (CT^{-\epsilon})^{nd_\nu^2}$, then $Y(\mathbb{Q}, T)$ is contained in an hypersurface in $\mathbb{C}^n$ of degree d_ν.

At this stage we have shown, irrespective of whether $\|g\|_{B^\eta} < T^{-\nu}$ or not, that B can be covered by at most $(2 \cdot n\eta)^n = c'T^{\frac{\epsilon}{2}}$ unitary images of polydiscs with $c' = \mathrm{poly}_n(\epsilon^{-1}, \nu)$. If we show, in a manner that does not depend on working around the origin, the choice of T and the polydisc D, that there are $c_1 = \mathrm{poly}_n(\epsilon^{-1}, \nu)$, $d_1 = \mathrm{poly}_n(\epsilon^{-1})$ such that $Y(T^{-\theta})(\mathbb{Q}, T)$ is contained in at most $c_1 T^{\frac{\epsilon}{2}}$ d_1-blocks in $X(T^{-\nu})$, then $\mathbf{u}(Z)(T^{-\theta})(\mathbb{Q}, T)$ will be contained in at most $c_1 c' T^\epsilon$ many d_1-blocks, and the proof will be complete.

We have that $D \subseteq \Delta^{14}$, since $\delta < C \leqslant 2^{-12}$, and working in the Δ-coordinates gives that $\|g \circ \mathbf{u}\|_{\Delta^7} > T^{-\nu}$. Our assumption on θ implies that implies that

$$\beta \leqslant T^{-\theta}\|g \circ \mathbf{u}\|_{\Delta^7}^{-20} \leqslant T^{-2^{8(n+1)}n^{10n+7}\epsilon^{-2(n+1)}\nu^2} < T^{-(\epsilon n + 12n^5)\left(\frac{4n}{\epsilon}\left(\frac{16n^5}{\epsilon}\right)^n \nu\right)^2}.$$

Note that $d_\nu^2 < \left(\frac{4n}{\epsilon}\left(\frac{(2n)^5}{\epsilon}\right)^n \nu\right)^2$ and $C = 2^{-12n^4} \geqslant T^{-12n^4}$. So we have that $\beta < (CT^{-\epsilon})^{nd_\nu^2}$, and as noted before this gives that $Y(T^{-\theta})(\mathbb{Q}, T)$ is contained in an hypersurface, say H_ν, of degree at most $d_\nu = \mathrm{poly}_n(\epsilon^{-1}, \nu)$.

The degree of H_ν, i.e. d_ν, depends on θ, which going forward will be an obstruction to our aim of covering $Y(T^{-\theta})(\mathbb{Q}, T)$ with blocks of degree depending on just ϵ and n. To remedy this, we apply Theorem 3.6 to obtain $c_2 = \mathrm{poly}_n(\epsilon^{-1}, \nu)$, $d_2 = \mathrm{poly}_n(\epsilon^{-1})$, and cover $H_\nu(\mathbb{Q}, T)$ by $c_2 T^{\frac{\epsilon}{8}}$ many algebraic subsets of H_ν of degree bounded by d_2.

Let W be an algebraic set from this cover, so the degree of W is bounded by d_2. Using that $\mathbf{u}(D)$ is the image of a polydisc by unitary transformation, we employ [9, Theorem 8] to get a real cellular cover of $\mathbf{u}(D) \cap W$ of size and complexity $d_3 = \mathrm{poly}_n(d_2)$ admitting $\frac{1}{2}$-extensions. Then apply [9, Lemma 94] to further refine each of the cells so that each cell $\mathscr{C}$ in the refinement admits a $\boldsymbol{\delta_*}$-extension. Here $\boldsymbol{\delta_*}$ is given by $\delta_* = C_* T^{\frac{\epsilon}{8n}}$ in the $D-$coordinates and $\delta_*^{E'_m \mu^{1+1/m}}$ in the A, D_0-coordinates of $\mathscr{C}$, where $\mu = L(n+1, d_*)$, and $C_* = 2^{-4n^4(n+1)^2}$, $d_* = \mathrm{poly}_n(\epsilon^{-1})$ are from Lemma 3.2; this choice will be recalled later. Note we are in the setting of Lemma 3.2 since the dimension $D \cap W$, and hence the number of D-fibers of $\mathscr{C}$, is at most $n-1$. We have arrived now at a covering of algebraic complexity $d_4 = \mathrm{poly}_n(d_3) = \mathrm{poly}_n(\epsilon^{-1})$ for $D \cap W$ by $\mathrm{poly}_n(\epsilon^{-1}, \log T) \cdot T^{\frac{\epsilon}{8}} < \mathrm{poly}_n(\epsilon^{-1}) \cdot T^{\frac{\epsilon}{4}}$ such cells.

