# UNIT- IV

## BETA AND GAMMA FUNCTIONS

Beta function: The definite integral $\int_0^1 x^{m-1}(1-x)^{n-1}dx$ is called beta function and it is denoted by $\beta(m,n)$ and read as $\beta m,n$. The above integral converges $m>0$, $n>0$

Properties:

1) $\beta(m,n) = \beta(n,m)$

By defn $\Rightarrow \beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$

put $1-x = y \Rightarrow x = 1-y \Rightarrow dx = -dy$

when $x=0$, $y=1$

$x=1$, $y=0$

$$\beta(m,n) = \int_1^0 (1-y)^{m-1}(y)^{n-1}(-dy)$$

$$= \int_0^1 (1-y)^{m-1} y^{n-1}(dy)$$

$$= \beta(n,m)$$

2) $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$

By defn $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$.

put $x = \sin^2\theta$.

$dx = 2\sin\theta\cos\theta \, d\theta$

when $x=0$, $\theta=0$.

$x=1$, $\theta=\frac{\pi}{2}$.

$$\beta(m,n) = \int_0^{\pi/2} (\sin^2\theta)^{m-1} (1-\sin^2\theta)^{n-1} \, 2\sin\theta\cos\theta \, d\theta$$

$$= 2\int_0^{\pi/2} (\sin\theta)^{2m-2} (\cancel{1-}\cos\theta)^{2n-2} \sin\theta\cos\theta \, d\theta$$

$$\beta(m,n) = 2\int_0^{\pi/2} (\sin\theta)^{2m-1} (\cos\theta)^{2n-1} \, d\theta.$$

3) Other forms of Beta functions:

$I^{st}$ form:

S.T $\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx.$

By def$^n$ $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx.$

put $x = 1/1+y$

$dx = -\frac{1}{(1+y)^2} dy$

when $x=0 \Rightarrow y=\infty$

$x=1 \Rightarrow y=0$

$$\beta(m,n) = \int_\infty^0 \left(\frac{1}{1+y}\right)^{m-1}\left(1-\frac{1}{1+y}\right)^{n-1}\left(-\frac{1}{(1+y)^2}\right)dy$$

$$= \int_\infty^0 \left(\frac{1}{1+y}\right)^{m-1}\left(\frac{y}{1+y}\right)^{n-1}\left(\frac{-1}{(1+y)^2}\right)dy$$

$$= \int_\infty^0 \frac{-y^{n-1}}{(1+y)^{m-\cancel{1}+n-\cancel{1}+\cancel{2}}} dy$$

$$= \int_0^\infty \frac{y^{n-1}}{(1+y)^{n+m}} dy$$

$II^{nd}$ form:

S.T $\beta(m$

we know t

$\Rightarrow$

put $x = \frac{1}{y}.$

$\therefore \int_1^\infty \frac{x^{m-}}{(1+x)^{n}}$

$\int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}$

Substitute (

$\therefore \beta(m,n) =$

$\underline{\text{II}^{nd}\text{ form}}$:

S.T $\beta(m,n) = \int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,dx.$

we know that $\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(x+1)^{m+n}}\,dx.$

$$\Rightarrow \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\,dx + \underbrace{\int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx}\ \rightarrow (1)$$

put $x = \frac{1}{y}$. $\Rightarrow dx = -\frac{1}{y^2}\,dy$. when $x=0 \Rightarrow y=\infty$

$x = 1 \Rightarrow y = 1$

$x = \infty \Rightarrow y = 0.$

$$\therefore \int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx = \int_1^0 \frac{\left(\frac{1}{y}\right)^{m-1}}{\left(1+\frac{1}{y}\right)^{m+n}}\left(-\frac{1}{y^2}\right)dy$$

$$= \int_1^0 \frac{y^{m+n}}{y^{m-1}(1+y)^{m+n}}\left(-\frac{1}{y^2}\right)dy.$$

$$= \int_1^0 \frac{-y^{m+n}}{(1+y)^{m+n}\,y^{m+1}}\,dy.$$

$$\int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\,dx \Leftarrow = \int_0^1 \frac{y^{n-1}}{(1+y)^{m+n}}\,dy \rightarrow (2)$$

Substitute (2) in place of (1)

$$\therefore \beta(m,n) = \int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,dx.$$

III$^{rd}$ form: S.T $\beta(m,n) = a^m b^m \int_0^\infty \frac{x^{m-1}}{(ax+b)^{m+n}} dx.$

Now $a^m b^n \int_0^\infty \frac{x^{m-1}}{(ax+b)^{m+n}} dx.$

$$= a^m b^n \int_0^\infty \frac{x^{m-1}}{b^{m+n}\left(1+\frac{ax}{b}\right)^{m+n}} dx.$$

put $\frac{ax}{b} = y \Rightarrow x = \frac{b}{a} y$

$$dx = \frac{b}{a} dy$$

when $x = 0 \Rightarrow y = 0$

$x = \infty \Rightarrow y = \infty$

$$\therefore \beta(m,n) = a^m b^n \int_0^\infty \frac{\left(\frac{b}{a}y\right)^{m-1}}{b^{m+n}(1+y)^{m+n}} \left(\frac{b}{a} dy\right)$$

$$= a^m b^m \int_0^\infty \frac{b^{m-1}\ y^{m-1}}{b^{m+n}\ a^{m-1}(1+y)^{m+n}} \left(\frac{b}{a}\right) dy$$

$$= a^m b^m \int_0^\infty \frac{b^m\ y^{m-1}}{b^{m+n}\ a^m\ (1+y)^{m+n}} dy$$

$$= a^m b^m \int_0^\infty \frac{b^m\ y^{m-1}}{b^m b^n a^m (1+y)^{m+n}} dy.$$

$$= \frac{\cancel{a^m}\cancel{b^n}\cancel{b^m}}{\cancel{b^m}\cancel{b^n}\cancel{a^m}} \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}} dy$$

$\beta(m,n) = \beta(m,n)$

$\underline{\text{IV}^{th}\text{ form}}$: S.T $\displaystyle\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(x+a)^{m+n}}\,dx = \frac{\beta(m,n)}{a^n(1+a)^m}$

By def$^n$ $\displaystyle\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

put $x = \dfrac{(1+a)t}{(t+a)} = \dfrac{t+at}{t+a} \Rightarrow dx = \dfrac{(t+a)(1+a) - (t+at)(1)}{(t+a)^2}$

$\cancel{dx = \dfrac{(t+a)(t) - (t+at)(1)}{(t+a)^2}\,da}$ $\quad = \dfrac{t+at+a+a^2 - t - at}{(t+a)^2}$

$dx = \dfrac{a+a^2}{(t+a)^2}\,dt$

when $x = 0 \Rightarrow t = 0$

$x = 1 \Rightarrow t = 1$

$$\Rightarrow \int_0^1 \left(\frac{(1+a)t}{(t+a)}\right)^{m-1}\left(1-\left(\frac{(1+a)t}{(t+a)}\right)\right)^{n-1}\left(\frac{a+a^2}{(t+a)^2}\right)dt = \beta(m,n)$$

$$\Rightarrow \int_0^1 \frac{(1+a)^{m-1}t^{m-1}}{(t+a)^{m-1}}\left(\frac{a^{n-1}(1-t)^{n-1}}{(t+a)^{n-1}}\right)\frac{a+a^2}{(t+a)^2}\,dt = \beta(m,n).$$

$$= \int_0^1 \frac{(1+a)^{m-1}a^{n-1}t^{m-1}(1-t)^{n-1}a(1+a)}{(t+a)^{m-\cancel{1}+n-\cancel{1}+\cancel{2}}}\,dt = \beta(m,n).$$

$$\Rightarrow \int_0^1 \frac{(1+a)^{m-1}(1+a)a^{n-1}a\,t^{m-1}(1-t)^{n-1}}{(t+a)^{m+n}}\,dt = \beta(m,n)$$

$$\Rightarrow (1+a)^m a^n \int_0^1 \frac{(t)^{m-1}(1-t)^{n-1}}{(t+a)^{m+n}}\,dt = \beta(m,n)$$

$$\Rightarrow \int_0^1 \frac{t^{m-1}(1-t)^{n-1}}{(t+a)^{m+n}}\,dt = \frac{\beta(m,n)}{(1+a)^m a^n}$$

$$\Rightarrow \int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(x+a)^{m+n}}\,dx = \frac{\beta(m,n)}{(1+a)^m a^n}$$

## $\underline{\text{V}^{th}\text{ form}}$: S.T $\int_b^a (x-b)^{m-1}(a-x)^{n-1}dx = (a-b)^{m+n-1}\beta(m,n)$

By defn $\Rightarrow \beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$

put $x = \frac{t-b}{a-b}$

$$dx = \frac{a-b(1)-(t-b)(0)}{(a-b)^2}dt$$

$$dx = \frac{\cancel{a-b}}{(a-b)^{\cancel{2}}}dt = \frac{dt}{a-b}$$

$a\cancel{-b}-t+\cancel{b}$

$\Rightarrow$ when $x=0 \Rightarrow t=b$

$x=1 \Rightarrow t=a$

$$\therefore \beta(m,n) = \int_b^a \left(\frac{t-b}{a-b}\right)^{m-1}\left(1-\frac{t-b}{a-b}\right)^{n-1}\left(\frac{dt}{a-b}\right)$$

$$= \int_b^a \frac{(t-b)^{m-1}}{(a-b)^{m-1}}\frac{(a-t)^{n-1}}{(a-b)^{n-1}}\frac{dt}{(a-b)^1}$$

$$= \int_b^a \frac{(t-b)^{m-1}(a-t)^{n-1}}{(a-b)^{m\cancel{-1}+n\cancel{-1}+\cancel{1}-1}}dt$$

$$\Rightarrow \int_b^a \frac{(t-b)^{m-1}(a-t)^{n-1}}{(a-b)^{m+n-1}}dt = \beta(m,n)$$

$$\Rightarrow \int_b^a \frac{(x-b)^{m-1}(a-x)^{n-1}}{(a+b)^{m+n-1}}dx = \beta(m,n)$$

$$\Rightarrow \int_b^a (x-b)^{m-1}(a-x)^{n-1}dx = (a+b)^{m+n-1}\beta(m,n)$$

S.T $\int_0^{\pi/2} \sin^m\theta \cos^n\theta \, d\theta = \frac{1}{2}\beta\left(\frac{m+1}{2}, \frac{n+1}{2}\right)$

we know that $2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta = \beta(m,n)$

Let $2m-1 = P$ , $2n-1 = Q$

$2m = P+1$ $\quad 2n = Q+1$

$m = \frac{P+1}{2}$ $\quad n = \frac{Q+1}{2}$.

$$\therefore \int_0^{\pi/2} \sin^P\theta \cos^Q\theta \, d\theta = \frac{1}{2}\beta\left(\frac{P+1}{2}, \frac{Q+1}{2}\right)$$

$$\therefore \int_0^{\pi/2} \sin^m\theta \cos^n\theta \, d\theta = \frac{1}{2}\beta\left(\frac{m+1}{2}, \frac{n+1}{2}\right)$$

Q) Express the following interms of β func^ns.

i) $\int_0^1 \frac{x}{\sqrt{1-x^2}} \, dx$.

put $x^2 = t$

$2x\,dx = dt$

$\Rightarrow x = 0 \rightarrow t = 0$

$x = 1 \rightarrow t = 1$

$\Rightarrow \int_0^1 \frac{\sqrt{t}}{\sqrt{1-t}} \left(\frac{dt}{2\sqrt{t}}\right)$

$\Rightarrow \frac{1}{2}\int_0^1 (1-t)^{-1/2} dt$

$\Rightarrow \frac{1}{2}\int_0^1 t^{1-1} (1-t)^{1/2-1} dt$

$\Rightarrow \frac{1}{2}\beta(1, +1/2)$

ii) $\int_0^3 \frac{dx}{\sqrt{9-x^2}}$

$x^2 = 9t$

$2x\,dx = 9\,dt$

$dx = \frac{9\,dt}{2x}$

$\Rightarrow x = 0 \rightarrow t = 0$

$x = 3 \rightarrow t = 1$

$\Rightarrow \int_0^1 \frac{1}{\sqrt{9-9t}} \left(\frac{9\,dt}{2x}\right)$

$\Rightarrow \int_0^1 \frac{1}{\cancel{3}(1-t)^{1/2}} \frac{\cancel{9}\,dt}{2\cancel{x}\,\cancel{3}\sqrt{t}}$

