Topic: Limits & Continuity

① $\lim_{x\to a}\left[\dfrac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}}\right]$ ② $\lim_{y\to 0}\left[\dfrac{\sqrt{a+y}-\sqrt{a}}{y\sqrt{a+y}}\right]$

③ $\lim_{x\to \frac{\pi}{6}}\left[\dfrac{\cos x-\sqrt{3}\sin x}{\pi-6x}\right]$ ④ $\lim_{x\to\infty}\left[\dfrac{\sqrt{x^2+5}-\sqrt{x^2-3}}{\sqrt{x^2+3}-\sqrt{x^2+1}}\right]$

⑤ Examine the continuity of following function at given points.

(i) $f(x)=\begin{cases}\dfrac{\sin 2x}{\sqrt{1-\cos 2x}} & \text{for } 0<x\le\frac{\pi}{2}\\[2mm] \dfrac{\cos x}{\pi-2x} & \text{for } \frac{\pi}{2}<x<\pi\end{cases}$ at $x=\frac{\pi}{2}$

(ii) $f(x)=\begin{cases}\dfrac{x^2-9}{x-3} & \text{for } 0<x<3\\ x+3 & 3\le x<6\\ \dfrac{x^2-9}{x+3} & 6\le x<9\end{cases}$ at $x=3$ & $x=6$

⑥ Find the value of K so that the function f(x) is continuous at indicated points

(i) $f(x)=\begin{cases}\dfrac{1-\cos 4x}{x^2} & x<0\\ K & x=0\end{cases}$ at $x=0$

(ii) $f(x)=\begin{cases}(\sec^2 x)^{\cot^2 x} & x\neq 0\\ K & x=0\end{cases}$ at $x=0$

(iii) $f(x)=\begin{cases}\dfrac{\sqrt{3}-\tan x}{\pi-3x} & x\neq\frac{\pi}{3}\\ K & x=\frac{\pi}{3}\end{cases}$ at $x=\frac{\pi}{3}$

⑦ Discuss the continuity of following functions. Which of these have removable discontinuity? Redefine the function so as to remove discontinuity.

(i) $f(x)=\begin{cases}\dfrac{1-\cos x}{x\tan x} & x\neq 0\\ 9 & x=0\end{cases}$ at $x=0$

(ii) $f(x)=\begin{cases}\dfrac{(e^{3x}-1)\sin x^\circ}{x^2} & x\neq 0\\ \dfrac{\pi}{60} & x=0\end{cases}$ at $x=0$

⑧ If $f(x)=\dfrac{e^{x^2}-\cos x}{x^2}$ for $x\neq 0$ is continuous at $x=0$. Find f(0)

⑨ If $f(x)=\dfrac{\sqrt{2}-\sqrt{1+\sin x}}{\cos^2 x}$ for $x\neq\frac{\pi}{2}$ is continuous at $x=\frac{\pi}{2}$.

Find $f\left(\frac{\pi}{2}\right)$

ANSWERS

① $\lim_{x\to a}\left[\dfrac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}}\right]$

$= \lim_{x\to a} \dfrac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}} \times \dfrac{\sqrt{3a+x}+2\sqrt{x}}{\sqrt{3a+x}+2\sqrt{x}}$

$= \lim_{x\to a} \dfrac{(\sqrt{a+2x}-\sqrt{3x})(\sqrt{3a+x}+2\sqrt{x})}{(\sqrt{3a+x})^2-(2\sqrt{x})^2}$

$= \lim_{x\to a} \dfrac{(\sqrt{a+2x}-\sqrt{3x})(\sqrt{3a+x}+2\sqrt{x})}{3a+x-4x}$

$= \dfrac{1}{3}\lim_{x\to a} \dfrac{(\sqrt{a+2x}-\sqrt{3x})(\sqrt{3a+x}+2\sqrt{x})}{a-x} \times \dfrac{(\sqrt{a+2x}+\sqrt{3x})}{(\sqrt{a+2x}+\sqrt{3x})}$

$= \dfrac{1}{3}\lim_{x\to a} \dfrac{(a+2x-3x)(\sqrt{3a+x}+2\sqrt{x})}{(a-x)(\sqrt{a+2x}+\sqrt{3x})}$

$= \dfrac{1}{3}\lim_{x\to a} \dfrac{(\sqrt{3a+x}+2\sqrt{x})}{\sqrt{a+2x}+\sqrt{3x}}$

$= \dfrac{1}{3}\ \dfrac{\sqrt{4a}+2\sqrt{a}}{\sqrt{3a}+\sqrt{3a}}$

$= \dfrac{1}{3}\ \dfrac{\overset{2}{\cancel{4}}\sqrt{a}}{\cancel{2}\sqrt{3a}}$

$= \dfrac{2}{3\sqrt{3}}$

② $\lim_{y\to 0}\left[\dfrac{\sqrt{a+y}-\sqrt{a}}{y\sqrt{a+y}}\right]$

$\lim_{y\to 0}\left[\dfrac{\sqrt{a+y}-\sqrt{a}}{y\sqrt{a+y}} \times \dfrac{\sqrt{a+y}+\sqrt{a}}{\sqrt{a+y}+\sqrt{a}}\right]$

$\lim_{y\to 0} \dfrac{a+y-a}{y\sqrt{a+y}(\sqrt{a+y}+\sqrt{a})}$

$\lim_{y\to 0} \dfrac{\cancel{y}}{\cancel{y}\sqrt{a+y}(\sqrt{a+y}+\sqrt{a})}$

$= \dfrac{1}{\sqrt{a}(\sqrt{a}+\sqrt{a})}$

$= \dfrac{1}{\sqrt{a}(2\sqrt{a})}$

$= \dfrac{1}{2a}$

③ $\lim_{x\to\frac{\pi}{6}}\left[\dfrac{\cos x-\sqrt{3}\sin x}{\pi-6x}\right]$

let $x-\dfrac{\pi}{6}=h$ $\qquad x\to\dfrac{\pi}{6}$

$\therefore x = h+\dfrac{\pi}{6}$ $\qquad h\to 0$

$\therefore \lim_{h\to 0}\left[\dfrac{\cos\left(h+\frac{\pi}{6}\right)-\sqrt{3}\sin\left(h+\frac{\pi}{6}\right)}{\pi-6\left(h+\frac{\pi}{6}\right)}\right]$

$\lim_{h\to 0}\left[\dfrac{\cos h\cos\frac{\pi}{6}-\sin h\sin\frac{\pi}{6}-\sqrt{3}\sin h\cos\frac{\pi}{6}-\sqrt{3}\cos h\sin\frac{\pi}{6}}{\pi-6h-\pi}\right]$

$= \lim_{h\to 0}\left[\dfrac{\frac{\sqrt{3}}{2}\cos h-\frac{1}{2}\sin h-\frac{3}{2}\sin h-\frac{\sqrt{3}}{2}\cos h}{-6h}\right]$

$= \lim_{h\to 0} \dfrac{\cancel{-}2\sin h}{\cancel{-}6h}$

$= \dfrac{1}{3}\lim_{h\to 0}\dfrac{\sin h}{h} = \dfrac{1}{3}$

Ans

(4) $\lim_{x \to \infty} \left[\frac{\sqrt{x^2+5} - \sqrt{x^2-3}}{\sqrt{x^2+3} - \sqrt{x^2-1}} \right]$

$\lim_{x \to \infty} \frac{\sqrt{x^2+5} - \sqrt{x^2-3}}{\sqrt{x^2+3} - \sqrt{x^2-1}} \times \frac{\sqrt{x^2+5} + \sqrt{x^2-3}}{\sqrt{x^2+5} + \sqrt{x^2-3}} \times \frac{\sqrt{x^2+3} + \sqrt{x^2-1}}{\sqrt{x^2+3} + \sqrt{x^2-1}}$

$= \lim_{x \to \infty} \frac{8}{2} \frac{\sqrt{x^2+3} + \sqrt{x^2-1}}{\sqrt{x^2+5} + \sqrt{x^2-3}}$

$= \frac{8}{2} \lim_{x \to \infty} \frac{\sqrt{x^2\left(1+\frac{3}{x^2}\right)} + \sqrt{x^2\left(1+\frac{1}{x^2}\right)}}{\sqrt{x^2\left(1+\frac{5}{x^2}\right)} + \sqrt{x^2\left(1-\frac{3}{x^2}\right)}}$

$= 4 \lim_{x \to \infty} \frac{x}{x} \times \frac{\sqrt{1+\frac{3}{x^2}} + \sqrt{1+\frac{1}{x^2}}}{\sqrt{1+\frac{5}{x^2}} + \sqrt{1-\frac{3}{x^2}}}$

$= 4 \quad \frac{\sqrt{1} + \sqrt{1}}{\sqrt{1} + \sqrt{1}}$

$= 4 \times \frac{2\sqrt{2}}{2\sqrt{2}}$

$= 4$

(5)

(i) $f(x) = \left\{ \begin{array}{ll} \frac{\sin 2x}{\sqrt{1-\cos 2x}} & \text{for } 0 < x \leq \frac{\pi}{2} \\ = \frac{\cos x}{\pi - 2x} & \text{for } \frac{\pi}{2} < x < \pi \end{array} \right\}$ at $x = \frac{\pi}{2}$

$f\left(\frac{\pi}{2}\right) = \frac{\sin 2\left(\frac{\pi}{2}\right)}{\sqrt{1-\cos 2\left(\frac{\pi}{2}\right)}} = \frac{\sin \pi}{\sqrt{1-\cos \pi}} = 0$

$L.H.L = \lim_{x \to \frac{\pi}{2}^-} f(x)$

$= \lim_{x \to \frac{\pi}{2}} \frac{\cos x}{\pi - 2x}$

Let $x - \frac{\pi}{2} = h$ $\qquad x \to \frac{\pi}{2}$

$x = \frac{\pi}{2} + h$ $\qquad h \to 0$

$\therefore \lim_{h \to 0} \frac{\cos\left(\frac{\pi}{2}+h\right)}{\pi - 2\left(\frac{\pi}{2}+h\right)}$

$= \lim_{h \to 0} \frac{-\sin h}{\cancel{\pi} - \cancel{\pi} + 2h}$

$= \frac{-1}{2} \lim_{h \to 0} \frac{\sin h}{h}$

$= \frac{-1}{2}$

$$R.H.L = \lim_{x \to \frac{\pi}{2}^+} f(x)$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{\sin 2x}{\sqrt{1-\cos 2x}}$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{2\sin x \cos x}{\sqrt{2\sin^2 x}}$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{2\cancel{\sin x}\cos x}{\cancel{\sin x}\sqrt{2}}$$

$$= \frac{2}{\sqrt{2}} \lim_{x \to \frac{\pi}{2}} \cos x$$

$$= 0$$

$\therefore L.H.L \neq R.H.L$

$\therefore f$ is not continuous at $x = \frac{\pi}{2}$

(ii) $f(x) = \begin{cases} \frac{x^2-9}{x-3} & 0 < x < 3 \\ x+3 & 3 \le x < 6 \\ \frac{x^2-9}{x+3} & 6 \le x < 9 \end{cases}$ at $x=3$ & $x=6$

At $x=3$

$f(3) = x+3 = 3+3 = 6$

$$L.H.L = \lim_{x \to 3^-} f(x) = \lim_{x \to 3} x+3$$

$$= 3+3$$

$$= 6$$

$$R.H.L = \lim_{x \to 3^+} f(x) = \lim_{x \to 3} \frac{x^2-9}{x-3}$$

$$= \lim_{x \to 3} \frac{\cancel{(x-3)}(x+3)}{\cancel{(x-3)}}$$

$$= \lim_{x \to 3} x+3$$

$$= 3+3$$

$$= 6$$

At $x=$

$\therefore L.H.L = R.H.L = f(3)$

$\therefore f$ is continuous at $x = 3$

At $x=6$

$$f(6) = \frac{x^2-9}{x+3} = \frac{(x-3)\cancel{(x+3)}}{\cancel{(x+3)}} = 6-3 = 3$$

$$L.H.L = \lim_{x \to 6^-} \frac{x^2-9}{x+3}$$

$$= \lim_{x \to 6} \frac{(x-3)\cancel{(x+3)}}{\cancel{(x+3)}}$$

$$= \lim_{x \to 6} x-3$$

$$= 6-3$$

$$= 3$$

$$R.H.L = \lim_{x \to 6^+} f(x) = \lim_{x \to 6} x+3$$

$$= 6+3$$

$$= 9$$

$\therefore L.H.L \neq R.H.L$

$\therefore f$ is not continuous at $x=6$

(6)

(i) $f(x) = \begin{cases} \dfrac{1-\cos 4x}{x^2} & x \neq 0 \\ = k & x = 0 \end{cases}$ at $x=0$