Pick a component of this cover, say $\mathbf{f} : \mathscr{C}^{\boldsymbol{\delta_*}} \to \mathbf{u}(D)^{\frac{1}{2}} \cap W$, so complexity of $\mathbf{f}$ is at most d_4. It suffices to prove that: there is $c_3 = \mathrm{poly}_n(\theta, \epsilon^{-1})$ and $d_5 = \mathrm{poly}_n(\epsilon^{-1})$ such that $\big(Y(T^{-\theta}) \cap f(\mathscr{C})\big)(\mathbb{Q}, T)$ is contained in at most $c_3 T^{\frac{\epsilon}{8}}$ d_5-blocks.

We proceed by induction on $k = \dim(\mathscr{C}) \leqslant n-1$. The base case $k = 0$ is trivial, and also implies our claim in the case of $n = 1$. Now let $n \geqslant 2$, and our inductive assumption is that we have the desired claim if $k \leqslant n-2$. Suppose $k = n-1$ and let m_* be the number of D-fibers of $\mathscr{C}$; so $m_* \leqslant n-1$. Our supposition implies that $\dim(W) = n-1$, and let I_W be an ideal of $\mathbb{C}[z_1, \ldots, z_n]$ with $Z(I_W) = W$. By Fact 3.4 we have a variable, say $z_W \in \{z_1, \ldots, z_n\}$, such that every element of I_W features z_W.

If $m_* < n-1$, then our choice of C_* and δ_* allows us to apply Lemma 3.2 with $l = m_* + 1 < n$, and choice of some $\boldsymbol{j} \subseteq \{z_1, \ldots, z_n\} \setminus \{z_W\}$. This gives a hypersurface, say H_*, of degree at most d_* in the $z_{\boldsymbol{j}}$ variables such that $\mathbf{f}(\mathscr{C})(\mathbb{Q}, T)$ is contained in H_*. Note this gives by Fact 3.4 that $\dim(W \cap H_*) \leqslant n-2$, and that the degree of $W \cap H_*$ is bounded by $d_6 = \operatorname{poly}_n(d_2, d_*) = \operatorname{poly}_n(\epsilon^{-1})$ by Bezout's theorem, see [9, subsubsection 2.2.3]. Now using that the complexity of $\mathbf{f}$ is bounded by d_4, we obtain a cellular covering of $\mathbf{u}(D) \cap W \cap H_*$ following a similar process and entailing similar properties as for the case of $\mathbf{u}(D) \cap W$ earlier. The size and complexity of this covering can clearly be controlled as per our requirements, and since cells in this covering are of dimension $\leqslant n-2$, we finish this case by invoking the inductive assumption.

