

UNIT-VCALCULUS OF SEVERAL VARIABLESLimit of funcn of 2 variables:

A funcn $f(x, y)$ is said to tend to the limit l , as (x, y) tends to (a, b) , i.e., $x \rightarrow a, y \rightarrow b$, corresponding to any +ve number ϵ there exists a +ve number δ such that,

$$|f(x, y) - l| < \epsilon \text{ whenever } |x - a| \leq \delta, |y - b| \leq \delta$$

It can be written as,

$$\begin{aligned} \lim_{\substack{x \rightarrow a \\ y \rightarrow b}} f(x, y) &= l \\ (\text{or}) \end{aligned}$$

$$\begin{aligned} \lim_{\substack{y \rightarrow b \\ x \rightarrow a}} f(x, y) &= l. \end{aligned}$$

Note: Here, if we want to find out the limit of the funcn we have to check along the path,

$$x \rightarrow 0$$

$$y \rightarrow 0$$

$$y \rightarrow mx$$

$$y \rightarrow mx^2$$

Continuity
we
continuous

Let
 $(x, y) \rightarrow$

Q) Evaluate

$\lim_{x \rightarrow 1}$

$\Rightarrow \lim_{x \rightarrow 1}$

\Rightarrow

\Rightarrow

$\lim_{y \rightarrow 2}$

$\Rightarrow \lim_{y \rightarrow 2}$

\Rightarrow

VARIABLES

end & said to tend to (a, b) , i.e., any +ve number such that,

$$|x-a| \leq \delta, |y-b| \leq \delta$$

out the limit of
the path,

Continuity of funcⁿ of 2 variables at a point:
We say that the funcⁿ $f(x, y)$ is continuous at a point then,

$$\lim_{(x,y) \rightarrow (a,b)} f(x, y) = f(a, b)$$

Q) Evaluate $\lim_{\substack{x \rightarrow 1 \\ y \rightarrow 2}} \frac{2x^2y}{x^2+y^2+1}$

$$\lim_{x \rightarrow 1} \left[\lim_{y \rightarrow 2} \frac{2x^2y}{x^2+y^2+1} \right]$$

$$\Rightarrow \lim_{x \rightarrow 1} \left[\frac{2x^2(2)}{x^2+(2)^2+1} \right]$$

$$\Rightarrow \lim_{x \rightarrow 1} \frac{4x^2}{x^2+5}$$

$$\Rightarrow \frac{4(1)^2}{(1)^2+5} = \frac{4}{6} = \frac{2}{3}$$

$$\lim_{y \rightarrow 2} \left[\lim_{x \rightarrow 1} \frac{2x^2y}{x^2+y^2+1} \right]$$

$$\Rightarrow \lim_{y \rightarrow 2} \left[\frac{2y}{y^2+2} \right]$$

$$\Rightarrow \frac{2(2)}{(2)^2+2} = \frac{4}{6} = \frac{2}{3}$$

$$\therefore \lim_{\substack{x \rightarrow 1 \\ y \rightarrow 2}} \frac{2x^2y}{x^2+y^2+1} = \frac{2}{3}$$

Q) If $f(x, y) = \frac{x-y}{2x+y}$, show that $\lim_{x \rightarrow 0} \left\{ \lim_{y \rightarrow 0} f(x, y) \right\} \neq \lim_{y \rightarrow 0} \left\{ \lim_{x \rightarrow 0} f(x, y) \right\}$.

$$\neq \lim_{y \rightarrow 0} \left\{ \lim_{x \rightarrow 0} f(x, y) \right\}.$$

$$\Rightarrow \lim_{x \rightarrow 0} \left\{ \lim_{y \rightarrow 0} \frac{x-y}{2x+y} \right\}$$

$$\Rightarrow \lim_{x \rightarrow 0} \left\{ \frac{x-0}{2x+0} \right\} = \lim_{x \rightarrow 0} \frac{x}{2x} = \frac{1}{2}.$$

$$\Rightarrow \cancel{\frac{0}{2(0)+0}} \neq 0.$$

$$\lim_{y \rightarrow 0} \left\{ \lim_{x \rightarrow 0} \frac{x-y}{2x+y} \right\}$$

$$\lim_{y \rightarrow 0} \left\{ \frac{0-y}{0+y} \right\} = \lim_{y \rightarrow 0} \frac{-y}{y} = -1$$

$$\therefore \lim_{x \rightarrow 0} \left\{ \lim_{y \rightarrow 0} f(x, y) \right\} \neq \lim_{y \rightarrow 0} \left\{ \lim_{x \rightarrow 0} f(x, y) \right\}$$

\therefore limit of the funcn doesn't exist at $(0, 0)$.

B) Discuss
 $= \begin{cases} \frac{2xy}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$

$$\Rightarrow \lim_{x \rightarrow 0} \lim_{y \rightarrow 0}$$

$$\Rightarrow \lim_{y \rightarrow 0} \left\{ \lim_{x \rightarrow 0} \right\}$$

\Rightarrow Along

$$\lim_{x \rightarrow 0} \lim_{y \rightarrow r}$$

Q) Discuss the continuity of the funcⁿ $f(x, y)$

$$= \begin{cases} \frac{2xy}{x^2+y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

$$\Rightarrow \lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} f(x, y), \lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} \frac{2xy}{x^2+y^2}.$$

$$\Rightarrow \lim_{x \rightarrow 0} \left\{ \begin{array}{l} \lim_{y \rightarrow 0} \frac{2xy}{x^2+y^2} \end{array} \right\}$$

$$= \lim_{x \rightarrow 0} \frac{0}{x^2} = 0 = f(0, 0)$$

$$\Rightarrow \lim_{y \rightarrow 0} \left\{ \begin{array}{l} \lim_{x \rightarrow 0} \frac{2xy}{x^2+y^2} \end{array} \right\}$$

$$\Rightarrow \lim_{y \rightarrow 0} \frac{0}{y^2} = 0 = f(0, 0).$$

't exist

\Rightarrow Along the line $y = mx$.

$$\lim_{\substack{x \rightarrow 0 \\ y \rightarrow mx}} \frac{2xy}{x^2+y^2} = \lim_{x \rightarrow 0} \left\{ \begin{array}{l} \lim_{y \rightarrow mx} \frac{2xy}{x^2+y^2} \end{array} \right\}$$

$$\Rightarrow \lim_{x \rightarrow 0} \left\{ \begin{array}{l} \frac{2mx^2}{x^2(1+m^2)} \end{array} \right\}$$

$$\Rightarrow \frac{2m}{1+m^2}$$

\therefore \lim \therefore limit not continuous at $(0, 0)$.

Q) For different values of m we are getting different limits.

\therefore Hence $\lim f(x,y)$ is not exist

$$x \rightarrow 0$$

$$y \rightarrow 0$$

Q) Examine the continuity at the origin of the function defined by $f(x,y) = \begin{cases} \frac{x^2}{\sqrt{x^2+y^2}}, & (x,y) \neq (0,0) \\ ? & (x,y) = (0,0) \end{cases}$

$$\lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} f(x,y) = \lim_{x \rightarrow 0} \left\{ \lim_{y \rightarrow 0} \frac{x^2}{\sqrt{x^2+y^2}} \right\}$$

$$= \lim_{x \rightarrow 0} \left\{ \frac{x^2}{\sqrt{x^2}} \right\}$$

$$= \lim_{x \rightarrow 0} \left\{ \frac{x^2}{|x|} \right\}$$

$$= \lim_{x \rightarrow 0} x = 0$$

$$\lim_{y \rightarrow 0} \left\{ \lim_{x \rightarrow 0} \frac{x^2}{\sqrt{x^2+y^2}} \right\}$$

$$\Rightarrow \lim_{y \rightarrow 0} \left\{ \frac{0}{y} \right\}$$

$$\Rightarrow 0$$

iii) Along
 $\lim_{x \rightarrow 0} \left\{ \dots \right\}$
 $\Rightarrow \lim_{x \rightarrow 0} \left\{ \dots \right\}$
 $\Rightarrow \dots$

Partial diff
Le

Variables

if it exists

$f(x,y)$ w

$\frac{\partial z}{\partial x}$ or

Similarly
to y is

by $\frac{\partial z}{\partial y}$

we are getting
exist.

iii) Along the $y = mx$.

$$\lim_{n \rightarrow 0} \left\{ \begin{array}{l} \text{lt} \\ y \rightarrow mx \end{array} \right. \frac{x^2}{\sqrt{x^2 + m^2 x^2}} \right\}$$

$$\Rightarrow \lim_{n \rightarrow 0} \frac{x}{\sqrt{1+m^2}}$$

$$\Rightarrow 0.$$

∴ Funcⁿ is continuous.

Partial differentiation:

Let $z = f(x, y)$ be a funcⁿ of two variables x and y then limit $\lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x, y) - f(x, y)}{\Delta x}$,

if it exists it is said to be partial derivative of $f(x, y)$ with respect to x . It is denoted by

$$\frac{\partial z}{\partial x} \text{ or } \frac{\partial f}{\partial x} \text{ or } f_x.$$

Similarly the partial derivatives of $f(x, y)$ w.r.t to y is $\lim_{\Delta y \rightarrow 0} \frac{f(x, y+\Delta y) - f(x, y)}{\Delta y}$. It is denoted

$$\text{by } \frac{\partial z}{\partial y} \text{ or } \frac{\partial f}{\partial y} \text{ or } f_y.$$

Higher order partial derivatives:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) = f_{xx}.$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) = f_{yy}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) = f_{xy}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) = f_{yx}.$$

Note:

$$\Rightarrow f_{xy} = f_{yx}.$$

$$\begin{aligned}\frac{\partial^2 f}{\partial y \partial x} &= \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) \\ &= \frac{\partial^2 f}{\partial x \partial y}\end{aligned}$$

Q) If $U = \sqrt{x^2 + y^2}$

that $\frac{\partial^2 U}{\partial x^2}$

$$\frac{\partial U}{\partial x} = \frac{\partial}{\partial x} (x^2 + y^2)^{1/2}$$

$$\frac{\partial^2}{\partial x^2}$$

Q) Find first & second order partial derivatives of $ax^2 + 2hxy + by^2$ and verify $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$.

$$\text{Given, } f(x, y) = ax^2 + 2hxy + by^2$$

$$\frac{\partial f}{\partial x} = 2ax + 2hy$$

$$\frac{\partial f}{\partial y} = 2hx + 2by$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) = 2a$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) = 2b$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial}{\partial x} (2hx + 2by) = 2h$$

Similarly,

$$\Rightarrow \frac{\partial^2 U}{\partial y^2} = 3y^2$$

$$\Rightarrow \frac{\partial^2 U}{\partial x^2} = 3x^2$$

$$= (x^2 +$$

=

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial}{\partial y} (2ax + 2by) = 2b$$

$$\therefore \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}.$$

Q) If $V = \frac{1}{\sqrt{x^2+y^2+z^2}}$, $x^2+y^2+z^2 \neq 0$ then prove

$$\text{that } \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

$\frac{-5}{2} + 1$

$$\begin{aligned} \frac{\partial V}{\partial x} &= \frac{\partial}{\partial x} (x^2+y^2+z^2)^{-1/2} = -\frac{1}{2} (x^2+y^2+z^2)^{-3/2} (2x) \\ &= -x (x^2+y^2+z^2)^{-3/2} \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 V}{\partial x^2} &= -x \frac{\partial}{\partial x} (x^2+y^2+z^2)^{-3/2} + (x^2+y^2+z^2)^{-3/2} (-1) \\ &= -x \left(-\frac{3}{2} (x^2+y^2+z^2)^{-5/2} (2x) \right) - (x^2+y^2+z^2)^{-3/2} \\ &= 3x^2 (x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2} \end{aligned}$$