$f(x)$ is continuous at $x=0$

$\therefore \lim_{x\to 0} f(x) = k$

$\lim_{x\to 0} \dfrac{1-\cos 4x}{x^2} = k$

$\lim_{x\to 0} \dfrac{2\sin^2 2x}{x^2} = k$

$2 \lim_{x\to 0} \left(\dfrac{\sin 2x}{x}\right)^2 = k$

$2 \times \dfrac{1}{4} = k$

$\therefore k = \dfrac{1}{2}$

(ii) $f(x) = \begin{cases} (\sec^2 x)^{\cot^2 x} & x \neq 0 \\ = k & x = 0 \end{cases}$ at $x=0$

f is continuous at $x=0$

$\therefore \lim_{x\to 0} f(x) = k$

$\lim_{x\to 0} (\sec^2 x)^{\cot^2 x} = k$

$\lim_{x\to 0} (1+\tan^2 x)^{\frac{1}{\tan^2 x}} = k$

Comparing with, $\lim (1+x)^{\frac{1}{x}}$

$\therefore k = e$

(iii) $f(x) = \begin{cases} \dfrac{\sqrt{3} - \tan x}{\pi - 3x} & x \neq \frac{\pi}{3} \\ = k & x = \frac{\pi}{3} \end{cases}$ at $x = \frac{\pi}{3}$

f is continuous at $x = \pi/3$

$\therefore \lim_{x\to \frac{\pi}{3}} f(x) = k$

$\therefore \lim_{x\to \frac{\pi}{3}} \dfrac{\sqrt{3} - \tan x}{\pi - 3x} = k$

Let, $x - \dfrac{\pi}{3} = h$ As $x \to \dfrac{\pi}{3}$

$\therefore x = h + \dfrac{\pi}{3}$ $h \to 0$

$\therefore \lim_{h\to 0} \dfrac{\sqrt{3} - \tan\left(h+\frac{\pi}{3}\right)}{\pi - 3\left(h+\frac{\pi}{3}\right)}$

$\therefore \dfrac{\lim_{h\to 0} \sqrt{3} - \dfrac{\tan h + \tan \pi/3}{1 - \tan h \tan \pi/3}}{\lim_{h\to 0} \cancel{\pi} - 3h - \cancel{\pi}}$

$\lim_{h\to 0} \dfrac{\sqrt{3}\left(1 - \tanh \tan \pi/3\right) - \tanh - \tan \pi/3}{-3h\left(1 - \tan h \tan \pi/3\right)}$

$= \dfrac{-\sqrt{3}}{3} \lim_{h\to 0} \dfrac{\left(1 - \sqrt{3}\tanh\right) - \tanh - \sqrt{3}}{h\left(1 - \sqrt{3}\tanh\right)}$

$$= -\frac{1}{3} \lim_{h\to 0} \frac{\cancel{\sqrt{3}} - 3\tan h - \tan h - \cancel{\sqrt{3}}}{h(1-\sqrt{3}\tan h)}$$

$$-\frac{1}{3} \times -4 \lim_{h\to 0} \frac{\tan h}{h} \times \lim_{h\to 0} \frac{1}{1-\sqrt{3}\tan h}$$

$$\frac{4}{3} \times 1 \times 1$$

$$= \frac{4}{3}$$

(7)

(i) $f(x) = \begin{cases} \dfrac{1-\cos x}{x\tan x} & x \neq 0 \\ = 9 & x = 0 \end{cases}$ at $x = 0$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{1-\cos x}{x\tan x}$$

$$= \lim_{x\to 0} \frac{2\sin^2(x/2) \cancel{\times x/2}}{x \frac{\tan x \cdot x}{x}}$$

$$= \frac{2\lim_{x\to 0}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2 \times \frac{x^2}{4}}{\lim_{x\to 0} x^2 \frac{\tan x}{x}}$$

$$= \frac{2 \times \cancel{x^2}}{\cancel{x^2} \times 4}$$

$$= \frac{2}{4}$$

$$= \frac{1}{2}$$

For f to be continuous

$\lim_{x\to 0} f(x) = \cancel{9}\ \frac{1}{2}$

$\therefore$ It is removable discontinuity at $x=0$

$\therefore$ Redefining the limit,

$f(x) = \begin{cases} \dfrac{1-\cos x}{x\tan x} & x \neq \frac{1}{2}0 \\ = \frac{1}{2} & x = 0 \end{cases}$

(ii) $f(x) = \begin{cases} \dfrac{(e^{3x}-1)\sin x^\circ}{x^2} & x \neq 0 \\ = \dfrac{\pi}{60} & x = 0 \end{cases}$ at $x = 0$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{(e^{3x}-1)\sin x^\circ}{x^2}$$

$$= \lim_{x\to 0} 3\frac{(e^{3x}-1)}{3x} \cdot \frac{\frac{\sin\left(\frac{\pi x}{180}\right)}{\frac{\pi x}{180}} \times \frac{\pi x}{180}}{x}$$

$$= \cancel{3}^1 \times 1 \cdot \frac{\pi \cancel{x}}{\cancel{180}_{60} \cancel{x}}$$

$$= \frac{\pi}{60}$$

$$\therefore \lim_{x\to 0} f(x) = \frac{\pi}{60}$$

$\therefore$ Function is continuous

(8) $f(x) = \dfrac{e^{x^2} - \cos x}{x^2}$

$\cdot$ f is continuous

$\therefore \lim_{x \to 0} \dfrac{e^{x^2} - \cos x}{x^2} = f(0)$

$\lim_{x \to 0} \dfrac{(e^{x^2} - 1) + (1 - \cos x)}{x^2} = f(0)$

$\lim_{x \to 0} \dfrac{(e^{x^2} - 1)}{x^2} + \lim_{x \to 0} \dfrac{1 - \cos x}{x^2} = f(0)$

$1 + \lim_{x \to 0} \dfrac{2\sin^2 x/2}{4\, \frac{x^2}{4}} = f(0)$

$1 + \dfrac{1}{2} = f(0)$

$\therefore f(0) = \dfrac{3}{2}$

(9) $f(x) = \dfrac{\sqrt{2} - \sqrt{1 + \sin x}}{\cos^2 x}$

f is continuous

$\therefore \lim_{x \to \frac{\pi}{2}} \dfrac{\sqrt{2} - \sqrt{1 + \sin x}}{\cos^2 x} = f\left(\frac{\pi}{2}\right)$

$\lim_{x \to \frac{\pi}{2}} \dfrac{\sqrt{2} - \sqrt{1 + \sin x}}{1 - \sin^2 x} \dfrac{\sqrt{2} + \sqrt{1 + \sin x}}{(\sqrt{2} + \sqrt{1 + \sin x})} = f\left(\frac{\pi}{2}\right)$

$\lim_{x \to \frac{\pi}{2}} \dfrac{\cancel{\sqrt{2}}\; 2 - 1 - \sin x}{(\cancel{1 - \sin x})(1 + \sin x)(\sqrt{2} + \sqrt{1 + \sin x})} = f\left(\frac{\pi}{2}\right)$

$\lim_{x \to \frac{\pi}{2}} \dfrac{1}{(1 + \sin x)(\sqrt{2} + \sqrt{1 + \sin x})} = f\left(\frac{\pi}{2}\right)$

$\therefore \dfrac{1}{(1+1)\;(\sqrt{2} + \sqrt{2})} = f\left(\frac{\pi}{2}\right)$

$= \dfrac{1}{2\,(2\sqrt{2})} = f\left(\frac{\pi}{2}\right)$

$\therefore f\left(\frac{\pi}{2}\right) = \dfrac{1}{4\sqrt{2}}$

6/12/19

PRACTICAL No. 02

TOPIC: Derivative

Q.1 Show that the following function defined from $\mathbb{R}$ to $\mathbb{R}$ are differentiable
(i) $\cot x$ (ii) $\operatorname{cosec} x$ (iii) $\sec x$

Q.2 If $f(x) = 4x+1, \quad x \leq 2$
$\quad = x^2+5, \quad x > 0 \quad$ at $x = 2$
Then find f is differentiable or not?

Q.3 If $f(x) = 4x+7, \quad x < 3$
$\quad = x^2+3x+1, \quad x \geq 3, \quad$ at $x = 3$
Then find f is differentiable or not?

Q.4 If $f(x) = 8x-5, \quad x \leq 2$
$\quad = 3x^2-4x+7, \quad x > 2 \quad$ at $x = 2$
Then find f is differentiable or not?

ANSWERS

1) (i) $f(x) = \cot x$

consider,

$$Df(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$$

$$= \lim_{x \to a} \frac{\cot x - \cot a}{x-a}$$

Put $(x-a) = h$

$\therefore x = a+h$

As, $x \to a \quad ; h \to 0$

$$= \lim_{h \to 0} \frac{\cot(a+h) - \cot a}{(a+h) - a}$$

$$= \frac{\lim_{h \to 0} \frac{\cos(a+h)}{\sin(a+h)} - \frac{\cos a}{\sin a}}{\lim_{h \to 0} h}$$

$$= \frac{1}{h} \lim_{h \to 0} \frac{\cos(a+h)\sin a - \cos a \sin(a+h)}{\sin(a+h)\sin a}$$

$$= \frac{1}{h} \lim_{h \to 0} \frac{\sin(a-(a+h))}{\sin(a+h)\sin a}$$

$$= -\lim_{h \to 0} \frac{\sin h}{h} \cdot \lim_{h \to 0} \operatorname{cosec}(a+h)\operatorname{cosec} a$$

$$= -1 \cdot \operatorname{cosec}(a+0)\operatorname{cosec} a$$

$$= -\operatorname{cosec}^2 a$$

$\therefore Df(a) = -\operatorname{cosec}^2 a$

$\therefore$ f is differentiable $\forall a \in \mathbb{R}$

(ii) $f(x) = \operatorname{cosec} x$

Consider,

$$Df(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$= \lim_{x \to a} \frac{\operatorname{cosec} x - \operatorname{cosec} a}{x - a}$$

Let,

$x - a = h$; $x = a + h$

as, $x \to a$, $h \to 0$

$$= \lim_{h \to 0} \frac{\operatorname{cosec}(a+h) - \operatorname{cosec} a}{a + h - a}$$

$$= \frac{\lim_{h \to 0} \frac{1}{\sin(a+h)} - \frac{1}{\sin a}}{\lim_{h \to 0} h}$$

$$= \frac{1}{h} \lim_{h \to 0} \frac{\sin a - \sin(a+h)}{\sin(a+h) \sin a}$$

$$= \frac{1}{h} \lim_{h \to 0} \left[\frac{\sin a - \sin a \cos h - \cos a \sin h}{(\sin a \cos h + \cos a \sin h) \sin a} \right]$$

$$= \frac{1}{h} \lim_{h \to 0} \sin a (1 - \cos h) - \cos a \sin h \,]$$

$$= \frac{\frac{1}{h} \lim_{h \to 0} 2 \cos\left(\frac{a+a+h}{2}\right) \sin\left(\frac{a-a-h}{2}\right)}{\lim_{h \to 0} \sin^2 a \cos h + \cos a \sin a \sin h}$$

$$= \frac{\frac{1}{h} \lim_{h \to 0} 2 \cos\left(\frac{2a+h}{2}\right) \sin\left(\frac{-h}{2}\right)}{\sin^2 a}$$

$$= - \lim_{h \to 0} \frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}} \cdot \frac{\lim_{h \to 0} \cos\left(\frac{2a+h}{2}\right)}{\sin^2 a}$$

$$= -1 \cdot \frac{\cos\left(\frac{2a}{2}\right)}{\sin^2 a}$$

$$= -1 \cdot \frac{\cos a}{\sin^2 a}$$

$$= -\cot a \operatorname{cosec} a$$

$\therefore Df(a) = -\cot a \operatorname{cosec} a$

$\therefore f$ is differentiable at $\forall x \in R$

(iii) $f(x) = \sec x$

Consider,

$$Df(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$= \lim_{x \to a} \frac{\sec x - \sec a}{x - a}$$

Let,

$x - a = h$; $x = a + h$

as, $x \to a$, $h \to 0$

$$= \lim_{h \to 0} \frac{\sec(a+h) - \sec a}{a + h - a}$$

$$= \frac{1}{h} \lim_{h \to 0} \frac{1}{\cos(a+h)} - \frac{1}{\cos a}$$

$$= \frac{1}{h} \lim_{h \to 0} \frac{\cos a - \cos(a+h)}{\cos(a+h) \cos a}$$

$$= \frac{\frac{1}{h} \lim_{h \to 0} -2 \sin\left(\frac{a+a+h}{2}\right) \sin\left(\frac{a-a-h}{2}\right)}{\lim_{h \to 0} (\cos a \cos h - \sin a \sin h) \cos a}$$

$$= -\frac{\cancel{2}}{\cancel{h}} \lim_{h\to 0} \frac{\sin\left(\frac{2a+h}{2}\right)\cos\left(\frac{-h}{2}\right)}{\lim_{h\to 0} \cos^2 a \cos h - \sin a \sin h \cos a}$$

$$= \frac{\cancel{-2}}{\cancel{h}} \lim_{h\to 0} \frac{\sin\left(\frac{2a+h}{2}\right)\cos\left(\frac{h}{2}\right)}{\cos^2 a}$$

$$= \frac{-2}{h} \lim_{h\to 0} \frac{\sin\left(\frac{2a+h}{2}\right)\sin\left(\frac{h}{2}\right)}{\cos^2 a}$$

$$= \frac{2}{\cos^2 a} \lim_{h\to 0} \sin\left(\frac{2a+h}{2}\right) \cdot \lim_{h\to 0} \sin\left(\frac{h}{2}\right) / h$$

$$= \frac{\cancel{2}}{\cos^2 a} \sin\left(\frac{2a+0}{2}\right) \cdot \frac{1}{\cancel{2}}$$