Now suppose that $m_* = n-1 = k$; then no fiber of $\mathscr{C}$ is of A or D_0 type. By renormalizing $\mathscr{C}$ we get a polydisc $\Delta_* \subset \mathbb{C}^{n-1}$, and an algebraic map $\mathbf{h} : \Delta_*^{\delta_*} \to \mathbf{u}(D)^{\frac{1}{2}} \cap W$ of complexity at most $d_7 = \operatorname{poly}_n(d_4)$ such that $\mathbf{f}(\mathscr{C}) \subseteq \mathbf{h}(\Delta_*)$. If $\mathbf{h}(\Delta_*) \subseteq X(T^{-\nu})$, then $\mathbf{h}(\Delta_*)$ is a d_7-block, since $\delta^* < \frac{1}{2}$. Otherwise consider the analytic set $X_* := \mathbf{h}^{-1}(X) = \{z \in \Delta_* :\ g_i(\mathbf{h}(z)) = 0,\ i = 1, \ldots, m\}$, and note that $\dim(X_*) \leqslant m_* - 1 < n-1$. We use $\delta_* < 2^{-n^2}$, to get a ball B_* around the origin with $\Delta_* \subset B_* \subset B_*^{\frac{1}{2}} \subset \Delta_*^{\delta}$. Now we are in the earlier situation with $B_*^{\frac{1}{2}}, X_*$ in the role of the $B^{\frac{1}{2}}, X$ respectively. Since $\dim(X_*) \leqslant m-1$, we use the same process as before, but now apply Corollary 4.4 with $l = \dim(X_*) + 1 \leqslant n-1$, to cover B_* by $\operatorname{poly}_n(\theta, \epsilon^{-1})$ many polydiscs, such that for every polydisc D_* in this collection, $\mathbf{h}(X_* \cap D_*)(\mathbb{Q}, T)$ is contained in an hypersurface of degree at most d_ν, say H_*, in $z_{\boldsymbol{j}}$ variables, where $\boldsymbol{j} := \{z_1, \ldots, z_n\} \setminus \{z_W\}$. Note this gives by Fact 3.4 that $\dim(W \cap H_*) \leqslant n-2$, and that the degree of $W \cap H_*$ is bounded by $d_8 = \operatorname{poly}_n(d_2, d_\nu)$ by Bezout's theorem, see [9, subsubsection 2.2.3]. So as we follow the rest of the process for $B_*^{\frac{1}{2}}, \mathbf{h}, X_*$, using that the complexity of $\mathbf{h}$ is bounded by $d_7 = \operatorname{poly}_n(\epsilon^{-1})$, we end up with a cellular cover of the appropriate size and complexity, but now the dimension of the cells in the covering is at most $n-2$. So we invoke induction again, and the proof is complete. □

Theorem 5.3. *For every ν there are $\theta, c = \operatorname{poly}_n\big(\nu, \log(T)\big)$ and $d = \operatorname{poly}_n\big(\log(T)\big)$ such that for every analytic set X and all T, we have that $X(T^{-\theta})(\mathbb{Q}, T)$ is contained in at most c many d-blocks in $X(T^{-\nu})$.*

6. Effective Pila-Wilkie counting

Throughout this section, we fix $X = \{z \in B :\ g(z) = 0\}$, g non-constant complex analytic on $B^{1/2} \subset \mathbb{C}^n$ with $n \geqslant 2$ and $\|g\|_{B^{1/2}} \leqslant 1$, and assume that X does not contain any semialgebraic set of positive dimension. To establish Theorem 1.1, it clearly suffices to prove the statement for such an X; note we already proved the $n = 1$ case in Section 2.

A *curve* is an algebraic variety of pure dimension 1, and we let $\mathcal{C}_{d,n}$ denote the chow space of curves in $\mathbb{C}^n$ of degree d. The space $\mathcal{C}_{d,n}$ is compact with respect to the chow metric [13, Chapter 4, Theorem 1.1]; we denote this metric by $\operatorname{dist}_{\mathcal{C}_{d,n}}(\cdot, \cdot)$ and topological notions referred in the context of $\mathcal{C}_{d,n}$ are with respect to this metric.

For a curve W of degree at most d, we abuse notation and use W to also denote its chow coordinates in $\mathcal{C}_{d,n}$.

In this section we consider polydiscs only of the type $\Delta = \Delta_z \times \Delta_w \subset \mathbb{C}^n$ with $\dim \Delta_z = 1$ and $\dim \Delta_w = n-1$, and $\pi_z : \Delta \to \Delta_z$ will denote the usual projection. Recall that Δ is a polydisc for a curve W if $W \cap (\bar{\Delta}_z \times \partial\Delta_w) = \emptyset$ (in this definition, we identify W with its unitary image in the co-ordinates of Δ). Also, we say that Δ is a η-gap polydisc for W if $(W + B^\eta) \cap (\bar{\Delta}_z \times \partial\Delta_w) = \emptyset$.