Similarly,

$$\frac{\partial^2 V}{\partial y^2} = 3y^2 (x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2}$$

$$\begin{aligned} \frac{\partial^2 V}{\partial z^2} &= 3z^2 (x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2} \\ &= (x^2+y^2+z^2)^{-5/2} (3z^2 - x^2 - y^2 - z^2) \end{aligned}$$

$$= (x^2+y^2+z^2)^{-5/2} (-x^2 - y^2 + 2z^2)$$

$$\frac{\partial^2 V}{\partial x^2} = (x^2 + y^2 + z^2)^{-5/2} (2x^2 - y^2 - z^2)$$

$$\frac{\partial^2 V}{\partial y^2} = (x^2 + y^2 + z^2)^{-5/2} (-x^2 - 2y^2 - z^2)$$

$$\frac{\partial^2 V}{\partial z^2} = (x^2 + y^2 + z^2)^{-5/2} (-x^2 - y^2 + 2z^2)$$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} =$$

$$\Rightarrow (x^2 + y^2 + z^2)^{-5/2} (2x^2 - y^2 - z^2 - x^2 - 2y^2 - z^2 - x^2 + y^2 + 2z^2)$$

$$\Rightarrow (x^2 + y^2 + z^2)^{-5/2} (0) = 0$$

$$\therefore \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

Q) If $V = \log(x^3 + y^3 + z^3 - 3xyz)$ Prove that

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right)^2 V = \frac{9}{(x+y+z)^2}$$

$$\frac{\partial}{\partial x} = \frac{1}{x^3 + y^3 + z^3 - 3xyz} (3x^2 - 3yz)$$

$$\frac{\partial}{\partial y} = \frac{1}{x^3 + y^3 + z^3 - 3xyz} (3y^2 - 3xz)$$

$$\frac{\partial}{\partial z} = \frac{1}{x^3 + y^3 + z^3 - 3xyz} (3z^2 - 3xy)$$

$$\begin{aligned} & \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right)^2 V \\ & \Rightarrow \frac{3x^2 - 3yz}{x^3 + y^3 + z^3 - 3xyz} \\ & \Rightarrow \frac{(x^2 + y^2 + z^2)}{(x+y+z)} \end{aligned}$$

$$\therefore \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right)^2 V$$

$$\Rightarrow \frac{\partial}{\partial x} \left(\frac{3}{(x+y+z)} \right)$$

$$\Rightarrow \frac{-3}{(x+y+z)}$$

$$z^2)$$

$$z^2)$$

$$z^2)$$

$$\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) f$$

$$\Rightarrow \cancel{\left(3x^2 - 3yz + 3y^2 - 3zx + 3z^2 - 3xyz \right)} \\ \cancel{\left(x^3 + y^3 + z^3 - 3xyz \right)}$$

$$\Rightarrow \cancel{\left(x^2 + y^2 + z^2 - xy - yz - xz \right)} \\ \cancel{\left((x+y+z)(x^2 + y^2 + z^2 - xy - yz - xz) \right)} f$$

$$\Rightarrow \cancel{\frac{x}{(x+y+z)^2}} \Rightarrow \frac{3}{(x+y+z)^2}$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right)^2 U =$$

$$\Rightarrow \frac{\partial}{\partial x} \left(\frac{3}{(x+y+z)^2} \right) + \frac{\partial}{\partial y} \left(\frac{3}{(x+y+z)^2} \right) + \frac{\partial}{\partial z} \left(\frac{3}{(x+y+z)^2} \right)$$

$$\Rightarrow -\frac{3}{(x+y+z)^3} - \frac{3}{(x+y+z)^3} - \frac{3}{(x+y+z)^3}$$

$$\Rightarrow \frac{-9}{(x+y+z)^3}$$

Q) If $x^x y^y z^z = e$, show that at $x=y=z$

$$\frac{\partial^2 z}{\partial x \partial y} = -(\log x)^{-1}$$

$$x^x y^y z^z = e$$

$$\log e = \log(x^x y^y z^z)$$

$$x \log x + y \log y + z \log z = 1$$

$$z \log z = 1 - x \log x - y \log y \rightarrow (1)$$

Differentiate eqn(1) partially w.r.t to x.

$$\text{Differentiate w.r.t } x \quad \frac{\partial z}{\partial x} \log z + z \times \frac{1}{z} \frac{\partial z}{\partial x} = -\left(1 \log x + x \times \frac{1}{x}\right)$$

$$(1 + \log z) \frac{\partial z}{\partial x} = - (1 + \log x)$$

$$\frac{\partial z}{\partial x} = \frac{-(1 + \log x)}{1 + \log z}$$

similarly

$$\frac{\partial z}{\partial y} = \frac{-(1 + \log y)}{1 + \log z}$$

$$\Rightarrow \frac{\partial}{\partial x} \left(\frac{-(1 + \log y)}{1 + \log z} \right)$$

$$\Rightarrow - (1 + \log y) \left(-\frac{1}{(1 + \log z)^2} \left(\frac{1}{z} \right) \frac{\partial z}{\partial x} \right)$$

$$= -(1+\log y) \left(-\frac{1}{(1+\log z)^2} \left(\frac{1}{z} \right) \left(\frac{-(1+\log y)}{1+\log z} \right) \right)$$

$$= -(1+\log x) \left(-\frac{1}{(1+\log x)^2} \left(\frac{1}{x} \right) (-1) \right)$$

$$\Rightarrow - (1+\log x) \left(+ \frac{1}{x(1+\log x)^2} \right)$$

$$= -\frac{1}{x(1+\log x)}.$$

$$\Rightarrow -\cancel{\log e(x(1+\log x))^{-1}}$$

$$\Rightarrow -\frac{-\cancel{\log e}}{x(1+\log x)} = \cancel{x} \cancel{-Y}$$

$$\Rightarrow \frac{-1}{x(\log e + \log x)} = \frac{-1}{x \log x} \\ = -(x \log x)^{-1}$$

Q) If $u = e^{xyz}$, show that $\frac{\partial^3 u}{\partial x \partial y \partial z} = (1 + 3xyz + x^2y^2z^2)e^{xyz}$

$$\frac{\partial u}{\partial x} = e^{xyz}(yz) \quad \frac{\partial u}{\partial z} = e^{xyz}(xy)$$

Q) If

$$\frac{\partial z}{\partial x} = f$$

$$\therefore \frac{\partial}{\partial x} \left(\frac{\partial}{\partial y} (e^{xyz}(xy)) \right)$$

$$\frac{d^2 z}{\partial x^2} = f$$

$$\Rightarrow \frac{\partial}{\partial x} \left(e^{xyz}(x) + xy e^{xyz}(xz) \right)$$

$$\frac{\partial z}{\partial y} = f'$$

$$= \cancel{\frac{\partial}{\partial x}} \left(xe^{xyz} + x^2y^2z e^{xyz} \right)$$

$$\frac{\partial^2 z}{\partial y^2} =$$

$$= xe^{xyz}(yz) + e^{xyz} + yz(2x)e^{xyz} \\ + x^2y^2z e^{xyz}(yz)$$

$$\frac{\partial^2 z}{\partial y^2} = a$$

$$\Rightarrow xyz e^{xyz} + e^{xyz} + e^{xyz} \cancel{\frac{\partial}{\partial y}} \\ + e^{xyz} x^2y^2z^2$$

Q) If $u = 1$

$$(x^2 + y^2)$$

$$\Rightarrow e^{xyz} (1 + 3xyz + x^2y^2z^2)$$

Q) If $u =$

Q) Verify

Q) If z

$$f(x+ay) + \phi(x-ay)$$

$$e^{xyz}$$

$$yz(xy)$$

Q) If $z = f(x+ay) + \phi(x-ay)$, P.T. $\frac{\partial^2 z}{\partial x^2} = a^2 \frac{\partial^2 z}{\partial y^2}$.

$$\frac{\partial z}{\partial x} = f'(x+ay)(1) + \phi'(x-ay)(1)$$

$$\frac{\partial^2 z}{\partial x^2} = f''(x+ay) + \phi''(x-ay)$$

$$\frac{\partial z}{\partial y} = f'(x+ay)a + \phi'(x-ay)(-a)$$

$$\frac{\partial^2 z}{\partial y^2} = f''(x+ay)a^2 + \phi''(x-ay)(a^2)$$

$$= a^2 (f''(x+ay) + \phi''(x-ay))$$

$$\frac{\partial^2 z}{\partial y^2} = a^2 \frac{\partial^2 z}{\partial x^2}$$

Q) If $v = \log(x^2 + y^2 + z^2)$, prove that

$$(x^2 + y^2 + z^2) \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) = 2.$$

Q) If $v = \tan^{-1}\left(\frac{xy}{x^2 - y^2}\right)$, prove that $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$

Q) Verify $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$ for the funcⁿ $u = \tan^{-1}(x/y)$

Q) If $z = \log(e^x + e^y)$, s.t. $xt - s^2 = 0$

$$x = \frac{\partial^2 z}{\partial x^2}, \quad t = \frac{\partial^2 z}{\partial y^2}, \quad s = \frac{\partial^2 z}{\partial x \partial y}$$

The chain rule of a partial differentiation:

- i) If $z = f(x, y)$, where $x = \phi(t)$, $y = \psi(t)$
then z is called composite funcⁿ of a variable t .
- ii) $z = f(x, y)$ where $x = \phi(u, v)$, $y = \psi(u, v)$
then z is called a composite funcⁿ of two variables u and v .

Note: $z = f(u, v)$ where $u = \phi(x, y)$, $v = \psi(x, y)$

In this case we can calculate,

$$\rightarrow \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial x}$$

$$\rightarrow \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial y}$$

Total differential coefficient:

$$\rightarrow z = f(x, y) \quad x = \phi(t), y = \psi(t)$$

$$* \frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}$$

$$(i) \quad \begin{aligned} & \text{if } u = \\ & s \cdot t \end{aligned}$$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}$$

$$\frac{\partial u}{\partial z} = \frac{\partial}{\partial z}$$

$$g_1 = \frac{x}{y}$$

$$\frac{\partial x}{\partial n} = \frac{1}{y}$$

$$\frac{\partial x}{\partial y} = -\frac{x}{y}$$

$$\frac{\partial x}{\partial z} = 0$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x}$$

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z}$$

differentiation:

$$y = \Psi(t)$$

of a variable t.

$$y = \Psi(u, v)$$

funcⁿ of two

$$v = \Psi(x, y)$$

Q) If $u = f(r, s, t)$, where $r = \frac{x}{y}$, $s = \frac{y}{z}$, $t = \frac{z}{x}$
s.t. $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z}$.

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial u}{\partial t} \frac{\partial t}{\partial x} + \frac{\partial u}{\partial s} \frac{\partial s}{\partial x}.$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r} \frac{\partial r}{\partial y} + \frac{\partial u}{\partial s} \frac{\partial s}{\partial y} + \frac{\partial u}{\partial t} \frac{\partial t}{\partial y}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r} \frac{\partial r}{\partial z} + \frac{\partial u}{\partial s} \frac{\partial s}{\partial z} + \frac{\partial u}{\partial t} \frac{\partial t}{\partial z}.$$

$$r = \frac{x}{y} \quad s = \frac{y}{z} \quad t = \frac{z}{x}$$

$$\frac{\partial r}{\partial x} = \frac{1}{y} \quad \frac{\partial s}{\partial x} = 0 \quad \frac{\partial t}{\partial x} = -\frac{z}{x^2}$$

$$\frac{\partial r}{\partial y} = -\frac{x}{y^2} \quad \frac{\partial s}{\partial y} = \frac{1}{z} \quad \frac{\partial t}{\partial y} = 0$$

$$\frac{\partial r}{\partial z} = 0 \quad \frac{\partial s}{\partial z} = -\frac{y}{z^2} \quad \frac{\partial t}{\partial z} = \frac{1}{x}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} \left(\frac{1}{y} \right) + \frac{\partial u}{\partial t} \left(-\frac{z}{x^2} \right) + \frac{\partial u}{\partial s} (0)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r} \left(-\frac{x}{y^2} \right) + \frac{\partial u}{\partial s} \left(\frac{1}{z} \right) + \frac{\partial u}{\partial t} (0)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r} (0) + \frac{\partial u}{\partial s} \left(-\frac{y}{z^2} \right) + \frac{\partial u}{\partial t} \left(\frac{1}{x} \right)$$

$$\Rightarrow x \frac{\partial u}{\partial x} = \frac{x}{y} \frac{\partial u}{\partial y} - \frac{z}{x} \frac{\partial u}{\partial z}$$

$$\Rightarrow y \frac{\partial u}{\partial y} = -\frac{x}{y} \frac{\partial u}{\partial x} + \frac{y}{z} \frac{\partial u}{\partial z}$$

$$\Rightarrow z \frac{\partial u}{\partial z} = -\frac{y}{z} \frac{\partial u}{\partial y} + \frac{z}{x} \frac{\partial u}{\partial x}$$

$$\therefore \frac{x \partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = \frac{x}{y} \frac{\partial u}{\partial y} - \frac{z}{x} \frac{\partial u}{\partial z}$$

$$+ \frac{y}{z} \frac{\partial u}{\partial z} - \frac{x}{y} \frac{\partial u}{\partial x} - \frac{y}{z} \frac{\partial u}{\partial y} + \frac{z}{x} \frac{\partial u}{\partial x}$$

$$\therefore \frac{x \partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = 0.$$

Q) If $u = f(y-z, z-x, x-y)$, s.t. $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

let $r_1 = y-z, r_2 = z-x, r_3 = x-y$.

$$\frac{\partial r_1}{\partial x} = 0$$

$$\frac{\partial r_1}{\partial y} = 1$$

$$\frac{\partial r_2}{\partial x} = 1$$

$$\frac{\partial r_2}{\partial y} = 0$$

$$\frac{\partial r_3}{\partial x} = 1$$

$$\frac{\partial r_3}{\partial y} = -1$$

$$\frac{\partial r_3}{\partial z} = -1$$

$$\frac{\partial r_1}{\partial z} = -1$$

$$\frac{\partial r_2}{\partial z} = 1$$

$$\frac{\partial r_3}{\partial z} = 0$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial x} + \frac{\partial u}{\partial s} \frac{\partial s}{\partial x} + \frac{\partial u}{\partial t} \frac{\partial t}{\partial x}$$

$$= \frac{\partial u}{\partial x} (0) + \frac{\partial u}{\partial s} (-1) + \frac{\partial u}{\partial t} (1)$$

$$\frac{\partial u}{\partial x} = - \frac{\partial u}{\partial s} + \frac{\partial u}{\partial t}$$

~~$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial y} \frac{\partial x}{\partial y} \frac{\partial u}{\partial x} \frac{\partial x}{\partial y} + \frac{\partial u}{\partial s} \frac{\partial s}{\partial y} + \frac{\partial u}{\partial t} \frac{\partial t}{\partial y}$$~~

$$= \frac{\partial u}{\partial x} (1) + \frac{\partial u}{\partial s} (0) + \frac{\partial u}{\partial t} (-1)$$