$$= \frac{\sin a}{\cos^2 a}$$

$= \tan a \sec a$

$\therefore Df(a) = \tan a \sec a$

f is differentiable at $\forall x \in R$

Q.2 $f(x) = 4x+1 \quad x \le 2$

$\quad\;\; = x^2+5 \quad x > 2 \quad$ at $x = 2$

→ R.H.D

$$Df(2^+) = \lim_{x\to 2^+} \frac{f(x)-f(a)}{x-a}$$

$$= \lim_{x\to 2} \frac{x^2+5-9}{x-2}$$

$$= \lim_{x\to 2} \frac{x^2-4}{x-2}$$

$$= \lim_{x\to 2} \frac{\cancel{(x-2)}(x+2)}{\cancel{(x-2)}}$$

$$= 2+2 = 4$$

$\therefore$ R.H.D $= 4$

L.H.D

$$Df(2^-) = \lim_{x\to 2^-} \frac{f(x)-f(a)}{x-a}$$

$$= \lim_{x\to 2} \frac{4x+1-9}{x-2}$$

$$= \lim_{x\to 2} \frac{4x-8}{x-2}$$

$$= \lim_{x\to 2} \frac{4\cancel{(x-2)}}{\cancel{x-2}}$$

$$= 4$$

$\therefore$ L.H.D $= 4$

$\therefore$ L.H.D = R.H.D

$\therefore$ f is differentiable at $x = 2$

SA

(3) $f(x) = 4x + 7 \quad , x < 3$ at $x = 3$

$\quad = x^2 + 3x + 1 \quad , x \geq 3$

→ R.H.D

$Df(3^+) = \lim_{x \to 3^+} \frac{f(x) - f(a)}{x - a}$

$= \lim_{x \to 3} \frac{x^2 + 3x + 1 - 19}{x - 3}$

$= \lim_{x \to 3} \frac{x^2 + 3x - 18}{x - 3}$

$= \lim_{x \to 3} \frac{x(x+6) - 3(x+6)}{x - 3}$

$= \lim_{x \to 3} \frac{\cancel{(x-3)}(x+6)}{\cancel{(x-3)}}$

$= 3 + 6$

R.H.D $= 9$

L.H.D

$Df(3^-) = \lim_{x \to 3^-} \frac{f(x) - f(a)}{x - a}$

$= \lim_{x \to 3} \frac{4x + 7 - 19}{x - 3}$

$= \lim_{x \to 3} \frac{4x - 12}{x - 3}$

$= 4 \lim_{x \to 3} \frac{\cancel{x-3}}{\cancel{x-3}}$

L.H.D $= 4$

R.H.D $\neq$ L.H.D

f is not differentiable at $x = 3$

(4) $f(x) = 8x - 5 \quad x \leq 2$ at $x = 2$

$\quad = 3x^2 - 4x + 7 \quad x > 2$

→ R.H.D

$Df(2^+) = \lim_{x \to 2^+} \frac{f(x) - f(a)}{x - a}$

$= \lim_{x \to 2} \frac{3x^2 - 4x + 7 - 11}{x - 2}$

$= \lim_{x \to 2} \frac{3x^2 - 4x - 4}{x - 2}$

$= \lim_{x \to 2} \frac{3x^2 - 6x + 2x - 4}{x - 2}$

$= \lim_{x \to 2} \frac{3x(x-2) + 2(x-2)}{x - 2}$

$= \lim_{x \to 2} \frac{(3x+2)\cancel{(x-2)}}{\cancel{(x-2)}}$

$= 3(2) + 2$

R.H.D $= 8$

L.H.D

$Df(2^-) = \lim_{x \to 2^-} \frac{f(x) - f(a)}{x - a}$

$= \lim_{x \to 2} \frac{8x - 5 - 11}{x - 2}$

$= \lim_{x \to 2} \frac{8x - 16}{x - 2}$

$= 8 \lim_{x \to 2} \frac{(x-2)}{(x-2)}$

PRACTICAL NO : 03

TOPIC : Application Of Derivative.

1. Find the intervals in which functions is increasing or decreasing

(i) $f(x) = x^3 - 5x - 11$ (ii) $f(x) = x^2 - 4x$

(iii) $f(x) = 2x^3 + x^2 - 20x + 4$ (iv) $f(x) = x^3 - 27x + 5$

(v) $f(x) = 69 - 24x - 9x^2 + 2x^3$

2. Find the intervals in which function is concave upwards and concave downwards.

(i) $y = 3x^2 - 2x^3$ (ii) $y = x^4 - 6x^3 + 12x^2 + 5x + 7$

(iii) $y = x^3 - 27x + 5$ (iv) $y = 69 - 24x - 9x^2 + 2x^3$

(v) $y = 2x^3 + x^2 - 20x + 4$

ANSWERS

Q.1

(i) $f(x) = x^3 - 5x - 11$

$\rightarrow f'(x) = 3x^2 - 5$

$\therefore$ f is increasing iff $f'(x) > 0$

$\therefore 3x^2 - 5 > 0$

$3x^2 > 5$

$x^2 > \frac{5}{3}$

$x > \pm\sqrt{\frac{5}{3}}$

$x \in \left(-\infty, -\sqrt{\frac{5}{3}}\right) \cup \left(\sqrt{\frac{5}{3}}, \infty\right)$

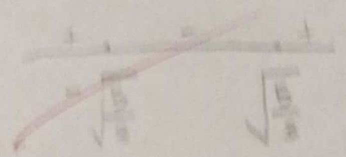
$\therefore$ f is decreasing iff $f(x) < 0$

$\therefore 3x^2 - 5 < 0$

$3x^2 < 5$

$x^2 < \frac{5}{3}$

$x < \pm\sqrt{\frac{5}{3}}$



$\therefore x \in \left(-\sqrt{\frac{5}{3}}, \sqrt{\frac{5}{3}}\right)$

(ii) $f(x) = x^2 - 4x$

$f(x) = 2x - 4$

f is increasing iff $f(x) > 0$

$\therefore 2x - 4 > 0$

$2(x-2) > 0$

$x - 2 > 0$

$x > 2$

$\therefore x \in (2, \infty)$

$\therefore$ f is decreasing iff $f(x) < 0$

$2x - 4 < 0$

$2(x-2) < 0$

$x - 2 < 0$

$x < 2$

$\therefore x \in (-\infty, 2)$

(iii) $f(x) = 2x^3 + x^2 - 20x + 4$

$f'(x) = 6x^2 + 2x - 20$

$\therefore$ f is increasing iff $f'(x) > 0$

$\therefore 6x^2 + 2x - 20 > 0$

$6x^2 + 12x - 10x - 20 > 0$

$6x(x+2) - 10(x+2) > 0$

$(6x - 10)(x+2) > 0$

$-2 \qquad \frac{10}{6}$

$\therefore x \in (-\infty, -2) \cup \left(\frac{10}{6}, \infty\right)$

f is decreasing iff $f'(x) < 0$

$6x^2 + 2x - 20 < 0$

$6x^2 + 12x - 10x - 20 < 0$

$6x(x+2) - 10(x-2) < 0$

$(6x-10)(x+2) < 0$

$-2 \qquad \frac{10}{3}$

$\therefore x \in \left(-2, \frac{10}{6}\right)$

$f(x) = x^3 - 27x + 5$

$f'(x) = 3x^2 - 27$

$= 3(x^2 - 9)$

f is increasing iff $f'(x) > 0$

$\therefore 3(x^2 - 9) > 0$

$x^2 - 9 > 0$

$(x-3)(x+3) > 0$

$-3 \qquad 3$

$\therefore x \in (-\infty, -3) \cup (3, \infty)$

f is decreasing iff $f'(x) < 0$

$3(x^2 - 9) < 0$

$x^2 - 9 < 0$

$(x-3)(x+3) < 0$

$-3 \qquad 3$

$\therefore x \in (-3, 3)$

v) $f(x) = 69 - 24x - 9x^2 + 2x^3$

$\therefore f'(x) = -24 - 18x + 6x^2$

i.e. $6x^2 - 18x - 24$

$6(x^2 - 3x - 4)$

$\therefore$ f is increasing iff $f'(x) > 0$

$\therefore 6(x^2 - 3x - 4) > 0$

$x^2 - 3x - 4 > 0$

$x^2 - 4x + x - 4 > 0$

$x(x-4) + 1(x-4) > 0$

$(x+1)(x-4) > 0$

$-1 \qquad 4$

$x \in (-\infty, -1) \cup (4, \infty)$

f is decreasing iff $f'(x) < 0$

$\therefore f\,6(x^2-3x-4) < 0$

$x^2-3x-4 < 0$

$x^2-4x+x-4 < 0$

$x(x-4)+1(x-4) < 0$

$(x+1)(x-4) < 0$

-1 4

$\therefore x \in (1, 4)$

Q.2

(i) $y = 3x^2-2x^3$

Let,

$f(x) = y = 3x^2-2x^3$

$\therefore f'(x) = 6x-6x^2$

$= 6x(1-x)$

$\therefore f''(x) = 6-12x$

$= 6(1-2x)$

$f''(x)$ is ~~concave~~ upwards iff,

$f''(x) > 0$

$6(1-2x) > 0$

$1-2x > 0$

$-2x > -1$

$2x < 1$

$x < \frac{1}{2}$

$\therefore x \in \left(-\infty, \frac{1}{2}\right)$

$f''(x)$ is concave downwards iff,

$f''(x) < 0$

$6(1-2x) < 0$

$1-2x < 0$

$-2x < -1$

$2x > 1$

$x > \frac{1}{2}$

$\therefore x \in \left(\frac{1}{2}, \infty\right)$

(ii) $y = x^4-6x^3+12x^2+5x+7$

Let,

$f(x) = y = x^4-6x^3+12x^2+5x+7$

$\therefore f'(x) = 4x^3-18x^2+24x+5$

$\therefore f''(x) = 12x^2-36x+24$

$= 12(x^2-3x+2)$

$f''(x)$ is concave upwards iff

$f''(x) > 0$

$12(x^2 - 3x + 2) > 0$

$x^2 - 3x + 2 > 0$

$x^2 - x - 2x + 2 > 0$

$x(x-1) - 2(x-1) > 0$

$(x-2)(x-1) > 0$

$\therefore x \in (-\infty, 1) \cup (2, \infty)$

$f''(x)$ is concave downwards iff.

$f''(x) < 0$

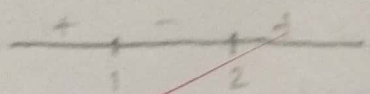
$12(x^2 - 3x + 2) < 0$

$x^2 - 3x + 2 < 0$

$x^2 - x - 2x + 2 < 0$

$x(x-1) - 2(x-1) < 0$

$(x-2)(x-1) < 0$



$\therefore x \in (1, 2)$

(iii) $y = x^3 - 27x + 5$

Let,

$f(x) = y = x^3 - 27x + 5$

$f'(x) = 3x^2 - 27$

$f''(x) = 6x$

$f(x)$ is concave ~~iff~~ upwards iff,

$f''(x) > \cancel{6x}\, 0$

$6x > 0$

$x > 0$

$\therefore x \in (0, \infty)$

$f''(x)$ is concave downwards iff,

$f''(x) < 0$

$6x < 0$

$x < 0$

$\therefore x \in (\infty, 0)$

(iv) $y = 69 - 24x - 9x^2 + 2x^3$

Let,

$f(x) = y = 69 - 24x - 9x^2 + 2x^3$

$\therefore f'(x) = -24 - 18x + 6x^2$

$\therefore f''(x) = -18 + 12x$

$f'(x)$ is concave upwards iff,

$f''(x) > 0$

$-18+12x > 0$

$12x > 18$

$x > \frac{18}{12}$

$\therefore x \in \left(\frac{3}{2}, \infty\right)$

$f''(x)$ is concave downwards iff

$f''(x) < 0$

$-18+12x < 0$

$12x < 18$

$x < \frac{18}{12}$

$\therefore x \in \left(-\infty, \frac{3}{2}\right)$

(v) $y = 2x^3 + x^2 - 20x + 4$

Let,

$f(x) = y = 2x^3 + x^2 - 20x + 4$

$f'(x) = 6x^2 + 2x - 20$

$f''(x) = 12x + 2$

$= 2(6x+1)$

$\therefore f''(x)$ is concave upwards iff,

$f''(x) > 0$

$2(6x+1) > 0$

$6x+1 > 0$

$6x > 1$

$x > \frac{-1}{6}$

$\therefore x \in \left(\frac{-1}{6}, \infty\right)$

$\therefore f''(x)$ is concave downwards iff,

$f''(x) < 0$

$2(6x+1) < 0$

$6x+1 < 0$

$6x < -1$

$x < \frac{-1}{6}$

$\therefore x \in \left(-\infty, \frac{-1}{6}\right)$

AK 20/12/19

PRACTICAL No :- 04

TOPIC :- Application of Derivative & Newton's Method.

~~Find maxima & minima~~

Q.1 Find maximum and minimum value of following functions.

(i) $f(x) = x^2 + \frac{16}{x^2}$ (ii) $f(x) = 3 - 5x^3 + 3x^5$

(iii) $f(x) = x^3 - 3x^2 + 1$ in $\left[\frac{-1}{2}, 4\right]$

(iv) $f(x) = 2x^3 - 3x^2 - 12x + 1$ in $[-2, 3]$

Q.2 Find the root of following equation by Newton's Method (Take 4 iteration only). Correct upto 4 decimal.