$\Rightarrow \frac{\cancel{3}1}{2}\int_0^1 t^{-1/2}(1-t)^{-1/2} dt$

$\Rightarrow \frac{1}{2}\int_0^1 t^{1/2-1}(1-t)^{1/2-1} dt$

$\therefore \frac{1}{2}\beta(1/2, 1/2)$

Q) Evaluate $\int_0^1 \frac{x^2}{\sqrt{1-x^5}}\,dx$ in terms of beta func$^n$.

put $x^5 = y$

$$5x^4\,dx = dy \Rightarrow x^2\,dx = \frac{dy}{5x^2} = \frac{dy}{5x^{2/5}}$$

when $x=0 \Rightarrow y=0$

$x=1 \Rightarrow y=1$

$$\therefore \int_0^1 \frac{1}{\sqrt{1-y}}\,\frac{dy}{5y^{2/5}}$$

$$\Rightarrow \frac{1}{5}\int_0^1 y^{-2/5}(1-y)^{-1/2}\,dy$$

$$\Rightarrow \frac{1}{5}\int_0^1 y^{3/5-1}(1-y)^{1/2-1}\,dy$$

$$\Rightarrow \frac{1}{5}\beta\left(\frac{3}{5},\frac{1}{2}\right).$$

Q) i) $\int_0^\infty \frac{x^{m-1}-x^{n-1}}{(1+x)^{m+n}}\,dx = 0$

W.K.T $\Rightarrow \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx = \beta(m,n)$

$$\therefore \int_0^\infty \frac{x^{m-1}-x^{n-1}}{(1+x)^{m+n}}\,dx = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx - \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

$$= \beta(m,n) - \beta(n,m)$$

$$= 0.$$

ii) $\int_0^\infty \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,dx = 2\beta(m,n)$

$$\Rightarrow \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx + \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

$$= \beta(m,n) + \beta(n,m) \qquad [\beta(m,n) = \beta(n,m)]$$

$$= \beta(m,n) + \beta(m,n)$$

$$\Rightarrow 2\beta(m,n).$$

## GAMMA FUNCTION:

The definite integral $\int_0^\infty e^{-x} x^{n-1}\,dx$ is called the gamma func$^n$ and is denoted by $\Gamma n$ and read as gamma n.

Properties:

1) $\Gamma 1 = 1$
2) $\Gamma \frac{1}{2} = \sqrt{\pi}$
3) $\Gamma n = (n-1)\Gamma (n-1)$, where $n$ is greater than 1.
4) If n is a non-negative integer $\Gamma (n+1) = n!$

$$\Rightarrow \Gamma \tfrac{9}{2} = \left(\tfrac{9}{2}-1\right)\Gamma\left(\tfrac{9}{2}-1\right)$$

$$\Rightarrow \tfrac{7}{2}\,\Gamma \tfrac{7}{2} \Rightarrow \tfrac{7}{2}\left(\tfrac{7}{2}-1\right)\Gamma\left(\tfrac{7}{2}-1\right)$$

$$\Rightarrow \tfrac{7}{2}\cdot\tfrac{5}{2}\,\Gamma\tfrac{5}{2}$$

$$\Rightarrow \tfrac{7}{2}\cdot\tfrac{5}{2}\cdot\left(\tfrac{5}{2}-1\right)\Gamma\left(\tfrac{5}{2}-1\right)$$

$$\Rightarrow \tfrac{7}{2}\cdot\tfrac{5}{2}\cdot\tfrac{3}{2}\,\Gamma\tfrac{3}{2}$$

$$\Rightarrow \tfrac{7}{2}\cdot\tfrac{5}{2}\cdot\tfrac{3}{2}\cdot\left(\tfrac{3}{2}-1\right)\Gamma\left(\tfrac{3}{2}-1\right)$$

$$\Rightarrow \tfrac{7}{2}\cdot\tfrac{5}{2}\cdot\tfrac{3}{2}\cdot\tfrac{1}{2}\,\Gamma\tfrac{1}{2}$$

$$\Rightarrow \frac{105}{16}\sqrt{\pi}$$

* Relaⁿ b/w Beta and Gamma funcⁿ.

$$\beta(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma m+n}$$

where m and n are greater than 0.

<u>Note</u>: $\Gamma n \, \Gamma 1-n = \dfrac{\pi}{\sin n\pi}$

Q) S.T $\Gamma 1/2 = \sqrt{\pi}$

W.K.T $\quad \beta(m,n) = \dfrac{\Gamma m \, \Gamma n}{\Gamma m+n} \rightarrow (1)$

put $m = 1/2$, $n = 1/2$ in eqⁿ (1)

$$\beta(1/2, 1/2) = \frac{\Gamma 1/2 \, \Gamma 1/2}{\Gamma 1/2+1/2}$$

$$= \frac{(\Gamma 1/2)^2}{\Gamma 1}$$

$$\beta(1/2, 1/2) = (\Gamma 1/2)^2 \rightarrow (2)$$

$$\beta(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} \, dx.$$

$$= \int_0^1 x^{-1/2} (1-x)^{-1/2} \, dx.$$

put $x = \sin^2\theta \qquad \Rightarrow x = 0 \rightarrow \theta = 0$

$dx = 2\sin\theta\cos\theta \, d\theta \qquad x = 1 \rightarrow \theta = \pi/2$

$$\beta(1/2, 1/2) = \int_0^{\pi/2} (\sin^2\theta)^{-1/2} (\cos^2\theta)^{-1/2} \, 2\sin\theta\cos\theta \, d\theta$$

$$= \int_0^{\pi/2} \frac{1}{\cancel{\sin\theta}} \frac{1}{\cancel{\cos\theta}} \, 2\cancel{\sin\theta}\cancel{\cos\theta} \, d\theta$$

$$= 2\int_0^{\pi/2} d\theta$$

$$= [\theta]_0^{\pi/2}$$

$$= \frac{\pi}{2} \times 2$$

$$= \pi \rightarrow$$

$$\therefore (\Gamma 1/2)^2 = \pi$$

$$\Rightarrow \Gamma 1/2 = \sqrt{\pi}$$

Q) S.T $\int_0^\infty e$

put $x^2$

$2x$

$\Rightarrow \int_0^\infty e^{-t}$

Q) P.T $\int_{-\infty}^{0}$

$\Rightarrow \int_{\infty}^{0} e^{-}$

$\Rightarrow \int_{0}^{\infty} e^{-}$

$$= 2\int_0^{\pi/2} d\theta$$

$$= [\theta]_0^{\pi/2}$$

$$= \frac{\pi}{2} \times 2$$

$$= \pi \rightarrow (3)$$

$$\therefore (\Gamma(1/2))^2 = \pi$$

$$\Rightarrow \Gamma(1/2) = \sqrt{\pi}$$

Q) S.T $\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}.$

put $x^2 = t$ $\Rightarrow x = 0 \rightarrow t = 0$

$2x\,dx = dt$ $x = \infty \rightarrow t = \infty.$

$dx = \frac{dt}{2x} = \frac{dt}{2\sqrt{t}}$

$$\Rightarrow \int_0^\infty e^{-t} \frac{dt}{2\sqrt{t}} = \frac{1}{2}\int_0^\infty e^{-t} (t)^{-1/2} dt$$

$$= \frac{1}{2}\int_0^\infty e^{-t} (t)^{1/2 - 1} dt$$

$$= \frac{1}{2}\Gamma(1/2) = \frac{\sqrt{\pi}}{2}.$$

Q) P.T $\int_{-\infty}^0 e^{-x^2} dx = \frac{\sqrt{\pi}}{2}.$

$$\Rightarrow \int_\infty^0 e^{-x^2}(-dx) = \int_0^\infty e^{-x^2} dx$$

put $x^2 = t$

$2x\,dx = dt$

$dx = \frac{dt}{2\sqrt{t}}$

$$\Rightarrow \int_0^\infty e^{-t} \frac{dt}{2\sqrt{t}} \Rightarrow \frac{1}{2}\int_0^\infty e^{-t} (t)^{1/2-1} dt$$

$$\Rightarrow \frac{1}{2} \times \Gamma(1/2) = \frac{\sqrt{\pi}}{2}.$$

Q) $\int_{-\infty}^{\infty} e^{-x^2}dx$

$\Rightarrow \int_{-\infty}^{0} e^{-x^2}dx + \int_{0}^{+\infty} e^{-x^2}dx$

$\Rightarrow \int_{\infty}^{0} e^{-x^2}(-dx) + \int_{0}^{\infty} e^{-x^2}dx$

$\Rightarrow 2\int_{0}^{\infty} e^{-x^2}(dx) = 2\int_{0}^{\infty} e^{-t}\frac{1}{2\sqrt{t}}dt$

$= 2 \times \frac{1}{2}\,\Gamma\frac{1}{2} = \sqrt{\pi}$

Q) Find the value of $\Gamma\frac{11}{2}$, $\Gamma\frac{-1}{2}$, $\Gamma\frac{5}{2}$, $\Gamma\frac{7}{2}$, $\Gamma 10$

i) $\Gamma\frac{11}{2} = \left(\frac{11}{2}-1\right)\Gamma\left(\frac{11}{2}-1\right) = \frac{9}{2}\cdot\left(\frac{9}{2}-1\right)\Gamma\left(\frac{9}{2}-1\right) = \frac{9}{2}\cdot\frac{7}{2}\cdot\left(\frac{7}{2}-1\right)\Gamma\left(\frac{7}{2}-1\right)$

$\Rightarrow \frac{9}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}\left(\frac{5}{2}-(1)\right)\Gamma\left(\frac{5}{2}-1\right)$

$\Rightarrow \frac{9}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\,\Gamma\frac{1}{2}$

$\Rightarrow \frac{945}{32}\sqrt{\pi}$

ii) $\Gamma\frac{-1}{2} \Rightarrow \Gamma(n+1) = n\Gamma n$

$\Gamma n = \frac{\Gamma(n+1)}{n}$

$\Rightarrow \Gamma\frac{-1}{2} = \frac{\Gamma\left(-\frac{1}{2}+1\right)}{-\frac{1}{2}} = \frac{\Gamma\frac{1}{2}}{-\frac{1}{2}} = \frac{\sqrt{\pi}}{-\frac{1}{2}} = -2\sqrt{\pi}$

iii) $\Gamma\frac{5}{2} \Rightarrow \left(\frac{5}{2}-1\right)\Gamma\left(\frac{5}{2}-1\right) \Rightarrow \frac{3}{2}\cdot\frac{1}{2}\,\Gamma\frac{1}{2} \Rightarrow \frac{3}{4}\sqrt{\pi}$

iv) $\Gamma\left(-\frac{7}{2}\right)$ $\quad \Gamma n = \frac{\Gamma(n+1)}{n}$

$$\Rightarrow \frac{\Gamma\left(-\frac{7}{2}+1\right)}{-\frac{7}{2}} \Rightarrow \frac{\Gamma\left(-\frac{5}{2}\right)}{-\frac{7}{2}} \Rightarrow \frac{\Gamma\left(-\frac{5}{2}+1\right)}{-\frac{5}{2}\cdot -\frac{7}{2}} = \frac{\Gamma\left(-\frac{3}{2}\right)}{\frac{35}{4}}$$

$$\Rightarrow \frac{\Gamma\left(-\frac{3}{2}+1\right)}{-\frac{3}{2}\cdot\frac{35}{4}} \Rightarrow \frac{\Gamma\left(-\frac{1}{2}\right)}{-\frac{3}{2}\cdot\frac{35}{4}} \Rightarrow \frac{\Gamma\left(-\frac{1}{2}+1\right)}{-\frac{3}{2}\cdot -\frac{1}{2}\cdot\frac{35}{4}} = \frac{\Gamma\left(\frac{1}{2}\right)}{\frac{105}{16}}$$

$$= \frac{16}{105}\sqrt{\pi}$$

v) $\Gamma 10 = 9\Gamma 9 = 9\cdot 8\,\Gamma 7 \Rightarrow 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1\cdot\Gamma 1$