~~$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial x} - \frac{\partial u}{\partial t}$$~~

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial z} + \frac{\partial u}{\partial s} \frac{\partial s}{\partial z} + \frac{\partial u}{\partial t} \frac{\partial t}{\partial z}$$

$$= \frac{\partial u}{\partial x} (-1) + \frac{\partial u}{\partial s} (1) + \frac{\partial u}{\partial t} (0)$$

$$\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$$

$$\frac{\partial u}{\partial z} = - \frac{\partial u}{\partial y} + \frac{\partial u}{\partial s}$$

$$\therefore \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = - \frac{\partial u}{\partial s} + \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} - \frac{\partial u}{\partial t} - \frac{\partial u}{\partial s}$$

$$\therefore \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$$

homework:

(Q) $U = \log(x^2 + y^2 + z^2)$, p.t. $(x^2 + y^2 + z^2) \left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} \right) = ?$

$$\begin{aligned}\frac{\partial U}{\partial x} &= \frac{1}{x^2 + y^2 + z^2} (2x) \Rightarrow \frac{\partial^2 U}{\partial x^2} = \frac{x^2 + y^2 + z^2 (2) - 2x(2x)}{(x^2 + y^2 + z^2)^2} \\ &= \frac{2x^2 + 2y^2 + 2z^2 - 4x^2}{(x^2 + y^2 + z^2)^2} \\ &= \frac{-2x^2 + 2y^2 + 2z^2}{(x^2 + y^2 + z^2)^2}\end{aligned}$$

∴ Similarly,

$$\frac{\partial^2 U}{\partial y^2} = \frac{2x^2 - 2y^2 + 2z^2}{(x^2 + y^2 + z^2)^2} \quad \frac{\partial^2 U}{\partial z^2} = \frac{2x^2 + 2y^2 - 2z^2}{(x^2 + y^2 + z^2)^2}$$

$$\therefore \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = \frac{-2x^2 + 2y^2 + 2z^2 + 2x^2 - 2y^2 + 2z^2 + 2x^2 + 2y^2 - 2z^2}{(x^2 + y^2 + z^2)^2}$$

$$= \frac{2x^2 + 2y^2 + 2z^2}{(x^2 + y^2 + z^2)^2}$$

$$= \frac{2(x^2 + y^2 + z^2)}{(x^2 + y^2 + z^2)^2}$$

$$= \frac{2}{x^2 + y^2 + z^2}$$

$$(x^2 + y^2 + z^2) \left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} \right) = x^2 + y^2 + z^2 \times \frac{2}{x^2 + y^2 + z^2} = 2.$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x} \quad \text{for } u = \tan^{-1}(x/y)$$

$$\begin{aligned}
 & \frac{\partial}{\partial x} \left(\frac{\partial}{\partial y} \left(\tan^{-1}(x/y) \right) \right) = \frac{\partial}{\partial x} \left(\frac{1}{1 + \left(\frac{x}{y}\right)^2} \left(-\frac{x}{y^2}\right) \right) \\
 &= \frac{\partial}{\partial x} \left(\frac{-x}{y^2 + x^2} \right) \\
 &= \frac{(x^2 + y^2)(-1) + x(2x)}{(x^2 + y^2)^2} \\
 &= \frac{-x^2 - y^2 + 2x^2}{(x^2 + y^2)^2} \\
 &= \frac{x^2 - y^2}{(x^2 + y^2)^2}.
 \end{aligned}$$

$$\begin{aligned}
 & \frac{\partial}{\partial y} \left(\frac{\partial}{\partial x} \left(\tan^{-1}(x/y) \right) \right) = \frac{\partial}{\partial y} \left(\frac{1}{1 + \frac{x^2}{y^2}} \times \left(\frac{1}{y} \right) \right) \\
 &= \frac{\partial}{\partial y} \left(\frac{1}{y + \frac{x^2}{y}} \right) \\
 &= \frac{\partial}{\partial y} \left(\frac{y}{x^2 + y^2} \right) \\
 &= \frac{x^2 + y^2(1) - y(2y)}{(x^2 + y^2)^2} \\
 &= \frac{x^2 + y^2 - 2y^2}{(x^2 + y^2)^2} \\
 &= \frac{x^2 - y^2}{(x^2 + y^2)^2}
 \end{aligned}$$

$$B) z = \log(e^x + e^y), \text{ so } t = xt - s^2 = 0,$$

$$g_1 = \frac{\partial^2 z}{\partial x^2}, \quad t = \frac{\partial^2 z}{\partial y^2}, \quad s = \frac{\partial^2 z}{\partial x \partial y}$$

$$\begin{aligned} \frac{\partial z}{\partial x} &= \frac{1}{e^x + e^y} (e^x) \Rightarrow \frac{\partial^2 z}{\partial x^2} = \frac{(e^x + e^y)(e^x) - e^x(e^x)}{(e^x + e^y)^2} \\ &= \frac{e^{2x} + e^{x+y} - e^{2x}}{(e^x + e^y)^2} \\ &= \frac{e^{x+y}}{(e^x + e^y)^2} \end{aligned}$$

$$\begin{aligned} \frac{\partial z}{\partial y} &= \frac{1}{e^x + e^y} (e^y) \Rightarrow \frac{\partial^2 z}{\partial y^2} = \frac{(e^x + e^y)(e^y) - e^y(e^y)}{(e^x + e^y)^2} \\ &= \frac{\cancel{e^x} + e^{2y} - \cancel{e^y} + e^{x+y}}{(e^x + e^y)^2} \\ &= \frac{e^{x+y}}{(e^x + e^y)^2} \end{aligned}$$

$$\frac{\partial}{\partial x} \left(\frac{\partial}{\partial y} (\log(e^x + e^y)) \right) = \frac{\partial}{\partial x} \left(\frac{1}{e^x + e^y} (e^y) \right)$$

$$= \frac{(e^x + e^y)(0) - e^y(e^x)}{(e^x + e^y)^2}$$

$$= \frac{0 - e^{x+y}}{(e^x + e^y)^2}$$

$$= \frac{-e^{x+y}}{(e^x + e^y)^2}$$

$$r = \frac{e^{x+y}}{(e^x+e^y)^2}, t = \frac{e^{x+y}}{(e^x+e^y)^2}, s^2 = \frac{(e^{x+y})^2}{(e^x+e^y)^4}$$

$$rt - s^2 = 0$$

$$\left(\frac{e^{x+y}}{(e^x+e^y)^2} \right) \left(\frac{e^{x+y}}{(e^x+e^y)^2} \right) - \frac{(e^{x+y})^2}{(e^x+e^y)^4} = 0$$

$$\Rightarrow \frac{(e^{x+y})^2}{(e^x+e^y)^4} - \frac{(e^{x+y})^2}{(e^x+e^y)^4} = 0.$$

$$\therefore rt - s^2 = 0 //$$

Q) If $u = f(r)$ and $x = r \cos \theta$, $y = r \sin \theta$

$$\text{P.T } \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r} f'(r)$$

$$u = f(r) \Rightarrow \frac{\partial u}{\partial r} = f'(r)$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} \frac{\partial r}{\partial x} = f'(r) \frac{\partial r}{\partial x}$$

$$\frac{\partial^2 u}{\partial x^2} = f''(r) \left(\frac{\partial r}{\partial x} \right)^2 + f'(r) \frac{\partial^2 r}{\partial x^2}$$

$$\text{Similarly, } \frac{\partial^2 u}{\partial y^2} = f''(r) \left(\frac{\partial r}{\partial y} \right)^2 + f'(r) \left(\frac{\partial^2 r}{\partial y^2} \right)$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) \left[\left(\frac{\partial r}{\partial x} \right)^2 + \left(\frac{\partial r}{\partial y} \right)^2 \right] + f'(r) \left[\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} \right]$$

$$x = r \cos \theta \quad y = r \sin \theta$$

$$x^2 + y^2 = r^2$$

$$\frac{\partial r}{\partial x} = \frac{x}{r} \Rightarrow \frac{\partial r}{\partial x} = \frac{1}{r} \Rightarrow \frac{\partial^2 r}{\partial x^2} = \frac{1}{r}$$

$$\Rightarrow f''(r) \left[\frac{x^2}{r^2} + \frac{y^2}{r^2} \right] + f'(r) \left[\frac{2}{r} \right]$$

$$\Rightarrow f''(r) \left[\frac{2}{r^2} \right] + f'(r) \left[\frac{2}{r} \right]$$

$$\text{Similarly, } \frac{\partial r}{\partial y} = \frac{y}{r} \Rightarrow \frac{\partial r}{\partial y} = \frac{1}{r} \Rightarrow \frac{\partial^2 r}{\partial y^2}$$

$$\Rightarrow f''(r) \left[\frac{x^2}{r^2} + \frac{y^2}{r^2} \right] + f'(r) \left[\frac{2}{r} \right]$$

$$\Rightarrow f''(r) \left[\frac{2}{r^2} \right] + f'(r) \left[\frac{2}{r} \right]$$

$$\Rightarrow f''(r) + \frac{1}{r} f'(r)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{d}{dx}\left(\frac{\partial u}{\partial x}\right) = \frac{u(1) - u\left(\frac{\partial u}{\partial x}\right)}{x^2} - \frac{u - x \frac{\partial u}{\partial x}}{x^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{u - y \frac{\partial u}{\partial y}}{y^2}.$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{u - x \frac{\partial u}{\partial x} + u - y \frac{\partial u}{\partial y}}{y^2}$$

$$= \frac{u - \frac{x^2}{u} + u - \frac{y^2}{u}}{u^2}$$

$$= \frac{u^2 - x^2 + u^2 - y^2}{u^3}$$

$$= \frac{2u^2 - (x^2 + y^2)}{u^3}$$

$$= \frac{2u^2 - u^2}{u^3} = \frac{u^2}{u^3} = \frac{1}{u}.$$

$$2) f''(u) \left[\frac{x^2}{u^2} + \frac{y^2}{u^2} \right] + f'(u) \left[\frac{1}{u} \right]$$

$$2) f''(u) \left[\frac{u^2}{u^2} \right] + f'(u) \left[\frac{1}{u} \right]$$

$$2) f''(u) + \frac{1}{u} f'(u) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$$

* Q) If $u = x \log xy$, where $x^2 + y^2 + 3xy = 1$ find $\frac{dy}{dx}$.

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \frac{dy}{dx}.$$

$$u = x \log x + x \log y$$

$$\begin{aligned}\frac{\partial u}{\partial x} &= x \cdot \frac{1}{x} + 1 \log x + \log y \\ &= 1 + \log(xy)\end{aligned}$$

$$\frac{\partial u}{\partial y} = \frac{x}{y}.$$

$$x^2 + y^2 + 3xy = 1.$$

$$2x + 2y \frac{dy}{dx} + 3y + 3x \left[x \frac{dy}{dx} + y \right] = 0$$

~~$$2y \frac{dy}{dx} = -3y - 2x.$$~~

~~$$\frac{dy}{dx} = \frac{-3y - 2x}{2y}.$$~~

$$\Rightarrow 2x + 2y \frac{dy}{dx} + 3x \frac{dy}{dx} + 3y = 0$$

$$2x + 3y + \frac{dy}{dx} (2y + 3x) = 0$$

$$\frac{dy}{dx} = \frac{-2x - 3y}{3x + 2y}$$

$$\frac{dy}{dx} = 1 + \log(xy) + \frac{x}{y} \left(\frac{-2x - 3y}{3x + 2y} \right)$$

Q) If $u =$

Jacobian:

two simul
from (x, y)

The deter

Jacobian

The deter

$$J \left(\frac{u, v}{x, y} \right)$$

If u

then,

$$J \left(\frac{u, v, w}{x, y, z} \right)$$

Q) If $u = x \log y$, where $x^3 + y^3 + 3xy = 1$ find $\frac{du}{dx}$.

Jacobian: Let $u = u(x, y)$, $v = v(x, y)$ then these two simultaneous relat's constitute a transformat' from (x, y) to (u, v) .

The determinant $\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$ is called the

Jacobian of (u, v) w.r.t (x, y) .