(i) $f(x) = x^3 - 3x^2 - 55x + 9.5$ (take $x_0 = 0$)

(ii) $f(x) = x^3 - 4x - 9$ in $[2, 3]$

(iii) $f(x) = x^3 - 1.8x^2 - 10x + 17$ in $[1, 2]$

Q.1

(i) $f(x) = x^2 + \frac{16}{x^2}$

$f'(x) = 2x - \frac{32}{x^3}$

Now consider,

$f'(x) = 0$

$\therefore 2x - \frac{32}{x^3} = 0$

$2x = \frac{32}{x^3}$

$x^4 = \frac{32}{2}$

$x^4 = 16$

$x = \pm 2$

$f''(x) = 2 + \frac{96}{x^4}$

$f''(2) = 2 + \frac{96}{2^4}$

$= 2 + \frac{96}{16}$

$= 2 + 6$

$= 8 > 0$

$\therefore$ f has minimum value at $x = 2$

$\therefore f(2) = 2^2 + \frac{16}{2^2}$

$= 4 + \frac{16}{4}$

$= 4 + 4$

$= 8$

$f''(-2) = 2 + \frac{96}{-2^4}$

$= 2 + \frac{96}{16}$

$= 2 + 6$

$= 8 > 0$

$\therefore$ f has minimum value at $x = -2$

$\therefore$ Function reaches minimum value at $x = 2$, and $x = -2$

$f(x) = 3 - 5x^3 + 3x^5$

$\therefore f'(x) = -15x^2 + 15x^4$

Consider,

$f'(x) = 0$

$\therefore -15x^2 + 15x^4 = 0$

$15x^4 = 15x^2$

$x^2 = 1$

$x = \pm 1$

$\therefore f''(x) = -30x + 60x^3$

$\therefore f''(1) = -30 + 60$

$= 30 > 0$

$\therefore f$ has minimum value at $x = 1$

$\therefore f(1) = 3 - 5(1)^3 + 3(1)^5$

$= 6 - 5$

$= 1$

$f''(-1) = -30(-1) + 60(-1)^3$

$= 30 - 60$

$= -30 < 0$

$\therefore f$ has maximum value at $x = -1$

$\therefore f(-1) = 3 - 5(-1)^3 + 3(-1)^5$

$= 3 + 5 - 3$

$= 5$

$\therefore f$ has the maximum value 5 at $x = -1$ and has the minimum value 1 at $x = 1$

(iii) $f(x) = x^3 - 3x^2 + 1$

$\therefore f'(x) = 3x^2 - 6x$

Consider,

$f'(x) = 0$

$\therefore 3x^2 - 6x = 0$

$3x(x-2) = 0$

$\therefore 3x = 0$ or $x - 2 = 0$

$\therefore x = 0$ or $x = 2$

$f''(x) = 6x - 6$

$\therefore f''(0) = 6(0) - 6$

$= -6 < 0$

$\therefore$ f has maximum value at $x = 0$

$\therefore f(0) = (0)^3 - 3(0)^2 + 1$

$= 1$

$f''(2) = 6(2) - 6$

$= 12 - 6$

$= 6 > 0$

$\therefore$ f has minimum value at $x = 2$

$f(2) = (2)^3 - 3(2)^2 + 1$

$= 8 - 3(4) + 1$

$= 9 - 12$

$= -3$

$\therefore$ f has maximum value 1 at $x = 0$ and f has minimum value -3 at $x = 2$.

(iv) $f(x) = 2x^3 - 3x^2 - 12x + 1$

$f'(x) = 6x^2 - 6x - 12$

Consider,

$f'(x) = 0$

$\therefore 6x^2 - 6x - 12 = 0$

$6(x^2 - x - 2) = 0$

$x^2 - x - 2 = 0$

$x^2 + x - 2x - 2 = 0$

$x(x+1) - 2(x+1) = 0$

$(x-2)(x+1) = 0$

$\therefore x = 2$ or $x = -1$

$\therefore f''(x) = 12x - 6$

$\therefore f''(2) = 12(2) - 6$

$= 24 - 6$

$= 18 > 0$

$\therefore$ f has minimum value at $x = 2$

$\therefore f(2) = 2(2)^3 - 3(2)^2 - 12(2) + 1$

$= 2(8) - 3(4) - 24 + 1$

$= 16 - 12 - 24 + 1$

$= -19$

$f''(-1) = 12(-1) - 6$

$= -12 - 6$

$= -18 < 0$

$\therefore$ f has maximum value at $x = -1$

$\therefore f(-1) = 2(-1)^3 - 3(-1)^2 - 12(-1) + 1$

$= -2 - 3 + 12 + 1$

$= 8$

$\therefore$ f has maximum value 8 at $x = -1$ and f has minimum value -19 at $x = 2$

Q.2 $\qquad x_0 = 0 \longrightarrow$ Given

(i) $f(x) = x^3 - 3x^2 - 55x + 9.5$

$f'(x) = 3x^2 - 6x - 55$

By Newton's Method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\therefore x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$\therefore x_1 = 0 + \frac{9.5}{55}$$

$$\therefore x_1 = 0.1727$$

$$\therefore f(x_1) = (0.1727)^3 - 3(0.1727)^2 - 55(0.1727) + 9.5$$
$$= 0.0051 - \overset{0.0895}{\cancel{0.8947}} - 9.4985 + 9.5$$
$$= -0.0829$$

$$f'(x_1) = 3(0.1727)^2 - 6(0.1727) - 55$$
$$= 0.0895 - 1.0362 - 55$$
$$= -55.9467$$

$$\therefore x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$
$$= 0.1727 - \frac{0.0829}{55.9467}$$
$$= 0.1712$$

$$f(x_2) = (0.1712)^3 - 3(0.1712)^2 - 55(0.1712) + 9.5$$
$$= 0.0050 - 0.0879 - 9.416 + 9.5$$
$$= 0.0011$$

$$f'(x_2) = 3(0.1712)^2 - 6(0.1712) - 55$$
$$= 0.0879 - 1.0272 - 55$$
$$= -55.9393$$

$$\therefore x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$
$$= 0.1712 + \frac{0.0011}{55.9393}$$
$$= 0.1712$$

$\therefore$ The root of the equation is 0.1712

(ii) $f(x) = x^3 - 4x - 9 \qquad [2,3]$

$f'(x) = 3x^2 - 4$

$$f(2) = 2^3 - 4(2) - 9$$
$$= 8 - 8 - 9$$
$$= -9$$

$$f(3) = 3^3 - 4(3) - 9$$
$$= 27 - 12 - 9$$
$$= 6$$

Let $x_0 = 3$ be the initial approximation,

$\therefore$ By Newton's Method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 3 - \frac{6}{23}$$

$$= 2.7392$$

$$f(x_1) = (2.7392)^3 - 4(2.7392) - 9$$
$$= 20.6528 - 10.9568 - 9$$
$$= 0.596$$

$$f'(x_1) = 3(2.7392)^2 - 4$$
$$= 22.5096 - 4$$
$$= 18.5096$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 2.7392 - \frac{0.596}{18.5096}$$

$$= 2.7071$$

$$f(x_2) = (2.7071)^3 - 4(2.7071) - 9$$
$$= 19.8386 - 10.8284 - 9$$
$$= 0.0102$$

$$f'(x_2) = 3(2.7071)^2 - 4$$
$$= 21.9851 - 4$$
$$= 17.9851$$

$$\therefore x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 2.7071 - \frac{0.0102}{17.9851}$$
$$= 2.7071 - 0.0056$$
$$= 2.7015$$

$$f(x_3) = (2.7015)^3 - 4(2.7015) - 9$$
$$= 19.7158 - 10.806 - 9$$
$$= -0.0901$$

$$f'(x_3) = 3(2.7015)^2 - 4$$
$$= 21.8943 - 4$$
$$= 17.8943$$

$$x_4 = 2.7015 + \frac{0.0901}{17.8943}$$
$$= 2.7015 + 0.0050$$
$$= 2.7065$$

(iii) $f(x) = x^3 - 1.8x^2 - 10x + 17$ $[1, 2]$

$f'(x) = 3x^2 - 3.6x - 10$

$$f(1) = (1)^3 - 1.8(1)^2 - 10(1) + 17$$
$$= 1 - 1.8 - 10 + 17$$
$$= 6.2$$

$$f(2) = (2)^3 - 1.8(2)^2 - 10(2) + 17$$
$$= 8 - 7.2 - 20 + 17$$
$$= -2.2$$

Let $x_0 = 2$ be initial approximation

$\therefore$ By Newton's Method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_1 = x_0 - \frac{f(x_2)}{f'(x_2)}$$

$$= 2 - \frac{2.2}{5.2}$$

$$= 2 - 0.4230$$

$$= 1.577$$

$$f(x_1) = (1.577)^3 - 1.8(1.577)^2 - 10(1.577) + 17$$

$$= 3.9219 - 4.4764 - 15.77 + 17$$

$$= 0.6755$$

$$f'(x_1) = 3(1.577)^2 - 3.6(1.577) - 10$$

$$= 7.4608 - 5.6772 - 16$$

$$= -8.2164$$

$$\therefore x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 1.577 + \frac{0.6755}{8.2164}$$

$$= 1.577 + 0.0822$$

$$= 1.6592$$

$$f(x_2) = (1.6592)^3 - 1.8(1.6592)^2 - 10(1.6592) + 17$$

$$= 4.5677 - 4.9553 - 16.592 + 17$$

$$= 0.0204$$

$$f'(x_2) = 3(1.6592)^2 - 3.6(1.6592) - 10$$

$$= 8.2688 - 5.97312 - 10$$

$$= -7.7143$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 1.6592 + \frac{0.0204}{7.7143}$$

$$= 1.6592 + 0.0026$$

$$= 1.6618$$

$$f(x_3) = (1.6618)^3 - 1.8(1.6618)^2 - 10(1.6618) + 17$$

$$= 4.5892 - 4.9708 - 16.618 + 17$$

$$= 0.0004$$

$$f'(x_3) = 3(1.6618)^2 - 3.6(1.6618) - 10$$

$$= 8.2847 - 5.9824 - 10$$

$$= -7.6977$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$$

$$= 1.6618 + \frac{0.0004}{7.6977}$$

$$= 1.6618$$

$\therefore$ The root of equation is 1.6618

PRACTICAL No:- 05

Topic : Integration

Q.1 Solve the following integration

(i) $\int \frac{dx}{\sqrt{x^2+2x-3}}$ (ii) $\int (4e^{3x}+1)dx$ (iii) $\int (2x^2 - 3\sin x + 5\sqrt{x})\,dx$

(iv) $\int \frac{x^3+3x+4}{\sqrt{x}}\,dx$ (v) $\int t^7 \sin(2t^4)\,dt$

(vi) $\int \sqrt{x}(x^2-1)\,dx$ (vii) $\int \frac{1}{x^3}\sin\left(\frac{1}{x^2}\right)dx$

(viii) $\int \frac{\cos x}{\sqrt[3]{\sin^2 x}}\,dx$ (ix) $\int e^{\cos^2 x}\sin 2x\,dx$

(x) $\int \frac{(x^2-2x)}{(x^3-3x^2+1)}\,dx$

(i) $\int \frac{dx}{\sqrt{x^2+2x-3}}$

$I = \int \frac{dx}{\sqrt{x^2+2x-3}}$

$= \int \frac{dx}{\sqrt{x^2+2x+1-4}}$

$= \int \frac{dx}{\sqrt{(x+1)^2-(2)^2}}$

Comparing with $\int \frac{dx}{\sqrt{x^2-a^2}}$; $x^2 = (x+1)^2$
$a^2 = (2)^2$

$I = \log\left|x+\sqrt{x^2-a^2}\right| + c$

$= \log\left|x+1+\sqrt{(x+1)^2-(2)^2}\right| + c$

(ii) $\int (4e^{3x}+1)\,dx$

$I = \int (4e^{3x}+1)\,dx$

$= \int 4e^{3x}dx + \int 1\,dx$

$= 4\frac{e^{3x}}{3} + x + C$

(iii) $\int (2x^2 - 3\sin x + 5\sqrt{x})\, dx$

$I = \int (2x^2 - 3\sin x + 5\sqrt{x})\,dx$

$= 2\int x^2 dx - 3\int \sin x\, dx + 5\int \sqrt{x}\, dx$

$= \frac{2x^3}{3} + 3\cos x + 5\times\frac{2}{3} x^{3/2} + C$

$= \frac{2}{3}x^3 + 3\cos x + \frac{10}{3}x^{3/2} + C$

(iv) $\int \frac{x^3+3x+4}{\sqrt{x}}\, dx$

$I = \int \frac{x^3+3x+4}{\sqrt{x}}\, dx$

$= \int \left(\frac{x^3}{\sqrt{x}} + \frac{3x}{\sqrt{x}} + \frac{4}{\sqrt{x}}\right) dx$

$= \int \left(\frac{x^3}{x^{1/2}} + \frac{3x}{x^{1/2}} + \frac{4}{x^{1/2}}\right) dx$

$= \int \left(x^3 \cdot x^{-1/2} + 3x \cdot x^{-1/2} + 4x^{-1/2}\right) dx$

$= \int x^{5/2} dx + 3\int x^{1/2} dx + 4\int x^{-1/2} dx$

$= \frac{2}{7} x^{7/2} + 3\cdot\frac{2}{3} x^{3/2} + 4\, x^{1/2}\cdot 2 + C$

$= \frac{2}{7} x^{7/2} + 2x^{3/2} + 8\sqrt{x} + C$

(v) $\int t^7 \sin(2t^4)\, dt$

$I = \int t^7 \sin(2t^4)\, dt$

Let, $t^4 = x$

$4t^3 dt = dx$

$\therefore I = \frac{1}{4}\int 4t^3 \cdot t^4 \sin(2t^4)\, dt$

$= \frac{1}{4}\int x \cdot \sin(2x)\, dx$

$= \frac{1}{4}\left[x \int \sin 2x - \int\left[\int \sin 2x \cdot \frac{d}{dx}(x)\right]\right]$