Lemma 6.1. *Given d and n, there is a computable function $f_{\mathrm{delta}} : (0,1) \to (0,1)$ with $f_{\mathrm{delta}}(t) \to 0$ as $t \to 0$ such that the following holds. Given a curve $W \in \mathcal{C}_{d,n}$, an η-gap polydisc Δ for W, and $\delta \leqslant 1/2\eta$, we have that $\mathrm{dist}_{\mathcal{C}_{d,n}}(W, W') < f_{\mathrm{delta}}(\delta)$ implies that*

(1) $\mathrm{dist}_H(W, W') < 1/2\eta$ and hence Δ is a polydisc for W', and

(2) for each $(a, b) \in W \cap \Delta$, there is $(a, b') \in W' \cap \Delta$ with $|b - b'| < \delta$.

Here $\mathrm{dist}_H(W, W')$ is the Hausdorff distance between $W \cap B^{1/2}$ and $W' \cap B^{1/2}$.

We say that $\Delta_1 = \Delta_{z_1} \times \Delta_{w_1}, \ldots, \Delta_n = \Delta_{z_n} \times \Delta_{w_n}$ are *orthogonal polydiscs* if under a unitary change of co-ordinates, we have that z_1 is the first co-ordinate in $\mathbb{C}^n$, z_2 is the second co-ordinate in $\mathbb{C}^n$ and so on. We shall employ the following easy fact later.

Fact 6.2. *Let $\Delta_1 = \Delta_{z_1} \times \Delta_{w_1}, \ldots, \Delta_n = \Delta_{z_n} \times \Delta_{w_n} \subset B^{1/2}$ be orthogonal polydiscs. Suppose for a connected set $K \subset B^{1/2}$ that $\pi_{z_j}(K)$ is contained in a set of disc of radius r around the center of Δ_{z_i} for every $j = 1, \ldots, n$. Then we have that $\mathrm{diam}(K) \leqslant 2\sqrt{n}r$.*

Lemma 6.3. *There is $\eta = \mathrm{poly}_n(d)$ such that for every curve $W \in \mathcal{C}_{d,n}$ and point $p \in \bar{B}$, we can obtain explicit orthogonal η-gap polydiscs $\Delta_1, \ldots, \Delta_n \subset B_p$ for W, each centered around p and containing B_p^η.*

Proof. Let $W \in \mathcal{C}_{d,n}$, fix $X := W \cap B$ and assume without loss of generality that p is the origin. We obtain the desired polydiscs by following [6, Corollary 18]. First, we recall the analogue of Crofton's formula [10, Proposition 14.6.3] used in the proof of Lemma 5.1 to get that

$$\mathrm{vol}(X) \;=\; O_n(1) \int_{G(n-1,n)} \#(X \cap L)\, dL,$$

where $G(n-1, n)$ denotes the space of all $n-1$-dimensional linear subspaces of $\mathbb{C}^n$ with standard measure. Since W is of degree d, we get that $\mathrm{vol}(X) = \mathrm{poly}_n(d)$.

Next we suitably modify the proof of [6, Proposition 17]. Set $X' = S^1 \cdot X$ and let $N(X, \epsilon)$ denote the size of the smallest ϵ-net in X'. By [6, Lemmas 15, 16] we have that

$$N(X', \epsilon) \;=\; O_n(\mathrm{vol}(X')\epsilon^{-2\cdot 1-1}) \;=\; \mathcal{O}_n(\mathrm{poly}_n(d)\epsilon^{-3}).$$

Let S be a 4ϵ-separated set in B^2 with $\#S = \mathcal{O}_n(1)\epsilon^{-2n}$ such that S is closed under n pairwise orthonormal rotations. Using $n \geqslant 2$ we see that for epsilon such that $\#S > nN(X', \epsilon)$, the proof of [6, Proposition 17] gives balls $B_1, \ldots B_n \subset B$ of radius ϵ with $1/\epsilon = \mathrm{poly}_n(d)$, each disjoint from X' such that the set of vectors from the origin to the centers of $B_1, \ldots, B_n$ form an orthogonal basis of $\mathbb{C}^n$.

