Numerical Techniques.

Topic Algebraic & Transcendental equation.

1. Polynomial function:- A function $f(x)$ is said to be a polynomial function if $f(x)$ is a polynomial in x.

i.e $f(x) = a_0x^n + a_1x^{n-1} \dots + a_{n-1}x + a_n$, where $a_0 \neq 0$.

the coefficients $a_0, a_1, \dots a_n$ are real constants and n is non-negative integer.

2. Algebraic function:- A function which is a sum or difference or product of two polynomials is called an algebraic function, otherwise, the function is called a transcendental or non-algebraic function.

Ex:- ① $f(x) = c_1e^x + c_2e^{-x}$, $f(x) = e^{5x} - \frac{x^3}{2} + 3 = 0$.
are examples of transcendental equations.

② $f(x) = x^3 - 2x + 1 = 0$; $f(x) = x^2 + 1 = 0$ are the examples of algebraic function.

Bisection method:- Bisection method is a simple iteration method to be solved is an equation

This method is also known as BOLZANO method of successive Bisection.

Suppose, equation of form $f(x) = 0$ has exactly one real root between two real numbers x_0, x_1.

1. Find a root of equation $x^3 - 5x + 1 = 0$ using bisection method in 5 stages.

Given $f(x) = 0$

$\Rightarrow x^3 - 5x + 1 = 0.$

Use trial and error method.

$x = 0, \quad f(0) = 1 > 0$

$x = 1, \quad f(1) = 1 - 5 + 1$

$= 2 - 5 = -3 < 0.$

Root lies b/w 0 & 1.

let $x_0 = 0, \; x_1 = 1$

+ve ——— -ve,

x_0 $\qquad$ x_1

By bisection method
first approximation is given by

$$x_2 = \frac{x_0 + x_1}{2} = \frac{1}{2} = 0.5.$$

$f(x_2) = f(0.5) = (0.5)^3 - 5(0.5) + 1$

$= -1.375 < 0$ $\underset{x_2}{-ve}$ ——— $\underset{x_0}{+ve}$

Root lies b/w x_2 & x_0.

Second approximation is

$$x_3 = \frac{x_2 + x_0}{2} = \frac{0.5 + 0}{2} = 0.25$$

$f(x_3) = f(0.25) = (0.25)^3 - 5(0.25) + 1.$

$= -0.2343 < 0$ $\underset{x_3}{-ve}$ ——— $\underset{x_0}{+ve}$

Root lies b/w x_3 & x_0.

Third approximation is $x_4 = \frac{x_3 + x_0}{2} = \frac{0.25 + 0}{2} = 0.12$

$f(x_4) = f(0.125) = (0.125)^3 - 5(0.125) + 1.$

$= 0.3769 > 0.$ $\underset{x_4}{+ve}$ ——— $\underset{x_3}{-ve}$

Root lies b/w x_2 & x_3.

fourth approx is

$$x_5 = \frac{x_3 + x_4}{2} = \frac{0.25 + 0.125}{2} = 0.1875$$

$f(x_5) = f(0.1875)$

$= (0.1875)^3 - 5(0.1875) + 1$

$= 0.0690 > 0$ $\underset{x_5}{+ve}$ ——— $\underset{x_3}{-ve}$

Root lies b/w x_5 & x_3.

Fifth approximation is

$x_6 = 0.1875 + 0.25/2.$

$x_6 = 0.2187.$

The root of $x^3 - 5x + 1$ is 0.2187.

2) Find a positive root of $x^3 - x = 1$ using bisection method.

Let $f(x) = x^3 - x - 1 = 0$.

By trail & error method.

$x = 0 \quad f(0) = -1 < 0.$

$x = 1 \quad f(1) = 1^3 - 1 - 1 = 0.$

$= -1 < 0.$

$x = 2 \quad f(2) = 8 - 2 - 1 = 5.$

$= 5 > 0.$

root lies b/w 1 and 2

$x_0 = 1 \quad x_1 = 2.$

+ve —— -ve
2 —— 1.

first approximations $x_2 = \frac{x_0 + x_1}{2} = \frac{1+2}{2} = 1.5.$

$f(x_2) = f(1.5) = (1.5)^3 - 1.5 - 1 = 0.875.$

root lies b/w x_2 & x_0.

+ve —— -ve
x_2 —— x_0.

Second approximation.

$x_0 = 1 \quad x_2 = 1.5.$

$$x_3 = \frac{1 + 1.5}{2} = \frac{2.5}{2} = 1.25.$$

$\boxed{x_3 = 1.25}$

$f(x_3) = -0.2968 < 0,$

(-) —— (+)
x_3 —— x_2.

Root lies b/w x_3 & x_2.

Third approximation is $x_4 = \frac{x_3 + x_2}{2} = \frac{1.25 + 1.5}{2}$

$= 1.375.$

$\boxed{x_4 = 1.375}$

$f(x_4) = 0.2246 > 0$ $\quad \overset{(+)}{x_4}$ —— $\overset{(-)}{x_3}$

Root lies b/w x_4 & x_3.

fourth approximation $x_5 = \frac{x_4 + x_3}{2}$.

$$x_5 = \frac{1.375 + 1.25}{2} = 1.3125$$

$\boxed{x_5 = 1.3125}$

$f(x_5) = -0.051 < 0$ $\quad \overset{(-)}{x_5}$ —— $\overset{(+)}{x_4}$.

Root lies b/w x_5 & x_4.

fifth approximation is $x_6 = \frac{x_5 + x_4}{2}$.

$$= \frac{1.3125 + 1.375}{2}.$$

$x_6 = 1.3437$.

$f(x_6) = 0.0826 > 0$ $\quad \overset{+ve}{x_6}$ —— $\overset{-ve}{x_5}$

Root lies b/w x_6 & x_5

Sixth approximation is $x_7 = \frac{x_6 + x_5}{2}$.

$$x_7 = \frac{1.3437 + 1.3125}{2} = 1.3281$$

$x_7 = 1.3281$

$f(x_7) = 0.0144 > 0$ $\quad \overset{+ve}{x_7}$ —— $\overset{-ve}{x_5}$.

Root lies b/w x_7 & x_5

seventh approximation $x_8 = \frac{x_7 + x_5}{2} = \frac{1.3281 + 1.3125}{2}$

$x_8 = 1.3203$.

$f(x_8) = -0.0187 < 0.$ $\quad$ $\underset{x_8}{-ve}$ ——— $\underset{x_7}{+ve}$

$\therefore$ Root lies b/w x_8 & x_7.

eighth approximation is.

$$x_9 = \frac{x_8 + x_7}{2} = \frac{1.3203 + 1.3281}{2}$$

$$x_9 = 1.3242.$$

$f(x_9) = f(1.3242) = -0.002 < 0.$ $\quad$ $\underset{x_9}{-}$ ——— $\underset{x_7}{+}$

Root lies b/w x_9 & x_7.

9th approximation is

$$x_{10} = \frac{x_9 + x_7}{2}.$$

$$x_{10} = 1.3261.$$

$f(x_{10}) = 0.006 > 0.$ $\quad$ $\underset{x_{10}}{(+)}$ ——— $\underset{x_9}{(-)}$

Root lies b/w x_{10} & x_9.

10th approximation is

$$x_{11} = \frac{x_{10} + x_9}{2} = \frac{1.3261 + 1.3242}{2}.$$

$$x_{11} = 1.3251$$

$f(x_{11}) = 0.0016 > 0.$ $\quad$ $\underset{x_{11}}{+}$ ——— $\underset{x_9}{-}$

11th approximation is

$$x_{12} = \frac{x_{11} + x_9}{2} = \frac{1.325 + 1.3242}{2}.$$

$$\boxed{x_{12} = 1.3246}$$

$f(x_{12}) = -0.0005 < 0$. $\overset{(-)}{x_{12}} \longrightarrow \overset{(+)}{x_{11}}$

Root lies b/w x_{12} & x_{11}

12th approximation.

$$x_{13} = \frac{x_{12} + x_{11}}{2} = 1.3248$$

$$x_{13} = 1.3248$$

$f(13) = 0.0003 > 0$.

root lies b/w x_{13} & x_{12}.

13th approximation.

$$x_{14} = \frac{x_{13} + x_{12}}{2}$$

$$x_{14} = 1.3247$$

$f(x_{14}) = -0.0007 < 0$. $\overset{-}{x_{14}} \longrightarrow \overset{+}{x_{13}}$.

14th approximation.

$$x_{15} = \frac{x_{14} + x_{13}}{2} = \frac{1.3247 + 1.3248}{2}$$

$$x_{15} = 1.3247$$

Q. Find real root of equation $x^3 - 6x - 4$ by bisection method.

Let $f(x) = x^3 - 6x - 4 = 0$.

By trial & error method.

$x = 0$, $f(0) = -4 < 0$.

$x = 1$, $f(1) = -9 < 0$.

$x = 2$, $f(2) = -8 < 0$.

$x = 3$, $f(3) = 27 - 18 - 4 = 5 > 0$.

$\therefore$ Root lies b/w 2 & 3.

Let $x_0 = 2$, $x_1 = 3$ $\qquad \overset{-}{x_0}$ ——— $\overset{+}{x_1}$

1st app. is $x_2 = \dfrac{x_0 + x_1}{2} = \dfrac{2+3}{2} = 2.5$.

$f(x_2) = f(2.5) = (2.5)^3 - 6(2.5) - 4 = -3.375 < 0$

root lies b/w x_2 & x_1.

2nd app. is

$$x_3 = \frac{x_2 + x_1}{2} = \frac{2.5 + 3}{2} = \frac{5.5}{2}$$

$$x_3 = 2.75$$

$f(x_3) = f(2.75) = 0.2968 > 0$.

root lies b/w x_2 & x_3 $\qquad \overset{+}{x_3}$ ——— $\overset{-}{x_2}$

3rd app is

$$x_4 = \frac{x_2 + x_3}{2} = \frac{2.5 + 2.75}{2} = 2.625.$$

$$x_4 = 2.625.$$

$f(x_4) = f(2.625) = -1.6621$

root lies b/w x_4 and x_3.

4th app is $x_5 = \frac{x_4 + x_3}{2} = \frac{2.625 + 2.75}{2}$

$x_5 = 2.6875$

root lies b/w x_5 & x_3 $\underset{x_5}{(-)}$ ——— $\underset{x_3}{(+)}$

5th app is $x_6 = \frac{x_5 + x_3}{2} = \frac{2.6875 + 2.75}{2}$

$x_6 = 2.7187$

$f(x_6) = f(2.7187) = -0.2173 < 0$ $\underset{x_6}{-}$ ——— $\underset{x_3}{+}$

root lies b/w x_6 & x_3

6th app. is $x_7 = \frac{x_6 + x_3}{2}$

$x_7 = 2.7343$

$f(x_7) = f(2.7343) = 0.0369 > 0$ $\underset{x_7}{+}$ ——— $\underset{x_6}{-}$

root lies b/w x_7 & x_6

7th app. is $x_8 = \frac{x_7 + x_6}{2} = 2.7265$

$x_8 = 2.7265$

$f(x_8) = f(2.7265) = -0.0907 < 0$ $\underset{x_8}{-}$ ——— $\underset{x_7}{+}$

root lies b/w x_9 & x_7

9th app is $x_{10} = \frac{x_9 + x_7}{2} = 2.7323$

$x_{10} = 2.7323$

$f(x_{10}) = f(2.7323) = 0.004 > 0$ $\underset{x_{10}}{+}$ ——— $\underset{x_9}{-}$

root lies b/w x_{10} & x_9.

10^{th} app is $x_{11} = \frac{x_{10} + x_9}{2} = 2.7323.$

$x_{11} = 2.7323.$

$\therefore$ 2.7323 is root of eqn $x^3 - 6x - 4$.

Q. Find a real root of the equation $x \log_{10} x = 1.2$ by bisection method.

$f(x) = x \log x - 1.2 = 0.$

By trial and error method.

$x = 1 \quad f(x) = 1 \log 1 - 1.2 = -1.2 < 0.$

$x = 2 \quad f(2) = 2 \log 2 - 1.2 = -0.59979 < 0.$

$x = 3 \quad f(3) = 3 \log 3 - 1.2 = 0.23136 > 0$

$\therefore$ root lies b/w 2 and 3.