$= \frac{1}{4}\left[-\frac{x\cos 2x}{2} + \frac{1}{2}\int \cos 2x \cdot 1\right]$

$= \frac{1}{4}\left[-\frac{x\cos 2x}{2} + \frac{1}{4}\sin 2x\right] + c$

$= -\frac{1}{8} x\cos 2x + \frac{1}{16}\sin 2x + c$

Resubstituting $x = t^4$

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$\therefore I = \frac{-1}{8} t^4 \cos(2t^4) + \frac{1}{16} \sin(2t^4) + c$

(vi) $\int \sqrt{x}\,(x^2-1)\,dx$

$I = \int \sqrt{x}\,(x^2-1)\,dx$

$I = \int (\sqrt{x}\cdot x^2 - \sqrt{x})\,dx$

$= \int (x^{5/2} - \sqrt{x})\,dx$

$= \int x^{5/2}dx - \int \sqrt{x}\,dx$

$= \frac{2}{7} x^{7/2} - \frac{2}{3} x^{3/2} + c$

(vii) $\int \frac{1}{x^3} \sin\left(\frac{1}{x^2}\right) dx$

$I = \int \frac{1}{x^3} \sin\left(\frac{1}{x^2}\right) dx$

Let,

$\frac{1}{x^2} = t$

$\therefore x^{-2} = t$

$\therefore \frac{-2}{x^3}\,dx = dt$

$I = \frac{1}{-2} \int \frac{-2}{x^3} \sin\left(\frac{1}{x^2}\right) dx$

$= \frac{-1}{2} \int \sin t\,dt$

$= -\frac{1}{2}[-\cos t] + c$

$= \frac{1}{2}\cos t + c$

Resubstituting $t = \frac{1}{x^2}$

$\therefore I = \frac{1}{2} \cos\left(\frac{1}{x^2}\right) + c$

(viii) $\int \frac{\cos x}{\sqrt[3]{\sin^2 x}}$

$I = \int \frac{\cos x\,dx}{\sqrt[3]{\sin^2 x}}$

Let $\sin x = t$

$\therefore \cos x\,dx = dt$

$\therefore I = \int \frac{dt}{\sqrt[3]{t^2}}$

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$\therefore I = \int \frac{dt}{t^{2/3}}$

$= \int t^{-2/3}\, dt$

$= 3t^{1/3} + C$

$= 3(\sin x)^{1/3} + C$

$= 3\sqrt[3]{\sin x} + C$

(ix) $\int e^{\cos^2 x} \sin 2x\, dx$

$I = \int e^{\cos^2 x} \sin 2x\, dx$

Let $\cos^2 x = t$

$-2\cos x \sin x\, dx = dt$

$-\sin 2x\, dx = dt$

$\therefore I =$

$I = -\int -\sin 2x\, e^{\cos^2 x}\, dx$

$= -\int e^t\, dt$

$= -e^t + c$

Resubstituting $t = \cos^2 x$

$\therefore I = -e^{\cos^2 x} + c$

(x) $\int \left(\frac{x^2 - 2x}{x^3 - 3x^2 + 1} \right) dx$

$I = \int \left(\frac{x^2 - 2x}{x^3 - 3x^2 + 1} \right) dx$

Let,

$x^3 - 3x^2 + 1 = t$

$\therefore (3x^2 - 6x)dx = dt$

$3(x^2 - 2x)\, dx = dt$

$(x^2 - 2x)\, dx = \frac{dt}{3}$

$\therefore I = \int \frac{1}{t} \frac{dt}{3}$

$= \frac{1}{3} \int \frac{dt}{t}$

$= \frac{1}{3} \log t + c$

$\frac{1}{3}$

Resubstituting $t = x^3 - 3x^2 + 1$,

$\therefore I = \frac{1}{3} \log(x^3 - 3x^2 + 1) + C$

AV 23/01/2020

PRACTICAL No:- 06

TOPIC : APPLICATION OF INTEGRATION & NUMERICAL INTEGRATION

Q.1 Find the length of following curve

1. $x = t - \sin t$, $y = 1 - \cos t$ $\quad t \in [0, 2\pi]$
2. $y = \sqrt{4-x^2}$ $\quad x \in [-2, 2]$
3. $y = x^{3/2}$ $\quad [0, 4]$
4. $x = 3\sin t$, $y = 3\cos t$ $\quad t \in [0, 2\pi]$
5. $x = \frac{1}{6}y^3 + \frac{1}{2y}$ $\quad y \in [1, 2]$

Q.2 Using Simpson's Rule Solve the following

1. $\int_0^2 e^{x^2}\,dx$ with $n = 4$
2. $\int_0^4 x^2\,dx$ with $n = 4$
3. $\int_0^{\pi/3} \sqrt{\sin x}\,dx$ with $n = 6$

Q.1

1. $x = t - \sin t$ $\quad y = 1 - \cos t$ $\quad t \in [0, 2\pi]$

$$L = \int_a^b \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}\,dt$$

$$\frac{dy}{dt} = \sin t \qquad \frac{dx}{dt} = 1 - \cos t$$

$$\therefore L = \int_0^{2\pi} \sqrt{(\sin t)^2 + (1-\cos t)^2}\,dt$$

$$= \int_0^{2\pi} \sqrt{\sin^2 t + 1 + \cos^2 t - 2\cos t}\,dt$$

$$= \int_0^{2\pi} \sqrt{2 - 2\cos t}\,dt$$

$$= \sqrt{2}\int_0^{2\pi} \sqrt{1-\cos t}\,dt$$

$$= \sqrt{2}\int_0^{2\pi} \sqrt{2\sin^2\left(\frac{t}{2}\right)}\,dt$$

$$= \sqrt{2}\cdot\sqrt{2} \; (\int 2\sin$$

$$= 2\times 2\left[-\cos\left(\frac{t}{2}\right)\right]_0^{2\pi}$$

$$= -4\,[-1-1]$$

$$= 8$$

2. $y = \sqrt{4-x^2}$ $\quad x \in [-2, 2]$

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

$$\frac{dy}{dx} = \frac{1(-\cancel{2}x)}{\cancel{2}\sqrt{4-x}}$$

$$= \frac{-x}{\sqrt{4-x}}$$

$$\therefore L = \int_{-2}^{2}\sqrt{1+\left(\frac{-x}{\sqrt{4-x^2}}\right)^2}\,dx$$

$$= \int_{-2}^{2}\sqrt{1+\frac{x^2}{4-x^2}}$$

$$= \int_{-2}^{2}\sqrt{\frac{4-x^2+x^2}{4-x^2}}\,dx$$

$$= \int_{-2}^{2}\sqrt{\frac{4}{4-x^2}}\,dx$$

$$= 2\int_{-2}^{2}\sqrt{\frac{1}{(2)^2-(x)^2}}\,dx$$

$$= 2\int_{-2}^{2}\frac{1}{\sqrt{(2)^2-(x)^2}}\,dx$$

$$= 2\left[\sin^{-1}\left(\frac{x}{2}\right)\right]_{-2}^{2}$$

$$= 2\left[\sin^{-1}(1) - \sin^{-1}(-1)\right]$$

$$= 2\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right]$$

$$= 2\pi$$

(3) $y = x^{3/2}$ $\qquad x \in [0,4]$

$$L = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$\frac{dy}{dx} = \frac{3}{2}\sqrt{x}$$

$$L = \int_0^4 \sqrt{1+\frac{9}{4}x}\,dx$$

$$= \frac{1}{2}\int_0^4 \sqrt{4+9x}\,dx$$

$$= \frac{1}{2}\int_0^4 \sqrt{(2)^2+(3x)^2}$$

$$= \frac{1}{2}\left[\frac{(4+9x)^{3/2}}{3/2}\times\frac{1}{9}\right]_0^4$$

$$= \frac{1}{2}\times\frac{2}{3}\times\frac{1}{9}\left[(4+9x)^{3/2}\right]_0^4$$

$$= \frac{1}{27}\left[40^{3/2} - 8\right]$$

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$x = 3\sin t \qquad y = 3\cos t \qquad t \in [0, 2\pi]$

$$L = \int_a^b \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}\, dt$$

$\frac{dy}{dt} = -3\sin t \qquad \frac{dx}{dt} = 3\cos t$

$$L = \int_0^{2\pi} \sqrt{(-3\sin^2 t) + (3\cos^2 t)^2}\, dt$$

$$L = 3\int_0^{2\pi} \sqrt{\sin^2 t + \cos^2 t}\, dt$$

$$= 3\int_0^{2\pi} \sqrt{1}\, dt$$

$$= 3\left[t\right]_0^{2\pi}$$

$$= 6\pi$$

$x = \frac{1}{6}y^3 + \frac{1}{2y} \qquad y \in [1, 2]$

$$L = \int_a^b \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

$$\frac{dx}{dy} = \frac{3}{6_2}y^2 - \frac{1}{2y^2}$$

$$L = \int_1^2 \sqrt{1 + \left(\frac{1}{2}y^2 - \frac{1}{2y^2}\right)^2}\, dy$$

$$= \int_1^2 \sqrt{1 + \left(\frac{y^4 - 1}{2y^2}\right)^2}\, dy$$

$$= \int_1^2 \sqrt{\frac{(y^4-1)^2 + 4 \times y^4 \times 1}{4y^4}}\, dy$$

$$= \int_1^2 \sqrt{\frac{(y^4+1)^2}{(2y^2)^2}}\, dy$$

$$= \int_1^2 \frac{y^4+1}{2y^2}\, dy$$

$$= \int_1^2 \frac{y^2}{2} dy + \int_1^2 \frac{1}{2y^2}\, dy$$

$$= \frac{1}{2}\left[\frac{y^3}{3}\right]_1^2 + \frac{1}{2}\left[\frac{1}{y}\right]_1^2$$

$$= \frac{1}{2}\left[\frac{8}{3} - \frac{1}{3}\right] + \frac{1}{2}\left[\frac{1}{2} - 1\right]$$

$$= \frac{7}{6} - \frac{1}{4}$$

$$= \frac{14 - 3}{12}$$

$$= \frac{11}{12}$$

Q-2 Sol :

① $\int_0^2 e^{x^2} dx$ with $n = 4$

$a = 0, b = 2, n = 4$

$h = \frac{b-a}{n} = \frac{2-0}{4} = \frac{1}{2}$

x	0	0.5	1	1.5	2
y	1	1.2840	2.7182	9.4877	54.5981

∴ By Simpson's Rule,

$$\int_0^2 e^{x^2} dx = \frac{0.5}{3}\left[(1+54.5981)+4(1.2840+9.4877)+2(2.7182)\right]$$

$$= 17.3535$$

② $\int_0^4 x^2 \, dx$ with $n = 4$

$a = 0, b = 4, n = 4$

$h = \frac{b-a}{n} = \frac{4-0}{4} = 1$

x	0	1	2	3	4
y	0	1	4	9	16

By Simpson's Rule,

$$\int_0^4 x^2 dx = \frac{1}{3}\left[(0+16)+4(1+9)+2(4)\right]$$

$$= \frac{1}{3}(16+40+8)$$

$$= \frac{1}{3}\,64$$

$$= \frac{64}{3}$$

③ $\int_0^{\pi/3} \sqrt{\sin x}\, dx$ with $n = 6$

$a = 0, b = \frac{\pi}{3}, n = 6$

$h = \frac{\frac{\pi}{3} - 0}{6} = \frac{\pi}{18}$

∴ x	0	$\frac{\pi}{18}$	$\frac{2\pi}{18}$	$\frac{3\pi}{18}$	$\frac{4\pi}{18}$	$\frac{5\pi}{18}$	$\frac{6\pi}{18}$
y	0	0.4167	0.5848	0.7071	0.8017	0.8752	0.9306

By Simpson's Rule,

$$\int_0^{\pi/3} \sqrt{\sin x} = \frac{h}{3}\left[(y_0+y_6)+4(y_1+y_3+y_5)+2(y_2+y_4)\right]$$