Now we follow the proof of [6, Corollary 18], and in turn [7, Theorem 7], to obtain $\eta = \mathrm{poly}_n(d)$ and explicit orthogonal η-gap polydiscs $\Delta_1, \ldots, \Delta_n \subset B$ for W, each centered around the origin and containing B^η. $\square$

Lemma 6.4. *Let d, n be given. We can explicitly obtain curves $W_1, \dots, W_m \in \mathcal{C}_{d,n}$, orthogonal polydiscs $\Delta_{l,1} = \Delta_{z;l,1} \times \Delta_{w;l,1}, \dots, \Delta_{l,n} = \Delta_{z;l,n} \times \Delta_{w;l,n} \subset B^{1/2}$ for each $l = 1, \dots, m$, and $\eta = \mathrm{poly}_n(d)$ with the following properties.*

1. *For each $l = 1, \dots, m$, $\Delta_{l,1}, \dots, \Delta_{l,n}$ are polydiscs for all $W \in \mathcal{C}_{d,n}$ satisfying $\mathrm{dist}_{\mathcal{C}_{d,n}}(W, W_l) < \eta_0 := f_{\mathrm{delta}}(\eta)$.*
2. *Let $W \in \mathcal{C}_{d,n}$ and $p \in \bar{B}$. Then there is a $l \in \{1, \dots, m\}$ such that $\mathrm{dist}_{\mathcal{C}_{d,n}}(W, W_l) < \eta_0/2$ and $p \in \Delta^8_{z;l,j} \times \Delta_{w;l,j}$ for all $j = 1, \dots, n$.*

Proof. Fix $\eta = \mathrm{poly}_n(d)$ from Lemma 6.3. We obtain an explicit covering of $\bar{B}$ by balls of radius $1/16\eta$, let $p_1, \dots, p_M$ be the centers of the balls in this cover. Lemma 6.1 gives $\eta_0 = f_{\mathrm{delta}}(\eta)$ such that $\mathrm{dist}_{\mathcal{C}_{d,n}}(W_1, W_2) < \eta_0$ implies $\mathrm{dist}_H(W_1, W_2) < 1/2\eta$ for all $W_1, W_2 \in \mathcal{C}_{d,n}$. We obtain an explicit covering of $\mathcal{C}_{d,n}$ by balls of radius $\eta_0/2$ with centers $W_1, \dots, W_N$.

For each combination of $p_k \in \{p_1, \dots, p_M\}$ and $W_l \in \{W_1, \dots, W_N\}$, apply Lemma 6.3 to obtain explicit η-gap polydiscs $\Delta_{kN,1}, \dots, \Delta_{kN+l,n} \subset B^{1/2}$ for W_l, each centered around p_k. For each $k = 1, \dots, M$ and $l = 1, \dots, N$, set $W_{kN+l} := W_l$ and observe that Lemma 6.1 and the choice of η_0 implies that $\Delta_{kM1+l,1}, \dots, \Delta_{kM_1+l,n}$ are polydiscs for each W satisfying $\mathrm{dist}_{\mathcal{C}_{d,n}}(W, W_l) < \eta_0 := f_{\mathrm{delta}}(\eta)$.

We finish the proof by noting that since the distance of any $p \in \bar{B}$ is at most $1/16\eta$ from a point $p_k \in \{p_1, \dots, p_M\}$, we have that p belongs to any polydisc $\Delta^8_z \times \Delta_w$, where $\Delta_z \times \Delta_w$ is an η-gap polydisc centered around p_k. □

Fixing a covering of $\mathcal{C}_{d,n}$. For every d, n, we fix an explicit set of curves $W_1, \dots, W_m$, and for each $l = 1, \dots, m$ explicit polydiscs $\Delta_{l,1} \dots, \Delta_{l,n}$, $\eta = \mathrm{poly}_n(d)$, and $\eta_0 = f_{\mathrm{delta}}(\eta)$ obtained via Lemma 6.4 throughout the rest of this section.

Resultants of g. Let $\Delta = \Delta_z \times \Delta_w$ be a polydisc for a curve W. The *analytic resultant* of g with respect to W, Δ is the function $g_W : \Delta_z \to \mathbb{C}$, given by

$$g_{W,\Delta}(z) \ := \prod_{(z,w) \in W \cap \Delta} g(z, w).$$

Note g_W is analytic over $\Delta_{z;W}$ by [8, Fact 5].