$\Rightarrow 362880$

* Relation b/w Beta and Gamma:

$$\beta(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma(m+n)}$$

By def $\Gamma m = \int_0^\infty e^{-x} x^{m-1}\,dx \rightarrow (1)$

$$x = yt \Rightarrow dx = y\,dt$$

when $x = 0 \rightarrow t = 0$

$x = \infty \rightarrow t = \infty$

$$\Rightarrow \Gamma m = \int_0^\infty e^{-yt} (yt)^{m-1} (y\,dt)$$

$$= \int_0^\infty e^{-yt} y^m t^{m-1}\,dt$$

$$\Rightarrow y^m \int_0^\infty e^{-yt} t^{m-1}\,dt$$

$$\frac{\Gamma m}{y^m} = \int_0^\infty e^{-yt} t^{m-1}\,dt \rightarrow (2)$$

Multiplying eqⁿ(2) with $e^{-y}\cdot y^{m+n-1}$ on both sides and integrating w.r.t $y$ from $0 \to \infty$

$$\frac{\Gamma m}{y^m}\times e^{-y}\cdot y^{m+n-1}\,\cancel{dy} = \int_0^\infty e^{-yx}\, x^{m-1}\times e^{-y}\times y^{m+n-1}\,\cancel{dy}\, dx$$

$$\Rightarrow \quad \Gamma m\, e^{-y} y^{n-1} = \int_0^\infty e^{-y(1+x)}\, y^{m+n-1}\, x^{m-1}\, dx$$

$\Rightarrow$ Integrating on both sides w.r.t ~~to~~ $y$ from $0 \to \infty$.

$$\Rightarrow \int_0^\infty \Gamma m\, e^{-y} y^{n-1}\, dy = \int_0^\infty \left[\int_0^\infty e^{-y(1+x)}\, y^{m+n-1}\, dy\right] x^{m-1}\, dx$$

$$\Rightarrow \Gamma m \int_0^\infty e^{-y}\, y^{n-1}\, dy = \int_0^\infty \frac{\Gamma(m+n)}{(1+x)^{m+n}}\, x^{m-1}\, dx.$$

$$\Rightarrow \Gamma m\, \Gamma n = \Gamma(m+n) \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\, dx$$

$$\Rightarrow \frac{\Gamma m\, \Gamma n}{\Gamma(m+n)} = \beta(m,n) \quad //.$$

Q) Evaluate i) $\int_0^1 x^5(1-x)^3\, dx$

i) $\cancel{\beta(m,n) = \int_0^\infty e^{-x} x^{n-1} d}$

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx.$$

$$\Rightarrow \int_0^1 x^{6-1}(1-x)^{4-1}\, dx$$

$$\Rightarrow \beta(6,4) = \frac{\Gamma 6\, \Gamma 4}{\Gamma 10} = \frac{5!\,3!}{9!} = \frac{1}{504}$$

ii) $\int_0^1 x^4 (1-x)^2 dx$

$\Rightarrow \int_0^1 x^{5-1}(1-x)^{3-1} dx = \beta(5,3) = \frac{\Gamma 5 \, \Gamma 3}{\Gamma 8} = \frac{4!\,2!}{7!}$

$= \frac{1}{105}$

iii) $\int_0^2 x(8-x^3)^{1/3} dx.$

$x^3 = 8y \Rightarrow 3x^2 dx = 8\,dy$

$dx = \frac{8\,dy}{3(8y)^{2/3}} = \frac{8\,dy}{3\times 4 (y)^{2/3}}$

$\Rightarrow x=0 \rightarrow y=0$

$x=2 \rightarrow y=1$

$= \frac{2}{3} y^{-2/3} dy$

$= \frac{2}{3} y^{-2/3} dy.$

$\Rightarrow \int_0^1 2y^{1/3}(8-8y)^{1/3} \times \frac{2}{3} \times y^{-2/3} dy$

$\Rightarrow \frac{4}{3}\int_0^1 y^{-1/3} \cdot 8^{1/3}(1-y)^{1/3} dy$

$\Rightarrow \frac{8}{3}\int_0^1 y^{-1/3}(1-y)^{1/3} dy = \frac{8}{3}\int_0^1 y^{2/3-1}(1-y)^{4/3-1} dy$

$\Rightarrow \frac{8}{3}\beta\left(\frac{2}{3},\frac{4}{3}\right)$

$\Rightarrow \frac{8}{3}\frac{\Gamma\frac{2}{3}\,\Gamma\frac{4}{3}}{\Gamma 2}$

$\Rightarrow \frac{8}{3}\frac{\Gamma\frac{2}{3}\,\Gamma\left(\frac{4}{3}-1\right)\left(\frac{4}{3}-1\right)}{1!}$

$\Rightarrow \frac{8}{3}\cdot\frac{1}{3}\frac{\Gamma\frac{2}{3}\,\Gamma\frac{1}{3}}{1!} \Rightarrow \frac{8}{9}\Gamma\frac{2}{3}\,\Gamma\left(1-\frac{2}{3}\right) = \frac{8}{9}\frac{\pi}{\sin 2\pi/3} = \frac{8}{9}\times\frac{2}{\sqrt{3}}\pi$

$\Rightarrow \frac{16\pi}{9\sqrt{3}}$

iv) $\int_0^1 x^{5/2}(1-x^2)^{3/2}\,dx$.

put $x^2 = y$ $\Rightarrow$ $x=0 \Rightarrow y=0$

$2x\,dx = dy$ $\quad x=1 \Rightarrow y=1$

$dx = \dfrac{dy}{2\sqrt{y}}$

$$\Rightarrow \int_0^1 (y^{1/2})^{5/2}(1-y)^{3/2}\frac{dy}{2y^{1/2}}.$$

$$\Rightarrow \int_0^1 y^{5/4}\, y^{-1/2}(1-y)^{3/2}\frac{dy}{2}$$

$$\Rightarrow \frac{1}{2}\int_0^1 y^{3/2}(1-y)^{3/2}\,dy = \frac{1}{2}\int_0^1 y^{5/2-1}(1-y)^{5/2-1}\,dy$$

$$\Rightarrow \frac{1}{2}\beta\left(\frac{5}{2},\frac{5}{2}\right) = \frac{1}{2}\frac{\Gamma\frac{5}{2}\,\Gamma\frac{5}{2}}{\Gamma 5} = \frac{1}{2}\frac{\left(\frac{5}{2}-1\right)\Gamma\left(\frac{5}{2}-1\right)\left(\frac{5}{2}-1\right)\Gamma\left(\frac{5}{2}-1\right)}{4!}$$

$$\Rightarrow \frac{1}{2}\cdot\frac{3}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\;\frac{\Gamma\frac{1}{2}\,\Gamma\frac{1}{2}}{4!}$$

$$\Rightarrow \frac{\cancel{9}^{\,1}}{{}_{4}\cancel{32}}\,\frac{\pi}{4!} \Rightarrow \frac{\pi}{96}.$$

f $\int_0^\infty e^{-kx}x^{n-1}\,dx = \dfrac{\Gamma n}{(k)^n}$

Q) compute

1) $\int_0^\infty e^{-x}x^3\,dx$

2) $\int_0^\infty x^4 e^{-2x}\,dx$

3) $\int_0^\infty e^{-4x}x^{3/2}\,dx$.

$y=0$
$y=1$

1) $\int_0^\infty e^{-x} x^3 dx.$

$\therefore = \int_0^\infty e^{-x} x^{4-1} dx$

$= \Gamma 4 = 3! = 6.$

2) $\int_0^\infty e^{-2x} x^4 dx$

$= \frac{\Gamma 5}{(2)^5} = \frac{4!}{(2)^5}$

3) $\int_0^\infty e^{-4x} x^{3/2} dx.$

$= \int_0^\infty e^{-4x} x^{\frac{5}{2}-1} dx$

$= \frac{\Gamma \frac{5}{2}}{(4)^{5/2}} = \frac{\frac{3}{2}!}{(4)^{5/2}}.$

$^{5/2-1} (1-y)^{5/2-1} dy$

$\frac{\left(\frac{5}{2}-1\right)\Gamma\frac{5}{2}-1 \left(\frac{5}{2}-1\right)\Gamma\frac{5}{2}}{4!}$

Q) S.T $\Gamma n = \int_0^1 \left(\log \frac{1}{x}\right)^{n-1} dx.$

put $x = \log\frac{1}{y} = \log y^{-1} = -\log y$

$y = e^{-x}.$

$x = 0 \Rightarrow y = 1$

$x = \infty \Rightarrow y = 0.$

$\frac{dy}{dx} = e^{-x}(-1)$

$dx = \frac{-1}{e^{-x}} dy$

$dx = -\frac{1}{y} dy$

By def $\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx$

$\Gamma n = \int_1^0 y (-\log y)^{n-1} \left(-\frac{1}{y}\right) dy$

$= \int_1^0 \cancel{y} \times \left(\log\frac{1}{y}\right)^{n-1} \times \left(-\frac{1}{\cancel{y}}\right) dy = \int_0^1 \left(\log\frac{1}{y}\right)^{n-1} dy.$

$\therefore \Gamma n = \int_0^1 \left(\log\frac{1}{y}\right)^{n-1} dy.$

Q) S.T $\Gamma n = \frac{1}{n} \int_0^\infty e^{-x^{1/n}} dx$.

Put $x^n = t$

$$n(x)^{n-1} dx = dt. \Rightarrow x^{n-1} dx = \frac{1}{n} dt$$

$$dx = \frac{1}{n} \frac{1}{(x)^{n-1}} dt.$$

$x = 0 \to t = 0$

$x = \infty \to t = \infty$

$$= \frac{1}{n} \frac{x}{x^n} dt$$

$$= \frac{1}{n} \frac{t^{1/n}}{t} dt.$$

$$= \frac{1}{n} t^{1/n - 1} dt.$$

$$\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx$$

$$= \int_0^\infty e^{-t^{1/n}} \frac{1}{n} dt$$

$$= \frac{1}{n} \int_0^\infty e^{-t^{1/n}} dt$$

$$\Gamma n. = \frac{1}{n} \int_0^\infty e^{-x^{1/n}} dx$$

Q) Evaluate i) $\int_0^{\pi/2} \sin^5\theta \cos^{7/2}\theta \, d\theta$.

ii) $\int_0^{\pi/2} \sin^7\theta \, d\theta$

iii) $\int_0^{\pi/2} \cos^{11}\theta \, d\theta$.

iv) $\int_0^{\pi/2} \sqrt{\cot\theta} \, d\theta$

i) ~~β(m)~~ $\int_0^{\pi/2} \sin^5\theta \cos^{7/2}\theta \, d\theta$.

$$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta.$$

$$\therefore \Rightarrow \int_0^{\frac{\pi}{2}} \sin^{2(3)-1}\theta \cos^{2\left(\frac{9}{4}\right)-1}\theta \, d\theta = \frac{1}{2}\beta\left(3,\frac{9}{4}\right)$$

$$\Rightarrow \beta\left(3,\frac{9}{4}\right)\times\frac{1}{2} = \frac{1}{2}\,\frac{\Gamma 3 \, \Gamma\frac{9}{4}}{\Gamma\left(3+\frac{9}{4}\right)} = \frac{1}{\cancel{2}}\,\frac{\cancel{2}!\,\Gamma\frac{9}{4}}{\Gamma\frac{21}{4}}$$

$$= \frac{\cancel{\Gamma\frac{9}{4}}}{\frac{17}{4}\cdot\frac{13}{4}\cdot\frac{9}{4}\,\cancel{\Gamma\frac{9}{4}}} = \frac{4\times4\times4}{17\times13\times9}$$

ii) $\int_0^{\frac{\pi}{2}} \sin^7\theta \, d\theta$.