$= \frac{\pi}{54}\Big[(0+0.9306)+4(0.4167+0.7071+0.8752)+2(0.5848+0.9...$

$+ \frac{\pi}{54}\Big]$

$= \frac{\pi}{54} \times 11.6996$

$= 0.6806$

$\int_0^{\pi/2} \sqrt{\sin x} = 0.6806$

PRACTICAL NO:- 07

TOPIC: DIFFERENTIAL EQUATION

Q) Solve the following differential equation

i) $x \frac{dy}{dx} + y = e^x$

ii) $e^x \frac{dy}{dx} + 2e^x y = 1$

iii) $x \frac{dy}{dx} = \frac{\cos x}{x} - 2y$

iv) $x \frac{dy}{dx} + 3y = \frac{\sin x}{x^2}$

v) $e^{2x} \frac{dy}{dx} + 2e^{2x} y = 2x$

vi) $\sec^2 x \tan y \, dx + \sec^2 y \tan x \, dy = 0$

vii) $\frac{dy}{dx} = \sin^2(x-y+1)$

viii) $\frac{dy}{dx} = \frac{2x+3y-1}{6x+9y+6}$

(i) $x\frac{dy}{dx} + y = e^x$

$\therefore \frac{dy}{dx} + \frac{1}{x} y = \frac{e^x}{x}$

Comparing with $\frac{dy}{dx} + P(x) y = Q(x)$

$P(x) = \frac{1}{x}$; $Q(x) = \frac{e^x}{x}$

$I.F = e^{\int 1/x \, dx}$

$= e^{\log x}$

$I.F = x$

$\therefore y(I.F) = \int Q(x)(I.F)\, dx$

$y(x) = \int \frac{e^x}{x} \cdot x \, dx$

$yx = \int e^x dx$

$xy = e^x + c$

(ii) $e^x \frac{dy}{dx} + 2e^x y = 1$

$e^x\left(\frac{dy}{dx} + 2y\right) = 1$

$\frac{dy}{dx} + 2y = \frac{1}{e^x}$

Comparing with $\frac{dy}{dx} + P(x) y = Q(x)$

$\therefore P(x) = 2$; $Q(x) = \frac{1}{e^x}$

$\therefore I.F = e^{\int 2 dx}$

$= e^{2x}$

$y(I.F) = \int Q(x)(I.F)\, dx$

$y e^{2x} = \int \frac{1}{e^x} e^{2x} dx$

$ye^{2x} = \int e^{-x} \cdot e^{2x} dx$

$y e^{2x} = \int e^x dx$

$y e^{2x} = e^x + c$

(iii) $x \frac{dy}{dx} = \frac{\cos x}{x} - 2y$

$x\frac{dy}{dx} + 2y = \frac{\cos x}{x}$

$\frac{dy}{dx} + \left(\frac{2}{x}\right) y = \frac{\cos x}{x^2}$

Comparing with $\frac{dy}{dx} + P(x) y = Q(x)$

$\therefore P(x) = \frac{2}{x}$, $Q(x) = \frac{\cos x}{x^2}$

$I.F = e^{\int \frac{2}{x} dx}$

$= e^{2\log x}$

$= e^{\log x^2}$

$I.F. = x^2$

$\therefore y(I.F.) = \int Q(x)(I.F.)\,dx$

$y(x^3) = \int \frac{\cos x}{x^3}(x^3)\,dx$

$x^3 y = \sin x + c$

(iv) $x\frac{dy}{dx} + 3y = \frac{\sin x}{x^2}$

$\frac{dy}{dx} + \left(\frac{3}{x}\right)y = \frac{\sin x}{x^3}$

Comparing with $\frac{dy}{dx} + P(x)y = Q(x)$

$P(x) = 3x^{-1}$ $\quad Q(x) = \frac{\sin x}{x^3}$

$\therefore I.F. = e^{\int \frac{3}{x}dx}$

$= e^{3\log x}$

$= e^{\log x^3}$

$= x^3$

$\therefore y(I.F.) = \int Q(x)(I.F.)\,dx$

$y(x^3) = \int \frac{\sin x}{\cancel{x^3}}(\cancel{x^3})\,dx$

$x^3 y = -\cos x + c$

$e^{2x}\frac{dy}{dx} + 2e^{2x}y = 2x$

$e^{2x}\left(\frac{dy}{dx} + 2y\right) = 2x$

$\frac{dy}{dx} + 2y = \frac{2x}{e^{2x}}$

Comparing with $\frac{dy}{dx} + P(x)y = Q(x)$

$\therefore P(x) = 2$ $\quad Q(x) = \frac{2x}{e^{2x}}$

$\therefore I.F. = e^{\int 2\,dx}$

$= e^{2x}$

$\therefore y(I.F.) = \int Q(x)(I.F.)\,dx$

$y(e^{2x}) = \int \frac{2x}{\cancel{e^{2x}}}(\cancel{e^{2x}})\,dx$

$ye^{2x} = \frac{\cancel{2}x^2}{\cancel{2}} + c$

$ye^{2x} = x^2 + c$

(vi) $\sec^2 x \tan y\,dx + \sec^2 y \tan x\,dy = 0$

$\sec^2 x \tan y\,dx = -\sec^2 y \tan x\,dy$

$\frac{\sec^2 x}{\tan x}dx = -\frac{\sec^2 y}{\tan y}dy$

$\therefore \int \frac{\sec^2 x}{\tan x}dx = -\int \frac{\sec^2 y}{\tan y}dy$

$\therefore \log|\tan x| = -\log|\tan y| + c$

$\log|\tan x| + \log|\tan y| = c$

$\log|\tan x \cdot \tan y| = c \Rightarrow \tan x \cdot \tan y = e^c$

(vii) $\frac{dy}{dx} = \sin^2(x-y+1)$

Put $x-y+1 = v$

$1 - \frac{dy}{dx} = \frac{dv}{dx}$

$\frac{dy}{dx} = 1 - \frac{dv}{dx}$

$\therefore 1 - \frac{dv}{dx} = \sin^2 v$

$1 - \sin^2 v = \frac{dv}{dx}$

$dx = \frac{dv}{1-\sin^2 v}$

$\int dx = \int \sec^2 v \, dv$

$x = \tan v + c$

But $v = x+y-1$

$\therefore x = \tan(x+y-1) + c$

(viii) $\frac{dy}{dx} = \frac{2x+3y-1}{6x+9y+6}$

$\frac{dy}{dx} = \frac{2x+3y-1}{3(2x+3y+2)}$

Put $2x+3y = v$

$2 + 3\frac{dy}{dx} = \frac{dv}{dx}$

$\frac{dy}{dx} = \frac{1}{3}\left(\frac{dv}{dx} - 2\right)$

$\therefore \frac{1}{\cancel{3}}\left(\frac{dv}{dx} - 2\right) = \frac{v-1}{\cancel{3}(v+2)}$

$\frac{dv}{dx} = \frac{v-1}{v+2} + 2$

$\frac{dv}{dx} = \frac{v-1+2v+4}{v+2}$

$\frac{dv}{dx} = \frac{3v+3}{v+2}$

$\frac{v+2}{3(v+1)} dv = dx$

$\frac{1}{3}\int \frac{(v+1+1)}{v+1} dv = \int dx$

$\frac{1}{3}\int 1 + \frac{1}{v+1} dv = \int dx$

$\frac{1}{3}\left(v + \log(v+1)\right) = x + c$

But $v = 2x+3y$

$\therefore 2x+3y+\log|2x+3y+1| = 3x+c$

$\therefore 3y = x - \log|2x+3y+1| + c$

AK 10/01/2020

PRACTICAL NO :- 08

TOPIC :- Euler's Method

(1) $\frac{dy}{dx} = y + e^{x} - 2$, $y(0) = 2$, $h = 0.5$, find $y(2)$

(2) $\frac{dy}{dx} = 1 + y^2$, $y(0) = 0$, $h = 0.2$ find $y(1)$

(3) $\frac{dy}{dx} = \sqrt{\frac{x}{y}}$, $y(0) = 1$, $h = 0.2$ find $y(1)$

(4) $\frac{dy}{dx} = 3x^2 + 1$, $y(1) = 2$, find $y(2)$ for $h = 0.5$, $h = 0.25$

(5) $\frac{dy}{dx} = \sqrt{xy} + 2$ $y(1) = 1$ Find $y(1.2)$ with $h = 0$

(1) $\frac{dy}{dx} = y + e^x - 2$

$f(x,y) = y + e^x - 2$, $y_0 = 2$, $x_0 = 0$, $h = 0.5$

n	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
0	0	2	1	2.5
1	0.5	2.5	2.1487	3.57435
2	1	3.5743	4.2925	5.3615

Using Euler's iteration formula

$$\boxed{y_{n+1} = y_n + h\, f(x_n, y_n)}$$

n	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
3	1.5	5.3615	7.8431	9.28305
4	2	9.2831		

$\therefore$ By Euler's formula,

$y(2) = 9.2831$

(2) $\frac{dy}{dx} = 1 + y^2$

$f(x,y) = 1 + y^2$, $y_0 = 0$, $x_0 = 0$, $h = 0.2$

using Euler's iteration formula,

$y_{n+1} = y_n + h\, f(x_n, y_n)$

n	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
0	0	0	1	0.2
1	0.2	0.2	1.04	0.408
2	0.4	0.408	1.1665	0.6413
3	0.6	0.6413	1.4113	0.9236
4	0.8	0.9236	1.8530	1.2942
5	1	1.2942		

$\therefore$ By Euler's formula,

$y(1) = 1.2942$

③ $\frac{dy}{dx} = \sqrt{\frac{x}{y}}$ $y(0) = 1$ $x_0 = 0$, $h = 0.2$

Using Euler's iteration formula,

$$y_{n+1} = y_n + h f(x_n, y_n)$$

n	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
0	0	1	0	1
1	0.2	1	0.4472	1.0895
2	0.4	1.0895	0.6059	1.1211
3	0.6	1.1211	0.7316	1.1463
4	0.8	1.1463	0.8354	1.16708
5	1	1.1671		

∴ By Euler's Formula,

$y(1) = 1.1671$

④ $\frac{dy}{dx} = 3x^2 + 1$ $y_0 = 2$, $x_0 = 1$, $h = 0$

For $h = 0.5$

Using Euler's iteration Formula,

$$y_{n+1} = y_n + h f(x_n, y_n)$$

n	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
0	1	2	4	4
1	1.5	4	49	28.5
2	2	28.5		

∴ By Euler's Formula,

$y(2) = 28.5$

For $h = 0.25$

n	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
0	1	2	4	3
1	1.25	3	5.6875	4.4219
2	1.5	4.4219	7.75	6.3594
3	1.75	6.3594	10.1817	8.9048
4	2	8.9048		

$\therefore$ By Euler's formula

$y(2) = 8.9048$

(5) $\frac{dy}{dx} = \sqrt{xy} + 2$ $\quad y_0 = 1, \; x_0 = 1, \; h = 0.2$

Using Euler's iteration formula,

$y_{n+1} = y_n + h f(x_n, y_n)$

n	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
0	1	1	3	1.6
1	1.2	1.6		

$\therefore$ By Euler's formula,

$y(1.2) = 1.6$

10/01/2020

PRACTICAL NO :- 09

TOPIC : Limits & Partial Order derivatives

(1) Evaluate the following limits

(i) $\lim_{(x,y)\to(-4,-1)} \frac{x^3 - 3y + y^2 - 1}{xy + 5}$ $\quad$ (ii) $\lim_{(x,y)\to(2,0)} \frac{(y+1)(x^2+y^2-4x)}{x+3y}$

(iii) $\lim_{(x,y,z)\to(1,1,1)} \frac{x^2 - y^2 z^2}{x^3 - x^2 y z}$

(2) Find f_x, f_y for each of the following f

(i) $f(x,y) = xy e^{x^2+y^2}$ $\quad$ (ii) $f(x,y) = e^x \cos y$ $\quad$ (iii) $f(x,y) = x^3y^2 - 3x^2y + y^3 + 1$

(3) Using definition find values of f_x, f_y at $(0,0)$ for $f(x,y) = \frac{2x}{1+y^2}$

(4) Find all second order partial derivatives of f. Also verify whether $f_{xy} = f_{yx}$

(i) $f(x,y) = \frac{y^2 - xy}{x^2}$ $\quad$ (ii) $f(x,y) = x^3 + 3x^2y^2 - \log(x^2+1)$

(iii) $f(x,y) = \sin(xy) + e^{x+y}$

(5) Find ~~all~~ the linearization of $f(x,y)$ at given point

(i) $f(x,y) = \sqrt{x^2+y^2}$ at $(1,1)$

(ii) $f(x,y) = 1 - x + y \sin x$ at $\left(\frac{\pi}{2}, 0\right)$

(iii) $f(x,y) = \log x + \log y$ at $(1,1)$

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Answers

Q.1

(i) $\lim_{(x,y)\to(-4,-1)} \dfrac{x^3 - 3y + y^2 - 1}{xy + 5}$

At $(-4,-1)$, Denominator $\neq 0$

$\therefore$ By applying limit

$= \dfrac{(-4)^3 - 3(-1) + (-1)^2 - 1}{-4(-1) + 5}$

$= \dfrac{-64 + 3 + 1 - 1}{4 + 5}$

$= -\dfrac{61}{9}$

(ii) $\lim_{(x,y)\to(2,0)} \dfrac{(y+1)\,(x^2 + y^2 - 4x)}{x + 3y}$

At $(2,0)$, Denominator $\neq 0$

$\therefore$ By applying limit,

$= \dfrac{(0+1)\,((2)^2 + 0 - 4(2))}{2+0}$

$= \dfrac{1\,(4 + 0 - 8)}{2}$

$= -\dfrac{4}{2}$

$= -2$

(iii) $\lim_{(x,y,z)\to(1,1,1)} \dfrac{x^2 - y^2 z^2}{x^3 - x^2 y z}$

At $(1,1,1)$, Denominator $= 0$

$\therefore \lim_{(x,y,z)\to(1,1,1)} \dfrac{x^2 - y^2 z^2}{x^3 - x^2 y z}$

$= \lim_{(x,y,z)\to(1,1,1)} \dfrac{\cancel{(x - yz)}\,(x + yz)}{x^2\,\cancel{(x - yz)}}$