The determinant value is denoted by $\frac{\partial(u, v)}{\partial(x, y)}$ or

$$J\left(\frac{u, v}{x, y}\right).$$

If $u = u(x, y, z)$, $v = v(x, y, z)$, $w = w(x, y, z)$

then,

$$J\left(\frac{u, v, w}{x, y, z}\right) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

Properties:

$$\rightarrow \frac{\partial(u, v)}{\partial(x, y)} \times \frac{\partial(x, y)}{\partial(u, v)} = 1.$$

→ chain rule for Jacobian:

If (u, v) are funcs of (η, s) and (η, s) are funcs of (x, y) , then:

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial(u, v)}{\partial(\eta, s)} & \frac{\partial(u, v)}{\partial(x, y)} \\ \frac{\partial(v, w)}{\partial(\eta, s)} & \frac{\partial(v, w)}{\partial(x, y)} \end{vmatrix}$$

$$\Rightarrow J\left(\frac{u, v}{x, y}\right) = \frac{\partial(u, v)}{\partial(\eta, s)} \frac{\partial(\eta, s)}{\partial(x, y)}$$

Q) If $u = \frac{yz}{x}$, $v = \frac{zx}{y}$, $w = \frac{xy}{z}$, so J

$$J\left(\frac{u, v, w}{x, y, z}\right) = ?$$

$$J\left(\frac{u, v, w}{x, y, z}\right) = \begin{vmatrix} -\frac{yz}{x^2} & \frac{z}{x} & \frac{y}{x} \\ \frac{z}{y} & -\frac{zx}{y^2} & \frac{x}{y} \\ \frac{y}{z} & \frac{x}{z} & -\frac{xy}{z^2} \end{vmatrix}$$

$$\Rightarrow -\frac{yz}{x^2} \left(\frac{x^2yz}{y^2z^2} - \frac{x^2}{yz} \right) - \frac{z}{x} \left(-\frac{xyz}{y^2z^2} - \frac{xy}{yz} \right)$$

$$+ \frac{y}{x} \left(\frac{xz}{yz} + \frac{xyz}{zyz} \right)$$

$$= -\frac{yz}{x^2} (0) - \frac{z}{x} \left(-\frac{x}{z} - \frac{x}{z} \right)$$

$$+ \frac{y}{x} \left(\frac{x}{y} + \frac{x}{zy} \right)$$

$$= -\frac{z}{x} \left(-\frac{2x}{z} \right) + \cancel{\frac{y}{x}} \left(\cancel{\frac{x}{y}} \right) \left(\frac{2x}{y} \right)$$

$$= 2 + 2$$

$$= 4$$

$$\Rightarrow J \left(\frac{u, v, w}{x, y, z} \right) = 4.$$

$$Q) \text{ If } x+y+z = u, \quad y+z = uv, \quad z = uvw$$

then evaluate $\frac{\partial(x,y,z)}{\partial(u,v,w)}$, $\frac{\partial(u,v,w)}{\partial(x,y,z)}$

$$y+z = uv$$

$$y + uvw = uv$$

$$\Rightarrow y = uv - uvw.$$

$$x+y+z = u$$

$$x + uv - uvw + uw = u$$

$$\Rightarrow x = u - uv$$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 1-v & -u & 0 \\ v-vw & u-uw & -uv \\ vw & uw & uv \end{vmatrix}$$

$$\Rightarrow 1-v(uv(u-uw) + uv(uw)) + u(uv(v-vw) + vw(uv))$$

$$\Rightarrow 1-v(u^2v - u^2vw + u^2vw) + u(uv^2 - uv^2vw + uv^2vw)$$

$$\Rightarrow u^2v - u^2v^2 + u^2v^2$$

$$\Rightarrow u^2v$$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = u^2v$$

$$\Rightarrow \frac{uv}{(x+y+z)} = \frac{y}{u}$$

$$u = x+y$$

$$\frac{\partial u}{\partial x} = 1$$

$$\frac{\partial v}{\partial x} = \underline{uv}$$

$$\frac{\partial v}{\partial y} = \underline{-}$$

$$\frac{\partial v}{\partial z} = \underline{(z)}$$

$$\frac{\partial w}{\partial x} = \underline{(y)}$$

$$\frac{\partial w}{\partial y} = \underline{1}$$

$$\frac{\partial w}{\partial z} = \underline{(y)}$$

$$\begin{aligned} uv &= y+z \\ (x+y+z)v &= y+z \\ v &= \frac{y+z}{x+y+z} \end{aligned}$$

$$u = x+y+z$$

$$\begin{aligned} z &= uvw \\ z &= \frac{(x+y+z)(y+z)}{(x+y+z)} w \\ w &= \frac{z}{y+z} \end{aligned}$$

$$\frac{\partial u}{\partial x} = 1 \quad \frac{\partial u}{\partial y} = 1 \quad \frac{\partial u}{\partial z} = 1.$$

$$\frac{\partial v}{\partial x} = \frac{(x+y+z)(0) - (y+z)(1)}{(x+y+z)^2} = \frac{-(y+z)}{(x+y+z)^2}$$

$$\frac{\partial v}{\partial y} = \frac{(x+y+z)(1) - (y+z)(1)}{(x+y+z)^2} = \frac{x+y+z-y-z}{(x+y+z)^2} = \frac{x}{(x+y+z)^2}$$

$$\frac{\partial v}{\partial z} = \frac{(x+y+z)(1) - (y+z)(1)}{(x+y+z)^2} = \frac{x}{(x+y+z)^2}$$

$$\frac{\partial w}{\partial x} = \frac{(y+z)(0) - z(0)}{(y+z)^2} = 0$$

$$\frac{\partial w}{\partial y} = \frac{(y+z)(0) - z(1)}{(y+z)^2} = \frac{-z}{(y+z)^2}$$

$$\frac{\partial w}{\partial z} = \frac{(y+z)(1) - z(1)}{(y+z)^2} = \frac{y}{(y+z)^2}$$

$$J \left(\begin{pmatrix} u, v, w \\ x, y, z \end{pmatrix} \right) = \begin{vmatrix} 1 & 1 & 1 \\ -\frac{(y+z)}{(x+y+z)^2} & \frac{x}{(x+y+z)^2} & \frac{x}{(x+y+z)^2} \\ 0 & \frac{-z}{(y+z)^2} & \frac{y}{(y+z)^2} \end{vmatrix}$$

$$\text{B) } g_f \quad x = 9$$

$$\frac{\partial (n, \theta)}{\partial (x, y)}$$

$$\therefore J \left(\begin{pmatrix} x, \\ n, \theta \end{pmatrix} \right)$$

$$1 \left(\frac{xy}{(x+y+z)^2(y+z)^2} + \frac{xz}{(y+z)^2(x+y+z)^2} \right) - 1$$

$$\left(\frac{-y(y+z)}{(x+y+z)^2(y+z)^2} \right) + 1 \left(\frac{(y+z)^2}{(x+y+z)^2(y+z)^2} \right)$$

$$\Rightarrow \frac{xy+xz}{(x+y+z)^2(y+z)^2} + \frac{y(y+z)}{(x+y+z)^2(y+z)^2} + \frac{z(y+z)}{(x+y+z)(y+z)^2}$$

$$\Rightarrow \frac{xy+xz+y^2+yz+yz+z^2}{(x+y+z)^2(y+z)^2}$$

$$\Rightarrow \frac{n(y+z)+(y+z)^2}{(x+y+z)^2(y+z)^2}$$

$$\Rightarrow \frac{(y+z)(x+y+z)}{(x+y+z)^2(y+z)^2}$$

$$J \left(\begin{pmatrix} u, v, w \\ x, y, z \end{pmatrix} \right) = \frac{1}{(y+z)(x+y+z)}$$

$$\Rightarrow x^2 + y^2 =$$

$$y \cancel{x} + x \cancel{y} = +$$

$$J \left(\begin{pmatrix} n, \theta \\ x, y \end{pmatrix} \right)$$

$$\frac{\partial n}{\partial x} \Rightarrow x$$

$$\frac{\partial n}{\partial y} = \frac{y}{n}$$

$$\frac{\partial \theta}{\partial x} = \frac{1}{1+}$$

$$\frac{\partial \theta}{\partial y} = \frac{1}{1+}$$

$$\frac{x}{x+y+z} \cdot \frac{y}{y+z} \cdot \frac{z}{z+x}$$

Q) If $x = r\cos\theta$, $y = r\sin\theta$ find $\frac{\partial(x,y)}{\partial(r,\theta)}$ and

$$\frac{\partial(r,\theta)}{\partial(x,y)} \text{ & so } \left| \begin{array}{cc} \frac{\partial(x,y)}{\partial(r,\theta)} & \frac{\partial(r,\theta)}{\partial(x,y)} \\ \end{array} \right| > 1.$$

$$\therefore J\left(\frac{x,y}{r,\theta}\right) = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r^2\sin^2\theta = r.$$

$$\Rightarrow x^2 + y^2 = r^2$$

$$\frac{y}{x} = \tan\theta \Rightarrow \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$J\left(\frac{r,\theta}{x,y}\right) = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix}$$

$$\frac{\partial r}{\partial x} \Rightarrow r \frac{\partial r}{\partial x} = rx \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}.$$

$$\frac{\partial r}{\partial y} = \frac{y}{r}.$$

$$\frac{\partial \theta}{\partial x} = \frac{1}{1 + \left(\frac{y}{x}\right)^2} \left(-\frac{y}{x^2}\right) = \frac{1}{y^2 + x^2} \left(-\frac{y}{x^2}\right) = \frac{-y}{x^2 + y^2} = \frac{-y}{r^2} = -\frac{r\sin\theta}{r^2} = -\frac{\sin\theta}{r}.$$

$$\frac{\partial \theta}{\partial y} = -\frac{\sin\theta}{r}.$$

$$\frac{\partial \theta}{\partial y} = \frac{1}{(1 + \frac{y^2}{x^2})} \left(\frac{1}{x}\right) = \frac{x^2}{x^2 + y^2} \left(\frac{1}{x}\right) = \frac{x}{r^2} = \frac{r\cos\theta}{r^2} = \frac{\cos\theta}{r}$$

$$\begin{aligned}
 J\left(\frac{r, \theta}{x, y}\right) &= \begin{vmatrix} \frac{x}{r} & \frac{y}{r} \\ -\frac{\sin \theta}{r} & \frac{\cos \theta}{r} \end{vmatrix} \\
 &= \frac{x \cos \theta}{r^2} + \frac{y \sin \theta}{r^2} \\
 &= \frac{r \cos^2 \theta + r \sin^2 \theta}{r^2} \\
 &= \frac{r}{r^2} = \frac{1}{r}
 \end{aligned}$$

$$\therefore J\left(\frac{x, y}{r, \theta}\right) J\left(\frac{r, \theta}{x, y}\right) = r \times \frac{1}{r} = 1$$

Q) If $x = r \sin \theta \cos \phi$ $y = r \sin \theta \sin \phi$ $z = r \cos \theta$

S.T $\frac{J(x, y, z)}{J(r, \theta, \phi)} = r^2 \sin \theta$.

$$J(x, y, z) = 1$$

$$\Rightarrow -r^2 \sin^2 \theta \sin \phi \cos \theta + r^2 \sin \theta \sin \phi \cos \phi (\sin^2 \theta)$$

$$\Rightarrow -r^2 \sin^3 \theta \sin \phi \cos \theta + r^2 \sin^3 \theta \sin \phi \cos \theta$$

Q) If $u = x^2 - y^2$, $v = 2xy$ where $x = r \cos \theta$, $y = r \sin \theta$

then find $\frac{\partial(u, v)}{\partial(r, \theta)} = 4r^3$.

$$u = (r^2 \cos^2 \theta) - (r^2 \sin^2 \theta) \quad v = 2(r^2 \sin \theta \cos \theta)$$
$$= r^2 (\cos^2 \theta - \sin^2 \theta)$$

$$\frac{\partial u}{\partial r} = \cos^2 \theta - \sin^2 \theta (2r)$$
$$= \cos 2\theta (2r)$$

$$\frac{\partial v}{\partial r} = 2 \sin \theta \cos \theta (2r)$$
$$= 2r \sin 2\theta$$

$$\frac{\partial u}{\partial \theta} = r^2 \left[\frac{\partial}{\partial \theta} (\cos 2\theta) \right]$$
$$= r^2 [-\sin 2\theta (2)]$$
$$= -2r^2 \sin 2\theta$$

$$\frac{\partial v}{\partial \theta} = \cancel{2r} r^2 \left(\frac{\partial}{\partial \theta} (\sin 2\theta) \right)$$
$$= r^2 (\cos 2\theta (2))$$
$$= 2r^2 \cos 2\theta$$

$$J\left(\frac{u, v}{r, \theta}\right) = \begin{vmatrix} 2r \cos 2\theta & -2r^2 \sin 2\theta \\ 2r \sin 2\theta & 2r^2 \cos 2\theta \end{vmatrix}$$

$$= 4r^3 (\cos^2 2\theta) + 4r^3 \sin^2 2\theta$$

$$J\left(\frac{u, v}{r, \theta}\right) = 4r^3$$

Q) If $x = e^\pi \sec \theta$ $y = e^\pi \tan \theta$ prove that $\frac{\partial(x, y)}{\partial(r, \theta)}$

$$\times \frac{\partial(r, \theta)}{\partial(x, y)} = 1.$$

$$\frac{\partial x}{\partial r} = \sec \theta (e^\pi) \quad \frac{\partial x}{\partial \theta} = e^\pi \sec \theta \tan \theta$$

$$\frac{\partial y}{\partial r} = \tan \theta (e^\pi) \quad \frac{\partial y}{\partial \theta} = e^\pi \sec^2 \theta$$

$$J\left(\frac{x, y}{r, \theta}\right) = \begin{vmatrix} e^\pi \sec \theta & e^\pi \sec \theta \tan \theta \\ e^\pi \tan \theta & e^\pi \sec^2 \theta \end{vmatrix}$$

$$= e^{2\pi} \sec^3 \theta - e^{2\pi} \sec \theta \tan^2 \theta$$

$$= e^{2\pi} \sec \theta.$$

$$x^2y^2 = e^{2\theta} (\sec \theta + \tan \theta) \quad \frac{x}{y} = \frac{\sec \theta}{\tan \theta}$$

$$x^2y^2 = e^{2\theta}$$

$$\log(x^2 - y^2) = 2\theta$$

$$\theta = \frac{\log(x^2 - y^2)}{2}$$

~~$$\frac{x}{y} = \frac{1}{\cos \theta} \times \frac{\cos \theta}{\sin \theta}$$~~