$$\Rightarrow \int_0^{\frac{\pi}{2}} \sin^{2(4)-1}\theta \cos^{2\left(\frac{1}{2}\right)-1}\theta \, d\theta$$

$$\Rightarrow \frac{1}{2}\left(\beta\left(4,\frac{1}{2}\right)\right) = \frac{1}{2}\times\frac{\Gamma 4\,\Gamma\frac{1}{2}}{\Gamma\frac{9}{2}} = \frac{1}{\cancel{2}}\times\frac{3\times2\times\sqrt{\pi}}{\frac{7}{2}\times\frac{5}{2}\times\frac{3}{2}\times\frac{1}{\cancel{2}}\,\Gamma\frac{1}{2}}$$

$$= \frac{\cancel{3}\times2\times2\times2\times2\times\cancel{\sqrt{\pi}}}{7\times5\times\cancel{3}\times\cancel{\sqrt{\pi}}}$$

$$= \frac{16}{35}$$

iii) $\int_0^{\pi/2} \cos^{11}\theta \, d\theta$

$\Rightarrow \int_0^{\frac{\pi}{2}} \sin^{2(\frac{1}{2})-1}\theta \cos^{2(6)-1}\theta \, d\theta$

$\Rightarrow \frac{1}{2} \beta\left(\frac{1}{2}, 6\right)$

$\Rightarrow \frac{1}{2} \times \frac{\Gamma\frac{1}{2}\, \Gamma 6}{\Gamma\frac{13}{2}}$

$\Rightarrow \cancel{\frac{1}{2}} \times \frac{\cancel{\sqrt{\pi}} \times 5 \times \cancel{3} \times 4 \times 2}{\frac{11}{2} \times \frac{9}{2} \times \frac{7}{2} \times \frac{5}{2} \times \frac{\cancel{3}}{2} \times \frac{\cancel{1}}{\cancel{2}}\, \cancel{\Gamma\frac{1}{2}}}$

$\Rightarrow \frac{5 \times 4 \times 2 \times 8 \times 4}{11 \times 9 \times 7 \times 5}$

iv) $\int_0^{\pi/2} \sqrt{\cot\theta} \, d\theta$

$\Rightarrow \int_0^{\pi/2} \sin^{-1/2}\theta \cos^{1/2}\theta \, d\theta$

$\Rightarrow \int_0^{\pi/2} \sin^{2(\frac{1}{4})-1}\theta \cos^{2(+\frac{3}{4})-1}\theta \, d\theta$

$\Rightarrow \frac{1}{2} \beta\left(\frac{1}{4}, \frac{3}{4}\right)$

$\Rightarrow \frac{1}{2} \frac{\Gamma\frac{1}{4}\, \Gamma\frac{3}{4}}{\Gamma 1} = \frac{1}{2} \times \Gamma\frac{1}{4}\, \Gamma\left(1-\frac{1}{4}\right)$

$= \frac{1}{2} \times \frac{\pi}{\sin\frac{\pi}{4}}$

$= \frac{1}{2} \times \frac{\pi}{\frac{1}{\sqrt{2}}}$

$= \frac{\sqrt{2}\,\pi}{\cancel{2}\sqrt{2}}$

$= \frac{\pi}{\sqrt{2}}$

Q) Evaluate 1) $\int_0^\infty 3^{-4x^2} dx$

2) $\int_0^\infty a^{-bx^2} dx$

3) $\int_0^1 x^4 \left(\log \frac{1}{x}\right)^3 dx$

4) $\int_0^1 x^2 \left(\log \frac{1}{x}\right)^3 dx.$

1) $3 = e^{\log 3}$

$3^{-4x^2} = e^{-4x^2 \log 3}.$

$$\int_0^\infty 3^{-4x^2} dx = \int_0^\infty e^{-4x^2 \log 3} dx$$

Put $2x\sqrt{\log 3} = t$

$2\sqrt{\log 3}\, dx = dt$

$dx = \frac{1}{2\sqrt{\log 3}} dt.$

$x = 0 \to t = 0$

$x = \infty \to t = \infty$

$$\int_0^\infty e^{-t^2} \frac{1}{2\sqrt{\log 3}} dt = \frac{1}{2\sqrt{\log 3}} \int_0^\infty e^{-t^2} dt$$

$$= \frac{1}{2\sqrt{\log 3}} \frac{\sqrt{\pi}}{2}.$$

$$= \frac{\sqrt{\pi}}{4\sqrt{\log 3}}$$

3) $\int_0^1 x^4 \left(\log \frac{1}{x}\right)^3 dx.$

Put $y = \log\frac{1}{x}$

$y = -\log x$ $\qquad\qquad x = 0 \rightarrow y = \infty$

$x = e^{-y}$ $\qquad\qquad x = 1 \rightarrow y = 0.$

$dx = -e^{-y} dy$

$$\int_\infty^0 (e^{-y})^4 y^3 (-e^{-y})\, dy = \int_0^\infty e^{-5y} y^{4-1} dy$$

$$= \frac{\Gamma 4}{(5)^4} = \frac{3!}{5^4} = \frac{6}{625}$$

2) $\int_0^\infty a^{-bx^2} dx$

Let $a = e^{\log a}$

$a^{-bx^2} = e^{-bx^2 \log a}$

$\therefore \Rightarrow \int_0^\infty e^{-bx^2 \log a}\, dx$

Put $bx^2 \log a = t^2$ $\qquad\qquad x = 0 \longrightarrow t = 0$

$\sqrt{b\log a}\; x = t$ $\qquad\qquad x = \infty \rightarrow t = \infty.$

$\sqrt{b\log a}\; dx = dt$

$dx = \frac{1}{\sqrt{b\log a}} dt$

$$\Rightarrow \int_0^\infty e^{-t^2} \left(\frac{1}{\sqrt{b\log a}}\right) dt = \frac{1}{\sqrt{b\log a}} \int_0^\infty e^{-t^2} dt$$

$$= \frac{\sqrt{\pi}}{2\sqrt{b\log a}}$$

4) $\int_0^1 x^2 \left(\log \frac{1}{x}\right)^3 dx$

Put $y = \log \frac{1}{x}$.

$y = -\log x \Rightarrow x = e^{-y}$.

$dx = -e^{-y} dy$.

$\therefore$ $x = 0 \Rightarrow y = \infty$

$x = 1 \Rightarrow y = 0$.

$\Rightarrow \int_\infty^0 (e^{-y})^2 (y)^3 (-e^{-y}) dy$

$\Rightarrow \int_\infty^0 e^{-2y} y^3 (-e^{-y}) dy$

$\Rightarrow \int_0^\infty e^{-3y} y^3 dy$

$\Rightarrow \int_0^\infty e^{-3y} y^{4-1} dy$

$\Rightarrow \frac{\Gamma 4}{(3)^4} = \frac{\cancel{3}! \cdot 2}{3^3 \times \cancel{3}} = \frac{2}{27}$.

8) S.T $\int_0^\infty x^4 e^{-x^2}dx = \frac{3\sqrt{\pi}}{8}$

put $x^2 = t \Rightarrow x = \sqrt{t}$

$2x\,dx = dt$ $\qquad x=0 \Rightarrow t=0$

$dx = \frac{1}{2x}dt$ $\qquad x=\infty \Rightarrow t=\infty$

$= \frac{1}{2\sqrt{t}}dt$

$\Rightarrow \int_0^\infty t^2 e^{-t}\left(\frac{1}{2\sqrt{t}}\right)dt$

$\Rightarrow \frac{1}{2}\int_0^\infty t^{3/2}e^{-t}\,dt$

$\Rightarrow \frac{1}{2}\int_0^\infty e^{-t}\,t^{\frac{5}{2}-1}dt$

$= \frac{1}{2}\Gamma\frac{5}{2} \quad \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdot\sqrt{\pi} \Rightarrow \frac{3\sqrt{\pi}}{4\cdot 2} = \frac{3}{8}\sqrt{\pi}$

9) Evaluate $4\int_0^\infty \frac{x^2}{1+x^4}dx$ using β-Γ func$^n$

$1-x^2 \Rightarrow x = \sin\theta$

$1+x^2 \Rightarrow x = \tan\theta$

Put $x^2 = \tan\theta \Rightarrow x = \sqrt{\tan\theta}$

~~$2x\,dx =$~~ $1+x^4 = 1+\tan^2\theta = \sec^2\theta$.

~~$\sec\tan\theta\,d\theta$~~

$\Rightarrow x^2 = \tan\theta$

$2x\,dx = \sec^2\theta\,d\theta$. $\qquad x=0 \Rightarrow \theta=0$

$dx = \frac{1}{2\sqrt{\tan\theta}}\sec^2\theta\,d\theta$ $\qquad x=\infty \Rightarrow \theta=\frac{\pi}{2}$.

$\frac{1}{4}\int_0^\infty \frac{x^2}{1+x^4}dx = \cancel{4}\,2\int_0^{\frac{\pi}{2}} \frac{\sqrt{\tan\theta}}{\cancel{1+\tan^2\theta}}\,\frac{1}{2\cancel{\sqrt{\tan\theta}}}\,\cancel{\sec^2\theta}\,d\theta$

$\Rightarrow 2\int_0^{\frac{\pi}{2}} \sin^{1/2}\theta\cos^{-1/2}\theta\,d\theta$.

$\Rightarrow \Gamma\frac{3}{4}\Gamma\frac{1}{4} \Rightarrow \cancel{\frac{1}{4}\Gamma\frac{1}{4}\Gamma\frac{1}{4}}$ $\quad = 2\int_0^{\frac{\pi}{2}} \sin^{2(\frac{3}{4})-1}\theta\cos^{2(\frac{1}{4})-1}\theta\,d\theta$

$\leftarrow \quad = \beta\left(\frac{3}{4},\frac{1}{4}\right) = \frac{\Gamma\frac{3}{4}\,\Gamma\frac{1}{4}}{\Gamma\frac{1}{2}} \Rightarrow$

Q) 1) $\int_0^{\pi/2} \sqrt{\cos x}\, dx \times \int_0^{\pi/2} \frac{1}{\sqrt{\cos x}}\, dx = \pi$

$\Rightarrow \int_0^{\pi/2} \cos^{1/2} dx = \int_0^{\pi/2} \sin^{2(\frac{1}{2})-1} x \cos^{2(\frac{3}{4})-1} x\, dx$

$$= \frac{1}{2}\beta\left(\frac{1}{2}, \frac{3}{4}\right) = \frac{1}{2} \frac{\Gamma\frac{1}{2}\,\Gamma\frac{3}{4}}{\Gamma\frac{5}{4}}$$

$$= \frac{1}{2} \times \frac{\sqrt{\pi} \times \Gamma\frac{3}{4}}{\Gamma\frac{5}{4}}$$

$\Rightarrow \int_0^{\pi/2} \cos^{-1/2} x\, dx = \int_0^{\pi/2} \sin^{2(\frac{1}{2})-(1)} x \cos^{2(\frac{1}{4})-1} x\, dx$

$$= \frac{1}{2}\beta\left(\frac{1}{2}, \frac{1}{4}\right)$$

$$= \frac{1}{2} \times \frac{\Gamma\frac{1}{2}\,\Gamma\frac{1}{4}}{\Gamma\frac{3}{4}}$$

$\Rightarrow \frac{1}{2} \times \frac{\Gamma\frac{1}{2}\,\Gamma\frac{3}{4}}{\Gamma\frac{5}{4}} \times \frac{1}{2} \times \frac{\Gamma\frac{1}{2}\,\Gamma\frac{1}{4}}{\Gamma\frac{3}{4}}$

$\Rightarrow \frac{1}{2} \times \frac{\Gamma\frac{1}{2}\,\cancel{\Gamma\frac{3}{4}}}{\frac{1}{4}\cancel{\Gamma\frac{1}{4}}} \times \frac{1}{2} \times \frac{\Gamma\frac{1}{2}\,\cancel{\Gamma\frac{1}{4}}}{\cancel{\Gamma\frac{3}{4}}}$

