UNIT - I

INTERPOLATION

→ Interpolation is the process of finding the most appropriate estimate for missing data for making most probable estimate it requires.

① The frequency distribution is normal and not marked by ups and downs.

② Interpolation is a technique is used in various disciplines like economics, business, population, studies etc. It's used to fill the gaps in the satisfied data for the sake of continuity of info.

Given tabular form.

x: x_0 x_1 x_2 --- x_n

y: y_0 y_1 y_2 --- y_n

satisfy the relation $y = f(x)$ and the explicit function of $f(x)$ is normal then $f(x)$ can be replaced by a simpler function $\phi(x)$ such that $f(x)$ & $\phi(x)$ agree with the set of tabulated point and any other value may be calculated from $\phi(x)$. The process of finding $\phi(x)$ is known as interpolation and $\phi(x)$ is known as interpolation function.

Here we find various interpolation polynomial using the concepts of forward difference, backward difference and central difference

Forward difference:-

Consider a function $y = f(x)$ of an independent variable x. Let $y_0, y_1, y_2 \cdots y_r$ be the values of y corresponding to values $x_0, x_1, x_2 \ldots x_r$ of x resp. then the difference $y_1 - y_0, y_2 - y_1, y_3 - y_2, \ldots y_{r+1} - y_r$ are called the first forward differences of y and we denote them by $\Delta y_0, \Delta y_1, \Delta y_2 \ldots \Delta y_r$ and Here, the symbol Δ is called forward difference operator.

$$\Delta y_0 = y_1 - y_0 , \Delta y_1 = y_2 - y_1 \ldots \Delta y_r = y_{r+1} - y_r .$$

where $r = 0, 1, 2 \ldots$

The difference of 1st order differences are called second order differences.

$$\Delta^2 y_0 = \Delta y_1 - \Delta y_0 , \Delta^2 y_1 = \Delta y_2 - \Delta y_1 \ldots$$

$$\Delta^2 y_r = \Delta y_{r+1} - \Delta y_r \quad \text{where } r = 0, 1, 2 \ldots$$

The n^{th} forward differences and defined by formula.

$$\Delta^n y_r = \Delta^{n-1} y_{r+1} - \Delta^{n-1} y_r \quad \text{where } r = 0, 1, 2 \ldots$$

Note:- $\Delta f(x) = f(x+h) - f(x)$, $h = 1, 2, 3 \ldots$

<u>forward difference table</u>:-

x	y			
x_0	y_0			
		$y_1 - y_0 = \Delta y_0$		
x_1	y_1		$\Delta y_1 - \Delta y_0 = \Delta^2 y_0$	
		$y_2 - y_1 = \Delta y_1$		$\Delta^2 y_1 - \Delta^2 y_0 = \Delta^3 y_0$
x_2	y_2		$\Delta y_2 - \Delta y_1 = \Delta^2 y_1$	
		$y_3 - y_2 = \Delta y_2$		
x_3	y_3			

Backward differences:-

Let $y_0, y_1, y_2 \cdots y_r$ be the values of function $y=f(x)$ corresponding to values of $x_0, x_1, x_2 \cdots x_r$ of x respectively thus

$\nabla y_1 = y_1 - y_0$, $\nabla y_2 = y_2 - y_1$ ---- are called the first backward differences.

In general $\nabla y_r = y_r - y_{r-1}$ $r=1,2 \cdots$

The symbol ∇ is called backward differences operator.

The difference of 1st order backward differences are called second order backward differences.

$$\nabla^2 y_2 = \nabla y_2 - \nabla y_1, \quad \nabla^2 y_3 = \nabla y_3 - \nabla y_2 \cdots$$

In general $\nabla^2 y_r = \nabla y_r - \nabla y_{r-1}$ $r = 1, 2, 3 \cdots$

The n^{th} backward difference are defined by

$$\nabla^n y_r = \nabla^{n-1} y_r - \nabla^{n-1} y_{r-1}, \quad r = n, n+1 \cdots$$
$$n = 1, 2, 3 \cdots$$

Note: $\nabla f(x) = f(x) - f(x-h)$

Backward difference table:-

x	y			
x_0	y_0			
		$\nabla y_1 = y_1 - y_0$		
x_1	y_1		$\nabla^2 y_2 = \nabla y_2 - \nabla y_1$	
		$\nabla y_2 = y_2 - y_1$		$\nabla^3 y_3 = \nabla^2 y_3 - \nabla^2 y_2$
x_2	y_2		$\nabla^2 y_3 = \nabla y_3 - \nabla y_2$	
		$\nabla y_3 = y_3 - y_2$		
x_3	y_3			

Central differences :-

Given $x: x_0 \quad x_1 \quad x_2 \quad x_3 \quad \cdots \quad x_r$

$y: y_0 \quad y_1 \quad y_2 \quad y_3 \quad \cdots \quad y_r$

We define the 1st central differences as follows.

$\delta y_{1/2}, \delta y_{3/2}, \delta y_{5/2} \cdots$ as follows.

$\delta y_{1/2} = y_1 - y_0, \quad \delta y_{3/2} = y_2 - y_1 \quad \cdots \quad \delta y_{r-1/2} = y_r - y_{r-1}$

The symbol δ is called central differences operator.

The second central difference are the difference of 1st order central differences.