$x_0 = 2 \quad x_1 = 3.$

first approximation.

$$x_2 = \frac{x_0 + x_1}{2} = \frac{2+3}{2} = \frac{5}{2} = 2.5$$

$\boxed{x_2 = 2.5}$

$f(2.5) = 2.5 \log 2.5 - 1.2 = -0.2051 < 0.$

Root lies b/w x_1 and x_2

2nd approximation

$$x_3 = \frac{x_1 + x_2}{2} = \frac{2.5 + 3}{2} = \frac{5.5}{2} = 2.75$$

$\boxed{x_3 = 2.75}$

$f(2.75) = 2.75 \log 2.75 - 1.2 = 0.008 > 0$ $\quad \frac{+}{x_3} \frac{(-)}{x_2}$

third approximation.

$$x_4 = \frac{x_3 + x_2}{2} = \frac{2.75 + 2.5}{2} = \frac{5.25}{2} = 2.625$$

$$\boxed{x_4 = 2.625}$$

$f(2.625) = f(x_4) = -0.0997 < 0.$ $\quad \frac{-}{x_4} \frac{+}{x_3}$

fourth approximation

$$x_5 = \frac{x_4 + x_3}{2} = \frac{2.75 + 2.625}{2} = \frac{5.375}{2}$$

$$\boxed{x_5 = 2.6875}$$

$f(2.8875) = f(x_5) = -0.0461$ $\quad \frac{-}{x_5} \frac{+}{x_3}$

fifth approximation.

$$x_6 = \frac{x_5 + x_3}{2} = \frac{2.6875 + 2.75}{2} =$$

$$\boxed{x_6 = 2.71875}$$

$\frac{-}{x_6} \frac{+}{x_3}$

$f(x_6) = f(2.7187) = -0.0191$

sixth approximation.

$$x_7 = \frac{x_6 + x_3}{2} = \frac{2.7187 + 2.75}{2} = 2.7343.$$

$$\boxed{x_7 = 2.7343}$$

$\frac{+}{x_3} \frac{-}{x_7}$

$f(x_7) = f(2.7343) = -0.0055$

seventh app.

$$x_8 = \frac{x_7 + x_3}{2} = \frac{2.7343 + 2.75}{2} = 2.7421$$

$x_8 = 2.7421$

$f(x_8) = 0.0012 > 0$ $\quad \frac{-}{x_7} \frac{+}{x_8}$

eigth app.

$$x_9 = \frac{x_7 + x_8}{2} = 2.7382.$$

$f(x_9) = f(2.7382) = -0.00215 < 0$

~~ninth~~ ~~ninth~~ ninth app.

roots lies b/w x_9 & x_8.

$$\underset{x_9}{-}\overline{\qquad\qquad}\underset{x_8}{+}$$

$$x_{10} = \frac{x_9 + x_8}{2} = \frac{2.7382 + 2.7421}{2}$$

$$\boxed{x_{10} = 2.7401}$$

$f(x_{10}) = f(2.7401) = -0.00047 < 0$

$$\underset{x_{10}}{-}\overline{\qquad\qquad}\underset{x_8}{+}$$

10^{th} approxi.

roots lies b/w x_{10} & x_8.

$$x_{11} = \frac{x_{10} + x_8}{2} = \frac{2.7401 + 2.7421}{2}$$

$$\boxed{x_{11} = 2.7411}$$

$f(x_{11}) = f(2.7411) = 0.00039 > 0$

$$\underset{x_{10}}{-}\overline{\qquad\qquad}\underset{x_{11}}{+}$$

12^{th} app.

$$x_{12} = \frac{x_{10} + x_{11}}{2} = \frac{2.7401 + 2.7411}{2}$$

$$\boxed{x_{12} = 2.7406}$$

$f(x_{12}) = f(2.7406) = -0.00004 < 0$

13^{th} app.

$$\underset{x_{12}}{-}\overline{\qquad\qquad}\underset{x_{11}}{+}$$

$$x_{13} = \frac{x_{12} + x_{11}}{2} = \frac{2.7411 + 2.7406}{2}$$

$$= \frac{5.4817}{2} = 2.74085$$

$x_{13} = 2.7408$

$f(x_{13}) = 0.00013 > 0$. $\quad \overset{+}{x_{13}} \text{———} \overset{-}{x_{12}}$

13^{th} app.

$$x_{14} = \frac{x_{13}+x_{12}}{2} = \frac{2.7406 + 2.7408}{2} = \frac{5.4813}{2}$$

$x_{14} = 2.740725$.

$f(x_{14}) = 0.00004 > 0$. $\quad \overset{+}{x_{14}} \text{———} \overset{-}{x_{12}}$

14^{th} app.

$$x_{15} = \frac{2.7407 + 2.7406}{2}$$

$x_{15} = 2.74065$.

$f(x_{15}) = -0.00004 < 0$. $\quad \overset{+}{x_{14}} \text{———} \overset{-}{x_{15}}$

15^{th} app

$$x_{16} = \frac{2.7406 + 2.7407}{2}$$

$= 2.74065$.

$\boxed{x_{15} = x_{16}}$

2.7406 is root of equation.

(5) By using bisection method, find appropriate root of $\sin x = \frac{1}{x}$.

$f(x) = x\sin x - 1 = 0$.

By trial & error method.

$x=1$, $f(1) = 1\sin 1 - 1 = -0.1585 < 0$

$x=2$, $f(2) = 2\sin 2 - 1 = 0.8185 > 0$.

$\therefore$ Root lies b/w 1 & 2.

Let $x_0 = 1$, $x_1 = 2$

First app.

$x_2 = \frac{x_0 + x_1}{2} = \frac{3}{2} = 1.5$

$x_2 = 1.5$

$f(x_2) = f(1.5) = 0.4962 > 0$

root lies b/w x_0 & x_2.

second app

$x_3 = \frac{x_2 + x_0}{2} = \frac{1.5+1}{2} = \frac{2.5}{2} = 1.25$

$x_3 = 1.25$

$f(x_3) = f(1.25) = 0.1862 > 0$

root lies b/w x_3 & x_0.

third app.

$x_4 = \frac{x_3 + x_0}{2} = \frac{1.25+1}{2} = \frac{2.25}{2} = 1.125$

$x_4 = 1.125$

$f(x_4) = f(1.125) = 0.0150 > 0.$

root lies b/w x_4 & x_0. $\quad \overset{+}{x_4} \text{———} \overset{-}{x_0}$

fourth app.

$$x_5 = \frac{x_4 + x_0}{2} = \frac{1.125 + 1}{2} = \frac{2.125}{2} = 1.0625$$

$x_5 = 1.0625.$

$f(x_5) = f(1.0625) = -0.0718 < 0.$

root lies b/w x_5 & x_4 $\quad \overset{-}{x_5} \text{———} \overset{+}{x_4}$

fifth app.

$$x_6 = \frac{x_5 + x_4}{2} = \frac{1.0625 + 1.125}{2} = 1.0937.$$

$f(x_6) = -0.0284 < 0.$ $\quad \overset{-}{x_6} \text{———} \overset{+}{x_4}$

root lies b/w x_6 & x_4.

sixth app.

$$x_7 = \frac{x_6 + x_4}{2} = 1.1093.$$

$x_7 = 1.1093.$

$f(x_7) = -0.0067 < 0.$

root lies b/w x_7 & x_4 $\quad \overset{-}{x_7} \text{———} \overset{+}{x_4}$

seventh app.

$$x_8 = \frac{x_7 + x_4}{2} = 1.1171$$

$x_8 = 1.1171.$

$f(x_8) = 0.0043 > 0$ $\quad \overset{+}{x_8} \text{———} \overset{-}{x_7}$

root lies b/w x_8 & x_7.

eight app.

$$x_9 = \frac{x_8 + x_7}{2} = 1.1132 \qquad x_9 = 1.1132$$

$f(x_9) = -0.0013 < 0$.

root lies b/w x_9 & x_8.

9^{th} app.

$$x_{10} = \frac{x_9 + x_8}{2} = 1.1151$$

$$x_{10} = 1.1151$$

$f(x_{10}) = 0.0013 > 0$. $\overset{+}{x_{10}} \quad \overset{-}{x_9}$

10^{th} app.

$$x_{11} = \frac{x_{10} + x_9}{2} = 1.1141$$

$$x_{11} = 1.1141$$

$f(x_{11}) = -0.00007 < 0$.

root lies b/w x_{11} & x_{10} $\overset{-}{x_{11}} \quad \overset{+}{x_{10}}$

11^{th} app.

$$x_{12} = \frac{x_{11} + x_{10}}{2} = 1.1146.$$

$$x_{12} = 1.1146.$$

$f(x_{12}) = 0.0006 > 0$.

root lies b/w x_{12} & x_{11}. $\overset{+}{x_{12}} \quad \overset{-}{x_{11}}$

12^{th} app.

$$x_{13} = \frac{x_{12} + x_{11}}{2} = 1.1143.$$

$$\boxed{x_{13} = 1.1143}$$

$f(x_{13}) = 0.0001 > 0$ $\quad \frac{+ \qquad -}{x_{13} \qquad x_{11}}$

root lies b/w x_{13} & x_{11}.

13th app.

$$x_{14} = \frac{x_{13} + x_{11}}{2} = 1.1142.$$

$$x_{14} = 1.1142.$$

$f(x_{14}) = 0.00005 > 0.$ $\quad \frac{+ \qquad -}{x_{14} \qquad x_{11}}$

root lies b/w x_{14} & x_{11}.

14th app.

$$x_{15} = \frac{x_{14} + x_{11}}{2} = 1.1141$$

$$\boxed{x_{15} = 1.1141}$$

$f(x_{15}) = -0.00007 < 0$ $\quad \frac{- \qquad +}{x_{15} \qquad x_{14}}$

root lies b/w x_{15} & x_{14}.

15th app.

$$x_{16} = \frac{x_{15} + x_{14}}{2} = 1.1141$$

$$\boxed{x_{16} = 1.1141}$$

Regular - falsi method :- (false position method)

① Let $f(x) = 0$ be the given equation

Find a and b such that $f(a) < 0$ & $f(b) > 0$.

or $f(a) \cdot f(b) < 0$.

Choose $x_0 = a$ & $x_1 = b$.

Root lies b/w x_0 and x_1.

② First approximation is

$$x_2 = \frac{x_0 f(x_1) - x_1 [f(x_0)]}{f(x_1) - f(x_0)}$$

ⓐ If $f(x_2) < 0$ then

second approximation $x_3 = \dfrac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)}$

ⓑ If $f(x_2) > 0$, second approximation is

$$x_3 = \frac{x_0 f(x_2) - x_2 f(x_0)}{f(x_2) - f(x_0)}$$

continue the above procedure until getting accurate value

Problem

① Find root of equation $x\log_{10}x = 1.2$ by using false position method.

sol Given $f(x) = x\log x - 1.2 = 0$.

At $x=1$, $f(1) = -1.2 < 0$.

$x=2$, $f(2) = 2\log 2 - 1.2 = -0.5979 < 0$.

$x=3$, $f(3) = 3\log 3 - 1.2 = 0.2313 > 0$.

root lies b/w 2 & 3.

choose $x_0 = 2$ $x_1 = 3$.

first approximation is $x_2 = \dfrac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} = \dfrac{2f(3) - 3f(2)}{f(3) - f(2)}$

$$x_2 = \frac{2(0.2313) - 3(-0.5979)}{0.2313 + 0.5979}$$

$\boxed{x_2 = 2.7210}$

$f(x_2) = 2.7210 \log 2.7210 - 1.2$

$= -0.0171$.

root lies b/w x_2 and x_1.

second app is $x_3 = \dfrac{x_1 \cdot f(x_2) - x_2 \cdot f(x_1)}{f(x_2) - f(x_1)}$

$$x_3 = \frac{3f(2.7210) - (2.7210)f(3)}{f(2.7210) - f(3)} = \frac{3(-0.0171) - (2.7210)(0.2313)}{-0.0171 - 0.2313}$$

$\boxed{x_3 = 2.7402}$

$f(x_3) = 2.7402 \log(2.7402) - 1.2$

$= -0.0003 < 0$.

root lies b/w x_3 and x_1

third app $x_4 = \dfrac{x_1 (f(x_3) - x_3 \cdot f(x_1))}{f(x_3) - f(x_1)}$

$$x_4 = \frac{3(-0.0003) - 2.7402(0.2313)}{-0.0003 - 0.2313}$$

$\boxed{x_4 = 2.7405}$

$f(x_4) = -0.0001 < 0$ $\quad \underset{x_4}{(-)}$ —— $\underset{x_1}{(+)}$

Root lies b/w x_4 & x_1

4th approximation is:

$$x_5 = \frac{x_1 f(x_4) - x_4 f(x_1)}{f(x_4) - f(x_1)}$$

$$= \frac{3(-0.0001) - 2.7405(0.2313)}{-0.0001 - 0.2313}$$

$\boxed{x_5 = 2.7406}$

$f(x_5) = -0.00004 < 0$ $\quad \underset{x_5}{(-)}$ —— $\underset{x_1}{(+)}$

Root lies b/w x_5 & x_1

5th approximation

$$x_6 = \frac{x_1 f(x_5) - x_5 f(x_1)}{f(x_5) - f(x_1)}$$

$$= \frac{3(-0.00004) - (2.7406)(0.2313)}{-0.00004 - 0.2313}$$

$\boxed{x_6 = 2.7406.}$

Since $x_5 = x_6 = 2.7406$.