$= \lim_{(x,y,z)\to(1,1,1)} \dfrac{x + yz}{x^2}$

On Applying limit

$= \dfrac{1 + 1(1)}{(1)^2}$

$= 2$

Q.2

(i) $f(x,y) = xy\,e^{x^2+y^2}$

$\therefore f_x = \dfrac{\partial}{\partial x}(f(x,y))$

$= \dfrac{\partial}{\partial x}\left(xy\,e^{x^2+y^2}\right)$

$= y e^{x^2+y^2}(2x)$

$\therefore f_x = 2xy e^{x^2+y^2}$

$f_y = \frac{\partial}{\partial y}(f(x,y))$

$= \frac{\partial}{\partial y}(xy e^{x^2+y^2})$

$= x e^{x^2+y^2}(2y)$

$\therefore f_y = 2yx e^{x^2+y^2}$

(ii) $f(x,y) = e^x \cos y$

$f_x = \frac{\partial}{\partial x}(f(x,y))$

$= \frac{\partial}{\partial x}(e^x \cos y)$

$\therefore f_x = e^x \cos y$

$f_y = \frac{\partial}{\partial y}(f(x,y))$

$= \frac{\partial}{\partial y}(\cancel{e^x} \cos y)$

$\therefore f_y = -e^x \sin y$

(iii) $f(x,y) = x^3y^2 - 3x^2y + y^3 + 1$

$f_x = \frac{\partial}{\partial x}(f(x,y))$

$= \frac{\partial}{\partial x}(x^3y^2 - 3x^2y + y^3 + 1)$

$\therefore f_x = 3x^2y^2 - 6xy$

$f_y = \frac{\partial}{\partial y}(f(x,y))$

$= \frac{\partial}{\partial y}(x^3y^2 - 3x^2y + y^3 + 1)$

$\therefore f_y = 2x^3y - 3x^2 + 3y^2$

Q3

(i) $f(x,y) = \frac{2x}{1+y^2}$

$f_x = \frac{\partial}{\partial x}\left(\frac{2x}{1+y^2}\right)$

$= \frac{1+y^2 \frac{\partial}{\partial x}(2x) - 2x \frac{\partial}{\partial x}(1+y^2)}{(1+y^2)^2}$

$= \frac{2+2y^2 - 0}{(1+y^2)^2}$

$$= \frac{2(1+y^2)}{\cancel{(1+y^2)}(1+y)^2}$$

$$= \frac{2}{1+y^2}$$

At $(0,0)$

$$= \frac{2}{1+0}$$

$$= 2$$

$$f_y = \frac{\partial}{\partial y}\left(\frac{2x}{1+y^2}\right)$$

$$= \frac{1+y^2 \frac{\partial}{\partial x}(2x) - 2x \frac{\partial}{\partial x}(1+y^2)}{(1+y^2)^2}$$

$$= \frac{1+y^2(0) - 2x(2y)}{(1+y^2)^2}$$

$$= \frac{-4xy}{(1+y^2)^2}$$

At $(0,0)$,

$$= -\frac{4(0)(0)}{(1+0)^2}$$

$$= 0$$

(ii) $f(x,y) = \dfrac{y^2-xy}{x^2}$

$$f_x = \frac{x^2 \frac{\partial}{\partial x}(y^2-xy) - (y^2-xy)\frac{\partial}{\partial x}(x^2)}{(x^2)^2}$$

$$= \frac{x^2(-y) - (y^2-xy)(2x)}{x^4}$$

$$= \frac{-x^2y - 2x(y^2-xy)}{x^4}$$

$$f_y = \frac{2y-x}{x^2}$$

$$f_{xx} = \frac{\partial}{\partial x}\left(\frac{-x^2y-2x(y^2-xy)}{x^4}\right)$$

$$= \frac{x^4\left(\frac{\partial}{\partial x}(-x^2y-2xy^2+2x^2y)\right) - (-x^2y-2xy+2x^2y)\frac{\partial}{\partial x}(x^4)}{(x^4)^2}$$

$$= \frac{x^4(-2xy-2y^2+4xy) - 4x^3(-x^2y-2xy+2x^2y)}{x^6} \quad \text{— (1)}$$

$$f_{yy} = \frac{\partial}{\partial y}\left(\frac{2y-x}{x^2}\right)$$

$$= \frac{2-0}{x^2} = \frac{2}{x^2} \quad \text{— (2)}$$

$$f_{xy} = \frac{\partial}{\partial y}\left(\frac{-x^2y-2xy^2+2x^2y}{x^4}\right) \quad \text{— (3)}$$

$$= \frac{-x^2-4xy+2x^2}{x^4}$$

$$f_{yx} = \frac{\partial}{\partial x}\left(\frac{2y-x}{x^2}\right)$$

$$= \frac{x^2\frac{\partial}{\partial x}(2y-x) - (2y-x)\frac{\partial}{\partial x}(x^2)}{(x^2)^2} \quad \text{— (4)}$$

$$= \frac{-x^2-4xy+2x^2}{x^4}$$

From (3) & (4);

$f_{xy} = f_{yx}$

(ii) $f(x,y) = x^3+3x^2y^2-\log(x^2+1)$

$f_x = \frac{\partial}{\partial x}(x^3+3x^2y^2-\log(x^2+1))$

$= 3x^2+6xy^2-\frac{2x}{x^2+1}$

$f_y = \frac{\partial}{\partial y}(x^3+3x^2y^2-\log(x^2+1))$

$= 0+6x^2y-0$

$= 6x^2y$

$f_{xx} = 6x+6y^2-\left(\frac{x^2+1\frac{\partial(2x)}{\partial x}-2x\frac{\partial(x^2+1)}{\partial x}}{(x^2+1)^2}\right)$

$= 6x+6y^2-\left(\frac{2(x^2+1)-4x^2}{(x^2+1)^2}\right)$ —— (1)

$f_{yy} = \frac{\partial}{\partial y}(6x^2y)$

$= 6x^2$ —— (2)

$f_{xy} = \frac{\partial}{\partial y}(3x^2+6xy^2-\frac{2x}{x^2+1})$

$= 0+12xy-0$

$= 12xy$ —— (3)

$f_{yx} = \frac{\partial}{\partial x}(6x^2y)$

$= 12xy$ —— (4)

From (3) & (4),

$\therefore f_{xy} = f_{yx}$

(iii) $f(x,y) = \sin(xy)+e^{x+y}$

$f_x = y\cos(xy)+e^{x+y}(1)$

$= y\cos(xy)+e^{x+y}$

$f_y = x\cos(xy)+e^{x+y}(1)$

$= x\cos(xy)+e^{x+y}$

$\therefore f_{xx} = \frac{\partial}{\partial x}(y\cos(xy)+e^{x+y})$

$= -y\sin(xy)\cdot(y)+e^{x+y}(1)$

$= -y^2\sin(xy)+e^{x+y}$ —— (1)

$f_{yy} = \frac{\partial}{\partial y}(x\cos(xy)+e^{x+y})$

$= -x\sin(xy)(x)+e^{x+y}(1)$

$= -x^2\sin(xy)+e^{x+y}$ —— (2)

$f_{xy} = \frac{\partial}{\partial y}\cancel{(x\cos(xy)+e^{x+y})}\ \frac{\partial}{\partial y}(y\cos(xy)+e^{x+y})$

$= -y^2\sin(xy)+\cos(xy)+e^{x+y}$ —— (3)

$f_{yx} = \frac{\partial}{\partial x}(x\cos(xy)+e^{x+y})$

$= -x^2\sin(xy)+\cos(xy)+e^{x+y}$ —— (4)

$\therefore$ From (3) & (4)

$f_{xy} \neq f_{yx}$

Q.5

(i) $f(x,y) = \sqrt{x^2+y^2}$ at $(1,1)$

$\rightarrow f(1,1) = \sqrt{(1)^2+(1)^2} = \sqrt{2}$

$f_x = \frac{1}{2\sqrt{x^2+y^2}}(2x)$ $\qquad f_y = \frac{1}{2\sqrt{x^2+y^2}}(2y)$

$= \frac{x}{\sqrt{x^2+y^2}}$ $\qquad = \frac{y}{\sqrt{x^2+y^2}}$

f_x at $(1,1) = \frac{1}{\sqrt{2}}$ $\qquad f_y$ at $(1,1) = \frac{1}{\sqrt{2}}$

$\therefore L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

$= \sqrt{2} + \frac{1}{\sqrt{2}}(x-1) + \frac{1}{\sqrt{2}}(y-1)$

$= \sqrt{2} + \frac{1}{\sqrt{2}}(x-1+y-1)$

$= \sqrt{2} + \frac{1}{\sqrt{2}}(x+y-2)$

$= \cancel{\sqrt{2}} + \frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y - \cancel{\frac{2}{\sqrt{2}}}$

$= \frac{x+y}{\sqrt{2}}$

) $f(x,y) = 1-x+y\sin x$ at $(\frac{\pi}{2},0)$

$f(\frac{\pi}{2},0) = 1 - \frac{\pi}{2} + 0 = 1-\frac{\pi}{2}$

$f_x = 0 - 1 + y\cos x$ $\qquad f_y = 0 - 0 + \sin x$

f_x at $(\frac{\pi}{2},0) = -1+0$
$= -1$ $\qquad f_y$ at $(\frac{\pi}{2},0) = \sin\frac{\pi}{2} = 1$

$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

$= 1-\frac{\pi}{2} + (-1)(x-\frac{\pi}{2}) + 1(y-0)$

$= 1-\cancel{\frac{\pi}{2}} - x + \cancel{\frac{\pi}{2}} + y$

$= 1 - x+y$

) $f(x,y) = \log x + \log y$ at $(1,1)$

$f(1,1) = \log(1) + \log(1) = 0$

$f_x = \frac{1}{x} + 0$ $\qquad f_y = 0+\frac{1}{y}$

f_x at $(1,1) = 1$ $\qquad f_y$ at $(1,1) = 1$

$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

$= 0 + 1(x-1) + 1(y-1)$

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$= x-1+y-1$

$= x+y-2$

PRACTICAL NO:- 10

TOPIC: Directional derivative, Gradient vector & maxima, minima Tangent & normal vectors

Q.1 Find the directional derivative of following function at given points & in the direction of given vector.

i) $f(x,y) = x+2y-3$, $a=(1,-1)$, $u=3i-j$

ii) $f(x,y) = y^2-4x+1$, $a=(3,4)$, $u=i+5j$

iii) $f(x,y) = 2x+3y$, $a=(1,2)$, $u=3i+4j$

Q.2 Find gradient vector for following function at given point

i) $f(x,y) = x^y + y^x$, $a=(1,1)$

ii) $f(x,y) = (\tan^{-1}x)\cdot y^2$, $a=(1,-1)$

iii) $f(x,y,z) = xyz - e^{x+y+z}$, $a=(1,-1,0)$

Q.3 Find the equation of tangent & normal to each of following curves at given points

i) $x^2\cos y + e^{xy} = 2$ at $(1,0)$

ii) $x^2+y^2-2x+3y+2=0$ at $(2,-2)$

Q.4 Find the equation of tangent & normal line to each of following surfaces

i) $x^2-2yz+3y+xz=7$ at $(2,1,0)$

ii) $3xyz-x-y+z=-4$ at $(1,-1,2)$

Q.5 Find local maxima & minima for the following functions

i) $f(x,y) = 3x^2+y^2-3xy+6x-4y$

ii) $f(x,y) = 2x^4+3x^2y-y^2$

iii) $f(x,y) = x^2-y^2+2x+8y-70$

ANSWERS

Q.1

i) $f(x,y) = x+2y-3$ $a=(1,-1)$, $u=3i-j$

→ Here,

$u=3i-j$ is not a unit vector

$\bar{u} = 3\bar{i} - \bar{j}$

$|u| = \sqrt{10}$

∴ Unit vector along u is $\frac{\bar{u}}{|u|} = \frac{1}{\sqrt{10}}(3i-j)$

$= \frac{1}{\sqrt{10}}(3,-1)$

$= \left(\frac{3}{\sqrt{10}}, \frac{-1}{\sqrt{10}}\right)$

Now,

$f(a+hu) = f\left((1,-1) + h\left(\frac{3}{\sqrt{10}}, \frac{-1}{\sqrt{10}}\right)\right)$

$= \cancel{1+2(-1)-3} + \left(\frac{3h}{\sqrt{10}}, \frac{-h}{\sqrt{10}}\right) \quad f\left(1+\frac{3h}{\sqrt{10}}, -1-\frac{h}{\sqrt{10}}\right)$

$= \cancel{-4+\frac{3h}{\sqrt{10}}} - 4 \quad 1+\frac{3h}{\sqrt{10}} + 2\left(-1-\frac{h}{\sqrt{10}}\right) - 3$