$\Rightarrow \frac{1}{\cancel{4}} \times \pi \times \cancel{4}$

$= \pi$

Q) S.T $\int_0^{\pi/2} \left(\sqrt{\tan\theta} + \sqrt{\sec\theta}\right) d\theta$

$$= \frac{1}{2}\Gamma\left(\tfrac{1}{4}\right)\left[\Gamma\left(\tfrac{3}{4}\right) + \frac{\sqrt{\pi}}{\Gamma\left(\tfrac{1}{4}\right)}\right]$$

Q) Evaluate $\int_0^{\infty} \frac{x^4(1+x^5)}{(1+x)^{15}}\,dx$

$$\Rightarrow \int_0^{\infty} \frac{x^4}{(1+x)^{15}}\,dx + \int_0^{\infty} \frac{x^9}{(1+x)^{15}}\,dx.$$

$$= \int_0^{\infty} \frac{x^{5-1}}{(1+x)^{5+10}}\,dx + \int_0^{\infty} \frac{x^{10-1}}{(1+x)^{10+5}}\,dx.$$

$$= \beta(5,10) + \beta(10,5)$$

$$= 2\beta(5,10)$$

$$= 2\,\frac{\Gamma 5\,\Gamma 10}{\Gamma 15}.$$

$$= 2\,\frac{4!\,9!}{14!}$$

Q) P.T $\int_0^{\infty} \frac{x^8(1-x^6)}{(1+x)^{24}}\,dx = 0$

$$\int_0^{\infty} \frac{x^8}{(1+x)^{24}}\,dx + \int_0^{\infty} \frac{-x^{14}}{(1+x)^{24}}\,dx$$

$$= \int_0^{\infty} \frac{x^{9-1}}{(1+x)^{9+15}}\,dx - \int_0^{\infty} \frac{x^{15-1}}{(1+x)^{15+9}}\,dx$$

$$= \beta(9,15) - \beta(15,9)$$

$$= 0.$$

Q) S.T $\int_0^\infty \sqrt{x}\, e^{-x^3} dx = \frac{\sqrt{\pi}}{3}$.

$x^3 = t$ $\qquad$ $x = 0 \Rightarrow t = 0$

$3x^2 dx = dt$ $\qquad$ $x = \infty \Rightarrow t = \infty$

$dx = \frac{1}{3t^{2/3}} dt$

$\Rightarrow \int_0^\infty (t^{1/3})^{1/2} e^{-t} \left(\frac{1}{3t^{2/3}}\right) dt$

$\Rightarrow \frac{1}{3} \int_0^\infty t^{1/6}\, t^{-2/3} e^{-t} dt$

$\Rightarrow \frac{1}{3} \int_0^\infty t^{-\cancel{3}/\cancel{6}_2 \, 1} e^{-t} dt$

$\Rightarrow \frac{1}{3} \int_0^\infty e^{-t}\, t^{\frac{1}{2}-1} dt$

$\Rightarrow$ $\frac{1}{3} \Gamma\frac{1}{2} \Rightarrow \frac{1}{3}\sqrt{\pi}$

$= \frac{\sqrt{\pi}}{3}$

## MULTIPLE INTEGRALS:

The double integral of $x, y$ over the region or is denoted by the symbol $\iint_R f(x,y)\, dR$ (or) $\iint_R f(x,y)\, dx\, dy$.

0

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## Evaluation:

Suppose that R can be described by inequalities of the form $a \le x \le b$, $y_1(x) \le y_2(x)$ so that $y = y_1(x)$ and $y = y_2(x)$ represents the boundary or Then integral $\int_a^b \int_{y_1(x)}^{y_2(x)} f(x,y)\,dy\,dx$

$$= \int_a^b \left[ \int_{y_1(x)}^{y_2(x)} f(x,y)\,dy \right]$$

Similarly if R can be described by the inequalities of the form $a \le y \le b$, $x_1(y) \le x_2(y)$.

$$\iint f(x,y)\,dx\,dy = \int_{y=a}^{b} \left[ \int_{x=x_1(y)}^{x=x_2(y)} f(x,y)\,dx \right] dy.$$

Note: If all the four limits of integratⁿ are constants then the double integral can be evaluated in either way, we first integrate w.r.t $x$ or $y$. (or)

We first integrating w.r.t & then w.r.t y.

over the

$\int f(x,y)\,dR$

R

Q) Evaluate. $\int_0^3 \int_1^2 xy(1+x+y)\,dy\,dx.$

$$\int_0^3 \left[\int_1^2 y(1+x+y)\,dy\right] x\,dx$$

$$\Rightarrow \int_0^3 \left[\int_1^2 y+xy+y^2\,dy\right] x\,dx$$

$$\Rightarrow \int_0^3 \left[\left[\frac{y^2}{2}+x\frac{y^2}{2}+\frac{y^3}{3}\right]_1^2 x\,dx\right.$$

$$\Rightarrow \int_0^3 \left[\left[\frac{4}{2}+\frac{4x}{2}+\frac{8}{3}\right]-\left[\frac{1}{2}+\frac{x}{2}+\frac{1}{3}\right]\right] x\,dx$$

$$\Rightarrow \int_0^3 \left[\frac{3}{2}+\frac{3x}{2}+\frac{7}{3}\right] x\,dx.$$

$$\Rightarrow \int_0^3 \frac{3x}{2}+\frac{3x^2}{2}+\frac{7x}{3}\,dx$$

$$\Rightarrow \int_0^3 \frac{9x+9x^2+14x}{6}\,dx$$

$$= \int_0^3 \frac{23x}{6}+\frac{9x^2}{6}\,dx.$$

$$\Rightarrow \left[\frac{23}{6}\frac{x^2}{2}+\frac{\cancel{9}^{3}}{\cancel{6}_{2}}\frac{x^3}{\cancel{3}_{1}}\right]_0^3$$

$$= \left[\frac{23}{12}\left(\frac{9}{2}\right)+\frac{1}{2}(27)-\frac{23}{12}(0)+\frac{1}{2}(0)\right]$$

$$= \frac{23}{\cancel{12}_{4}}\times\frac{\cancel{9}^{3}}{2}+\frac{27}{2} \Rightarrow \frac{23\times3+27\times2}{4} = \frac{69+54}{4} = \frac{123}{4}$$

Q) Evaluate $\int_0^2 \int_0^x y \, dy \, dx$.

$$\Rightarrow \int_{x=0}^{2} \left[ \int_{y=0}^{x} y \, dy \right] dx$$

$$\Rightarrow \int_0^2 \left[ \frac{y^2}{2} \right]_0^x dx$$

$$\Rightarrow \int_0^2 \frac{x^2}{2} dx$$

$$\Rightarrow \frac{1}{2} \left[ \frac{x^3}{3} \right]_0^2 = \frac{1}{2} \left[ \frac{8}{3} \right] = \frac{4}{3}.$$

Q) $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{1}{1+x^2+y^2} \, dy \, dx$

$$\Rightarrow \int_0^1 \left[ \int_0^{\sqrt{1+x^2}} \frac{1}{1+x^2+y^2} dy \right] dx$$

Put $1+x^2 = a^2$

$\sqrt{1+x^2} = a$

$\Rightarrow$

Q) $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{1}{1+x^2+y^2}\, dy\, dx$

$\Rightarrow \int_0^1 \left[ \int_0^{\sqrt{1+x^2}} \frac{1}{1+x^2+y^2} dy \right] dx$   put $1+x^2 = a^2$
$\sqrt{1+x^2} = a$

$\Rightarrow \int_0^1 \left[ \int_0^a \frac{1}{a^2+y^2} dy \right] dx$

$\Rightarrow \int_0^1 \left[ \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) \right]_0^a dx$

$\Rightarrow \int_0^1 \left[ \frac{1}{a} \frac{\pi}{4} - 0 \right] dx.$

$\Rightarrow \frac{\pi}{4} \int_0^1 \frac{1}{\sqrt{1+x^2}} \Rightarrow \frac{\pi}{4} \left( \log(x+\sqrt{1+x^2}) \right)_0^1$

$\Rightarrow \frac{\pi}{4} \left( \log(1+\sqrt{2}) \right)$

Q) Evaluate $\int_0^2 \int_0^x e^{x+y}\, dy\, dx.$

$\Rightarrow \int_{x=0}^2 \left[ \int_0^x e^y dy \right] e^x dx.$

$\Rightarrow \int_{x=0}^2 [e^y]_0^x e^x dx$

$= \int_0^2 (e^x - 1)e^x dx = \int_0^2 (e^{2x} - e^x) dx$

$1+x^2=a^2$

$\sqrt{1+x^2}=a$

$x^2))_0^1$

$$= \left[\frac{e^{2x}}{2} - e^x\right]_0^2$$

$$= \frac{e^4}{2} - e^2 - \left[\frac{1}{2} - 1\right]$$

$$= \frac{e^4}{2} - e^2 + \frac{1}{2} = \frac{1}{2}(e^4 - 2e^2 + 1)$$

$$= \frac{1}{2}(e^2-1)^2$$

Q) Evaluate $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dx\, dy$.

$$= \int_0^\infty \left[\int_0^\infty e^{-y^2} dy\right] e^{-x^2} dx$$

$$= \int_0^\infty \frac{\sqrt{\pi}}{2} e^{-x^2} dx$$

$$= \frac{\sqrt{\pi}}{2} \int_0^\infty e^{-x^2} dx$$

$$= \frac{\sqrt{\pi}}{2} \cdot \frac{\sqrt{\pi}}{2} = \frac{\pi}{4}$$

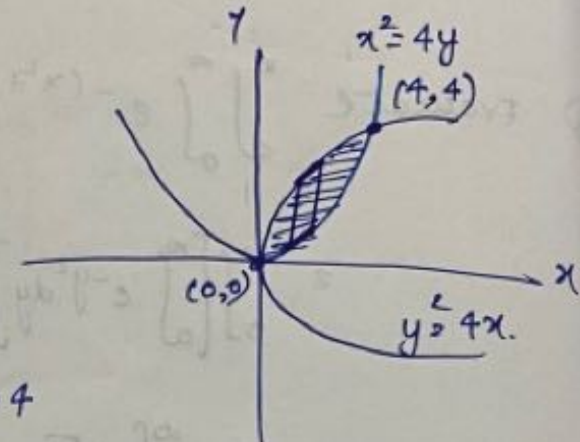
Q) Evaluate $\int_0^4 \int_0^{x^2} e^{y/x}\, dy\, dx$.

" $\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2}\, dy\, dx$.