2.7406 is root of given functions

$x \log x = 1.2$.

② Find the root of the equation $x \cdot e^x = 2$ using false position method.

Given $f(x) = x \cdot e^x - 2 = 0$.

At $x = 1$ $f(1) = +0.7182$. — + (x_0, x_1)

At $x = 0$ $f(0) = -2$.

root lies b/w 0 & 1. $f(x_0) = -2$.

choose $x_0 = 0$ & $x_1 = 1$ $f(x_1) = 0.718$

first app $$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

$$x_2 = \frac{0(0.7182) - 1.(-2)}{0.7182 + 2} = \frac{2}{2.7182}$$

$\boxed{x_2 = 0.7897}$ $\boxed{x_2 = 0.7357}$

$f(x_2) = -0.4646 < 0$. — + (x_2, x_1)

root lies b/w x_2 & x_1.

second app:

$$x_3 = \frac{x_2 \cdot f(x_1) - x_1 f(x_2)}{f(x_1) - f(x_2)}$$

$\boxed{x_3 = 0.8395}$ — + (x_3, x_1)

$f(x_3) = -0.0563$

root lies b/w x_3 & x_1.

third app

$$x_4 = \frac{x_3 \cdot f(x_1) - x_1 f(x_3)}{f(x_1) - f(x_3)}$$

$\boxed{x_4 = 0.8511}$

$f(x_4) = -0.0065 < 0$

$-$ $+$
x_4 x_1

root lies b/w x_4 & x_1

fourth app.

$$x_5 = \frac{x_4 \cdot f(x_1) - x_1 \cdot f(x_4)}{f(x_1) - f(x_4)}$$

$$\boxed{x_5 = 0.8524}$$

$f(x_5) = -0.0008$

x_5 x_1

root lies b/w x_5 & x_1

5th app.

$$x_6 = \frac{x_5 \cdot f(x_1) - x_1 \cdot f(x_5)}{f(x_1) - f(x_5)}$$

$$\boxed{x_6 = 0.8525}$$

$f(x_6) = -0.0004 < 0$

$-$ $+$
x_6 x_1

root lies b/w x_6 & x_1

6th app.

$$x_7 = \frac{x_6 \cdot f(x_1) - x_1 f(x_6)}{f(x_1) - f(x_6)}$$

$$\boxed{x_7 = 0.8525}$$

Q) find the root of the equation $x^3-2x-5=0$ using false position method.

Given $f(x) = x^3-2x-5=0$.

At $x=0$ $f(0) = -5<0$.

At $x=1$ $f(1) = 1-2-5 = -6<0$

x_0 At $x=2$ $f(2) = 8-4-5 = -1<0$.

x_1 At $x=3$ $f(3) = 27-6-5 = \boxed{16} > 0$.

root lies b/w 2 & 3.

-ve ——— +ve
$2\ x_0$ ——— $3\ x_1$

choose $\boxed{x_0 = 2}$ and $\boxed{x_1 = 3}$.

first approximation

$$x_2 = \frac{x_0 \cdot f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} = 2.0588.$$

$$\boxed{x_2 = 2.0588}$$

$f(x_2) = -0.3910$.

− ——— +
x_2 ——— x_1

second approximation.

$$x_3 = \frac{x_2 \cdot f(x_1) - x_1 \cdot f(x_2)}{f(x_1) - f(x_2)}$$

$$\boxed{x_3 = 2.0812}$$

$f(x_3) = -0.1479$

− ——— +
x_3 ——— x_1

third approximation.

root lies b/w x_3 & x_1

$$x_4 = \frac{x_3 \cdot f(x_1) - x_1 f(x_3)}{f(x_1) - f(x_3)}$$

$$\boxed{x_4 = 2.0896}$$

$f(x_4) = -0.0551$.

− ——— +
x_4 ——— x_1

fourth approximation.

$$x_5 = \frac{x_4 \cdot f(x_1) - x_1 f(x_4)}{f(x_1) - f(x_4)}$$

$$\boxed{x_5 = 2.0927}$$

$f(x_5) = -0.0206.$

fifth approximation.

root lies b/w x_5 & x_1

$$\begin{array}{ccc} - & & + \\ \hline x_5 & & x_1 \end{array}$$

$$x_6 = \frac{x_5 f(x_1) - x_1 f(x_5)}{f(x_1) - f(x_5)}$$

$$\boxed{x_6 = 2.0938}$$

$$\begin{array}{ccc} - & & + \\ \hline x_6 & & x_1 \end{array}$$

$f(x_6) = -0.0083.$

sixth approximation.

root lies b/w x_6 & x_1

$$x_7 = \frac{x_6 \cdot f(x_1) - x_1 f(x_6)}{f(x_1) - f(x_6)}$$

$$\boxed{x_7 = 2.0942}$$

$f(x_7) = -0.0039.$

seventh approximation.

root lies b/w x_7 & x_1

$$\begin{array}{ccc} - & & + \\ \hline x_7 & & x_1 \end{array}$$

$$x_8 = \frac{x_7 \cdot f(x_1) - x_1 f(x_7)}{f(x_1) - f(x_7)}$$

$$\boxed{x_8 = 2.0944}$$

$f(x_8) = -0.0016.$

root lies b/w x_8 & x_1

eigth approximation, x_9

$$\begin{array}{ccc} - & & + \\ \hline & & x_1 \end{array}$$

$$x_9 = \frac{x_8 \cdot f(x_1) - x_1 f(x_8)}{f(x_1) - f(x_8)}$$

$$\boxed{x_9 = 2.0944.}$$

$x_8 = x_9 = 2.0944.$

i.e root is 2.0944.

⑤ Find the root of equation: $x^4 - x - 10 = 0$ using false position method.

Given $f(x) = x^4 - x - 10 = 0$.

At $x = 0$ $\quad f(0) = -10 < 0$

$f(x_0)$ At $x = 1$ $\quad f(1) = 1 - 1 - 10 = -10 < 0.$

$f(x_1)$ At $x = 2$ $\quad f(2) = 16 - 2 - 10 = 4 > 0$

root lies b/w 1 & 2

choose $\boxed{x_0 = 1}$ and $\boxed{x_1 = 2}$.

$\underset{x_0}{-} \quad \underset{x_1}{+}$

first approximation.

root lies b/w x_0 & x_1

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}.$$

$$\boxed{x_2 = 1.714}$$

$f(x_2) = -3.0833.$

$\underset{x_2}{-} \quad \underset{x_1}{+}$

Second approximation.

root lies b/w x_2 & x_1

$$x_3 = \frac{x_2 f(x_1) - x_1 f(x_2)}{f(x_1) - f(x_2)}.$$

$$\boxed{x_3 = 1.8384}$$

$f(x_3) = -0.0459.$

third approximation $\quad \underset{x_3}{-} \quad \underset{x_1}{+}$

root lies b/w x_3 & x_1

$$x_4 = \frac{x_3 \cdot f(x_1) - x_1 f(x_3)}{f(x_1) - f(x_3)}$$

$$\boxed{x_4 = 1.8402.}$$

$f(x_4) = -0.3729$ $\quad \underset{x_4}{-} \quad \underset{x_1}{+}$

4th approximation.

$$x_5 = \frac{x_4 \cdot f(x_1) - x_1 \cdot f(x_4)}{f(x_1) - f(x_4)}$$

$\boxed{x_5 = 1.8539}$

$f(x_5) = 0.0413$

$\underset{x_5}{-} \overline{\qquad\qquad} \underset{x_1}{+}$

fifth approximation.

root lies b/w x_5 & x_1

$$x_6 = \frac{x_5 \cdot f(x_1) - x_1 \cdot f(x_5)}{f(x_1) - f(x_5)}$$

$$\boxed{x_6 = 1.8553.}$$

$f(x_6) = -0.0069$

Sixth approximation.

$\underset{x_6}{-} \overline{\qquad\qquad} \underset{x_1}{+}$

root lies b/w x_6 & x_1

$$x_7 = \frac{x_6 \cdot f(x_1) - x_1 f(x_6)}{f(x_1) - f(x_6)}$$

$$\boxed{x_7 = 1.8555}.$$

$f(x_7) = -0.0020.$

Seventh approximation.

$\underset{x_7}{-} \overline{\qquad\qquad} \underset{x_1}{+}$

root lies b/w x_7 & x_1

$$x_8 = \frac{x_7 \cdot f(x_1) - x_1 f(x_7)}{f(x_1) - f(x_7)}$$

$$\boxed{x_8 = 1.85557}$$

(3) $e^x \sin x = 1$

$f(x) = e^x \sin x - 1 = 0.$

at $x=0$, $f(0) = e^0 \sin 0 - 1 = -1 < 0 = f(x_0)$

at $x=1$, $f(1) = e^1 \sin 1 - 1 = 1.2873 > 0 = f(x_1)$

root lies b/w 0 & 1.

choose $x_0 = 0$, $x_1 = 1$.

1st app is $x_2 = \dfrac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$

$= \dfrac{0 f(1) - 1 f(0)}{f(1) - f(0)}$

$\boxed{x_2 = 0.4371.}$

$f(x_2) = -0.3446 < 0.$

root lies b/w x_2 & x_1

2nd app is $x_3 = \dfrac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)}$

$\boxed{x_3 = 0.5559.}$

$f(x_3) = -0.0799 < 0$ $\quad \underset{x_3}{(-)} \text{———} \underset{x_1}{(+)}$

root lies b/w x_3 & x_1.

3rd app is $x_4 = \dfrac{x_1 f(x_3) - x_3 f(x_1)}{f(x_3) - f(x_1)}$

$\boxed{x_4 = 0.5818.}$

$f(x_4) = -0.0167 < 0$.

root lies b/w x_4 & x_1

4^{th} app is $x_5 = \dfrac{x_1 f(x_4) - x_4 f(x_1)}{f(x_4) - f(x_1)}$

$\boxed{x_5 = 0.5871}$

$f(x_5) = -0.0035 < 0$.

root lies b/w x_5 & x_1.

5^{th} app.

$x_6 = \dfrac{x_1 f(x_5) - x_5 f(x_1)}{f(x_5) - f(x_1)}$

$\boxed{x_6 = 0.5882}$

$f(x_6) = -0.0008 < 0$.

root lies b/w x_6 & x_1.

6^{th} app.

$x_7 = \dfrac{x_1 f(x_6) - x_6 f(x_1)}{f(x_6) - f(x_1)}$

$\boxed{x_7 = 0.5884}$

$f(x_7) = -0.0003 < 0$.

7^{th} app.

$x_8 = \dfrac{x_1 f(x_8) - x_7 f(x_1)}{f(x_8) - f(x_1)}$ $\boxed{x_8 = 0.5884}$

Iteration method:-

1. set $f(x) = 0$.

2. $x = a$, $x = b$.

$$\left.\begin{array}{l} f(a) < 0 \\ f(b) > 0 \end{array}\right\}.$$

$x_0 = a$ or $x_0 = b$.

or $x_0 = \frac{a+b}{2}$.

1. let $f(x) = 0$ be the equation.
2. Find a & b such that $f(a) < 0$ & $f(b) > 0$.
   or $f(a)\,f(b) < 0$.
3. choose $x_0 = \frac{a+b}{2}$.
4. Rewrite $f(x) = 0$ as $x = \phi(x)$ $\forall$ $a \le x \le b$.
   $a \le \phi(x) \le b$.
5. If $|\phi'(x)|$ then equation can be used as iteration formula.
   i.e. $x_i = \phi(x_{i-1})$

Problems

① Find positive root of $x^4 - x - 10 = 0$ by iteration

let $f(x) = x^4 - x - 10 = 0$

$x = 0$ $f(0) = -10 < 0$.