$$\frac{y}{x} = \sin \theta$$

$$\theta = \sin^{-1}\left(\frac{y}{x}\right)$$

$$\frac{\partial \theta}{\partial x} = \frac{1}{x^2 - y^2} \quad (\cancel{x})$$

$$\frac{\partial \theta}{\partial x} = \frac{1}{\sqrt{1 - \frac{y^2}{x^2}}} \left(-\frac{y}{x^2}\right)$$

$$\frac{\partial \theta}{\partial x} = \frac{x}{x^2 - y^2}$$

$$= \frac{1}{\sqrt{x^2 - y^2}} \left(-\frac{y}{x^2}\right)$$

$$\frac{\partial \theta}{\partial y} = \frac{1}{x^2 - y^2} \quad (\cancel{xy})$$

$$= \frac{-y}{\sqrt{x^2 - y^2}}$$

$$= \frac{-y}{x^2 - y^2}$$

$$= \frac{-y}{x\sqrt{(e^\theta)^2}}$$

$$\frac{\partial \theta}{\partial y} = \frac{y}{y^2 - x^2}$$

$$= \frac{-y}{x e^\theta}$$

$$\frac{\partial \theta}{\partial y} = \frac{1}{\sqrt{\frac{x^2 - y^2}{x^2}}} \left(\frac{1}{x}\right)$$

$$= \frac{x}{e^\theta} \left(\frac{1}{x}\right)$$

$$= \frac{1}{e^\theta}$$

$$J \left(\begin{pmatrix} \theta, r \\ x, y \end{pmatrix} \right) = \begin{vmatrix} \frac{x}{x^2 - y^2} & \frac{y}{y^2 - x^2} \\ \frac{-y}{xe^{\theta}} & \frac{1}{e^{\theta}} \end{vmatrix}$$

$$= e^{\theta} \left(\frac{x}{x^2 - y^2} \right) + \frac{y^2}{xe^{\theta}(y^2 - x^2)}$$

$$= \frac{1}{e^{\theta}} \left(\frac{x^2 - y^2}{x(x^2 - y^2)} \right)$$

$$= \frac{1}{e^{\theta}} \left(\frac{1}{x} \right)$$

$$= \frac{1}{e^{\theta} (e^{\theta} \sec \theta)}$$

$$= \frac{1}{e^{2\theta} \sec \theta}$$

$$J \left(\begin{pmatrix} x, y \\ r, \theta \end{pmatrix} \right) J \left(\begin{pmatrix} \theta, r \\ x, y \end{pmatrix} \right) = e^{2\theta} \sec \theta \times \frac{1}{e^{2\theta} \sec \theta} = 1$$

$$\text{Q) } f$$

$$J \left(\begin{pmatrix} x, y \\ u, v \end{pmatrix} \right)$$

$$\frac{\partial y}{\partial u} = v$$

$$\therefore J \left(\begin{pmatrix} x \\ u \end{pmatrix} \right)$$

$$\Rightarrow v = \frac{y}{u}$$

$$\Rightarrow v = \frac{1}{u - }$$

$$\frac{\partial u}{\partial x} =$$

$$\frac{\partial v}{\partial y} = \frac{1}{}$$

$$\text{Q) } J \left(\begin{pmatrix} x \\ u \end{pmatrix} \right)$$

J

$$8) \text{ If } x = u(1-v), y = uv \text{ P.T. } J J' = 1$$

$$J\left(\frac{x, y}{u, v}\right) \Rightarrow \frac{\partial x}{\partial u} = (1-v) \quad \frac{\partial x}{\partial v} = \cancel{u} \cancel{v} \frac{\partial}{\partial v} (u - uv) \\ = 0 - u \\ = -u.$$

$$\frac{\partial y}{\partial u} = v \quad \frac{\partial y}{\partial v} = u$$

$$J\left(\frac{x, y}{u, v}\right) = \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix} = u - uv + vu \\ = u$$

$$\Rightarrow v = \frac{y}{u}, \quad x = u(1 - \frac{y}{u}) \Rightarrow x = u - y \Rightarrow x + y = u$$

$$\Rightarrow v = \frac{y}{x+y}$$

$$\frac{\partial u}{\partial x} = 1, \quad \frac{\partial u}{\partial y} = 1, \quad \frac{\partial v}{\partial x} = \frac{(x+y)(0) - y(1)}{(x+y)^2} = \frac{-y}{(x+y)^2}$$

$$\frac{\partial v}{\partial y} = \frac{(x+y)1 - y(1)}{(x+y)^2} = \frac{x+y-y}{(x+y)^2} = \frac{x}{(x+y)^2}$$

$$J\left(\frac{u, v}{x, y}\right) = \begin{vmatrix} 1 & 1 \\ -\frac{y}{(x+y)^2} & \frac{x}{(x+y)^2} \end{vmatrix} = \frac{x}{(x+y)^2} + \frac{y}{(x+y)^2} \\ = \frac{1}{x+y} = \frac{1}{u}$$

$$\therefore J J' = u \times \frac{1}{u} = 1$$

$$Q) \text{ If } x = uv, y = \frac{u}{v} \text{ so that } \frac{\partial(x, y)}{\partial(u, v)} \times \frac{\partial(u, v)}{\partial(x, y)} = 1$$

$$Q) \text{ If } x = r \sin \theta \cos \phi, y = r \sin \theta \sin \phi, z = r \cos \theta$$

$$\text{Find } \frac{\partial(r, \theta, \phi)}{\partial(x, y, z)}$$

Functional Dependence:

If the funcn $u \& v$ of the independent variables $x \& y$ are functionally dependent,

then $\frac{\partial(u, v)}{\partial(x, y)} = 0$, then $u \& v$ are said to be

functionally ~~dependent~~ independent.

If $\frac{\partial(u, v)}{\partial(x, y)} \neq 0$, then $u \& v$ are said to be functionally independent.

$$\times \frac{\partial(u, v)}{\partial(x, y)} = 1$$

$$z = r \cos \theta$$

if $x=uv$, $y=\frac{u}{v}$ show that $\frac{\partial(x, y)}{\partial(u, v)} \frac{\partial(u, v)}{\partial(x, y)} = 1$

$$\begin{aligned} \frac{\partial x}{\partial u} &= v & \frac{\partial x}{\partial v} &= u & \frac{\partial y}{\partial u} &= \frac{v(1)-u(0)}{v^2} = \frac{v}{v^2} = \frac{1}{v} \\ \frac{\partial y}{\partial v} &= \frac{u(0)-v(1)}{v^2} = \frac{-u}{v^2} \end{aligned}$$

$$J\left(\frac{x, y}{u, v}\right) = \begin{vmatrix} v & u \\ \frac{1}{v} & -\frac{u}{v^2} \end{vmatrix}$$

$$= \sqrt{\left(-\frac{u}{v^2}\right)} - \frac{u}{v} = -\frac{2u}{v}$$

the independent

pendent,

said to be

to be

$$\begin{aligned} \Rightarrow u = vy & \quad u = uv = vy(v) \Rightarrow v^2 y = x \\ & \Rightarrow v^2 = \frac{x}{y} \Rightarrow v = \sqrt{\frac{x}{y}} \end{aligned}$$

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{1}{2\sqrt{xy}}(y) = \frac{y}{2\sqrt{xy}} & \frac{\partial u}{\partial y} &= \frac{1}{2\sqrt{xy}}(x) = \frac{x}{2\sqrt{xy}} = \frac{x}{2u} \\ \frac{\partial u}{\partial x} &= \frac{y}{2u}. & \frac{\partial u}{\partial y} &= \frac{x}{2u}. \end{aligned}$$

$$2) \frac{\partial v}{\partial x} = \frac{1}{2\sqrt{\frac{x}{y}}} \left(\frac{1}{y}\right) = \frac{1}{2y\sqrt{x}} \quad \frac{\partial v}{\partial y} = \frac{1}{2\sqrt{\frac{x}{y}}} \left(-\frac{x}{y^2}\right) = \frac{-x}{2\sqrt{xy^2}}$$

$$J\left(\frac{u, v}{x, y}\right) = \begin{vmatrix} \frac{y}{2u} & \frac{x}{2u} \\ \frac{1}{2vy} & \frac{-x}{2vy^2} \end{vmatrix}$$

$$= -\frac{x}{2vy^2} \times \frac{y}{2u} - \frac{x}{2u} \times \frac{1}{2vy}$$

$$= \frac{-x}{4uvy} - \frac{x}{4uvy} \Rightarrow \frac{-2x}{4uvy} = \frac{-2x}{2aux} = \frac{-v}{2u}$$

$$J\left(\frac{x, y}{r, \theta, \phi}\right) J\left(\frac{u, v}{x, y}\right) = \frac{-2u}{y} \times \frac{v}{2u} = 1.$$

Q) If $x = r \sin \theta \cos \phi$, $y = r \sin \theta \sin \phi$, $z = r \cos \theta$

Show that $\underline{J(x, y, z)} = r^2 \sin \theta$
 $J(r, \theta, \phi)$

$$\frac{\partial x}{\partial r} = \sin \theta \cos \phi \quad \frac{\partial x}{\partial \theta} = r \cos \theta \cos \phi \quad \frac{\partial x}{\partial \phi} = r \cos \theta (-\sin \phi) \\ = -r \sin \theta \sin \phi$$

$$\frac{\partial y}{\partial r} = \sin \theta \sin \phi \quad \frac{\partial y}{\partial \theta} = r \sin \theta \cos \phi \quad \frac{\partial y}{\partial \phi} = r \sin \theta (\cos \theta \cos \phi) \\ = r \sin \theta \cos \phi$$

$$\frac{\partial z}{\partial r} = \cos \theta \quad \frac{\partial z}{\partial \theta} = r (-\sin \theta) \quad \frac{\partial z}{\partial \phi} = 0 \\ = -r \sin \theta$$

$$J\left(\frac{x, y, z}{r, \theta, \phi}\right) = \begin{vmatrix} \sin \theta \cos \phi & r \cos \theta \cos \phi & -r \sin \theta \sin \phi \\ \sin \theta \sin \phi & r \sin \theta \cos \phi & r \sin \theta \cos \phi \\ \cos \theta & -r \sin \theta & 0 \end{vmatrix}$$

$$= \sin \theta \cos \phi (+r^2 \sin^2 \theta \cos^2 \phi) - r \cos \theta \cos \phi (-r \sin \theta \cos \theta \cos \phi) \\ - r \sin \theta \sin \phi (-r \sin^2 \theta \sin \phi - r \sin \theta \cos^2 \theta)$$

$$= r^2 \sin^3 \theta \cos^2 \phi + r^2 \sin^2 \theta \cos^2 \phi \cos^2 \theta + r^2 \sin^3 \theta \sin^2 \phi \\ + r^2 \sin^2 \theta \sin \theta \cos^2 \theta.$$

$$\begin{aligned}
 & r^2 \sin^3 \theta (\cos^2 \phi + \sin^2 \phi) + r^2 \sin \theta \cos^2 \theta (\sin^2 \phi + \cos^2 \phi) \\
 &= r^2 \sin^3 \theta + r^2 \sin \theta \cos^2 \theta \\
 &= r^2 \sin \theta (\sin^2 \theta + \cos^2 \theta) \\
 &\quad J\left(\frac{x, y, z}{r, \theta, \phi}\right) = r^2 \sin \theta
 \end{aligned}$$

~~$r \cos \theta (\sin \phi - r \sin \theta \sin \phi)$~~

$$J\left(\frac{x, y, z}{r, \theta, \phi}\right) \times J\left(\frac{r, \theta, \phi}{x, y, z}\right) = 1.$$

$$\therefore J\left(\frac{r, \theta, \phi}{x, y, z}\right) = \frac{1}{J\left(\frac{x, y, z}{r, \theta, \phi}\right)}$$

$$\begin{aligned}
 &\stackrel{2}{=} \frac{1}{r^2 \sin \theta} \\
 &= \frac{\text{cosec } \theta}{r^2}
 \end{aligned}$$

$$\therefore J\left(\frac{r, \theta, \phi}{x, y, z}\right) = \frac{\text{cosec } \theta}{r^2}$$

~~$\sin \theta \cos \phi \cos \theta$~~

B) If $u = \frac{x+y}{1-xy}$ and $v = \tan^{-1}x + \tan^{-1}y$

P.T. u & v are F.D. & find the relatⁿ b/w them.

$$\frac{\partial u}{\partial x} = \frac{(1-xy)(1) - (x+y)(-y)}{(1-xy)^2}$$

$$\frac{\partial u}{\partial y} = \frac{(1-xy)(1) - (x+y)(-x)}{(1-xy)^2}$$

$$\frac{\partial u}{\partial x} = \frac{1-xy+xy+y^2}{(1-xy)^2}$$

$$= \frac{1-xy+x^2+xy}{(1-xy)^2}$$

$$\frac{\partial u}{\partial x} = \frac{1+y^2}{(1-xy)^2}$$

$$\frac{\partial u}{\partial y} = \frac{1+x^2}{(1-xy)^2}$$

$$\frac{\partial v}{\partial x} = \frac{1}{1+x^2}$$

$$\frac{\partial v}{\partial y} = \frac{1}{1+y^2}$$

$$\therefore J \left(\frac{u, v}{x, y} \right) = \begin{vmatrix} \frac{1+y^2}{(1-xy)^2} & \frac{1+x^2}{(1-xy)^2} \\ \frac{1}{1+x^2} & \frac{1}{1+y^2} \end{vmatrix}$$

$$= \frac{1}{(1-xy)^2} - \frac{1}{(1-xy)^2}$$

$$J \left(\frac{u, v}{x, y} \right) = 0$$

$\therefore u$ & v are F.D.