$= 1-2-3+\frac{3h}{\sqrt{10}} - \frac{2h}{\sqrt{10}}$

$= -4 + \frac{h}{\sqrt{10}}$

$$\therefore D_u f(a) = \lim_{h\to 0} \frac{f(a+hu) - f(a)}{h}$$

$$= \lim_{h\to 0} \frac{-\cancel{4} + \frac{h}{\sqrt{10}} - (\cancel{-4})}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}\sqrt{10}}$$

$$= \frac{1}{\sqrt{10}}$$

ii) $f(x,y) = y^2 - 4x + 1$, $a = (3,4)$, $u = i + 5j$

→ Here,

$u = i + 5j$ is not a unit vector

$\bar{u} = \bar{i} + 5\bar{j}$

$|\bar{u}| = \sqrt{26}$

$\therefore$ Unit vector along u is $\frac{\bar{u}}{|\bar{u}|} = \frac{1}{\sqrt{26}}(\bar{i} + 5\bar{j})$

$$= \frac{1}{\sqrt{26}}(1,5)$$

$$= \left(\frac{1}{\sqrt{26}}, \frac{5}{\sqrt{26}}\right)$$

Now

$$f(a+hu) = f\left((3,4) + h\left(\frac{1}{\sqrt{26}}, \frac{5}{\sqrt{26}}\right)\right)$$

$$= f\left(3 + \frac{h}{\sqrt{26}}, 4 + \frac{5h}{\sqrt{26}}\right)$$

$$= \left(4 + \frac{5h}{\sqrt{26}}\right)^2 - 4\left(3 + \frac{h}{\sqrt{26}}\right) + 1$$

$$= 16 + \frac{25h^2}{26} + \frac{40h}{\sqrt{26}} - 12 - \frac{4h}{\sqrt{26}} + 1$$

$$= \frac{25h^2}{26} + \frac{36h}{\sqrt{26}} + 5$$

$$D_u f(a) = \lim_{h\to 0} \frac{f(a+hu) - f(a)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{25h^2}{26} + \frac{36h}{\sqrt{26}} + 5 - 5}{h}$$

$$= \lim_{h\to 0} \frac{25h^2}{26} + \frac{36h}{\sqrt{26}}$$

$$= \lim_{h\to 0} \frac{\cancel{h}\left(\frac{25h}{26} + \frac{36}{\sqrt{26}}\right)}{\cancel{h}}$$

$$= \frac{25(0)}{26} + \frac{36}{\sqrt{26}}$$

$$= \frac{36}{\sqrt{26}}$$

iii) $f(x,y) = 2x + 3y$, $a = (1,2)$, $u = 3i + 4j$

→ Here,

$u = 3i + 4j$ is not a unit vector

$\bar{u} = 3\bar{i} + 4\bar{j}$

$|\bar{u}| = \sqrt{25} = 5$

$\therefore$ Unit vector along $u = \frac{\bar{u}}{|\bar{u}|} = \frac{1}{5}(3\bar{i} + 4\bar{j})$

$$= \frac{1}{5}(3,4)$$

$$= \left(\frac{3}{5}, \frac{4}{5}\right)$$

Now,

$$f(a+hu) = f\left((1,2) + h\left(\frac{3}{5}, \frac{4}{5}\right)\right)$$

$$= f\left(1+\frac{3h}{5}, 2+\frac{4h}{5}\right)$$

$$= 2\left(1+\frac{3h}{5}\right) + 3\left(2+\frac{4h}{5}\right)$$

$$= 2+\frac{6h}{5} + 6 + \frac{12h}{5}$$

$$= 8 + \frac{18h}{5}$$

$$\therefore D_u f(a) = \lim_{h\to 0} \frac{f(a+hu) - f(a)}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{8}+\frac{18h}{5} - \cancel{8}}{h}$$

$$= \lim_{h\to 0} \frac{18\cancel{h}}{5\cancel{h}}$$

$$= \frac{18}{5}$$

Q.2

i) $f(x,y) = x^y + y^x \qquad a = (1,1)$

$f_x = y(x^{y-1}) + y^x \log y$

$f_y = x(y^{x-1}) + x^y \log x$

$$\nabla f(x,y) = (f_x, f_y)$$

$$= \left(yx^{y-1} + y^x \log y, xy^{x-1} + x^y \log x\right)$$

$\nabla f(x,y)$ at $(1,1)$

$$= \left(1 (1)^0 + 1^1 \log(1), 1(1)^{(1-1)} + 1^1 \log 1\right)$$

$$= (1,1)$$

ii) $f(x,y) = (\tan^{-1} x) \cdot y^2 \quad , \quad a = (1,-1)$

$$f_x = y^2\left(\frac{1}{1+x^2}\right) = \frac{y^2}{1+x^2}$$

$$f_y = 2y \tan^{-1} x$$

$$\nabla f(x,y) = (f_x, f_y)$$

$$= \left(\frac{y^2}{1+x^2}, 2y \tan^{-1} x\right)$$

$\nabla f(x,y)$ at $(1,-1)$

$$= \left(\frac{(-1)^2}{1+(1)^2}, 2(-1) \tan^{-1}(1)\right)$$

$$= \left(\frac{1}{2}, \frac{-2\pi}{4}\right)$$

$$= \left(\frac{1}{2}, -\frac{\pi}{2}\right)$$

(ii) $f(x,y,z) = xyz - e^{x+y+z}$ $\quad a = (1,-1,0)$

$f_x = yz - e^{x+y+z}$
$f_y = xz - e^{x+y+z}$
$f_z = xy - e^{x+y+z}$

$\nabla f(x,y,z) = (f_x, f_y, f_z)$
$= (yz - e^{x+y+z},\ xz - e^{x+y+z},\ xy - e^{x+y+z})$

$\nabla f(x,y,z)$ at $(1,-1,0)$
$= (-1(0) - e^{1-1+0},\ 1(0) - e^{1-1+0},\ 1(-1) - e^{1-1+0})$
$= (0-1,\ 0-1,\ -1-1)$
$= (-1,-1,-2)$

Q-3

i) $x^2\cos y + e^{xy} = 2$ $\quad$ at $(1,0)$

$f(x,y) = x^2\cos y + e^{xy} - 2$
$f_x = 2x\cos y + ye^{xy}$
$f_y = -x^2\sin y + xe^{xy}$

$(x_0,y_0) = (1,0)$
f_x at $(1,0) = 2(1)\cos 0 + 0$
$= 2$
f_y at $(1,0) = -(1)^2\sin(0) + 1(e)^{1(0)}$
$= 1$

$f_x(x-x_0) + f_y(y-y_0) = 0$
$2(x-1) + 1(y-0) = 0$
$2x - 2 + y = 0$
$2x + y - 2 = 0$ $\longrightarrow$ Equation of Tangent

Now,
For equation of Normal;
$bx + ay + d = 0$
$x + 2y + d = 0$

$\therefore (1) + 2(0) + d = 0$ $\quad$ At $(1,0)$
$1 + d = 0$
$d = -1$

$\therefore x + 2y - 1 = 0$ $\longrightarrow$ Equation of Normal

ii) $x^2 + y^2 - 2x + 3y + 2 = 0$ $\quad$ at $(2,-2)$

$f(x,y) = x^2 + y^2 - 2x + 3y + 2$
$f_x = 2x + 0 - 2 + 0 + 0$ $\quad \therefore f_x$ at $(2,-2) = 2(2) - 2$
$= 2x - 2$ $\quad = 2$
$f_y = 0 + 2y - 0 + 3 + 0$ $\quad f_y$ at $(2,-2) = 2(-2) + 3$
$= 2y + 3$ $\quad = -1$

$\therefore$ Equation of tangent
$f_x(x-x_0) + f_y(y-y_0) = 0$
$2(x-2) + (-1)(y+2) = 0$
$2x - 4 - y - 2 = 0$
$2x - y - 6 = 0$ $\longrightarrow$ Equation of Tangent

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For Equation of Normal;

$bx + ay + d = 0$

$-x + 2y + d = 0$ at $(2, -2)$

$-(2) + 2(-2) + d = 0$

$-2 - 4 + d = 0$

$d = +6$

$\therefore -x + 2y + 6 = 0 \longrightarrow$ Equation of Normal

Q.4

i) $x^2 - 2yz + 3y + xz = 7$ at $(2, 1, 0)$

$f(x, y, z) = x^2 - 2yz + 3y + xz - 7$

$f_x = 2x - 0 + 0 + z - 0$

$= 2x + z$

$\therefore f_x$ at $(2, 1, 0) = 2(2) + 0 = 4$

$f_y = -2z + 3 + 0 - 0$

$= -2z + 3$

f_y at $(2, 1, 0) = -2(0) + 3 = 3$

$f_z = 0 - 2y + 0 + x - 0$

$= -2y + x$

f_z at $(2, 1, 0) = -2(1) + 2 = 0$

Equation of tangent;

$f_x(x - x_0) + f_y(y - y_0) + f_z(z - z_0) = 0$

$4(x - 2) + 3(y - 1) + 0(z - 0) = 0$

$4x - 8 + 3y - 3 = 0$

$\therefore 4x + 3y - 11 = 0$ — Equation of tangent

Equation of normal,

$$\frac{x - x_0}{f_x} = \frac{y - y_0}{f_y} = \frac{z - z_0}{f_z}$$

$$\frac{x - 2}{4} = \frac{y - 1}{3} = \frac{z - 0}{0}$$ — Equation of normal

ii) $3xyz - x - y + z = -4$ at $(1, -1, 2)$

$f(x, y, z) = 3xyz - x - y + z + 4$

$f_x = 3yz - 1 - 0 + 0 + 0$

$= 3yz - 1$

f_x at $(1, -1, 2) = 3(-1)(2) - 1 = -7$

$f_y = 3xz - 0 - 1 + 0 + 0$

$= 3xz - 1$

f_y at $(1, -1, 2) = 3(1)(2) - 1 = 5$

$f_z = 3xy - 0 - 0 + 1 - 0$

$= 3xy + 1$

f_z at $(1, -1, 2) = 3(1)(-1) + 1 = -2$

$\therefore$ Equation of tangent;

$f_x(x - x_0) + f_y(y - y_0) + f_z(z - z_0) = 0$

$-7(x - 1) + 5(y + 1) + (-2)(z - 2) = 0$

$-7x + 7 + 5y + 5 - 2z + 4 = 0$

$-7x + 5y - 2z + 16 = 0$ — Equation of tangent

Equation of normal;

$$\frac{x - x_0}{f_x} = \frac{y - y_0}{f_y} = \frac{z - z_0}{f_z}$$

$$\frac{x - 1}{-7} = \frac{y + 1}{5} = \frac{z - 2}{-2}$$ — Equation of normal

Q5

i) $f(x,y) = 3x^2+y^2-3xy+6x-4y$

$\therefore f_x = 6x+0-3y+6-0$
$= 6x-3y+6$ — ①

$f_y = 2y-3x+0-4$
$= 2y-3x-4$ — ②

Multiplying ② by 2

$f_x = 0$
$6x-3y+6=0$
$3(2x-y+2)=0$
$2x-y+2=0$
$2x-y=-2$ — ③

$f_y = 0$
$2y-3x-4=0$
$2y-3x=4$ — ④

Multiplying ③ by 2 and subtracting ④ from ③

$\therefore 4x-\cancel{2y}=-4$
$-\ \cancel{2y}-3x=4$
$7x = 0$
$x = 0$

Substituting value of x in ③

$\therefore 2(0)-y=-2$
$-y=-2$
$y=2$

$\therefore$ Critical points are $(0,2)$

Now,
$r = f_{xx} = 6$
$t = f_{yy} = 2$
$s = f_{xy} = -3$
$rt-s^2 = 12-9$
$= 3 > 0$

Here, $r>0$ and $rt-s^2>0$
$\therefore f$ has minimum at $(0,2)$

$\therefore f(0,2) = 3(0)^2+(2)^2-3(0)(2)+6(0)-4(2)$
$= 0+4-0+0-8$
$= -4$

ii) $f(x,y) = 2x^4+3x^2y-y^2$

$f_x = 8x^3+6xy-0$
$= 8x^3+6xy$

$f_y = 0+3x^2-2y$
$= 3x^2-2y$

Now,

$f_x = 0$
$8x^3+6xy=0$
$2x(4x^2+3y)=0$
$4x^2+6y=0$ — ①

$f_y = 0$
$3x^2-2y=0$
$3x^2-2y=0$
$3x^2-2y=0$ — ②

Multiply in ① by 3 and ② by 4 and Subtracting ② from ①

$12x^2+18y=0$
$-\ 12x^2-8y=0$
$24y=0$
$y=0$ — ③

Substituting ③ in ②

$3x^2 - 2(0) = 0$

$3x^2 = 0$

$x^2 = 0$

$x = 0$ — ④

$\therefore$ Critical points are $(0,0)$

Now,

$r = f_{xx} = 24x^2 + 6y$

$t = f_{yy} = -2$

$s = f_{xy} = 6x$

$rt - s^2 = (24x^2 + 6y)(-2) - (6x)^2$

$= -48x^2 - 12y - 36x^2$

$= -84x^2 - 12y$

At $(0,0)$

$r = 24(0)^2 + 6(0)$

$= 0$

$s = 6(0) = 0$

$rt - s^2 = -84(0)^2 - 12(0) = 0$

$\therefore r = 0$ and $rt - s^2 = 0$

$\therefore$ Nothing can be said.

iii) $f(x,y) = x^2 - y^2 + 2x + 8y - 70$

$f_x = 2x - 0 + 2 + 0 - 0$

$= 2x + 2$

$f_y = -2y + 0 + 8 - 0$

$= -2y + 8$

$f_x = 0$

$2x + 2 = 0$

$2(x+1) = 0$

$x + 1 = 0$

$x = -1$

$f_y = 0$

$-2y + 8 = 0$

$-2(y-4) = 0$

$y - 4 = 0$

$y = +4$

$\therefore$ Critical points are $(-1, 4)$

$\therefore r = f_{xx} = 2$

$t = f_{yy} = -2$

$s = f_{xy} = 0$

$rt - s^2 = 2(-2) - 0^2$

$= -4 < 0$

Here $r > 0$ and $rt - s^2 < 0$

$\therefore$ Nothing can be said.

AK 07/02/2020