$x = 1$ $f(1) = -10 < 0$

$x = 2$ $f(2) = 4 > 0$.

Root lies b/w 1 & 2.

$x_0 = 1.5 \left(\frac{1+2}{2}\right)$

write $f(x) = 0$ as $x = \phi(x)$

$$x^4 - x - 10 = 0.$$

$$x^4 = x + 10$$

$$x = (x+10)^{1/4}.$$

Here $\phi(x) = (x+10)^{1/4}$.

$$\phi'(x) = \frac{1}{4}(x+10)^{-3/4}.$$

$$\phi'(x) = \frac{1}{4(x+10)^{3/4}}.$$

$$|\phi'(x)| = \frac{1}{4(x+10)^{3/4}}.$$

$$|\phi'(x)|_{x=1} = \frac{1}{4(1+10)^{3/4}}.$$

$$= 0.041 < 1.$$

$$|\phi'(x)|_{x=1} < 1$$

$$|\phi'(x)|_{x=2} = \frac{1}{4(2+10)^{3/4}}$$

$$\doteq 0.038 < 1$$

$$|\phi'(x)|_{x=2} < 1$$

$$|\phi'(x)| < 1 \ \forall \ 1 \leq x < 2.$$

By iteration method

$$x_i = \phi(x_{i-1})$$

$$x_1 = \phi(x_0)$$

$$= \phi(1.5)$$

$$= (1.5+10)^{1/4}$$

$$\boxed{x_1 = 1.8415}$$

$$x_2 = \phi(x_1)$$

$$= \phi(1.8415)$$

$$= (1.8415+10)^{1/4} = 1.8550.$$

$x_3 = \phi(x_2)$

$= \phi(1.8550)$

$= (1.8550+10)^{1/4}$.

$\boxed{x_3 = 1.8555}$

$x_4 = \phi(x_3)$

$= \phi(1.8555)$

$= (1.8555+10)^{1/4}$

$\boxed{x_4 = 1.8555.}$

$\therefore x_3 = x_4$ are same

Hence 1.8555 is the root of function

$x^4 - x - 10 = 0$.

② Find the positive root of $x^3 - 2x - 5 = 0$ by iteration.

$x=0, \; f(0) = -5 < 0$.

$x=1, \; f(1) = -6 < 0$.

$x=2, \; f(2) = -1 < 0$.

$x=3, \; f(3) = 16 > 0$.

Root lies b/w 2 & 3.

$x_0 = \frac{2+3}{2} = \frac{5}{2} = 2.5$

write $f(x) = 0$ as $x = \phi(x)$

$x^3 - 2x - 5 = 0$.

$x^3 = 2x+5$.

$x = (2x+5)^{1/3}$.

Here, $\phi(x) = (2x+5)^{1/3}$.

$\phi'(x) = \frac{1}{3}(2x+5)^{1/3-1} = \frac{1}{3}(2x+5)^{-2/3}$

$$|\phi'(x)| = \frac{1}{3(2x+5)^{2/3}}$$

$$|\phi'(x)|_{x=2} = \frac{1}{3[2(2)+5]^{2/3}} = \frac{1}{3(4+5)^{2/3}} = \frac{1}{3(9)^{2/3}}$$

$$|\phi'(x)|_{x=2} = \frac{1}{3(4.3267)} = 0.0770 < 2.$$

$$|\phi'(x)|_{x=3} = \frac{1}{3[2(3)+5]^{2/3}} = \frac{1}{3(11)^{2/3}} = \frac{1}{14.8382}.$$

$$|\phi'(x)|_{x=3} = 0.0673 < 2,$$

$$|\phi'(x)| < 2 \ \forall \ 2 < x < 3.$$

By iteration:

$$x_i = \phi(x_{i-1})$$

$$x_1 = \phi(x_{i-1})$$

$$x_1 = \phi(2.5)$$

$$= (2 \times 2.5 + 5)^{1/3}.$$

$$\boxed{x_1 = 2.1544}$$

$$x_2 = \phi(x_1)$$

$$= \phi(2.1544)$$

$$= (2 \times 2.1544 + 5)^{1/3}.$$

$$\boxed{x_2 = 2.1036.}$$

$$x_3 = \phi(x_2)$$

$$= 2(2.1036)$$

$$= (2 \times 2.1036)^{1/3}$$

$$\boxed{x_3 = 2.0959.}$$

$x_4 = \phi(x_3)$

$= \phi(2.0959)$

$\boxed{x_4 = 2.0947}$

$x_5 = \phi(x_4)$

$x_5 = \phi(2.0947)$

$\boxed{x_5 = 2.0945}$

$x_6 = \phi(x_5)$

$x_6 = \phi(2.0945)$

$\boxed{x_6 = 2.0945}$

(3) Evaluate $\sqrt{12}$ by iteration method.

$x = \sqrt{12}$.

$x^2 = 12$.

$f(x) = x^2 - 12 = 0$.

$x = 0$

Root lies b/w 3 & 4

$x = 0 \quad f(0) = -12 < 0$

$x = 1 \quad f(1) = 1 - 12 = -11 < 0$.

$x = 2 \quad f(2) = 4 - 12 = -8 < 0$

$x = 3 \quad f(3) = 9 - 12 = -3 < 0$.

$x = 4 \quad f(4) = 16 - 12 = 4 > 0$.

choose $x_0 = 3.5$.

$x^2 - 12 = 0$.

$x^2 = 12 \Rightarrow x \cdot x = 12$.

$x = \frac{12}{x} \Rightarrow x = \phi(x)$

where $\phi(x) = \frac{12}{x}$.

$\phi'(x) = \frac{-12}{x^2}$.

$|\phi'(x)| = \frac{12}{x^2}$.

at $x = 4$, $|\phi'(x)| = \frac{12}{16} = 0.75 < 1$.

$x_i = \phi(x_{i-1})$

$x_1 = \phi(x_0)$

$= \phi(3.5)$

$x_1 = \frac{12}{3.5} = 3.4285$.

$x_2 = \phi(3.4285) = \frac{12}{3.4285} = 3.5000$.

$x_3 = \phi(3.5000)$

$= 3.4285$.

(4) Find a +ve root of $3x = \cos x + 1$ by iteration

$f(x) = 3x - \cos x - 1 = 0$.

$x = 0$, $f(0) = -2 < 0$.

$x = 1$, $f(1) = 3 - \cos 1 - 1 = 1.4596 > 0$.

root lies b/w 0 & 1

choose $x_0 = \frac{0+1}{2} = 0.5$

$3x - \cos x - 1 = 0$.

$3x = \cos x + 1$

$\phi(x) = \frac{1}{3}(\cos x + 1)$

$\phi'(x) = \frac{1}{3}(-\sin x)$

$|\phi'(x)| = \frac{1}{3}\sin x$

$|\phi'(x)|$

at $x = 0 \Rightarrow \frac{1}{3}(0) = 0 < 1$.

at $x = 1$

$|\phi'(x)| = \frac{1}{3}\sin 1$.

$= 0.2804 < 1$.

$x_i = \phi(x_{i-1})$

$x_1 = \phi(x_0)$

$= \phi(0.5)$

$= \frac{1}{3}(\cos(0.5)+1)$

$x_1 = 0.6258$.

$x_2 = \phi(x_1)$

$= \phi(0.6258)$

$\boxed{x_2 = 0.6034}$

$x_3 = \phi(x_2) = \phi(0.6034)$

$x_3 = \frac{1}{3}(\cos(0.6034))$

$\boxed{x_3 = 0.6078}$

$x_4 = \frac{1}{3}(\cos(0.6078)+1)$

$= 0.6069$.

$x_5 = \frac{1}{3}(\cos(0.6069)+1)$

$\boxed{x_5 = 0.6071}$

$x_6 = \frac{1}{3}(\cos(0.6071)+1)$

$\boxed{x_6 = 0.6071}$

$\therefore$ The root of given function $3x = \cos x + 1$ is 0.6071 since $x_5 = x_6 = 0.607$

Newton's raphson method

Let $f(x) = 0$.

1. choose initial root x_0.
2. First approximation by newton's raphson is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Second approximation is

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

Repeat above procedure until getting the accurate root upto 4 decimal places.

1. Using Newton - Raphson method, find the root of equation $f(x) = e^x - 3x$.

$$f(x) = e^x - 3x = 0.$$

$x = 0$ $\quad f(0) = e^0 - 0 = 1$

$x = 1$ $\quad f(1) = e^1 - 3 = -0.2817 < 0.$

Root lies b/w 0 & 1.

choose $x_0 = \frac{1}{2} = 0.5.$

$f(x) = e^x - 3x.$

$f'(x) = e^x - 3.$

first approximaton.

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0.5 - \frac{f(0.5)}{f'(0.5)}$$

$x_1 = 0.5 + 0.1100.$

$\boxed{x_1 = 0.6100.}$

2nd app.

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 0.6100 - \frac{f(0.6100)}{f'(0.6100)}$$

$$= 0.6100 - \left(\frac{e^{0.6100} - 3(0.6100)}{e^{0.6100} - 3}\right)$$

$\boxed{x_2 = 0.6189}$

3rd approximation.

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 0.6189 - \frac{f(0.6189)}{f'(0.6189)}$$

$\boxed{x_3 = 0.6190}$

4th approximation

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$$

$$= 0.6190 - \frac{f(0.6190)}{f'(0.6190)}$$

$\boxed{x_4 = 0.6190}$

0.6190 is root of $e^x - 3x$.

since $x_3 = x_4 = 0.6190.$

(2) Using newton - Raphson method find the root of equation $f(x) = x\sin x + \cos x = 0$.

$$f(x) = x\sin x + \cos x = 0$$

$x = 0 \quad f(0) = 1 > 0$,

$x = 1 \quad f(1) = 1.3817 > 0$.

$x = 2 \quad f(2) = 1.4024 > 0$.

$x = 3 \quad f(3) = -0.5666 < 0$.

Root lies b/w 2 & 3.

choose $x_0 = \frac{5}{2} = 2.5$.

$$f(x) = x\sin x + \cos x = 0.$$

$$f'(x) = x.\cos x + \sin x - \sin x$$

first approximation -

choose $x_0 = 2.5$.

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 2.5 - \frac{2.5\sin(2.5) + \cos(2.5)}{2.5\cos(2.5) + \sin(2.5) - \sin(2.5)}$$

$x_1 = 2.7671$.

2^{nd} app:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$x_2 = 2.7671 - \frac{0.0815}{-2.9411}$$

$$x_2 = 2.7671 + 0.0277.$$

$$\boxed{x_2 = 2.7948}$$

3^{rd} app.

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 2.7948 + 0.0031$$

$$\boxed{x_3 = 2.7979}.$$

4^{th} app:

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$$

$$= 2.7979 + 0.0004$$

$$\boxed{x_4 = 2.7983}$$

5^{th} app:

$$x_5 = x_4 - \frac{f(x_4)}{f'(x_4)}$$

$$= 2.7983 + 0.00006.$$

$$x_5 = 2.79836.$$

$\therefore$ 2.7983 is root of $x \sin x + \cos x = 0.$

since $x_4 = x_5 = 2.7983$

(5) Find root of $e^x \sin x = 1$ using NRM

$f(x) = e^x \sin x - 1 = 0$.

$x = 0 \Rightarrow f(0) = -1 < 0$

$x = 1 \Rightarrow f(1) = 1.2873 > 0$

root lies b/w 0 & 1.

$x_0 = \frac{0+1}{2} = 0.5$

$f'(x) = e^x \sin x - 1$

$= e^x \sin x + e^x \cos x$.

1st approx.

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0.5 - \frac{e^{(0.5)} \sin(0.5) - 1}{e^{(0.5)} \sin(0.5) + e^{(0.5)} \cos(0.5)}$$

$= 0.5936$.

2nd approx.

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.5885$$

3rd approx

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.5885$$

$\therefore$ 0.5885 is root of $e^x \sin x - 1 = 0$.

⑥ Using Newton Rhapson method find ⓐ square root of a number. ⓑ Find reciprocal of a number.

Sol

ⓐ N be the number whose square root is to be found

$x = \sqrt{N}$.

$x^2 - N = 0$. (or) $f(x_i) = x_i^2 - N$.