$$V = \tan^{-1} \left(\frac{x+y}{1-xy} \right) = \tan^{-1} u \Rightarrow u = \tan V$$

B) P.T. the

are func
n relatⁿ b/w

$$\frac{\partial u}{\partial x} = 1$$

$$\frac{\partial v}{\partial x} = y + 2$$

$$\frac{\partial w}{\partial x} = 2x$$

$$J \left(\frac{u, v, w}{x, y, z} \right)$$

$$= 1$$

$$+ 1$$

$$2x$$

$$+ 2$$

$$J /$$

tan⁻¹ y

refer brother

$$\frac{(xy)(1) - (x+y)(-x)}{(1-xy)^2}$$

$$\frac{1-xy+x^2+xy}{(1-xy)^2}$$

$$\frac{1+x^2}{(1-xy)^2}$$

A.P.T the func's $u = x + y + z$, $v = xy + yz + zx$
 $w = x^2 + y^2 + z^2$

are functionally dependent & we find the relat'n b/w them.

$$\frac{\partial u}{\partial x} = 1 \quad \frac{\partial u}{\partial y} = 1 \quad \frac{\partial u}{\partial z} = 1$$

$$\frac{\partial v}{\partial x} = y + z \quad \frac{\partial v}{\partial y} = x + z \quad \frac{\partial v}{\partial z} = y + x$$

$$\frac{\partial w}{\partial x} = 2x \quad \frac{\partial w}{\partial y} = 2y \quad \frac{\partial w}{\partial z} = 2z$$

$$J \left(\begin{array}{c|ccc} \frac{u, v, w}{x, y, z} \\ \hline 1 & 1 & 1 \\ y+z & x+z & x+y \\ 2x & 2y & 2z \end{array} \right)$$

$$= 1 (2xz + 2z^2 - 2xy - 2y^2)$$

$$- 1 (2yz + 2z^2 - 2x^2 - 2xy)$$

$$+ 1 (2y^2 + 2yz - 2x^2 - 2xz)$$

$$= 2xz + 2z^2 - 2xy - 2y^2 - 2yz - x^2 + 2x^2 + 2xy + 2y^2 + 2yz - 2x^2 - 2xz$$

$$\therefore J \left(\begin{array}{c|ccc} \frac{u, v, w}{x, y, z} \\ \hline 1 & 1 & 1 \\ y+z & x+z & x+y \\ 2x & 2y & 2z \end{array} \right) = 0 \quad \therefore u, v, w \text{ are F.D.}$$

$$(u)^2 = w + 2v$$

$$\Rightarrow u^2 = w + 2v$$

Q) Prove that $u = \frac{x^2 - y^2}{x^2 + y^2}$ $v = \frac{2xy}{x^2 + y^2}$ are P.D

and find the relat' b/w them.

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{x^2 + y^2(2x) - (x^2 - y^2)(2x)}{(x^2 + y^2)^2} & \frac{\partial u}{\partial y} &= \frac{(x^2 + y^2)(2y) - (x^2 - y^2)(2y)}{(x^2 + y^2)^2} \\ &= \frac{2x^3 + 2y^2 - 2x^3 + 2xy^2}{(x^2 + y^2)^2} & &= \frac{-2y^3 + 2y - 2y^3 + 2y}{(x^2 + y^2)^2} \\ &= \frac{4xy^2}{(x^2 + y^2)^2} & &= \frac{-4y^3}{(x^2 + y^2)^2} \end{aligned}$$

$$\begin{aligned} \frac{\partial v}{\partial x} &= \frac{x^2 + y^2(2y) - 2xy(2x)}{(x^2 + y^2)^2} & \frac{\partial v}{\partial y} &= \frac{x^2 + y^2(2x) - 2xy(2y)}{(x^2 + y^2)^2} \\ &= \frac{2y^3 + 2y^3 - 4x^2y}{(x^2 + y^2)^2} & &= \frac{2x^3 + 2xy^2 - 4xy^2}{(x^2 + y^2)^2} \\ &= \frac{2y^3 - 2x^2y}{(x^2 + y^2)^2} & &= \frac{2x^3 - 2xy^2}{(x^2 + y^2)^2} \end{aligned}$$

$$J(u, v) = \frac{u}{v}$$

$$x^2/y^2 = \frac{x}{y}$$

$$u^2 + v^2 = \frac{(x^2 + y^2)^2}{(x^2 + y^2)^2}$$

$$u^2 + v^2 = \frac{(x^2 + y^2)^2}{(x^2 + y^2)^2}$$

$$J = \begin{vmatrix} \frac{4xy^2}{(x^2 + y^2)^2} & \frac{-4y^3}{(x^2 + y^2)^2} \\ \frac{2y^3 - 2x^2y}{(x^2 + y^2)^2} & \frac{2x^3 - 2xy^2}{(x^2 + y^2)^2} \end{vmatrix}$$

$$\Rightarrow \underline{8^2}$$

$$\left| \begin{array}{cc} \frac{4xy^2}{(x^2+y^2)^2} & \frac{-4yx^2}{(x^2+y^2)^2} \\ \frac{2y^3-2xy}{(x^2+y^2)^2} & \frac{2x^3-2xy^2}{(x^2+y^2)^2} \end{array} \right|$$

z^2 are R.D

$$\Rightarrow \frac{8x^4y^2 - 8x^2y^4 + 8y^4x^2 - 8x^4y^2}{(x^2+y^2)^4}$$

$$\begin{pmatrix} u \\ v \end{pmatrix}, \quad \begin{matrix} u, v \\ u \\ v \end{matrix} = 0 \quad \therefore u, v \text{ are F.D}$$

$$x^2y^2 = \frac{x^2-y^2}{u} \Rightarrow v = \frac{2xy}{x^2-y^2} \times u$$

$$v = u \left(\frac{2xy}{x^2-y^2} \right).$$

$$U^2 = \frac{(x^2-y^2)^2}{(x^2+y^2)^2} + \frac{4x^2y^2}{(x^2+y^2)^2}$$

$$= \frac{(x^2-y^2)^2 + 4x^2y^2}{(x^2+y^2)^2}$$

$$= \frac{(x^2+y^2)^2}{(x^2+y^2)^2} = 1$$

$$Q) P.O.T \quad u = 2x - y + 3z \quad v = 2x - y - z \quad w = 2x - y + z$$

are F.o.D and find functional relation.

$$\frac{\partial u}{\partial x} = 2 \quad \frac{\partial u}{\partial y} = -1 \quad \frac{\partial u}{\partial z} = 3$$

$$\frac{\partial v}{\partial x} = 2 \quad \frac{\partial v}{\partial y} = -1 \quad \frac{\partial v}{\partial z} = -1$$

$$\frac{\partial w}{\partial x} = 2 \quad \frac{\partial w}{\partial y} = -1 \quad \frac{\partial w}{\partial z} = 1$$

$$J \left(\begin{array}{c} u, v, w \\ x, y, z \end{array} \right) = \begin{vmatrix} 2 & -1 & 3 \\ 2 & -1 & -1 \\ 2 & -1 & 1 \end{vmatrix}$$

$$= 2(-1 - 1) + 1(2 + 2) + 3(-2 + 2)$$

$$= -4 + 4 = 0$$

$$\therefore J \left(\begin{array}{c} u, v, w \\ x, y, z \end{array} \right) = 0 \quad \therefore u, v, w \text{ are F.o.D}$$

$$u + v - 2w = 2x - y + 3z + 2x - y - z - 4x + 2y - 2z$$

$$= 4x - 2y + 2z - 4x + 2y - 2z$$

$$= 0$$

$$u + v - 2w = 0$$

$$\Rightarrow u + v = 2w$$

of Maximum and Minimum of Func's of two variables.

→ Let $f(x,y)$ be a func' of two variables x & y , at $x=a, y=b, f(x,y)$ is said to have maximum or minimum value if $f(a,b) > f(a+h, b+k)$ or $f(a,b) < f(a+h, b+k)$ respectively, where h & k are small values.

→ Extreme value: $f(a,b)$ is said to be an extreme value of f if it is a maximum or minimum value.

Working rule:

i) Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ and equate each to zero &

solve these two eqn we get the pair of points

$(a_1, b_1), (a_2, b_2), \dots$ these are called stationary points.

$$2) \quad g_1 = \frac{\partial^2 f}{\partial x^2} \quad g_2 = \frac{\partial^2 f}{\partial x \partial y} \quad t = \frac{\partial^2 f}{\partial y^2}$$

3) i) if $rt - s^2 > 0$ and $s < 0$ at (a_1, b_1) then (a_1, b_1) is a point of maximum and $f(a_1, b_1)$ is a maximum value.

ii) if $rt - s^2 > 0$ and $s > 0$ at (a_1, b_1) then (a_1, b_1) is a point of minimum and $f(a_1, b_1)$ is a minimum value.

iii) if $rt - s^2 \cancel{<} 0$ at (a_1, b_1) then neither a maximum nor a minimum at a_1, b_1 exists. In this case the point (a_1, b_1) is a saddle point.

iv) if $rt - s^2 = 0$ at (a_1, b_1) , no conclusion can be drawn about maximum or minimum as
 ↳ similarly find the other pair of values

$(a_2, b_2), (a_3, b_3), \dots$

Q) Find the maximum and minimum values of

$$f(x, y) = x^3 + 3xy^2 - 3x^2 - 3y^2 + 4.$$

$$\frac{\partial f}{\partial x} = 3x^2 + 3y^2 - 6x$$

$$\frac{\partial f}{\partial y} = 3x(2y) - 6y = 6xy - 6y$$

$$\Rightarrow 3x^2 + 3y^2 - 6x = 0$$

$$6xy - 6y = 0.$$

$$y(6x - 6) = 0.$$

$$\Rightarrow (y-0)(6x-6)=0$$

$$y=0, x=1$$

$$i) 3x^2 - 6x = 0$$

$$x(3x-6)=0$$

$$x=0, x=2$$

$$ii) 3 + 3y^2 - 6 = 0.$$

$$3y^2 - 3 = 0$$

$$y^2 - 1 = 0$$

$$y^2 = 1$$

$$y=1, y=-1$$

⇒ Stationary

points are, $(0,0), (2,0), (1,1), (1,-1)$

values of

$$y = 6x - 6 \quad s = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) \quad t = 6x - 6.$$

$$s = 6y \\ s^2 = 36y^2$$

$$st - s^2 \Rightarrow 6x - 6 - 36y^2$$

$$\Rightarrow 36x^2 + 36 - 72x - 36y^2 \\ \Rightarrow 36x^2 - 36y^2 - 72x + 36.$$

i) at $(0, 0)$

$$\Rightarrow (6x - 6)^2 - (36y^2) \Rightarrow 36 > 0 \quad \& \quad 6(0) - 6 = -6 \\ \therefore st < 0.$$

$(0, 0)$ is a maximum pt.

maximum value is $f(0, 0) = 4$.

ii) at $(2, 0)$

$$\Rightarrow (12 - 6)^2 - 36(0) = 36 > 0 \quad \& \quad 6(2) - 6 = 6 > 0.$$

$(2, 0)$ is a minimum pt

$$\therefore \text{minimum value is } f(2, 0) = 8 + 4 - 12 \\ = 0.$$

iii) at $(1, 1)$

$$(6 - 6)^2 - 36(1) = -36 < 0$$

at $(1, 1)$ $f(x, y)$ has neither max nor min.

iv) at $(1, -1)$

$$\Rightarrow (6(1)-6)^2 - 36(-1)^2 = -36 < 0.$$

\therefore at $(1, -1)$ no $f(x, y)$ has neither max nor min.