Lemma 6.5. *Let $\Delta = \Delta_z \times \Delta_w$ be an η-gap polydisc for some $W \in \mathcal{C}_{d,n}$. Then there is $\tau_W > 0$ such that for all $W' \in \mathcal{C}_{d,n}$, $\mathrm{dist}_{\mathcal{C}_{d,n}}(W, W') < \eta_0/2$ implies that*

$$\max_{z \in \Delta^8_z} |g_{W,\Delta}(z)| > \tau_W.$$

Moreover, if g is computable then we can compute a τ_W satisfying the above.

Proof. Let Δ and $W \in \mathcal{C}_{d,n}$ be given as in the statement above. Let U be the ball of radius η_0 with center W. For $W' \in U$ Lemma 6.1 gives that Δ is a polydisc for W', and for such W' we set

$$\mathrm{dist}(X, W') := \max_{z \in \Delta^8_z} |g_{W',\Delta}(z)|$$

Clearly $\mathrm{dist}(X, W') > 0$ for all $W' \in U$ since X is assumed to not contain any semialgebraic set. Towards a contradiction suppose there is a sequence $W_1, W_2, \dots$ in U^2 such that $\mathrm{dist}(X, W_m) < 1/m$ for all $m \in \mathbb{N}^{\geqslant 1}$. Using that $\mathcal{C}_{d,n}$ is compact we get a subsequence $W_{k_1}, W_{k_2}, \dots$ which converges in $\mathcal{C}_{d,n}$ to some $W_0 \in \bar{U}^2 \subset U$. So Δ is a polydisc for W_0, and applying Lemma 6.1 for W_0 we get that $\mathrm{dist}(X, W_0) = 0$

using the continuity of g and that $f_\delta(t) \to 0$ as $t \to 0$. We have arrived at a contradiction and the proof of the first assertion is complete.

Towards a proof of the second assertion, suppose that g is computable. Let d_g be the computable function given by Definition 2.1. We iterate on $k \in \mathbb{N}^{\geqslant 1}$, starting with the smallest k with $1/d_g(k) \leqslant 1/2\eta$, and show that our process stops at a finite step to yield a number τ_W satisfying the desired inequality.

Given a k, we obtain $Y_1, \dots, Y_{m_k} \in U^2$ such that for all $Y' \in U^2$ there is a $\mu \in \{1, \dots, m_k\}$ with $\mathrm{dist}_{\mathcal{C}_{d,n}}(Y', Y_\mu) < f_{\mathrm{delta}}(1/d_g(k))$. For each $\mu \in \{1, \dots, m_k\}$ we compute $\tau_{\mu,k}$ as the maximum of the absolute values of $g_{Y_\mu,\Delta}$ on a computable $(1/k)$-separated set of size $k/4$ on $\partial\Delta_z^8$.

Set $\tau_k := \min_{\mu \leqslant m_k} \tau_{\mu,k}$, and it follows that $\mathrm{dist}(X, Y_\mu) \geqslant \tau_k$ for all $\mu \leqslant m_k$. Now Lemma 6.1 gives that $\mathrm{dist}^{\mathcal{U}}(X, Y') \geqslant \tau_k - 2^d/k$ for all $Y' \in U^2$. The first assertion implies that there is k for which $\tau_k - 2^d/k > 0$ and the proof is complete. □

The following is an immediate corollary of Lemmas 6.4 and 6.5.

Corollary 6.6. *There is $\tau = \tau(d, X) > 0$ such that for all $W \in \mathcal{C}_{d,n}$ we have that*

$$\mathrm{dist}(X, W) := \min_{l: W \in U_l} \min_j \max_{z \in \Delta^8_{z;l,j}} |g_{W,\Delta_{l,j}}(z)| \;\geqslant\; 2^{-\tau},$$

where U_l is ball of radius $\eta_0/2$ with center W_l.

This parameter $\tau(d, X)$ serves as a relative Bernstein index measure for X. We arrange that $\tau(d, X)$ is a non-increasing function of d.