By Newton's Rhapson method

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

$$f'(x_i) = 2x_i.$$

$$x_{i+1} = x_i - \left[\frac{x_i^2 - N}{2x_i}\right]$$

$$= \frac{2x_i^2 - x_i^2 + N}{2x_i}$$

$$= \frac{x_i^2 + N}{2x_i}$$

$$\boxed{x_{i+1} = \frac{x_i}{2} + \frac{N}{2x_i}}$$

ⓑ $x = \frac{1}{N}$.

$\frac{1}{x} = N$.

$f(x) = \frac{1}{x} - N = 0$.

$f(x_i) = \frac{1}{x_i} - N = 0$.

$f'(x_i) = \frac{-1}{x_i^2}$.

By Newton - Raphson method.

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

$$= x_i - \frac{\frac{1}{x_i} - N}{\frac{-1}{x_i^2}}$$

$$= x_i + x_i^2\left(\frac{1}{x_i} - N\right)$$

$$= x_i + \frac{x_i^{\cancel{2}}}{\cancel{x_i}} - N x_i^2$$

$$= 2x_i - N x_i^2.$$

$$\boxed{x_{i+1} = x_i\left[2 - N x_i\right]}$$

b) Using Newton's Rhapson method.

① find square root of 10.

② Find a reciprocal of 22.

Sol ① square root of 10.

$f(x) = x^2 - 10 = 0.$ Here $N = 10.$

Root lies b/w 3 & 4.

$x_0 = 3$

By newton's rhapson method.

$$x_{i+1} = \frac{x_i}{2} + \frac{N}{2x_i} = \frac{1}{2}\left[x_i + \frac{N}{x_i}\right]$$

first approximation.

$$x_1 = \frac{1}{2}\left(3 + \frac{10}{3}\right) = 3.1666.$$

second approximation.

$$x_2 = \frac{1}{2}\left[3.1666 + \frac{10}{3.1666}\right] = \underline{3.1622.}$$

third approximation.

$$x_3 = \frac{1}{2}\left[3.1622 + \frac{10}{3.1622}\right] = \underline{3.1622.}$$

Hence square root of 10 = 3.1622.

② Reciprocal :-

$x = \frac{1}{29}$.

or $f(x) = \frac{1}{x} - 22 = 0$.

0.04545

choose $x_0 = 0.04$.

By newton rhapson method.

$$x_{i+1} = x_i\,[2 - N x_i]$$

$$x_1 = x_0\,[2 - N x_0]$$

$$= 0.04\,[2 - 22(0.04)]$$

$$= 0.04\,[2 - 0.88]$$

$$\boxed{x_1 = 0.0448}$$

$$x_2 = x_1\,[2 - N x_1]$$

$$= 0.0448\,[2 - 22(0.0448)]$$

$$\boxed{x_2 = 0.04544}$$

$$x_3 = x_2\,[2 - N x_2]$$

$$= 0.0454\,[2 - N(0.0454)]$$

$$\boxed{x_3 = 0.0454}$$

0.0454 is the value of reciprocal of 22.

④ Using newton's rhapson method.

① Using find a root of 24.

② find reciprocal of 18.

① square root of 24.

$f(x) = x^2 - 24 = 0$.

$N = 24$.

root lies b/w 4 & 5.

$x_0 = 5$.

By newton's rhapson method.

$x_{i+1} = \frac{1}{2}\left[x_i + \frac{N}{x_i}\right]$

$x_1 = \frac{1}{2}\left(x_0 + \frac{N}{x_0}\right) = \frac{1}{2}\left[5 + \frac{24}{5}\right]$

$\boxed{x_1 = 4.9}$

$x_2 = \frac{1}{2}\left(x_1 + \frac{N}{x_1}\right) = \frac{1}{2}\left[4.9 + \frac{24}{4.9}\right]$

$\boxed{x_2 = 4.8989}$

$x_3 = \frac{1}{2}\left[x_2 + \frac{N}{x_2}\right] = \frac{1}{2}\left[4.8989 + \frac{24}{4.8989}\right]$

$\boxed{x_3 = 4.8989}$

Hence square root of 24 = 4.8989.

② Reciprocal

$x = \frac{1}{18}$

α $f(x) = \frac{1}{x} - 18 = 0$

choose $x_0 = 0.05$

By newton Rhapson method:

$x_{i+1} = x_i [2 - N x_i]$

$x_{0+1} = x_0 [2 - N x_0]$

$x_1 = 0.05 [2 - 18(0.05)]$

$= 0.055$

$x_2 = 0.055 (2 - 18 \times 0.055)$

$x_2 = 0.0555$

$x_3 = 0.0555 (2 - 18 \times 0.0555)$

$x_3 = 0.0555$

0.0555 is reciprocal of 18.

Numerical integration:-

1) Trapezoidal rule.

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + y_3 - - -)\right]$$

Here $h = \frac{b-a}{n}$.

$x_1 = x_0 + h \qquad x_2 = x_1 + h \qquad x_3 = x_2 + h$

or $x_0 + 2h \qquad x_0 + 3h$.

2) Simpson's $1/3^{rd}$ rule:-

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[(y_0 + y_n) + 2(y_2 + y_4 + y_6 - - -) + 4(y_1 + y_3 + y_5 + - - -)\right]$$

3) Simpson's $3/8^{th}$ rule:-

$$\int_a^b f(x)\,dx = \frac{3h}{8}\left[(y_0 + y_n) + 2(y_3 + y_6 + y_9 + - - -) + 4(y_1 + y_2 + y_4 + y_5 + y_7 - - -)\right]$$

Note:-

Simpson's $1/3^{rd}$ rule is used only when n is multiple of 2.

Simpson's $3/8^{th}$ rule is used only when n is multiple of 3.

Problems:

① Evaluate $\int_0^1 x^3 dx$ with five sub intervals by trapezoidal rule.

given $n = 5, b = 1, a = 0$.

$h = \frac{b-a}{n} = \frac{1-0}{5} = \frac{1}{5}$.

$x_0 = 0$ $\quad$ $y_0 = x_0^3 = 0$.

$x_1 = x_0 + h$ $\quad$ $y_1 = x_1^3 = \frac{1}{(5)^3} = \frac{1}{125} = 0.008$

$= 0 + 1/5$

$x_2 = x_1 + h$ $\quad$ $y_2 = x_2^3 = \left(\frac{2}{5}\right)^3 = 0.064$.

$= \frac{1}{5} + \frac{1}{5}$

$= 2/5$,

$x_3 = x_2 + h$ $\quad$ $y_3 = (3/5)^2 = 0.216$

$= 3/5$

$x_4 = x_3 + h$ $\quad$ $y_4 = (4/5)^3 = 0.512$.

$= 4/5$.

$x_5 = x_4 + h$. $\quad$ $y_4 = (5/5)^3 = 1$.

$= 1$

Trapezoidal rule:

$\int_0^1 x^3 dx = \frac{h}{2}\left[(y_0 + y_5) + 2(y_1 + y_2 + y_3 + \cdots)\right]$

$= \frac{1}{10}\left[1 + 2(0.008 + 0.064 + 0.216 + 0.512)\right]$

$= 0.26$.

② Evaluate $\int_a^b \frac{1}{1+x} dx$ using ① Simpson's 1/3 rule.
② Simpsons 3/8 rule

→ compare result with initial value.

Simpson's 1/3 rule.

$$\int_a^b = f(x)\, dx = \frac{h}{3}\left((y_0+y_n) + 2(y_2+y_4+y_6+---) + 4(y_1+y_3+y_5+---)\right)$$

$$h = \frac{b-a}{h} = \frac{6-0}{6} = 1$$

$x_a = a = 0 \qquad x_n = b = 6.$

$x_0 = 0 \qquad y_0 = \frac{1}{1+x_0}.$

$y_0 = 1$

$y_1 = 0.5$

$x_1 = 1$

$x_2 = x_1 + h. \qquad y_2 = 1/3 = 0.333$

$= 1+1 \qquad y_3 = 1/4 = 0.25$

$= 2$

$x_3 = x_2 + h$

$= 3$

$x_4 = x_3 + h. \qquad y_4 = 1/5 = 0.2.$

$= 4.$

$x_5 = x_4 + h \qquad y_5 = 1/6 = 0.1666$

$= 5$

$x_6 = x_5 + h. \qquad y_6 = \frac{1}{7} = 0.1428.$

$= 6$

Simpson's 1/3 rule.

$$\int_0^6 \frac{1}{1+x}\,dx = \frac{h}{3}\left[(y_0+y_6) + 2(y_2+y_4) + 4(y_1+y_3+y_5)\right]$$

$$= \frac{1}{3}\left(1.1428 + 2(0.5333) + 4(0.5+0.25+0.1666)\right)$$

$$= 1.9586.$$

② Simpson's 3/8th rule.

$$\int_0^6 \frac{1}{1+x}\,dx = \frac{3h}{8}\left((y_0+y_6) + 2(y_3) + 3(y_1+y_2+y_4+y_5 \ldots\right)$$

$$= \frac{3}{8}\left(1.1428 + 2(0.25) + 3(0.5 + 0.3333 + 0.2 + 0.1666)\right)$$

$$= 5.24178 \times \frac{3}{8}$$

$$= 1.965.$$

③ $h = \frac{b-a}{n} \Rightarrow n = \frac{1}{0.1}$

$n = 10.$

$\int_0^1 \sqrt{1+x^3}\,dx$ $\quad h = 0.1$ using.

① 1/3 rd rule ② Trapezoidal rule.

Simple 1/3 rule.

$$\int_0^1 \sqrt{1+x^3}\,dx = \frac{h}{3}\left[(y_0+y_{10}) + 2(y_2+y_4+y_6+y_8) + 4(y_1+y_3+y_5+y_7+y_9)\right]$$

$x_0 = 0$ $\quad y_0 = 1$

$x_1 = x_0 + h$ $\quad y_1 = \sqrt{1+(0.1)^3}$

$= 0.1$ $\quad = 1.0004$

$x_2 = x_1 + h.$ $\qquad y_2 = \sqrt{1+(0.2)^3}$

$= 0.2$ $\qquad = 1.0039.$

$x_3 = 0.3.$ $\qquad y_3 = 1.0134.$

$x_4 = 0.4$ $\qquad y_4 = 1.0315.$

$x_5 = 0.5$ $\qquad y_5 = 1.0606.$

$x_6 = 0.6$ $\qquad y_6 = 1.1027.$

$x_7 = 0.7$ $\qquad y_7 = 1.1588.$

$x_8 = 0.8$ $\qquad y_8 = 1.2296.$

$x_9 = 0.9$ $\qquad y_9 = 1.31491$

$x_{10} = 1$ $\qquad y_{10} = 1.4142.$

$$① \Rightarrow \frac{0.1}{3}\Big[(1 + 1.4142) + 2(1.0039 + 1.0315 + 1.1027 + 1.2296)$$

$$+ 4(1.0004 + 1.0134 + 1.0606 + 1.1588 + 1.3149)\Big]$$

$$= 1.1114.$$

$$② \Rightarrow \int_a^b \sqrt{1+x^3}\,dx = \frac{h}{2}\Big[(y_0 + y_{10}) + 2(y_1 + y_2 + y_3 + y_4 \cdots + y_9)\Big]$$

$$= 1.1122.$$

④ Evaluate $\int_0^1 \frac{dx}{1+x^2}$ using simpson $\frac{3}{8}$ rule $h = \frac{1}{6}$

$h = \frac{b-a}{n} \Rightarrow \frac{1}{6} = \frac{1-0}{n} \Rightarrow n = 6.$

$x_0 = 0 \qquad y_0 = 1$

$x_1 = x_0 + h. \qquad y_1 = \frac{1}{1+\left(\frac{1}{6}\right)^2} = \frac{1}{1+0.0277}.$

$= \frac{1}{6}. \qquad y_1 = 0.9730$

$x_2 = x_1 + h. \qquad y_2 = \frac{1}{1+\left(\frac{1}{3}\right)^2} = \frac{1}{1+0.1111}$

$= \frac{2}{6} = \frac{1}{3}. \qquad y_2 = 0.9000$

$x_3 = x_2 + h. \qquad y_3 = \frac{1}{1+\left(\frac{1}{2}\right)^2} = \frac{1}{1+0.25}$

$= \frac{3}{2} = \frac{1}{2}. \qquad y_3 = 0.8$

$x_4 = x_3 + h. \qquad y_4 = \frac{1}{1+\left(\frac{2}{3}\right)^2} = \frac{1}{1+0.4444}$

$= \frac{4}{6} = \frac{2}{3}. \qquad y_4 = 0.6923.$

$x_5 = x_4 + h. \qquad y_5 = \frac{1}{1+\left(\frac{5}{6}\right)^2} = \frac{1}{1+0.4444}.$

$= \frac{5}{6}. \qquad y_5 = 0.5901.$

$x_6 = x_5 + h. \qquad y_6 = \frac{1}{1+(1)^2} = \frac{1}{1+1}$

$= 1 \qquad y_6 = 0.5.$

Simpson's $\frac{3}{8}^{th}$ rule.

$$\int_a^b f(x)dx = \frac{3h}{8}\left[(y_0+y_n)+2(y_3+y_6+y_9+\cdots)+3(y_1+y_2+y_4\cdots)\right]$$

$$= \frac{3\times\frac{1}{6}}{8}\left[(1+0.5)+2(0.8)+3(0.9729+0.9+0.6923+0.5901)\right]$$

$= 0.7853.$

(5) Dividing the range into 10 equal parts, find an approximate value of $\int_0^{\pi} \sin x \, dx$.

by (1) Trapezoidal rule.