Q) Discuss the maxima and minima of

$$x^2y + xy^2 - axy.$$

$$f(x, y) = x^2y + xy^2 - axy.$$

$$\frac{\partial f}{\partial x} = 2xy + y^2 - ay \quad \frac{\partial f}{\partial y} = x^2 + 2xy - ax$$

$$2xy + y^2 - ay = 0$$

$$\underline{2xy + x^2 - ax = 0}$$

$$y^2 - x^2 - ay + ax = 0$$

$$\Rightarrow x^2 - y^2 + ay - ax = 0.$$

$$x(x-a) + y(a-y) = 0,$$

$$(x-a)(x-a) + (y-a)(a-y) = 0$$

$$(y^2 - x^2) - a(y-x) = 0$$

$$(y-x)(y+x) - a(y-x) = 0$$

$$(y-x)(y+x-a) = 0$$

$$\therefore y-x=0, \quad y+x-a=0$$

$$y=x, \quad y+x=a.$$

either max

i) $2xy + y^2 - ay = 0$
 $2x^2 + y^2 - ax = 0$
 $3x^2 - ax = 0$
 $x(3x-a) = 0$
 $x=0, x=\frac{a}{3}$

stationary points are $(0,0), (\frac{a}{3}, \frac{a}{3})$

ii) $2xy + y^2 - ay = 0$
 $2x(a-x) + (a-x)^2 - ay = 0$
 $2ax - 2x^2 + a^2 + x^2 - 2ax - ay = 0$
 $-x^2 + a^2 - ay = 0$
 $x^2 - a^2 + ay = 0$
 $x^2 - ay = x^2 - a^2 + a(a-x) = 0$
 $x^2 - a^2 + a^2 - ax = 0$
 $x^2 - ax = 0$
 $x(x-a) = 0$
 $x=0, x=a$

stationary pts are $(0,a), (a,0)$

$\eta_1 = 2y \quad S = 2x + 2y - a \quad t = 2x$
 $\eta_2 = 2x \quad S^2 = (2x+2y-a)^2$

at pt $(0,0)$

$\Rightarrow \eta_1 t - S^2 = 4(0)(0) - (2(0)+2(0)-a)^2$
 $= -a^2 < 0.$

at $(0,0)$ neither $f(x,y)$ has neither max nor min.

ii) at $(\frac{a}{3}, \frac{a}{3})$

$$rt - s^2 = 4\left(\frac{a}{3}\right)\left(\frac{a}{3}\right) - \left(2\left(\frac{a}{3}\right) + 2\left(\frac{a}{3}\right) - a\right)^2$$

$$= \frac{4a^2}{9} - \left(\frac{4a - 3a}{3}\right)^2$$

$$= \frac{4a^2}{9} - \frac{a^2}{9} = \frac{3a^2}{9} = \frac{a^2}{3} > 0$$

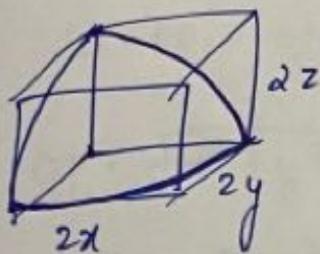
$$g_1 = 2\left(\frac{a}{3}\right) = \frac{2a}{3}$$

If $a > 0$ then $g_1 > 0 \therefore$ minimum

If $a < 0$ then $g_1 < 0 \therefore$ maximum

Q) Find the value of the largest parallelopiped that can be inscribed in the ellipsoid,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$



$$\text{length} = 2x$$

$$\text{breadth} = 2y$$

$$\text{height} = 2z$$

$$f(x, y, z) = V = (2x)(2y)(2z)$$

$$= 8xyz$$

$$\phi(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

$$F(x, y, z) = 8xyz + \lambda \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \right)$$

$$F_x = 8yz + \frac{\lambda}{a^2}(2x) = 0$$

$$F_y = 8xz + \frac{\lambda}{b^2}(2y) = 0$$

$$F_z = 8xy + \frac{\lambda}{c^2}(2z) = 0.$$

$$\therefore \lambda = \frac{-8yz(a^2)}{2x}, \lambda = \frac{-8xz(b^2)}{2y}, \lambda = \frac{-8xy(c^2)}{2z}$$

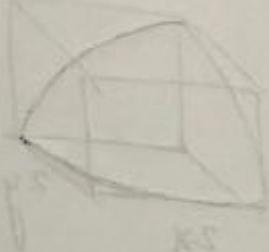
$$\therefore \frac{-8yz(a^2)}{2x} = \frac{-8xz(b^2)}{2y} = \frac{-8xy(c^2)}{2z}$$

$$\Rightarrow y^2a^2 = x^2b^2 \quad \Rightarrow z^2a^2 = x^2c^2.$$

$$\Rightarrow z^2b^2 = y^2c^2$$

$$\begin{aligned} \therefore y^2 a^2 &= x^2 b^2 \rightarrow ① & z^2 a^2 &= x^2 c^2 \\ z^2 b^2 &= y^2 c^2 & \frac{x^2}{a^2} &= \frac{z^2}{c^2} \\ \therefore \frac{y^2}{b^2} &= \frac{z^2}{c^2} \\ \Rightarrow \text{From } ① & \quad \frac{x^2}{a^2} = \frac{y^2}{b^2} \end{aligned}$$

Q) Find +
that i:
Let PC



$$\therefore \frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c^2}.$$

$$\therefore \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

$$\Rightarrow \frac{x^2}{a^2} + \frac{x^2}{a^2} + \frac{x^2}{a^2} = 1$$

$$\Rightarrow \frac{3x^2}{a^2} = 1$$

$$\therefore 3x^2 = a^2$$

$$x = \pm \frac{a}{\sqrt{3}}$$

$$y = \pm \frac{b}{\sqrt{3}}$$

$$z = \pm \frac{c}{\sqrt{3}}$$

$$\therefore (x, y, z) = \left(\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}}, \frac{c}{\sqrt{3}} \right)$$

$$\text{max value} = V = xyz = 8 \left(\frac{a}{\sqrt{3}} \right) \left(\frac{b}{\sqrt{3}} \right) \left(\frac{c}{\sqrt{3}} \right)$$

$$= \frac{8abc}{3\sqrt{3}}$$

Q Find the pt on the plane $x+2y+3z=4$
that is closest to the origin.

Let $P(x, y, z)$ be any point on the plane.

$$O = (0, 0, 0)$$

$$OP = \sqrt{x^2 + y^2 + z^2}$$

$$OP^2 = x^2 + y^2 + z^2$$

$$f(x, y, z) = x^2 + y^2 + z^2$$

$$\phi(x, y, z) = x + 2y + 3z = 4$$

$$F(x, y, z) = x^2 + y^2 + z^2 + \lambda x + 2\lambda y + 3\lambda z$$

$$F_x = 2x + \lambda$$

$$F_y = 2y + 2\lambda$$

$$F_z = 2z + 3\lambda$$

$$\therefore \lambda = -2x, \quad \lambda = -\frac{2y}{2}, \quad \lambda = -\frac{2z}{3}.$$

$$\lambda = -y \quad \lambda = -\frac{2z}{3}$$

$$-2x = -y$$

$$-y = -\frac{2z}{3}$$

$$-2x = -\frac{2z}{3}$$

$$2x = y$$

$$3y = 2z$$

$$3x = z$$

$$x = \frac{y}{2}$$

$$\frac{y}{2} = \frac{z}{3}$$

$$\frac{z}{3} = x$$

$$\therefore x = \frac{y}{2} = \frac{z}{3} \Rightarrow 6x = 3y = 2z$$

$$\therefore x + 2y + 3z = 4$$

$$\Rightarrow x + 2(2x) + 3(3x) = 4$$

$$\Rightarrow x + 4x + 9x = 4$$

$$14x = 4$$
$$x = \frac{4}{14} = \frac{2}{7}$$

$$y = 2x = 2\left(\frac{4}{14}\right) = \frac{4}{7}$$

$$z = 3\left(\frac{4}{14}\right) = \frac{12}{14} = \frac{6}{7}$$

$$\therefore f(x, y, z) = x^2 + y^2 + z^2$$

$$= \frac{4}{49} + \frac{16}{49} + \frac{36}{49}$$

$$f(x, y, z) = \frac{56}{49} = \frac{8}{7}$$

$$\Rightarrow (x, y, z) = \left(\frac{2}{7}, \frac{4}{7}, \frac{6}{7}\right) = pt$$

2Q) Find the max & min distances of the pt

(3, 4, 12) from the sphere $x^2 + y^2 + z^2 = 1$.

3Q) Find the pt on the plane $3x + 2y + z = 12$
which is nearest to the origin.

3Q) Find the max value of $u = x^2 y^3 z^4$ if
 $2x + 3y + 4z = a$

Q) Find the shortest distance from the origin
to the surface $xyz^2=2$

Q) Find the pts on the surface $z=xy+1$
that are nearest to the Origin.

Q) Find three +ve numbers whose sum is 100
and whose product is max.

$$\phi(x, y, z) = x + y + z - 100$$

$$f(x, y, z) = xyz$$

$$F(x, y, z) = xyz + (x+y+z)\lambda$$

$$F_x = yz + \lambda \quad F_y = xz + \lambda \quad F_z = xy + \lambda$$

$$\lambda = -yz, \lambda = -xz, \lambda = -xy.$$

$$y = x, z = y, \text{ and } x = z$$

$$\therefore x = y = z.$$

$$x + x + x = 100$$

$$3x = 100$$

$$\Rightarrow x = \frac{100}{3}, y = \frac{100}{3}, z = \frac{100}{3}.$$

$$f(x, y, z) = xyz = \left(\frac{100}{3}\right)^3 = \text{max value.}$$

if

78) Find the maxima and minima of

$$xy + \frac{a^3}{x} + \frac{a^3}{y}.$$

88) Find the max and min values of $f(x, y)$

$$= x^3 y^2 (1 - x - y).$$

98) A Rectangular box open at the top is to have volume 32 cubic feet. Find the dimension of the box required least material for its construct.

$$f(x, y, z) = 2xy + 2zx + yz.$$

$$\phi(x, y, z) = xyz = 32.$$

$$\Rightarrow F(x, y, z) = 2xy + 2zx + yz + \lambda(xyz)$$

$$F_x = 2y + 2z + \lambda yz = 0$$

$$F_y = 2x + z + \lambda xz = 0$$

$$F_z = 2x + y + \lambda xy = 0$$

$$\therefore \lambda = \frac{-2y - 2z}{yz}, \lambda = \frac{-2x - z}{xz}, \lambda = \frac{-2x - y}{xy}$$

$$\cancel{\frac{-2y - 2z}{yz}} = \cancel{\frac{-2x - z}{xz}}$$

$$\cancel{-2xy - 2z} = \cancel{-2xy}$$

$$\frac{-2x - z}{xz} = \cancel{\frac{-2x - y}{xy}}$$

$$\cancel{-2x = -2}$$
$$2x = 2.$$

$$\begin{aligned} \frac{-2y-2z}{yz} &= \frac{-2x-z}{xz}, & \frac{-2x-z}{xz} &= \frac{-2x-y}{xy} \\ \Rightarrow -2yz - 2xz &= -2xy - yz & \Rightarrow -2xy - 2yz - 2xz - yz & \\ \Rightarrow -2xz &= -yz & -2xy &= -2xz \\ \Rightarrow 2x &= y & y &= z \\ \therefore 2x &= y = z & & \end{aligned}$$

$$\phi(x, y, z) = xyz = 32$$

$$\Rightarrow x(2x)(2x) = 32$$

$$4x^3 = 32$$

$$x = 2, y = 4, z = 4$$

∴ Dimensions of the box are ($x = 2, y = 4, z = 4$)

$$Q1) \phi(x, y, z) = x^2 + y^2 + z^2 = 1$$

$$P = (3, 4, 12)$$

$$\begin{aligned}QP &= \sqrt{(x-3)^2 + (y-4)^2 + (z-12)^2} \\&= \sqrt{x^2 + 9 - 6x + y^2 + 16 - 8y + z^2 + 144 - 24z} \\&= \sqrt{x^2 + y^2 + z^2 - 6x - 8y - 24z + 169} \\&\therefore QP^2 = x^2 + y^2 + z^2 - 6x - 8y - 24z + 169.\end{aligned}$$

$$\therefore f(x, y, z) = x^2 + y^2 + z^2 - 6x - 8y - 24z + 169$$

$$\begin{aligned}F(x, y, z) &= x^2 + y^2 + z^2 - 6x - 8y - 24z + 169 \\&\quad + \lambda(x^2 + y^2 + z^2)\end{aligned}$$

$$F_x = 2x - 6 + 2\lambda x = 0.$$

$$F_y = 2y - 8 + 2y\lambda = 0$$

$$F_z = 2z - 24 + 2\lambda z = 0$$

$$\therefore \lambda = -\frac{2x+6}{2x}, \lambda = -\frac{2y+8}{2y}, \lambda = -\frac{2z+24}{2z}$$

$$-\frac{2x+6}{2x} = -\frac{2y+8}{2y}$$

$$-\frac{2y+8}{2y} = -\frac{2z+24}{2z}$$

$$-4xy + 12y = -4xy + 16x.$$

$$-8yz + 8z = -8yz + 24y$$

$$\frac{12y}{3} = \frac{16x}{2}$$

$$8z = \frac{24y}{3}$$

$$3y = 2x$$

$$z = 3y$$

$$\frac{4}{2} = \frac{x}{3}$$

$$\frac{z}{3} = y$$

$$\Rightarrow \frac{z}{3} = \frac{y}{2}$$

$$\frac{x}{3} = \frac{y}{2} = \frac{z}{6}$$

$$2x = 3y = z$$

$$\phi(x, y, z) = x^2 + y^2 + z^2 = 1$$

$$\Rightarrow x^2 + \left(\frac{2x}{3}\right)^2 + (2x)^2 = 1$$

$$x^2 + \frac{4x^2}{9} + 4x^2 = 1$$

$$9x^2 + 4x^2 + 36x^2 = 9$$

$$49x^2 = 9$$

$$x^2 = \frac{9}{49} \Rightarrow x = \pm \frac{3}{7}$$

$$y = \frac{2}{3} \left(\frac{3}{7} \right) = \pm \frac{2}{7}, \quad z = 2 \left(\frac{3}{7} \right) = \pm \frac{6}{7}$$

$$\therefore Q = \left(\frac{3}{7}, \frac{2}{7}, \frac{6}{7} \right) \text{ or } Q = \left(-\frac{3}{7}, -\frac{2}{7}, -\frac{6}{7} \right)$$