Q) Evaluate i) $\iint_R y \, dx \, dy$

ii) $\iint_R y^2 \, dx \, dy$

Where R is the region bounded by the parabola as $y^2 = 4x$ & $x^2 = 4y$




$$\frac{x^4}{4^2} = 4x$$

$$x^4 = 4^3 x \Rightarrow x(x^3 - 4^3) = 0$$

$$\Rightarrow x = 0 \text{ (or) } x = 4$$

when $x = 0 \Rightarrow y = 0$

$$x = 4 \Rightarrow y = \frac{4^2}{4} = 4.$$

$\therefore (0,0), (4,4)$

$\Rightarrow$ Fix $x : 0 \to 4$

$$y : \frac{x^2}{4} \to 2\sqrt{x}$$

$$\Rightarrow \int_0^4 \left[ \int_{\frac{x^2}{4}}^{2\sqrt{x}} y \, dy \right] dx$$

$$\Rightarrow \int_0^4 \left[ \frac{y^2}{2} \right]_{\frac{x^2}{4}}^{2\sqrt{x}} dx = \int_0^4 \left[ \frac{\cancel{4}^{2} x}{\cancel{2}} - \frac{x^4}{32} \right] dx$$

$$\frac{\left(\frac{x^2}{4}\right)^2}{2}$$

the parabola


$$\Rightarrow \int_0^4 \left(2x - \frac{x^4}{32}\right) dx$$

$$\Rightarrow \left[\frac{\cancel{2}x^2}{\cancel{2}} - \frac{x^5}{32\times5}\right]_0^4$$

$$\Rightarrow \left[16 - \frac{(4)^5}{32\times5} - 0\right]$$

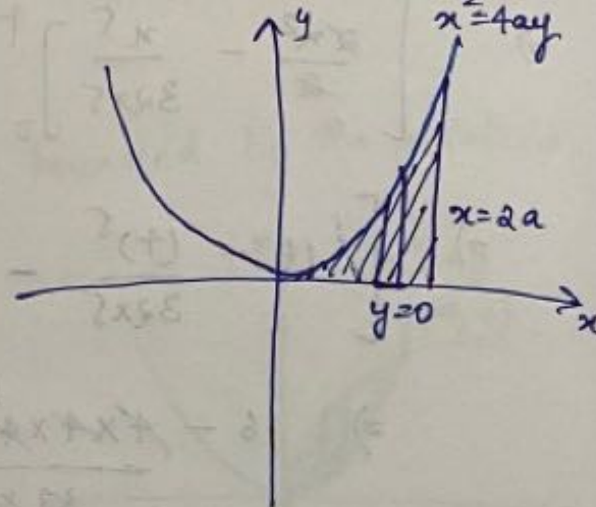
$$\Rightarrow 16 - \frac{\cancel{4}\times\cancel{4}\times\cancel{4}^{\,2}\times4\times4}{\cancel{32}_{\cancel{8}\,\cancel{2}}\times5}$$

$$\Rightarrow 16 - \frac{32}{5} \Rightarrow \frac{80-32}{5} = \frac{48}{5}$$

Q) Evaluate $\iint xy\,dx\,dy$, where R is the region bounded by x-axis ~~oo~~ ordinate $x=2a$ and the curve $x^2=4ay$.

$x=2a$

$x^2=4ay$




$\Rightarrow 4a^2=4ay$

$4a(a-y)=0$

$\Rightarrow y=a$

when $y=a \Rightarrow x=2a$. On x-axis $y=0 \Rightarrow x=0$

$\therefore$ point of intersectⁿ $(0,0)\,(2a,a)$

Fix $x$, $x: 0 \to 2a$

$y: 0 \to x^2/4a$

$$\iint xy\,dy\,dx = \int_{x=0}^{2a}\left[\int_0^{\frac{x^2}{4a}} y\,dy\right] x\,dx.$$

$$\Rightarrow \int_0^{2a}\left[\frac{y^2}{2}\right]^{x^2/4a} x\,dx$$

$$\Rightarrow \frac{1}{2}\int_0^{2a}\frac{x^4}{16a^2}\,x\,dx.$$

$$\Rightarrow \frac{1}{32a^2}\int_0^{2a} x^5\,dx$$

$$= \frac{1}{32a^2}\left[\frac{x^6}{6}\right]_0^{2a}$$

$$= \frac{2^6 a^6}{32\,a^2\times 6} = \frac{a^4}{3}$$

Q) bounded by

when $y=0 \Rightarrow$

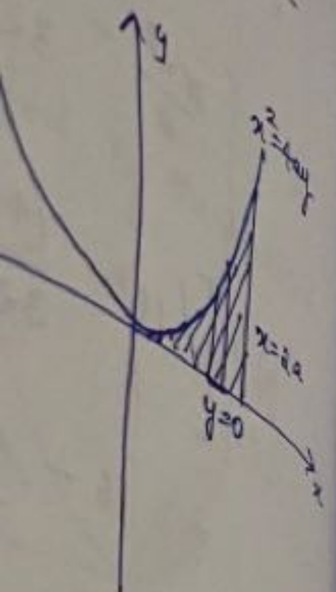
$\Rightarrow x$

$\frac{y^2}{b^2} = 1-\frac{x^2}{a^2}$

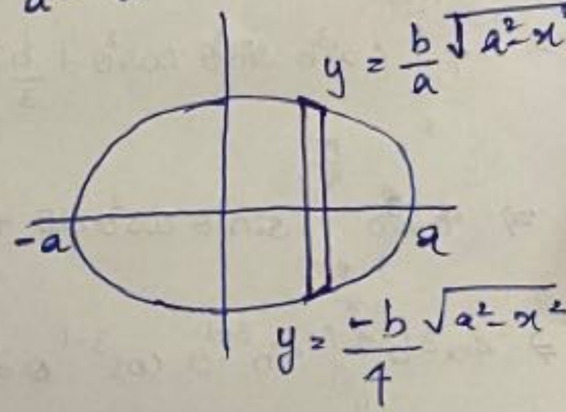
$y^2 =$

$\Rightarrow \iint (x^2+y^2)$

Pu




8) Evaluate $\iint (x^2+y^2)\,dx\,dy$. over the area bounded by $\cancel{\frac{x^2}{a^2}}$ ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$




when $y=0 \Rightarrow \frac{x^2}{a^2}=1$

$\Rightarrow x^2=a^2$

$x=\pm a$, --

$$\frac{y^2}{b^2}=1-\frac{x^2}{a^2}=\frac{a^2-x^2}{a^2}$$

$$y^2=\frac{b^2}{a^2}(a^2-x^2) \Rightarrow y=\pm\frac{b}{a}\sqrt{a^2-x^2}$$

$$\Rightarrow \iint (x^2+y^2)\,dx\,dy = \int_{x=a}^{a}\int_{y=\frac{-b}{a}\sqrt{a^2-x^2}}^{y=\frac{b}{a}\sqrt{a^2-x^2}} (x^2+y^2)\,dx\,dy.$$

$$=2\int_{-a}^{a}\left[\int_0^{\frac{b}{a}\sqrt{a^2-x^2}} (x^2+y^2)\,dy\right]dx$$

$$=2\int_{-a}^{a}\left[x^2y+\frac{y^3}{3}\right]_0^{\frac{b}{a}\sqrt{a^2-x^2}} dx.$$

$$=2\int_{-a}^{a}\left[x^2\frac{b}{a}\sqrt{a^2-x^2}+\frac{1}{3}\frac{b^3}{a^3}(a^2-x^2)^{3/2}\right]dx$$

$$=4\int_0^a\left[x^2\frac{b}{a}(a^2-x^2)^{1/2}+\frac{1}{3}\frac{b^3}{a^3}(a^2-x^2)^{3/2}\right)dx$$

Put $x=a\sin\theta$

$dx=a\cos\theta\,d\theta$

when $x=0 \Rightarrow \theta=0$

$x=a \Rightarrow \theta=\pi/2$

xis $y=0 \Rightarrow x=0$

a,a)

x dx.

x

$5\int^a$

$2a$

$$\Rightarrow 4\int_0^{\frac{\pi}{2}} \left[a^2 \frac{b}{a} \sin^2\theta (a\cos\theta) + \frac{1}{3}\frac{b^3}{a^3} a^3\cos^3\theta\right] a\cos\theta$$

$$\Rightarrow 4\int_0^{\pi/2} \left(a^3 b \sin^2\theta \cos^2\theta + \frac{b^3}{3} a\cos^4\theta\right] d\theta.$$

$$\Rightarrow 4a^3b \int_0^{\frac{\pi}{2}} \sin^2\theta \cos^2\theta\, d\theta + \frac{4ab^3}{3}\int_0^{\frac{\pi}{2}} \cos^4\theta\, d\theta.$$

$$\Rightarrow 4a^3b\int_0^{\frac{\pi}{2}} \sin^{3-1}\theta \cos^{3-1}\theta\, d\theta + \frac{4ab^3}{3}\int_0^{\frac{\pi}{2}} \sin^{2(\frac{1}{2})-1}\theta \cos^{5-1}\theta\, d\theta$$

$$\Rightarrow 4a^3b \times \frac{1}{2}\beta(2,2) + \frac{4ab^3}{3}\beta\left(\frac{1}{2},5\right)\times\frac{1}{2}$$

$$\Rightarrow 4a^3b \times \frac{1}{2} \times \frac{\Gamma 2\, \Gamma 2}{\Gamma 4} + \frac{4ab^3}{3} \times \frac{1}{2} \times \frac{\Gamma\frac{1}{2}\, \Gamma 5}{\Gamma\frac{11}{2}}.$$

$$\Rightarrow 4a^3b \times \frac{1}{2} \times \frac{1!\,1!}{3!} + \frac{4ab^3}{3} \times \frac{1}{2} \times \frac{\sqrt{\pi}\, 4!}{\frac{9}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}$$

$$\Rightarrow \cancel{4}a^3b \times \frac{1}{\cancel{2}} \times \frac{1}{3\times\cancel{2}} + \frac{4ab^3}{3} \times \frac{1}{\cancel{2}} \times \frac{4\times\cancel{3}\times\cancel{2}\times 2\times 2\times 2\times 2\times 2}{9\times 7\times 5\times \cancel{3}}$$

$$\Rightarrow \frac{a^3b}{3} + \frac{4ab^3}{3} \times \frac{4\times 8\times 4}{315}$$

$$\begin{array}{r} 45 \\ \times 7 \\ \hline 315 \end{array}$$

Q) Evaluate $\iint (x^2+y^2)\,dx\,dy$ where R is the region in the +ve quadrant $x+y \le 1$.

Q) Evaluate $\iint y\,dx\,dy$, when R is the domain bounded by y-axis, the curve $y=x^2$ and the line $x+y=2$.

## Double Integrals in polar-coordinates:

1) Evaluate $\int_0^{\pi}\int_0^{a\sin\theta} r\,dr\,d\theta$

$$\Rightarrow \int_0^{\pi}\left[\int_{r=0}^{a\sin\theta} r\,dr\right]d\theta = \int_0^{\pi}\left[\frac{r^2}{2}\right]_0^{a\sin\theta}d\theta$$

$$\Rightarrow \frac{1}{2}\int_0^{\pi} a^2\sin^2\theta\,d\theta$$

$$\Rightarrow \frac{a^2}{2}\int_0^{\pi}\sin^2\theta\,d\theta = \frac{a^2}{2}\int_0^{\pi}\frac{1-\cos 2\theta}{2}d\theta$$

$$\Rightarrow \frac{a^2}{4}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi}$$

$$\Rightarrow \frac{a^2}{4}\left[\pi - 0\right] \Rightarrow \frac{\pi a^2}{4}$$

2) $\int_0^{\infty}\int_0^{\frac{\pi}{2}} e^{-r^2}\, r\,d\theta\,dr$

$$\Rightarrow \int_0^{\infty}\left[\int_0^{\frac{\pi}{2}} d\theta\right]e^{-r^2}r\,dr.$$

$$\Rightarrow \int_0^{\infty}[\theta]_0^{\pi/2} \Rightarrow \frac{\pi}{2}\int_0^{\infty}e^{-r^2}r\,dr \Rightarrow \frac{\pi}{2}\int_0^{\infty}e^{-t}\frac{dt}{2}$$

$$\Rightarrow \frac{\pi}{4}\int_0^{\infty}e^{-t}\,dt = \frac{\pi}{4}\left[e^{-\infty}-e^{-0}\right]$$

$$= \frac{\pi}{4}$$

Q) Evaluate $\int_0^{\frac{\pi}{4}}\int_0^{a\sin\theta} \frac{1}{\sqrt{a^2-r^2}}\, r\,dr\,d\theta$

$$\Rightarrow \int_0^{\frac{\pi}{4}}\left[\int_0^{a\sin\theta} \frac{r}{\sqrt{a^2-r^2}}\,dr\right]d\theta$$

$$\Rightarrow \int_0^{\pi/4}\left[\int_{t=a^2}^{a^2\cos^2\theta} \frac{1}{\sqrt{t}}\left(-\frac{dt}{2}\right)\right]d\theta$$