$h = \frac{b-a}{n} = \frac{\pi-0}{10} = \frac{\pi}{10}.$

$x_0 = 0$ $\qquad$ $y_0 = 0.$

$x_1 = x_0 + h$ $\qquad$ $y_1 = \sin x_1$

$= 0 + \frac{\pi}{10}$ $\qquad$ $= \sin \frac{\pi}{10}$

$= \frac{\pi}{10}$ $\qquad$ $y_1 = 0.3090.$

$x_2 = x_1 + h$ $\qquad$ $y_2 = \sin x_2.$

$= \frac{\pi}{10} + \frac{\pi}{10}$ $\qquad$ $= \sin \frac{\pi}{5}$

$= \frac{\pi}{5}$ $\qquad$ $= 0.5877$

$x_3 = x_2 + h$ $\qquad$ $y_3 = \sin x_3$

$= \frac{3\pi}{10}$ $\qquad$ $= \sin \frac{3\pi}{10} = 0.8090.$

$x_4 = \frac{4\pi}{10}$ $\qquad$ $y_4 = \sin \frac{4\pi}{10}$

$y_4 = 0.9510.$

$x_5 = \frac{5\pi}{10}$ $\qquad$ $y_5 = \sin \frac{5\pi}{10}$

$y_5 = 1.$

$x_6 = \frac{6\pi}{10}$ $\qquad$ $y_6 = 0.9510.$

$x_7 = \frac{7\pi}{10}$ $\qquad y_7 = \sin\frac{7\pi}{10}$

$y_7 = 0.8090$

$x_8 = \frac{8\pi}{10}$ $\qquad y_8 = \sin\frac{8\pi}{10} = 0.5877$

$x_9 = \frac{9\pi}{10}$ $\qquad y_9 = \sin\frac{9\pi}{10} = 0.3090$

$x_{10} = \frac{10\pi}{10}$ $\qquad y_{10} = 0$

① Trapezoid rule.

$$\int_0^{\pi} \sin x\,dx = \frac{h}{2}\left[(y_0+y_{10}) + 2[y_1+y_2+y_3+y_4 - - - + y_9]\right]$$

$$= \frac{\frac{\pi}{10}}{2}\left[(0+0) + 2(0.3090+0.5877+0.8090+0.9510+1+0.9510+0.8090+0.5877+0.3090)\right]$$

$$\int_0^{\pi} \sin x\,dx = 1.9843$$

② Simpson's rule.

$$\int_0^{\pi} \sin x\,dx = \frac{h}{3}\left[(y_0+y_{10}) + 2(y_2+y_4+y_6+y_8) + 4(y_1+y_3+y_5+y_7+y_9)\right]$$

$$= 2.00002$$

Numerical solution of ordinary D.E.

Taylor's Series:

To find solution of ordinary DE $\frac{dy}{dx} = f(x,y)$ —① given initial condition are $y(x_0) = y_0$ then.

$$y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \cdots$$

Here $h = x - x_0$.

To find ~~x=x+h~~ y_2 at x_2.

$$y_2 = y_1 + \frac{h}{1!} y_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \cdots$$

Problem:- ① Using Taylor's method, solve $\frac{dy}{dx} = x^2 + y^2$ for $x = 0.4$, given that $y=0$ at $x=0$

Given $\frac{dy}{dx} = f(x,y)$

$f(x,y) = x^2+y^2$.

$y' = x^2+y^2$.

$y_0 = 0$, $x_0 = 0$, Take $h = 0.4$.

$y_0' = x^2 + y^2$.

$y_0' = 0$

$y'' = 2x + 2yy'$.

$y_0'' = 2x_0 + 2y_0 y_0'$

$= 0$.

$y''' = 2 + 2[yy'' + (y')^2]$

$y_0''' = 2 + 2[y_0 y_0'' + (y_0')^2]$

$y_0''' = 2$.

$y_0^{IV} = 0 + 2[yy''' + y'y'' + 2(y')(y'')]$

$y_0^{IV} = 2[y_0 y_0''' + y_0' y_0'' + 2y_0' y_0'']$

$= 2[0+0+0]$

$y_0^{IV} = 0$

By taylor's series method.

$y_1 = y_0 + \frac{h}{1!} + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \cdots$

$= 0 + (0.4)(0) + \frac{(0.2)^2}{2}(0) + \frac{(0.4)^3}{6} 2 + \cdots$

$y_1 = \frac{(0.4)^3}{3} = 0.02133.$

② Solve $y' = x - y^2$, $y(0) = 1$ using Taylor's series method and compute $y(0.1)$, $y(0.2)$.

Given $f(x, y) = x - y^2$.

$y' = x - y^2$

Initial condition is $y(0) = 1$.

$x_0 = 0$, $y_0 = 1$

$y' = x - y^2$

$y'' = 1 - 2yy'$

$y''' = 0 - 2[yy'' + y'y']$

$y''' = -2[yy'' + (y')^2]$

$y_0' = x_0 - y_0^2$

$y_0' = 0 - (1)^2$

$= -1$

$y_0'' = 1 - 2 y_0 y_0'$

$= 1 - 2(1)(-1)$

$y_0'' = 3$

$y_0''' = -2[y_0 y_0'' + (y_0')^2]$

$= -2[(1)(3) + (-1)^2]$

$= -2[3+1] = -8.$

By taylor's series,

y_1 at $x_1 = x_0 + h = 0 + 0.1$

$= 0.1$

$y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \cdots$

$= 1 + (0.1)(-1) + \frac{(0.1)^2}{2}(3) + \frac{(0.1)^3}{6}(-8)$

$= 0.9137.$

$y_1 = 0.9137$ at $x_1 = 0.1$

To find y_2 at $x_2 = x_1 + h$.

$= 0.1 + 0.1$

$= 0.2$.

$y_1' = x_1 - y_1^2$

$= 0.1 - (0.9137)^2$

$= -0.7348$

$y_1'' = 1 - 2y_1 y_1'$

$= 1 - 2(0.9137)(-0.7348)$

$= 2.3427$.

$y_1''' = -2[y_1 y_1'' + (y_1')^2]$

$= -2[(0.9137)(2.3427) + (-0.7348)^2]$

$= -2[2.1405 + 0.5399]$

$= -2[2.6804]$

$= -5.3609$.

$y_2(0.2) = y_1 + \frac{h}{1!} y' + \frac{h^2}{2!} y'' + \frac{h^3}{3!} y_1''' + \cdots$

$= 0.9137 + 0.1(-0.7348) + \frac{(0.1)^2}{2}(2.3427) +$

$\frac{(0.1)^3}{3!}(-5.3609)$

$y_2 = 0.85104$.

(3) Solve $y' - 2y = 3e^x$, $y(0) = 0$ using Taylor's series and compute $y(0.2)$.

Given $f(x, y) = 2y + 3e^x$.

$y' = 2y + 3e^x$.

Initial condition is $y(0) = 0$.

$x_0 = 0$, $y_0 = 0$.

$y' = 2y + 3e^x$

$y'' = 2y' + 3e^x$

$y''' = 2y'' + 3e^x$

$y_0' = 2y_0 + 3e^{x_0}$
$= 2(0) + 3e^0$
$= 3$

$y_0'' = 2y_0' + 3e^{x_0}$
$= 2(3) + 3e^0$
$= 9$

$y_0''' = 2y'' + 3e^{x_0}$
$= 2(9) + 3e^0$
$= 21$

By Taylor's series

y_1 at $x_1 = x_0 + h$
$= 0 + 0.2 = 0.2$

$$y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \dots$$

$$= 0 + \frac{0.2}{1}(3) + \frac{(0.2)^2}{2}(9) + \frac{(0.2)^3}{6}(21)$$

$$= 0.6 + 0.18 + 0.028$$

$y_1(0.2) = 0.808$

(4) Tabulate $y(0.1)$, $y(0.2)$ & $y(0.3)$ using Taylor's series method given that $y' = y^2 + x$ and $y(0) = 1$

Given $f(x,y) = y^2 + x$

$y' = y^2 + x$

Initial condition is $y(0) = 1$

$x_0 = 0$, $y_0 = 1$, $h = 0.1$

$y' = y^2 + x$

$y'' = 2yy' + 1$

$y''' = 2[yy'' + (y')^2]$

$y_0' = y_0^2 + x_0$

$y_0' = 1$

$y_0'' = 2y_0y_0' + 1$
$= 2(1)(1) + 1$

$y_0'' = 3$

$y_0''' = 2[y_0 y_0'' + (y_0')^2]$

$= 2[(1)(3) + (1)^2]$

$y_0''' = 8.$

By Taylor's series.

y_1 at $x_1 = x_0 + h.$

$= 0 + 0.1$

$x_1 = 0.1$

$y_1(0.1) = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \cdots$

$= (1) + \frac{0.1}{1}(1) + \frac{(0.1)^2}{2}(3) + \frac{(0.1)^3}{6}(8)$

$y_1(0.1) = 1.1163.$

$y_1 = 1.1163$ at $x_1 = 0.1$

To find y_2 at $x_2 = x_1 + h = 0.1 + 0.1$

$= 0.2.$

$y_1' = y_1^2 + x_1$

$= (1.1163)^2 + 0.1$

$y_1' = 1.3461$

$y_1'' = 2 y_1 y_1' + 1$

$= 2(1.1163)(1.3461) + 1$

$y_1'' = 4.0053.$

$y_1''' = 2[y_1 y_1'' + (y_1')^2]$

$= 12.5662.$

$y_2 = y_1 + \frac{h}{1!} y_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \cdots$

$= 1.1163 + (0.1)(1.3461) + \frac{(0.1)^2}{2}(4.0053) + \frac{(0.1)^3}{6}(12.5662)$

$\boxed{y_2 = 1.2730}$ at $x_2 = x_1 + h = 0.2$

To find y_3 at $x_3 = x_2 + h = 0.2 + 0.1$
$= 0.3.$

$y_2' = y_2^2 + x_2$
$= (1.2730)^2 + (0.2)$

$y_2' = 1.8205$

$y_2'' = 2y_2 y_2' + 1 = 2(1.2730)(1.8205) + 1$

$y_2'' = 5.6349.$

$y_2''' = 2(y_2 y_2'' + (y_2')^2) = 2[(1.2730)(5.6349) + (1.8205)^2]$

$= 20.9748.$

$y_3 = 1.2730 + (0.1)(1.8205) + \frac{(0.1)^2}{2}(5.6349) + \frac{(0.1)^3}{6}(20.9748)$

$y_3 = 1.4867.$

(b) Solve $y' = x + y$ given $y(1) = 0$, find $y(1.1)$ $y(1.2)$ by Taylor's series method.