~~$$PQ^2 = x^2 + y^2 + z^2 - 6x - 8y - 24z + 169$$~~

$$PQ^2 = \frac{9}{49} + \frac{4}{49} + \frac{36}{49} - \frac{18}{7} - \frac{16}{7} - \frac{144}{7} + 169$$

$$PQ = \sqrt{\frac{1012}{7}} \Rightarrow \begin{matrix} \text{minimum} \\ \text{maximum distance} \end{matrix}$$

$$PQ^2 = \frac{9}{49} + \frac{4}{49} + \frac{36}{49} + \frac{18}{7} + \frac{16}{7} + \frac{144}{7} + 169$$

$$PQ = \sqrt{\frac{1368}{7}} \Rightarrow \text{maximum distance}$$

$$Q2) \text{ Let } P = (x, y, z)$$

$$OP = \sqrt{(x-0)^2 + (y-0)^2 + (z-0)^2}$$

$$= \sqrt{x^2 + y^2 + z^2}$$

$$f(x, y, z) = OP^2 = x^2 + y^2 + z^2$$

$$\phi(x, y, z) = 3x + 2y + z = 12$$

$$\therefore F(x, y, z) = x^2 + y^2 + z^2 + \lambda(3x + 2y + z)$$

$$F_x = 2x + 3\lambda = 0$$

$$F_y = 2y + 2\lambda = 0$$

$$F_z = 2z + \lambda = 0$$

$$\therefore \lambda = -\frac{2x}{3}, \lambda = -\frac{2y}{2}, \lambda = -\frac{2z}{1}$$

$$-\frac{2x}{3} = -y$$

$$-y = -2z$$

$$-\frac{2x}{3} = 2z$$

$$-2x = -3y$$

$$\Rightarrow y = 2z$$

$$x = 3z$$

$$\Rightarrow 2x = 3y$$

$$\Rightarrow$$

$$\Rightarrow \frac{x}{3} = \frac{y}{2}$$

$$\therefore 2x = 3y = 6z$$

$$\Rightarrow 3y = 3z \Rightarrow 3y = 6z$$

$$\frac{2x}{3} = \frac{z}{2}$$

$$\therefore \phi(x, y, z) = 3x + 2y + z = 12$$

$$3x + 2\left(\frac{2x}{3}\right) + 1\left(\frac{2x}{3}\right) = 12$$

$$3x + \frac{4x}{3} + \frac{2x}{3} = 12$$

$$9x + 4x + 2x = 36$$

$$14x = 36$$

$$x = \frac{36}{14}$$

$$y = \frac{2}{3} \left(\frac{36}{14}\right) = \frac{24}{14}$$

$$\therefore P = \left(\frac{36}{14}, \frac{24}{14}, \frac{12}{14}\right)$$

$$\therefore P = \left(\frac{18}{7}, \frac{12}{7}, \frac{6}{7}\right)$$

$$Q3) f(x, y, z) = x$$

$$fx = 2xy^3$$

$$fy = 3y^2x$$

$$fz = 4z^3x$$

$$\lambda = -\frac{2xy^2}{x}$$

$$-xy^2z^4 = -$$

$$y = x$$

$$\phi(x, y, z) =$$

Max value

$$(3) f(x, y, z) = x^2 y^3 z^4 \quad g(x, y, z) = 2x + 3y + 4z = a$$

$$f(x, y, z) = x^2 y^3 z^4 + \lambda (2x + 3y + 4z)$$

$$F_x = 2x y^3 z^4 + 2\lambda$$

$$F_y = 3y^2 x^2 z^4 + 3\lambda$$

$$F_z = 4z^3 x^2 y^3 + 4\lambda$$

$$\lambda = -\frac{2xy^3z^4}{2}, \quad \lambda = -\frac{3y^2x^2z^4}{3}, \quad \lambda = -\frac{4z^3x^2y^3}{4}$$

$$\therefore -xy^3z^4 = -yz^2x^2z^4, \quad y^2x^2z^4 = z^2x^2y^2$$

$$y = x \quad z = y$$

$$\therefore x = y = z$$

$$g(x, y, z) = 2x + 3y + 4z = a$$

$$2x + 3x + 4x = a$$

$$9x = a$$

$$x = \frac{a}{9}, \quad y = \frac{a}{9}, \quad z = \frac{a}{9}$$

Max value of $x^2 y^3 z^4$ is $\left(\frac{a}{9}\right)^2 \left(\frac{a}{9}\right)^3 \left(\frac{a}{9}\right)^4$

$$\therefore \left(\frac{a}{9}\right)^9$$

$$Q4) \phi(x, y, z) = xyz^2 = 2$$

$$\text{let } P = (x, y, z)$$

$$OP^2 = x^2 + y^2 + z^2$$

$$\therefore f(x, y, z) = x^2 + y^2 + z^2$$

$$F(x, y, z) = x^2 + y^2 + z^2 + \lambda(xyz^2)$$

$$F_x = 2x + \lambda yz^2 = 0$$

$$F_y = 2y + \lambda xz^2 = 0$$

$$F_z = 2z + \lambda xy^2 = 0$$

$$\therefore \lambda = \frac{-2x}{yz^2}, \lambda = \frac{-2y}{xz^2}, \lambda = \frac{-2z}{xy^2}$$

$$\therefore -\frac{\cancel{\lambda}x}{yz^2} = -\frac{\cancel{\lambda}y}{xz^2}$$

$$+\frac{\cancel{\lambda}y}{xz^2} = \frac{z^2}{xyz}$$

$$\Rightarrow x^2 = y^2$$

$$2y^2 = z^2$$

$$\therefore x^2 = y^2 = \frac{z^2}{2}$$

$$\Rightarrow \phi(x, y, z) = xyz^2 = 2.$$

$$\Rightarrow \cancel{x^2}y^2z^4 = 4$$

$$x^2(x^2)(2x^2)^2 = 4$$

$$x^4(4x^4) = 4$$

$$4x^8 = 4$$

$$x^8 = 1$$

$$x = \pm 1$$

$$\therefore \text{Point } (x, y, z) = (1, 1, \frac{1}{2}) \text{ or } (-1, -1, -\frac{1}{2})$$

$$\therefore OP^2 = 1 + 1 + \frac{1}{4} = \frac{4+4+1}{4} = \frac{9}{4}$$

$$\therefore OP = \frac{3}{2} \quad \therefore \text{Shortest distance} = \frac{3}{2}$$

$$\phi(x, y, z)$$

$$P = (x, y, z)$$

$$OP^2 = x^2 + y^2 + z^2$$

$$f(x, y, z) =$$

$$F_x = 2x -$$

$$F_y = 2y -$$

$$F_z = 2z +$$

$$\therefore \lambda = \frac{-2x}{-y}$$

$$-\frac{2x}{-y} = \frac{-2y}{-x}$$

$$\therefore x^2 = 2y^2$$

$$\therefore x^2 = y^2$$

$$f(x, y) = xy + \frac{a^3}{x} + \frac{a^3}{y}$$

$$\frac{\partial f}{\partial x} = y + a^3 \left(-\frac{1}{x^2} \right)$$

$$\frac{\partial f}{\partial x} = y - \frac{a^3}{x^2}$$

$$\frac{\partial f}{\partial y} = x + a^3 \left(-\frac{1}{y^2} \right)$$

$$\frac{\partial f}{\partial y} = x - \frac{a^3}{y^2}$$

$$y - \frac{a^3}{x^2} = 0$$

$$y = \frac{a^3}{x^2}$$

$$x^2 y = a^3$$

$$\Rightarrow x = \frac{a^3}{y^2}$$

$$xy^2 = a^3$$

$$x^2 y^2 = xy^2$$

$$\cancel{x=y}$$

$$\text{i)} \cancel{ay^2 = a^3}$$

$$\Rightarrow \boxed{y^2 = a^2} \Rightarrow y = \pm a$$

$$\text{ii)} \cancel{x(a^2) = a^3}$$

$$\boxed{x = a}$$

$$\Rightarrow x^2(x) = a^3$$

$$x^3 = a^3$$

$$x=a, y=a$$

∴ stationary point is $(a, a), (a, -a)$.

$$n = \frac{\partial^2 f}{\partial x^2} = -a^3 \left(-\frac{2}{x^3} \right) \quad t = \frac{\partial^2 f}{\partial y^2} = -a^3 \left(-\frac{2}{y^3} \right)$$

$$n = \frac{2a^3}{x^3}$$

$$t = \frac{2a^3}{y^3}$$

$$S^2 = \frac{\partial^2 f}{\partial x \partial y} = \frac{2}{\partial x} \left(\cancel{x} - \frac{a^3}{y^2} \right) = 1 \Rightarrow S^2 = 1.$$

$$nt - S^2 \Rightarrow \frac{2a^3}{x^3} \times \frac{2a^3}{y^3} - 1 \Rightarrow \frac{4a^6}{x^3 y^3} - 1$$

i) at (a, a)

$$nt - S^2 \Rightarrow \frac{4a^6}{a^3 a^3} - 1 \Rightarrow 3 > 0$$

$$n = \frac{2a^3}{x^3} = \frac{2a^3}{a^3} = 2 \Rightarrow n > 0$$

∴ $f(x, y)$ has minimum at pt (a, a)

at point $(a, -a)$.

$$\Rightarrow f(a, a) = a^2 + \frac{a^5}{a} + \frac{a^8}{a} = 3a^2 \rightarrow \text{maxima/minima.}$$

$$Q8) f(x, y) = x^3y^2(1-x-y) = x^3y^2 - x^4y^2 - x^3y^3$$

$$\frac{\partial f}{\partial x} = 3x^2y^2 - 4x^3y^2 - 3x^2y^3$$

$$\frac{\partial f}{\partial y} = 2yx^3 - 2yx^4 - 3y^2x^3$$

$$\therefore \frac{\partial f}{\partial x} = 0 \quad \& \quad \frac{\partial f}{\partial y} = 0$$

$$\Rightarrow x^2y^2(3 - 4x - 3y) = 0.$$

$$\Rightarrow 3 - 4x - 3y = 0. \rightarrow (1)$$

$$\Rightarrow xy(2x^2 - 2x^3 - 3y^2) = 0$$

$$x^2(2 - 2x - 3y) = 0$$

$$2 - 2x - 3y = 0 \rightarrow (2)$$

$$\Rightarrow 2 - 2x - 3y = 0$$

$$\frac{3 - 4x - 3y = 0}{-1 + 2x = 0}$$

$$2x = 1$$

$$x = \frac{1}{2}$$

$$2 - 2(\frac{1}{2}) - 3(\frac{1}{3}) = 0$$

$$2 - 1 - 3y = 0$$

$$3y = 1$$

$$y = \frac{1}{3}$$

$$3\sqrt{4}\left(\frac{1}{2}\right) - 3y = 0.$$

stationary point is $(\frac{1}{2}, \frac{1}{3})$

$$n = -4$$

$$t = -3$$

$$\begin{aligned} s &= \frac{\partial}{\partial x} (2 - 2x - 3y) \\ &= -2. \\ s^2 &\leq 4 \end{aligned}$$

$$\begin{aligned} x^4 y^2 - x^3 y^3 &: \frac{\partial}{\partial x} (3x^2 y^2 - 4x^3 y^2 - 3x^2 y^3) \quad t = \frac{\partial}{\partial y} (2y x^3 - 2y x^4 - 3x^3 y^2) \\ &= 6xy^2 - 12x^2 y^2 - 6xy^3 \quad = 2x^3 - 2x^4 - 8x^3 y \end{aligned}$$

$$\begin{aligned} s &= \frac{\partial}{\partial x} (2y x^3 - 2y x^4 - 3x^3 y^2) \\ &= 6yx^2 - 8yx^3 - 9x^2 y^2 \end{aligned}$$

at pt $(\frac{1}{2}, \frac{1}{3})$

$$\begin{aligned} nt - s^2 &\Rightarrow \left(8\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)^2 - 12\left(\frac{1}{2}\right)\left(\frac{1}{3}\right) - 6\left(\frac{1}{2}\right)\left(\frac{1}{3}\right) \right) \\ &\quad \left(2\left(\frac{1}{2}\right) - 2\left(\frac{1}{2}\right) - 8\left(\frac{1}{2}\right)\left(\frac{1}{3}\right) \right) \\ &\quad - \left(8\left(\frac{1}{2}\right)\left(\frac{1}{3}\right) - 8\left(\frac{1}{2}\right)\left(\frac{1}{3}\right) - 8\left(\frac{1}{2}\right)\left(\frac{1}{3}\right) \right)^2 \\ &\Rightarrow \left(\frac{1}{3} - \frac{1}{3} - \frac{1}{9} \right) \left(\frac{1}{2} - \frac{1}{2} - \frac{1}{6} \right) - \left(\frac{1}{2} - \frac{1}{3} - \frac{1}{6} \right)^2 \end{aligned}$$

$$\Rightarrow \left(\frac{1}{72} \right) - \left(\frac{12 - 8 - 6}{24} \right)^2$$

$$\Rightarrow \left(\frac{1}{72} \right) - \left(-\frac{1}{24} \right)^2 \Rightarrow \frac{1}{72} - \left(\frac{1}{144} \right) \Rightarrow \frac{1}{144} > 0.$$

$$n = -\frac{1}{9} < 0 \quad \therefore$$

nor minimum

minima.

$$x^4 y^2 - x^3 y^3$$

EQUATIONS

EX

1 = 0

1

1

$\therefore f(x, y)$ has maxima at pt $(\frac{1}{2}, \frac{1}{3})$

$$f(\frac{1}{2}, \frac{1}{3}) = (\frac{1}{2})^3 (\frac{1}{3})^2 (1 - \frac{1}{2} - \frac{1}{3})$$

$$= (\frac{1}{8})(\frac{1}{9}) \left(\frac{6-3-2}{6} \right)$$

$$= \frac{1}{8} \times \frac{1}{9} \times \frac{1}{6}$$

$\Rightarrow \frac{1}{432}$ is the maxima.