Put $a^2-r^2=t$

$-2r\,dr=dt$

$r\,dr=\frac{dt}{-2}$

$r=0 \Rightarrow t=a^2$

$r=a\sin\theta$

$\Rightarrow t=a^2-a^2\sin^2\theta$

$=a^2\cos^2\theta$

$$\Rightarrow \int_0^{\pi/4}\left[2\sqrt{t}\left(-\frac{dt}{2}\right)\right]d\theta$$

$$\Rightarrow -\int_0^{\pi/4}\left[\sqrt{a^2\cos^2\theta}-\sqrt{a^2}\right]d\theta$$

$$\Rightarrow -\int_0^{\pi/4}(a\cos\theta-a)\,d\theta$$

$$\Rightarrow -a\int_0^{\pi/4}(\cos\theta-1)\,d\theta$$

$$\Rightarrow -a\left[\sin\theta-\theta\right]_0^{\pi/4} = -a\left[\sin\frac{\pi}{4}-\frac{\pi}{4}\right]$$

$$= -a\left[\frac{1}{\sqrt{2}}-\frac{\pi}{4}\right]$$

$$= a\left[\frac{\pi}{4}-\frac{1}{\sqrt{2}}\right]$$

Q) Evaluate

$r=a(1-c$

$\Rightarrow \int_0^{\pi}\left[\int_0^{a(1-c}\right.$

$=\int_0^{\pi}\left[\right.$

$=\frac{1}{2}$

$\Rightarrow \frac{a^2}{2}$

$\Rightarrow \frac{a^2}{2}$

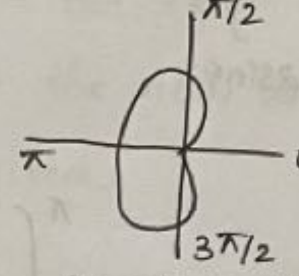
$\Rightarrow 4$

Q) Evaluat

b/w the

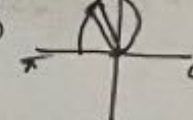
$\oint$

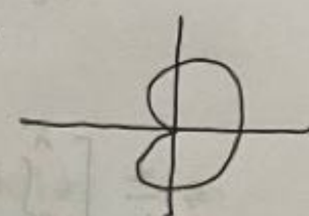
Q) Evaluate $\iint r\sin\theta \, dr \, d\theta$ over the cardiod $r = a(1-\cos\theta)$ above the initial line.




$\Rightarrow \int_0^{\pi}\left[\int_0^{a(1-\cos\theta)} r \, dr\right]\sin\theta \, d\theta$

$= \int_0^{\pi}\left[\frac{r^2}{2}\right]_0^{a(1-\cos\theta)} \sin\theta \, d\theta \Rightarrow$



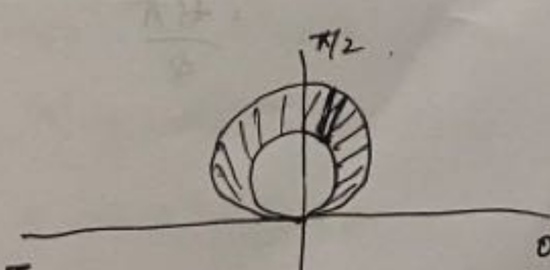

$= \frac{1}{2}\int_0^{\pi} a^2(1-\cos\theta)^2 \sin\theta \, d\theta.$

$\Rightarrow \frac{a^2}{2}\int_0^{\pi}(1-\cos\theta)^2 \sin\theta \, d\theta \quad \Rightarrow 1-\cos\theta = t$

$\Rightarrow \frac{a^2}{2}\int_0^{2} t^2 dt \Rightarrow \frac{a^2}{2}\left[\frac{t^3}{3}\right]_0^2 . \Rightarrow \cancel{\frac{a^2}{2}\left[\frac{1}{\sqrt{2}} - \frac{\pi}{4}\right]}$

$\Rightarrow \frac{4a^2}{3}. \quad \Rightarrow \cancel{\frac{a^2}{2} \times \frac{8}{3} \Rightarrow \frac{4a^2}{3}} \quad = \cancel{a\left[\frac{\pi}{4} - \frac{1}{\sqrt{2}}\right]}$

Q) Evaluate $\iint r^3 \, dr \, d\theta$ over the area included b/w the circles $r = 2\sin\theta$ and $r = 4\sin\theta$



$$\int_0^{\pi}\int_{2\sin\theta}^{4\sin\theta} r^3\,dr\,d\theta = \int_0^{\pi}\left[\int_{2\sin\theta}^{4\sin\theta} r^3\,dr\right]d\theta$$

$$= \int_0^{\pi}\left[\frac{r^4}{4}\right]_{2\sin\theta}^{4\sin\theta} d\theta$$

$$\Rightarrow \frac{1}{4}\left[\int_0^{\pi}(4^4\sin^4\theta - 2^4\sin^4\theta)\,d\theta\right.$$

$$\Rightarrow \frac{1}{4}\int_0^{\pi} 16\sin^4\theta\,(16-1)\,d\theta.$$

$$\Rightarrow \frac{1}{4}\times 16\times 15\int_0^{\pi}\sin^4\theta\,d\theta$$

$$\Rightarrow \frac{1}{4}\times 16\times 15\times 2\int_0^{\pi/2}\sin^4\theta\,d\theta$$

$(\sin\theta)^4$
$(\sin(-\theta))^4$
$\Rightarrow(-\sin\theta)^4$
$=\sin\theta.$

$$\Rightarrow \frac{1}{\cancel{4}_1}\times \cancel{16}^{4}\times 15\times 2\int_0^{\pi/2}\sin^{2(\frac{5}{2})-1}\theta\cos^{2(\frac{1}{2})-1}\theta\,d\theta$$

$$\Rightarrow 120\times\frac{\Gamma\frac{5}{2}\,\Gamma\frac{1}{2}}{\Gamma 3} \Rightarrow 120\times\frac{\frac{3}{2}\times\frac{1}{2}\times\Gamma\frac{1}{2}\,\Gamma\frac{1}{2}}{\Gamma 3}$$

$$= \frac{45\pi}{2}.$$

Q Change

→ Let $x = f($

b/w the

$u, v$ of

$\iint_R F(x$

Note: To ch

put $x =$

Q Evaluate

into polar

$\int_0^a\int_0^{\sqrt{a^2-x}}$

Given $y$

To change

$x = r\cos\theta$

$x^2 = r^2 c$

$\Rightarrow$

# Change of Variables in double integrals.

→ Let $x = f(u, v)$ and $y = g(u, v)$ be the relatⁿ b/w the old variables $x, y$ with the new variables $u, v$ of the new coordinate system.

$$\iint_R F(x, y)\, dx\, dy = \iint F(f, g)\, |J|\, du\, dv.$$

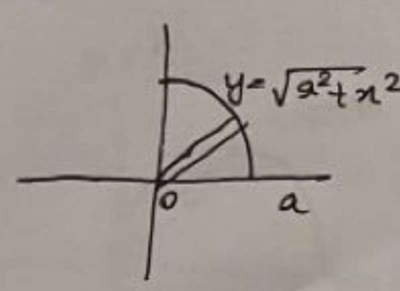
**Note**: To change the cartesian to polar form,
put $x = r\cos\theta$, $y = r\sin\theta$, $dx\,dy = r\,dr\,d\theta$

Evaluate the following integral by transforming into polar coordinates.

$$\int_0^a \int_0^{\sqrt{a^2 - x^2}} y\sqrt{x^2 + y^2}\, dx\, dy.$$

Given $y = 0$, $y = \sqrt{a^2 - x^2} \Rightarrow x^2 + y^2 = a^2$

$x = 0$, $x = a$.




To change cartesian to polar put

$x = r\cos\theta$ $\quad y = r\sin\theta$

$x^2 = r^2\cos^2\theta$ $\quad y^2 = r^2\sin^2\theta.$

$\Rightarrow x^2 + y^2 = r^2.$

$\Rightarrow r^2 = a^2 \Rightarrow r = \pm a.$

$$\Rightarrow \int_0^a \int_0^{\sqrt{a^2-x^2}} y\sqrt{x^2+y^2}\,dx\,dy = \int_0^{\frac{\pi}{2}}\int_0^a r\sin\theta\,(r)\,r\,dr\,d\theta$$

$$= \int_0^{\pi/2}\left[\int_0^a r^3\,dr\right]\sin\theta\,d\theta$$

$$= \int_0^{\pi/2}\left[\frac{r^4}{4}\right]_0^a \sin\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{a^4}{4}\sin\theta\,d\theta$$

$$= \frac{a^4}{4}\left[-\cos\theta\right]_0^{\pi/2}$$

$$= -\frac{a^4}{4}\left(\cos\pi/2 - \cos 0\right)$$

$$= \frac{-a^4}{4}(0-1) = \frac{a^4}{4}$$

Q) Transform the integral into polar coordinates and hence find the integral.

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{x^2+y^2}\,dx\,dy.$$

Q) Evaluate the double integral ~~θ~~ to

$$\int_0^a \int_0^{\sqrt{a^2-y^2}} (x^2+y^2)\,dx\,dy.$$

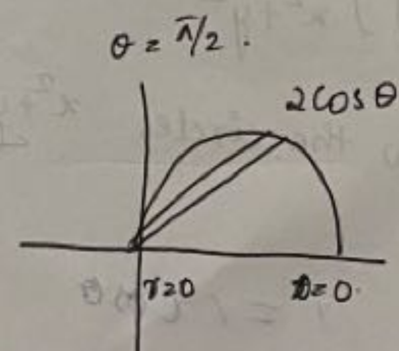
Q) Evaluate D.I,

$$\int_0^2 \int_0^{\sqrt{2x-x^2}} (x^2+y^2)\,dx\,dy.$$

Given $y=0 \Rightarrow y=\sqrt{2x-x^2}$, $y^2=2x-x^2 \Rightarrow x^2+y^2=2x$

$x=0 \Rightarrow x=2.$

$x^2+y^2 = r^2(\sin^2\theta+\cos^2\theta) = r^2$




$\theta : 0 \to \frac{\pi}{2}$ $\qquad r^2 = 2r\cos\theta$

$r : 0 \to 2\cos\theta$ $\qquad r = 2\cos\theta$

$$\int_0^2 \int_0^{\sqrt{2x-x^2}} (x^2+y^2)\,dx\,dy = \int_0^{\frac{\pi}{2}} \int_0^{2\cos\theta} r^2\, r\,dr\,d\theta$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \left[\frac{r^4}{4}\right]_0^{2\cos\theta} d\theta.$$

$$\Rightarrow \int_0^{\pi/2} \frac{16\cos^4\theta}{4}\, d\theta$$

$$\Rightarrow 4\int_0^{\frac{\pi}{2}} \cos^4\theta\, d\theta$$

$$\Rightarrow 4\int_0^{\frac{\pi}{2}} \sin^{2\left(\frac{1}{2}\right)-0}\theta \cos^{2\left(\frac{5}{2}\right)-1}\theta\, d\theta$$

$$\Rightarrow 4\,\frac{\Gamma\frac{1}{2}\,\Gamma\frac{5}{2}}{\Gamma 3}$$

$$\Rightarrow \frac{\cancel{4}^{2}\times \Gamma\frac{1}{2}\;\frac{3}{\cancel{2}}\cdot\frac{1}{2}\,\Gamma\frac{1}{2}}{\cancel{2}!}$$

$$\Rightarrow (\sqrt{\pi})^2\times\frac{3}{\cancel{4}\,2} \Rightarrow \frac{3(\sqrt{\pi})^2}{\cancel{4}\,2} \Rightarrow \frac{3\pi}{\cancel{4}\,2}.$$

9) By changing into polar coordinates, evaluate

$\iint \frac{x^2y^2}{x^2+y^2}\, dx\, dy$ over the angular region

b/w the circles $x^2+y^2=a^2$, $x^2+y^2=b^2$ $(b>0)$

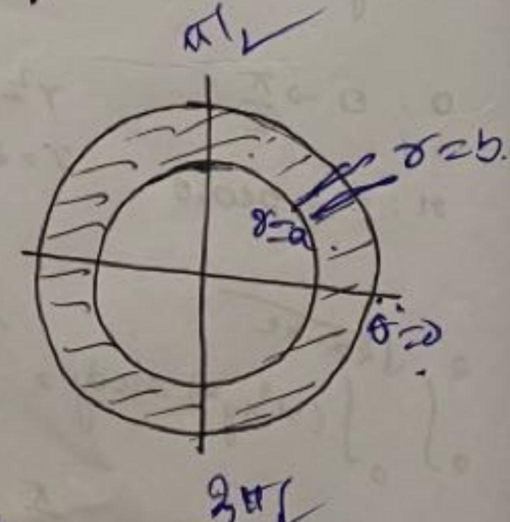
$x = r\cos\theta$

$y = r\sin\theta$

$x^2+y^2 = r^2$

$r^2 = a^2$

$r = a$




$\theta : 0 \to 2\pi$

$r : a \to b.$

$$\int_0^a \int_0^{\sqrt{a^2-y^2}}$$

Given.