Given $f(x, y) = x + y.$
$y' = x + y.$

$y(1) = 0$

$x_0 = 1$, $y_0 = 0$, $h = 0.1$

$y' = x + y.$	$y_0' = x_0 + y_0$
	$y_0' = 1 + 0 = 1$
$y'' = 1 + y'$	$y_0'' = 1 + y_0' = 1 + 1 = 2$
$y''' = 0 + y''$	$y_0''' = 0 + y_0'' = 0 + 2 = 2$
$y^{iv} = y'''$	$y_0^{iv} = 2.$

By Taylor's series

y_1 at $x_1 = x_0 + h$
$= 1 + 0.1$
$= 1.1$

$$y_1(1.1) = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \frac{h^4}{4!} y_0^{iv}$$

$$= 0 + \frac{0.1}{1}(1) + \frac{(0.1)^2}{2}(2) + \frac{(0.1)^3}{6}(2) + \frac{(0.1)^4}{24}(2)$$

$y_1(1.1) = 0.1103$

$y_1 = 0.1103$ at $x_1 = 1.1$

To find y_2 at $x_2 = x_1 + h$
$= 1.1 + 0.1$
$= 1.2$

$y_1' = x_1 + y_1 = 1.1 + 0.1103 = 1.2103$

$y_1'' = 1 + y_1' = 1 + 1.2103 = 2.2103$

$y_1''' = y_1'' = 2.2103$

$y_1^{iv} = 2$

$$y_2 = y_1 + \frac{h}{1!} y_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \frac{h^4}{4!} y_1^{iv}$$

$$= 0.1103 + \frac{0.1}{1}(1.2103) + \frac{(0.1)^2}{2}(2.2103) + \frac{(0.1)^3}{6}(2.2103) + \frac{(0.1)^4}{24}(2)$$

$y_2 = 0.2427$

Euler's method :-

To find the solution of ordinary D.E $\frac{dy}{dx} = f(x,y)$ subject to the condition $y(x_0) = y_0$ is given by

y_1 at $(x_1 = x_0 + h) = y_0 + h f(x_0, y_0)$

To find $y_2 (x_2) = y_1 + h f(x_1, y_1)$

In general.

$$\boxed{y_{n+1} = y_n + h f(x_n, y_n)}$$

Problems

① Solve by euler's method $y' = x + y$, $y(0) = 1$ and find $y(0.3)$ taking step size $h = 0.1$.

Given

$f(x,y) = x + y$, $x_0 = 0$ $y_0 = 1$. $h = 0.1$

By euler's method.

To find y_1 at $x_1 = x_0 + h = 0 + 0.1 = 0.1$

$$y_1(0.1) = y_0 + h f(x_0, y_0)$$
$$= 1 + 0.1 f(0, 1)$$
$$= 1 + 0.1(0+1) = 1 + 0.1 \; [\because f(x,y) = x+y]$$
$$= 1.1$$

To find y_2 at $x_2 = x_1 + h = 0.1 + 0.1$
$= 0.2$.

$$y_2(0.2) = y_1 + h f(x_1, y_1)$$
$$= 1.1 + 0.1 f(0.1, 1.1)$$
$$= 1.1 + 0.1 (0.1 + 1.1)$$
$$= 1.1 + 0.1 [1.2]$$
$$y_2 = 1.22 \text{ at } x_2 = 0.2.$$

To find y_3 at $x_3 = x_2 + h$
$= 0.2 + 0.1$
$= 0.3$

$$y_3(0.3) = y_2 + hf(x_2, y_2)$$
$$= 1.22 + 0.1 f(0.2, 1.22)$$
$$= 1.22 + 0.1(1.42)$$

$$\boxed{y_3(0.3) = 1.362}$$

② Using euler's method, solve for y at $x=2$ from $\frac{dy}{dx} = 3x^2+1$, $y(1) = 2$ taking step size
① $h = 0.5$ ② $h = 0.25$ $x_0 = 1$ $y_0 = 2$.

To find y_1 at $x_1 = x_0 + h$
$= 1 + 0.5$
$x_1 = 1.5$

$$y_1(1.5) = y_0 + hf(x_0, y_0)$$
$$= 2 + (0.5)(3(1)^2+1)$$
$$= 2 + (0.5)(4)$$

$$\boxed{y_1(1.5) = 4.}$$

To find y_2 at $x_2 = x_1 + h$.
$= 1.5 + 0.5$
$= 2$.

$$y_2(2) = y_1 + hf(x_1, y_1)$$
$$= 4 + (0.5)(3(1.5)^2+1)$$
$$= 4 + (0.5)(3(2.25)+1)$$
$$= 4 + (0.5)(6.75+1) = 4 + \frac{1}{2}(\cancel{7.75}) \quad 3.8725$$
$$= 7.8705.$$

(2) $h = 0.25$.

To find y_1 at $x_1 = x_0 + h$.
$= 1 + 0.25$.
$x_1 = 1.25$.

$$y_1(1.25) = y_0 + hf(x_0, y_0)$$
$$= 2 + (0.25)(3(1)^2+1) = 2 + (0.25)(4)$$
$$\boxed{y_1(1.25) = 3}$$

To find y_2 at $x_2 = x_1 + h$.
$= 1.25 + 0.25 = 1.5$.

$$y_2(1.5) = y_1 + hf(x_1, y_1)$$
$$= 3 + (0.25)(3(1.25)^2+1)$$
$$= 3 + (0.25)(5.6875)$$
$$\boxed{y_2 = 4.421875}$$

To find y_3 at $x_3 = x_2 + h$.
$= 1.5 + 0.25$.
$= 1.75$.

$$y_3(1.75) = y_2 + hf(x_2, y_2)$$
$$= 4.421875 + (0.25)[3(1.5)^2+1]$$
$$= 4.421875 + (0.25)(7.75)$$
$$\boxed{y_3 = 6.3593}$$

To find y_4 at $x_4 = x_3 + h$.
$= 1.75 + 0.25$.
$= 2$.

$$y_4(2) = y_3 + hf(x_3, y_3)$$
$$= 6.3593 + (0.25)(3(1.75)^2+1)$$
$$\boxed{y_4(2) = 8.9061}$$

③ Solve numerically using euler's method

$y' = y^2 + x$, $y(0) = 1$, find $y(0.1)$ & $y(0.2)$

let $\boxed{h = 0.1}$ $x_0 = 0$ $y_0 = 1$.

To find y_1 at $x_1 = x_0 + h$.

$= 0 + 0.1$

$= 0.1$.

$y_1(0.1) = y_0 + h.f(x_0, y_0)$

$= 1 + (0.1)(1^2 + 0)$

$= 1 + (0.1)(1)$

$\boxed{y_1(0.1) = 1.1}$

To find y_2 at $x_2 = x_1 + h$.

$= 0.1 + 0.1$

$= 0.2$.

$y_2(0.2) = y_1 + hf(x_1, y_1)$

$= 1.1 + (0.1).(1.21 + 0.1)$

$= 1.1 + (0.1)(1.31)$

$= 1.1 + 0.131$

$\boxed{y_2(0.2) = 1.231}$

Modified euler's method.

To find the solution of ordinary D.E $\frac{dy}{dx} = f(x,y)$, give $y(x_0) = y_0$ is given by.

$$y_1^{(0)} = y_0 + hf(x_0, y_0)$$

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$

and so on.

Repeat above procedure until two successive values are similar upto 4 decimal places.

To find y_2 at x_2.

$$y_2^{(0)} = y_1 + hf(x_1, y_1)$$

$$y_2^{(1)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(0)})\right]$$

$$y_2^{(2)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(1)})\right]$$

— — — —

Repeat the above procedure until two successive values are similar upto 4 decimal places.

① Using modified euler's method find $y(0.2)$ and $y(0.4)$ given $y' = y + e^x$, $y(0) = 0$.

Sol: Given $f(x,y) = y + e^x$, $x_0 = 0$, $y_0 = 0$, $h = 0.2$.

By modified euler's method:

$$y_1^{(0)} = y_0 + h f(x_0, y_0)$$
$$= 0 + 0.2 f(0,0)$$
$$= 0.2[0 + e^0].$$
$$y_1^{(0)} = 0.2.$$

$$y_1^{(1)} = y_0 + \frac{h}{2}\left(f(x_0, y_0) + f(x_1, y_1^{(0)})\right)$$
$$= 0 + \frac{0.2}{2}\left[f(0,0) + f(0.2, 0.2)\right].$$
$$= \frac{0.2}{2}\left[1 + 0.2 + e^{0.2}\right]$$
$$= 0.1\left[1 + 0.2 + 1.2214\right].$$
$$= 0.2421.$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left(f(x_0, y_0) + f(x_1, y_1^{(1)})\right)$$
$$= 0 + \overset{0.1}{\cancel{\frac{0.2}{2}}}\left[1 + 0.2421 + e^{0.2}\right].$$
$$= 0.2463.$$

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(2)})\right)$$
$$= 0 + 0.1\left[1 + 0.2463 + e^{0.2}\right]$$
$$= 0.2467.$$

$$y_1^{(4)} = y_0 + \frac{h}{2}\left(f(x_0, y_0) + f(x_1, y_1^{(3)})\right)$$
$$= 0 + 0.2\left(1 + 0.2467 + e^{0.2}\right] = 0.2468.$$

$$y_2^{(5)} = y_0 + \frac{h}{2}\left(f(x_0, y_0) + f(x_1, y_1^{(4)})\right]$$
$$= 0 + 0.1\left(1 + 0.2468 + e^{0.2}\right] = 0.2468.$$

$$\boxed{y_1(0.2) = 0.2468}$$

To find y_2 at $x_2 = x_1 + h = 0.2 + 0.2 = 0.4$

$$y_2^{(0)} = y_1 + hf(x_1, y_1)$$
$$= 0.2468 + 0.2 f(0.2, 0.2468)$$
$$= 0.2468 + 0.2(0.2468 + e^{0.2})$$
$$= 0.5404$$

$$y_2^{(1)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^{(0)})]$$
$$= 0.2468 + \frac{0.2}{2}[f(0.2, 0.2468) + f(0.4, 0.5404)]$$
$$= 0.2468 + \frac{0.2}{2}[0.2468 + e^{0.2} + 0.5404 + e^{0.4}]$$
$$= 0.5968$$

$$y_2^{(2)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^{(1)})]$$
$$= 0.2468 + \frac{0.2}{2}[0.2468 + e^{0.2} + 0.5968 + e^{0.4}]$$
$$= 0.6024$$

$$y_2^{(3)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^{(2)})]$$
$$= 0.2468 + \frac{0.2}{2}[0.2468 + e^{0.2} + 0.6024 + e^{0.4}]$$
$$= 0.6030$$

$$y_2^{(4)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^{(3)})]$$
$$= 0.2468 + \frac{0.2}{2}[0.2468 + e^{0.2} + 0.6030 + e^{0.4}]$$
$$= 0.6031$$

$$y_2^{(5)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^{(4)})]$$
$$= 0.2468 + \frac{0.2}{2}[0.2468 + e^{0.2} + 0.6031 + e^{0.4}]$$

$$\boxed{y_2^{(5)} = 0.6031}$$

$$\boxed{y_2(0.4) = 0.6031}$$

② Solve $\frac{dy}{dx} = x^2+y$, $y(0)=1$ by modified euler's method. and compute $y(0.02)$ & $y(0.04)$

Given $f(x,y) = x^2+y$ $x_0 = 0$ $y_0 = 1$. $h = 0.02$.

By modified euler's method.

$$y_1^{(0)} = y_0 + hf(x_0, y_0)$$
$$= 1 + (0.02)[1]$$
$$= 1.02.$$

$$\left[\begin{array}{l} x_1 = x_0 + h. \\ = 0 + 0.02. \\ = 0.02. \end{array}\right]$$

$$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(0)})]$$
$$= 1 + \frac{0.02}{2}(1 + (0.02)^2 + 1.02)$$
$$= 1 + 0.01(1 + 0.0004 + 1.02)$$
$$= 1.0202.$$

$$y_1^{(2)} = y_0 + \frac{h}{2}(f(x_0, y_0) + f(x_1, y_1^{(1)}))$$
$$= 1 + 0.01(1 + (0.02)^2 + 1.0202)$$
$$= 1.0202.$$

$$y_1(0.02) = 1.0202$$

$\boxed{y(0.04)}$ $x_2 = 0.04.$ $y_1 = 1.0202$

$$y_2^{(0)} = y_1 + h.f(x_1, y_1)$$
$$= 1.0202 + 0.02(1.0218)$$
$$= 1.0406.$$

$$y_2^{(1)} = y_1 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(1)})]$$
$$= 1.0202 + 0.01[1.0218 +$$

Runge - kutta method (R-K method)

To solve the ordinary D.E $\frac{dy}{dx} = f(x,y)$, $y(x_0) = y_0$.

by R-K method.

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

where - $k_1 = hf(x_0, y_0)$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

<u>Problems</u>:- Using R-K method compute y at $x = 0.1, 0.2$ from $y' + y = 0$. $y(0) = 1$.

$$y' = -y.$$

<u>Sol</u> $f(x,y) = -y$, $y_0 = 1$, $x_0 = 0$, $h = 0.1$.