Lemma 6.7. *Let W be a curve of degree at most d and let $p \in W \cap B$. Then there are orthogonal polydiscs $\Delta_1 = \Delta_{z;1} \times \Delta_{w;1}, \dots, \Delta_n = \Delta_{z;n} \times \Delta_{w;n} \subset B^{1/2}$ for W such that $\pi_{z;j}(p)$ is the center of $\Delta_{z;j}$ and $\|g_{W,\Delta_j}\|_{\Delta^{1/r}_{z;j}} \;\geqslant\; r^\tau$ for all $r \leqslant 1/2$ and $j = 1, \dots, n$.*

Proof. Let W and p be given as in the statement of the lemma. By Lemma 6.4 and Corollary 6.6 we have orthogonal polydiscs $\Delta'_1, \dots, \Delta'_n \subset B^{1/2}$ for W such that $\mathrm{dist}(X, W) \geqslant 2^{-\tau}$ and $p \in \Delta'^8_{z;j} \times \Delta'_{w;n}$ for all $j = 1, \dots, n$. For each j, set $\Delta_{z;j} := (\Delta'^2_{z;j})_{\pi_{z;j}(p)}$ and $\Delta_j := \Delta_{z;j} \times \Delta'_{w;j} \subset \Delta_j$, and hence $g_{W,\Delta_j}(z)$ is analytic on the open disc $\Delta_{z;j}$. Fix $j \in \{1, \dots, n\}$. The fact that $\pi_{z;j}(p) \in \Delta'^8_{z;j}$ implies $\Delta'^8_{z;j} \subset \Delta^2_{z;j}$, and hence

$$\mathrm{dist}(X, W) \;\geqslant\; 2^{-\tau} \quad\Longrightarrow\quad \|g_{W,\Delta_j}\|_{\Delta^2_{z;j}} \;\geqslant\; 2^{-\tau}.$$

Since $\|g\|_{B^{1/2}} \leqslant 1$, we have that $\|g_{W,\Delta_j}\|_{\Delta_{z;j}} \leqslant 1$, and our desired conclusion follows by a direct application of the Hadamard three circle theorem. □

Theorem 6.8. *Let X and ϵ be given. Then there are constants $d = \mathrm{poly}_n(\epsilon^{-1})$ and $c = \mathrm{poly}_n\big(\epsilon^{-1}, \tau(d, X)\big)$ such that for all T we have that*

$$\#X(\mathbb{Q}, T) \;\leqslant\; cT^\epsilon.$$

Proof. For X and ϵ, fix $d = \mathrm{poly}_n(\epsilon^{-1})$ as given by Theorem 1.2 and let $\tau = \tau(d, X)$. Applying Theorem 1.2 for $\nu = 3\tau$, we get $c = \mathrm{poly}_n(\tau, \epsilon^{-1})$ such that for all T, $X(\mathbb{Q}, T)$ is covered by cT^ϵ many d-blocks in $X(T^{-3\tau})$. Note here we used that the value of d given by Theorem 1.2 does not depend on ν.

Let K be a d-block in $X(T^{-3\tau})$. We shall show that $\mathrm{diam}(K) < T^{-2}$ for all large enough T, hence it suffices to only consider the case of $\dim K = 1$. Take a curve W of degree at most d with $K \subseteq W$, and fix a point $p \in K$. Apply Lemma 6.7 to obtain orthogonal polydiscs $\Delta_1 = \Delta_{z;1} \times \Delta_{w;1}, \ldots, \Delta_n = \Delta_{z;n} \times \Delta_{w;n} \subset B^{1/2}$ for W such that $\pi_{z;j}(p)$ is the center of $\Delta_{z;j}$ and $\|g_{W,\Delta_j}\|_{\Delta_{z,j}^{1/r}} \geqslant r^\tau$ for all $r \leqslant 1/2$ and $j = 1, \ldots, n$.

Then $\|g\|_K \leqslant T^{-3\tau}$ implies that the $\pi_{z;j}\big(\operatorname{diam}(K)\big) \leqslant 2T^{-3}$ for all $j = 1, \ldots, n$. Then Fact 6.2 implies that $\operatorname{diam}(K) < T^{-2}$ and hence $\#K(\mathbb{Q}, T) \leqslant 1$, for all $T > 2\sqrt{n}$. This completes the proof. $\square$

Proof of Theorem 1.1, Case $n \geqslant 2$: Lemma 6.5 implies that if g is computable then $\tau(d, X)$ is also computable for every d. Hence we are done by Theorem 6.8. $\square$

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