$x = r\cos\theta$

$r^2\cos^2\theta + r$

$r^2(1) =$

$r =$

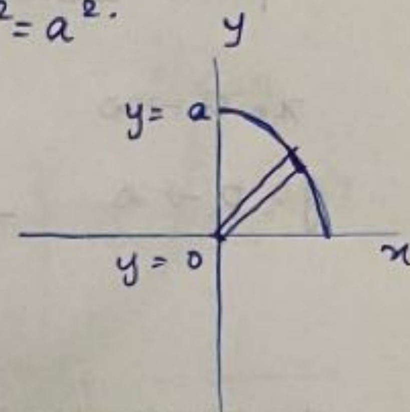
$$\Rightarrow \int_0^a \int_0^{\sqrt{a^2-y^2}} (x$$

Homework.

2) $\int_0^a \int_0^{\sqrt{a^2-y^2}} (x^2+y^2)\,dx\,dy.$

Given. $x=0 \rightarrow x=\sqrt{a^2-y^2} = x^2+y^2=a^2.$

$y=0 \rightarrow y=a.$




$x = r\cos\theta \quad y = r\sin\theta.$

$r^2\cos^2\theta + r^2\sin^2\theta = a^2$

$r^2(1) = a^2$

$r = \pm a.$

$$\Rightarrow \int_0^a \int_0^{\sqrt{a^2-y^2}} (x^2+y^2)\,dx\,dy = \int_0^a \int_0^{\frac{\pi}{2}} r^2\,dr\,d\theta.$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \left[\int_0^a r^2\,dr\right] d\theta$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \left[\frac{r^3}{3}\right]_0^a d\theta.$$

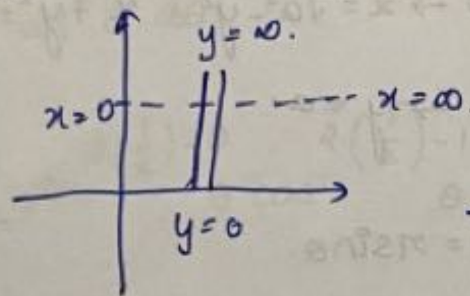
$$\Rightarrow \int_0^{\frac{\pi}{2}} \frac{a^3}{3}\,d\theta = \frac{a^3}{3}\left[\theta\right]_0^{\frac{\pi}{2}}$$

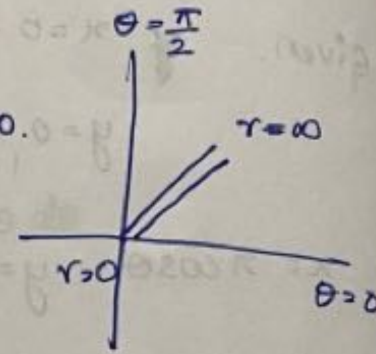
$$\Rightarrow \frac{a^3}{3}\left[\frac{\pi}{2}\right] = \frac{a^3\pi}{6}.$$

Q) Evaluate $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$ by changing to polar coordinates.

$x : 0 \to \infty$

$y : 0 \to \infty$




$x = r\cos\theta \quad y = r\sin\theta.$

$dx\,dy = r\,dr\,d\theta$

$$\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dx\,dy = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} e^{-r^2}\, r\,dr\,d\theta.$$

$$= \int_{\theta=0}^{\frac{\pi}{2}} \left[\int_{r=0}^{\infty} e^{-r^2}\, r\,dr\right] d\theta$$

$r^2 = t$

$2r\,dr = dt$

$r\,dr = \frac{dt}{2}$

$r = 0 \Rightarrow t = 0$

$r = \infty \Rightarrow t = \infty$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \left[\frac{1}{2}\int_0^\infty e^{-t}\,dt\right] d\theta$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \left[\frac{1}{2}\left[-e^{-t}\right]_0^\infty\right] d\theta$$

$$\Rightarrow \frac{1}{2}\int_0^{\frac{\pi}{2}} -1\,d\theta.$$

$$\Rightarrow -\frac{1}{2}\left[\theta\right]_0^{\frac{\pi}{2}}$$

$$= -\frac{\pi}{4}.$$

of CHANGE OF

Q) change the

$\int_0^{4a}$

$y = \frac{x^2}{4a} \Rightarrow$

$y = 2\sqrt{ax}$

$x = 0 \to x =$

$y^2 = 4ax$

$x = \frac{y^2}{4a}$

$x = \frac{4a \times 4a}{4a}$

$x = 4a$

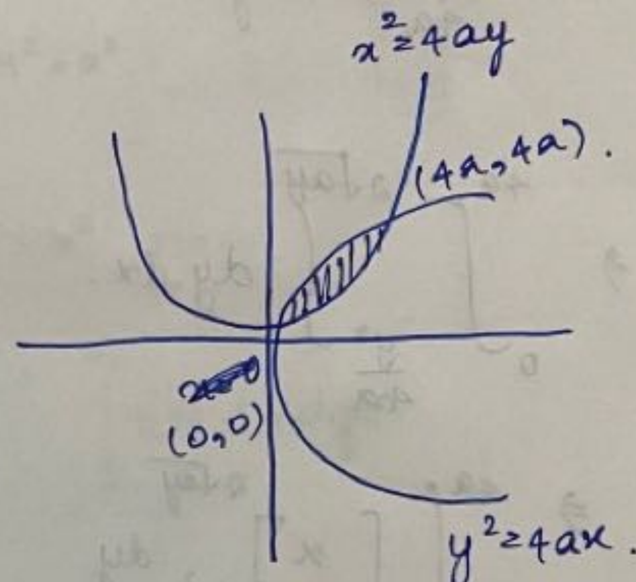
$x = \frac{0}{4a}$

$x = 0.$

## CHANGE OF ORDER OF INTEGRATION:

Q) change the order of integration and evaluate

$$\int_0^{4a}\int_{\frac{x^2}{4a}}^{2\sqrt{ax}} dy\,dx$$




$y = \frac{x^2}{4a} \Rightarrow x^2 = 4ay$

$y = 2\sqrt{ax} \Rightarrow y^2 = 4ax$.

$x=0 \rightarrow x = 4a$.

$y^2=4ax \quad , \quad x^2 = 4ay$

$x = \frac{y^2}{4a} \qquad \left(\frac{y^2}{4a}\right)^2 = 4ay. \Rightarrow y^2\left(\frac{y^3}{4a} - 4a\right)=0.$

$x = \frac{4a \times \cancel{4a}}{\cancel{4a}} \qquad \frac{y^{\cancel{4}3}}{16a^2} = 4a\cancel{y} \qquad y=0$ or

$\Rightarrow y^3 = 4^3 \times a^3 \qquad \left(\frac{y^3}{4a} = 4a\right)$

$x = 4a \qquad \Rightarrow y = 4a. \qquad y = 4a.$

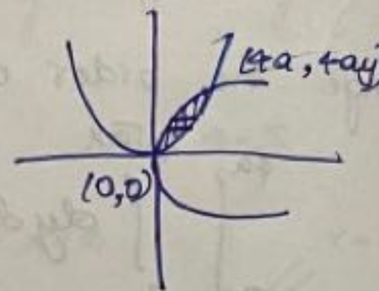
$x = \frac{0}{4a}$

$x=0$.

To change the order of integrat$^n$ (fix $y$).

$y : 0 \to 4a$

$x : \frac{y^2}{4a} \to 2\sqrt{ay}$




$$\Rightarrow \int_0^{4a} \int_{\frac{y^2}{4a}}^{2\sqrt{ay}} dy\, dx.$$

$$\Rightarrow \int_0^{4a} [x]_{\frac{y^2}{4a}}^{2\sqrt{ay}} dy$$

$$\Rightarrow \int_0^{4a} \left(2\sqrt{ay} - \frac{y^2}{4a}\right) dy$$

$$\Rightarrow \left[2\sqrt{a}\,\frac{y^{3/2}}{3/2} - \frac{y^3}{3 \times 4a}\right]_0^{4a}$$

$$\Rightarrow \left[2\sqrt{a} \times \frac{2}{3} \times (4a)^{3/2} - \frac{(4a)^3}{3 \times 4a}\right]$$

$$\Rightarrow \left[2\sqrt{a} \times \frac{2}{3} \times (2\sqrt{a})^3 - \frac{(4a)^3}{3 \times 4a}\right]$$

$$\Rightarrow \left[(4a)^2 \left(\frac{2}{3} - \frac{4a}{3 \times 4a}\right)\right] = 4(a)^2\left(\frac{2}{3} - \frac{1}{3}\right) = \frac{4a^2}{3}$$

$$\Rightarrow \cancel{(4a)^2 \left(\frac{8a - 1}{3 \times 4a}\right) \Rightarrow \frac{32a^2 - 4a}{3}}$$

Home work:

1) $\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{x^2+y$

$\Rightarrow y = 0 \to y = \sqrt{}$

$x = 0 \to x = a$

$x = r\cos\theta,\ y =$

$\Rightarrow \int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{x^2+y$

$\Rightarrow \int_0^{\frac{\pi}{2}} [$

$\Rightarrow \int_0^{\frac{\pi}{2}} [\frac{r}{3}$

2) $\iint \frac{x^2y^2}{x^2+y^2}$

$x^2+y^2 = b^2$

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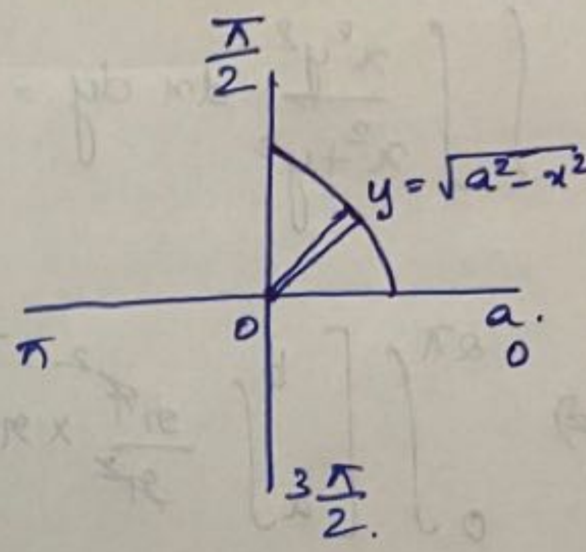
## Home Work:




1) $$\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{x^2+y^2}\, dx\, dy.$$

$\Rightarrow$ $y=0 \to y=\sqrt{a^2-x^2} \Rightarrow y^2+x^2=a^2$.

$x=0 \to x=a.$

$x=r\cos\theta,\ y=r\sin\theta,\ x^2+y^2=r^2,\ r=a.$

$$\Rightarrow \int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{x^2+y^2}\, dx\, dy = \int_0^{\frac{\pi}{2}} \int_0^a r\, r\, dr\, d\cancel{y}\theta$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \left[ \int_0^a [r^2]\, dr \right] d\theta$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \left[\frac{r^3}{3}\right]_0^a d\theta \Rightarrow \int_0^{\frac{\pi}{2}} \frac{a^3}{3}\, d\theta$$

$$\Rightarrow \frac{a^3}{3}\left[\theta\right]_0^{\frac{\pi}{2}} \Rightarrow \frac{\pi a^3}{6}.$$

2) $\iint \frac{x^2y^2}{x^2+y^2}\, dx\, dy$. over the region $x+y^2=a^2$,

$x^2+y^2=b^2\ (b>0)$.