By R-K method

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

where $k_1 = hf(x_0, y_0)$

$= 0.1\ f(0,1)$

$= -0.1.$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$= 0.1\ f\left(0 + \frac{0.1}{2}, 1 + \frac{(-0.1)}{2}\right)$$

$= -0.095.$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$= 0.1\ f\left(0 + \frac{0.1}{2}, 1 + \frac{(-0.095)}{2}\right)$$

$= -0.0952.$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

$= 0.1\ f(0 + 0.1, 1 + (-0.0952))$

$= -0.0904.$

$y_1(x_1 = x_0 + h = 0.1) = 1 + \frac{1}{6}[-0.1 + 2(-0.095) + 2(-0.0952) + (-0.0904)]$

$y_1 = 0.9048$ at $x_1 = 0.1$.

y_2 at $x_2 = x_1 + h = 0.1 + 0.1 = 0.2$.

$y_2 = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$k_1 = hf(x_1, y_1)$
$= 0.1 f(0.1, 0.9048)$
$= -0.0904$.

$k_2 = hf(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2})$
$= 0.1 f(0.1 + \frac{0.1}{2}, 0.9048 + (\frac{-0.0904}{2})$
$= -0.0859$.

$k_3 = hf(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2})$
$= 0.1 f(0.1 + \frac{0.1}{2}, 0.9048 + (-\frac{0.0859}{2}))$
$= -0.0861$.

$k_4 = hf(x_1 + h, y_1 + k_3)$
$= 0.1 f(0.1 + 0.1, 0.9048 - 0.0801)$

$k_4 = -0.0818$.

$y_2 = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$y_2 =$ at $x_2 = 0.2$.

Apply fourth order R-K method, to find an approx value of y when $x = 1.2$, in steps of 0.1, given that $y' = x^2 + y^2$.

$y(1) = 1.5$. $f(x, y) = x^2 + y^2$) $x_0 = 1$ $y_0 = 1.5$.

$h = 0.1$.

By R-K method:

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

where $k_1 = hf(x_0, y_0) = 0.1 \times 3.25$.

$k_1 = 0.325$.

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$= 0.1 f(1.05, 1.6625)$

$k_2 = 0.3866$.

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$= 0.1 f(1.05, 1.6933)$

$k_3 = \cancel{0.6930}$ 0.3969.

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

$= 0.1 f(1.1, 1.8969)$

$k_4 = 0.4808$.

$(x_1 = x_0 + h = 1.1)$ $y_1 = 1.5 + \frac{1}{6}(0.325 + 2(0.3866) + 2(0.3969) + 0.4808)$

$$\boxed{y_1 = 1.8954}$$

$$y_2 = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$k_1 = h.f(x_1, y_1)$

$= 0.1 f(1.1, 1.8954)$

$= 0.4802$.

$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = 0.1 f(1.15, 2.1355)$

$= 0.5882$.

$$k_3 = h f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$$

$$= 0.1 f(1.15, 2.1895)$$

$k_3 = 0.6116.$

$$k_4 = h f(x_1 + h, y_1 + k_3)$$

$$= 0.1 (1.2, 2.507)$$

$$= 0.7725.$$

$$y_2 = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$\boxed{y_2 = 2.5041}$$

(3) Using R-K method, find $y(0.2)$ for the equation

$\frac{dy}{dx} = \frac{y-x}{y+x}$, $y(0) = 1$. Take $h = 0.2$. Ans 1.1678.

$f(x,y) = \frac{y-x}{y+x}$. $x_0 = 0$, $y_0 = 1$, $h = 0.2$

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$k_1 = h \cdot f(x_0, y_0) = 0.2 f(0, 1) = 0.2\left(\frac{1-0}{1}\right)$

$k_1 = 0.2.$

$k_2 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.2 f(0.1, 1.1)$

$= \left(\frac{1.1 - 0.1}{1.1 + 0.1}\right) 0.2 = \left(\frac{1}{1.2}\right) 0.2.$

$k_2 = 0.1666.$

$k_3 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.2 f(0.1, 1.0833)$

$= \left(\frac{1.0833 - 0.1}{1.0833 + 0.1}\right) 0.2 = 0.1661.$

$k_4 = h \cdot f(x_0 + h, y_0 + k_3) = 0.2 f(0.2, 1.1661)$

$= 0.1414.$

$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 1.1678.$

(4) Apply fourth order R-K method to find $y(0.1)$ and $y(0.2)$ given $y' = xy + y^2$, $y(0) = 1$

$f(x,y) = xy + y^2 \quad x_0 = 0 \quad y_0 = 1 \quad h = 0.1$

$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$k_1 = hf(x_0, y_0)$

$= 0.1 f(0,1) = 0.1(1)$

$= 0.1$

$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.1 f(0.05, 1.05)$

$= 0.1155$

$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.1 f(0.05, 1.05775)$

$= 0.1171$

$k_4 = hf(x_0 + h, y_0 + k_3) = 0.1 f(0.1, 1.1171)$

$= 0.1359$

$y_1 = 1 + \frac{1}{6}(0.1 + (0.1155)2 + (0.1171)2 + 0.1359)$

$= 0.1168 + 1 = 1.1168$

$x_1 = x_0 + h = 0 + 0.1 = 0.1 \quad y_1 = 1.1168$

$k_1 = hf(x_1, y_1) = 0.1 f(0.1, 1.1168)$

$k_1 = 0.1358$

$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = 0.1 f(0.15, 1.1847)$

$k_2 = 0.1581$

$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = 0.1 f(0.15, 1.1958)$

$k_3 = 0.1609$

$k_4 = hf(x_1 + h, y_1 + k_3) = 0.1 f(0.2, 1.2777)$

$k_4 = 0.1888$

$y_2 = 1.1168 + \frac{1}{6}(0.1358 + 2 \times 0.1581 + 2 \times 0.1609 + 0.1888) = 1.2772$

(5) Apply RK method, to find $y(0.2)$ & $y(0.4)$ given $dy/dx = x^2+y^2$ $y(0)=1$ take $h=0.1$

$f(x,y) = x^2+y^2$ $h=0.1$ $x_0=0$, $y_0=1$

By R-K method.

$$y_1 = y_0 + \frac{1}{6}(k_1+2k_2+2k_3+k_4)$$

$$k_1 = hf(x_0, y_0) = 0.1 f(0,1)$$
$$= 0.1$$

$$k_2 = hf\left(x_0+\frac{h}{2}, y_0+\frac{k_1}{2}\right) = 0.1 f(0.05, 1.05)$$
$$= 0.1105$$

$$k_3 = hf\left(x_0+\frac{h}{2}, y_0+\frac{k_2}{2}\right) = 0.1 f(0.05, 1.0552)$$
$$= 0.1115$$

$$k_4 = hf(x_0+h, y_0+k_3) = 0.1 f(0.1, 1.1115)$$
$$= 0.1245$$

$$y_1 = y_0 + \frac{1}{6}(k_1+2k_2+2k_3+k_4)$$
$$= 1.1114.$$

$x_1 = x_0+h = 0+0.1 = 0.1$ $y_1 = 1.1114$. $h=0.1$

$$k_1 = hf(x_1, y_1) = 0.1 f(0.1, 1.1114)$$
$$= 0.1245$$

$$k_2 = hf\left(x_1+\frac{h}{2}, y_1+\frac{k_1}{2}\right) = 0.1 f(0.15, 1.1736)$$
$$= 0.1399.$$

$$k_3 = hf\left(x_1+\frac{h}{2}, y_1+\frac{k_2}{2}\right) = 0.1 f(0.15, 1.1813)$$
$$= 0.1417.$$

$$k_4 = hf(x_1+h, y_1+k_3) = 0.1 f(0.2, 1.2531)$$
$$= 0.1610.$$

$$y_2 = y_1 + \frac{1}{6}(k_1+2k_2+2k_3+k_4)$$
$$= 1.2528.$$

(6) Using R-K method find,

(i) $y' = x - 2y$, $y(0) = 1$ taking $h = 0.1$ find y at $x = 0.1, 0.2$

$x_0 = 0 \quad y_0 = 1 \quad h = 0.1$

$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$k_1 = hf(x_0, y_0) = 0.1 f(0, 1)$

$= -0.2$

$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.1 f(0.05, 0.9)$

$= -0.175$

$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.1 f(0.05, 0.9125)$

$= -0.1775$

$k_4 = hf(x_0 + h, y_0 + k_3) = 0.1 f(0.1, 0.8225)$

$= -0.1545$

$y_1 = 1 + \frac{1}{6}(-0.2 - 2 \times 0.175 - 2 \times 0.1775 - 0.1545)$

$y_1 = 0.8234$

$x_1 = 0.1 \quad y_1 = 0.8234$

$y_2 = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$k_1 = h \cdot f(x_1, y_1) = 0.1 f(0.1, 0.8234)$

$= -0.1546$

$k_2 = h \cdot f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = 0.1 f(0.15, 0.7461)$

$= -0.1342$

$k_3 = h \cdot f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = 0.1 f(0.15, 0.7563)$

$= -0.1362$

$k_4 = h \cdot f(x_1 + h, y_1 + k_3) = 0.1 f(0.2, 0.6872)$

$= -0.1174$

$y_2 = 0.8234 - \frac{1}{6}(0.1546 + 2 \times 0.1342 + 2 \times 0.1362 + 0.1174)$

$= 0.6879$ //

(ii) $y' = x^2 - y \quad y(0) = 1 \quad y(0.1)$ & $(y(0.2))$

$y_1 = y_0 + 1/6(k_1 + 2k_2 + 2k_3 + k_4)$

$k_1 = hf(x_0, y_0) = 0.1 f(0,1)$

$= -0.1$

$k_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}) = 0.1 f(0.05, 0.95)$

$= -0.094775$

$k_3 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}) = 0.1 f(0.05, 0.9526)$

$= -0.095$

$k_4 = hf(x_0 + h, y_0 + k_3) = 0.1 f(0.1, 0.905)$

$= -0.0895$

$y_1 = 1 + \frac{1}{6}(-0.1 - 0.1895 - 0.19 - 0.0895)$

$\boxed{y_1 = 0.9052}$

$x_1 = x_0 + h, \quad h = 0.1, \quad y_1 = 0.9052$

$= 0.1$

$k_1 = hf(x_1, y_1) = 0.1 f(0.1, 0.9052)$

$= -0.08952$

$k_2 = hf(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}) = 0.1 f(0.15, 0.8604)$

$= -0.0837$

$k_3 = hf(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}) = 0.1 f(0.15, 0.8633)$

$= -0.0841$

$k_4 = hf(x_1 + h, y_1 + k_3) = 0.1 f(0.2, 0.8211)$

$= -0.0781$

$y_2 = 0.9052 + \frac{1}{6}(-0.0895 - 0.1675 - 0.1682 - 0.0781)$

$\boxed{y_2 = 0.8213}$

(iii) $y' = f(x, y) = 3x + y^2$ $y(1) = 1.2$ find $y(1.1)$

$f(x, y) = 3x + y^2$ $x_0 = 1$. $y_0 = 1.2$. $y(1.2) = ?$

$h = 0.1$

$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$k_1 = h \cdot f(x_0, y_0) = 0.1 f(1, 1.2)$

$= 0.444$.

$k_2 = h \cdot f(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}) = 0.1 f(1.05, 1.422)$

$= 0.5172$.

$k_3 = h \cdot f(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}) = 0.1 f(1.05, 1.4586)$

$= 0.5277$.

$k_4 = h \cdot f(x_0 + h, y_0 + k_3) = 0.1 f(1.1, 1.7277)$

$= 0.6284$.

$y_1 = 1.2 + \frac{1}{6}(0.444 + 2 \times 0.5172 + 2 \times 0.5277 + 0.6284)$

$\boxed{y_1 = 1.7270}$

(iv) $y' = x - y$, $y(1) = 0.4$ $y(1.2) = ?$

$x_0 = 1$ $y_0 = 0.4$ $h = 0.2$.

$k_1 = hf(x_0, y_0) = 0.2 f(1, 0.4)$

$= 0.12$.

$k_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}) = 0.2 f(1.1, 0.46)$

$= 0.128$.

$k_3 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}) = 0.2 f(1.1, 0.464)$

$= 0.1272$.

$k_4 = hf(x_0 + h, y_0 + k_3) = 0.2 f(1.2, 0.5272)$

$= 0.1345$

$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$= 0.4 + \frac{1}{6}(0.12 + 2 \times 0.128 + 2 \times 0.1272 + 0.134$

$y(1.2) = y_1 = 0.5273$ //