$\delta^2 y_1 = \delta y_{3/2} - \delta y_{1/2} \quad \cdots \quad \delta^2 y_r = \delta y_{r+\frac{1}{2}} - \delta y_{r-1/2}$

In general, nth central differences are given by

① for odd n: $\delta^n y_{r-1/2} = \delta^{n-1} y_r - \delta^{n-1} y_{r-1}$

② for even n: $\delta^n y_r = \delta^{n-1} y_{r+\frac{1}{2}} - \delta^{n-1} y_{r-\frac{1}{2}}$

Note:- $\delta f(x) = f(x + \frac{1}{2}h) - f(x - \frac{1}{2}h)$

Central difference table.

x	y			
x_0	y_0			
		$\delta y_{1/2} = y_1 - y_0$		
x_1	y_1		$\delta^2 y_1 = \delta y_{3/2} - \delta y_{1/2}$	
		$\delta y_{3/2} = y_2 - y_1$		$\delta^3 y_{3/2} = \delta^2 y_2 - \delta^2 y_1$
x_2	y_2		$\delta^2 y_2 = \delta y_{5/2} - \delta y_{3/2}$	
		$\delta y_{5/2} = y_3 - y_2$		
x_3	y_3			

Average operator :- The average operator 'μ' is defined by

$$\mu y_r = \frac{1}{2}\left[y_{r+\frac{1}{2}} + y_{r-\frac{1}{2}}\right]$$

Note :- $\mu f(x) = \frac{1}{2}\left[f(x+\frac{h}{2}) + f(x-\frac{h}{2})\right]$.

Ex :- $\mu y_1 = \frac{1}{2}\left[y_{3/2} + y_{1/2}\right]$.

Shift operator :- Shift operator is defined by

$$E y_r = y_{r+1}$$

$$E^2 y_r = y_{r+2}$$

$$E^n y_r = y_{r+n}$$

Inverse shift operator E^{-1} is defined by

$$E^{-n} y_r = y_{r-n}$$

Note :- $E f(x) = f(x+h)$.

Relations of symbols.

① Prove that $\Delta = E - 1$

Proof :- we have $\Delta y_0 = y_1 - y_0$.

$$\Delta y_0 = E y_0 - y_0$$

$$\Delta y_0 = (E-1) y_0$$

$$\boxed{\Delta = E - 1}$$

$$\boxed{E = 1 + \Delta}$$

Ex :- $\Delta^2 y_0 = (E-1)^2 y_0$.

$$= (E^2+1-2E) y_0.$$

$$= E^2 y_0 + y_0 - 2E y_0.$$

$$= y_2 + y_0 - 2y_1.$$

② P.T $\nabla = 1 - E^{-1}$.

proof :- we have $\nabla y_1 = y_1 - y_0$.

$$\nabla y_1 = y_1 - E^{-1} y_1$$

$$\nabla y_1 = y_1 (1 - E^{-1})$$

$$\boxed{\nabla = 1 - E^{-1}}$$

$$\boxed{E^{-1} = 1 - \nabla}$$

③ P.T $\delta = E^{1/2} - E^{-1/2}$.

we have $\delta y_{1/2} = y_1 - y_0$.

$$\delta y_{1/2} = E^{1/2} y_{1/2} - E^{-1/2} y_{1/2}.$$

$$\delta \cancel{y_{1/2}} = \cancel{y_{1/2}} [E^{1/2} - E^{-1/2}]$$

$$\boxed{\delta = E^{1/2} - E^{-1/2}}$$

④ S.T $\mu = \frac{1}{2}[E^{1/2} + E^{-1/2}]$.

$$\mu y_r = \frac{1}{2}[y_{r+\frac{1}{2}} + y_{r-\frac{1}{2}}]$$

$$\mu y_r = \frac{1}{2}[E^{1/2} y_r + E^{-1/2} y_r]$$

$$\mu \cancel{y_r} = \cancel{y_r} [\tfrac{1}{2}[E^{1/2} + E^{-1/2}]]$$

$$\boxed{\mu = \frac{1}{2}[E^{1/2} + E^{-1/2}]}$$

⑤ P.T $\mu^2 = 1 + \frac{1}{4}\delta^2$.

We have $\mu = \frac{1}{2}[E^{1/2} + E^{-1/2}]$

$$\mu^2 = \left[\frac{1}{2}[E^{1/2} + E^{-1/2}]\right]^2$$

$$= \frac{1}{4}[E^{1/2} + E^{-1/2}]^2$$

$$= \frac{1}{4}\left[[E^{1/2} - E^{-1/2}]^2 + 4\right]$$

$$= \frac{1}{4}[E^{1/2} - E^{-1/2}]^2 + 1$$

$$\boxed{\mu^2 = 1 + \frac{1}{4}\delta^2}$$

operator D :-

The operator D is defined by

$$D f(x) = \frac{d}{dx} f(x).$$

We know that $y(x+h) = y(x) + h y'(x) + \frac{h^2}{2!} y''(x) + \text{---}$

$$= y(x) + h D(y(x)) + \frac{h^2}{2!} D^2 y(x) + \text{------}$$

$$E y(x) = \left[1 + hD + \frac{h^2 D^2}{2!} + \frac{h^3 D^3}{3!} + \text{----}\right] y(x)$$

$$E = 1 + hD + \frac{(hD)^2}{2!} + \frac{(hD)^3}{3!} - \text{-----}$$

$$\boxed{E = e^{hD}}$$

$$\boxed{1 + \Delta = e^{hD}}$$

$$\boxed{\log(1+\Delta) = hD}$$

Problems:-

① Proof:- $E\nabla = \Delta = \nabla E$.

$$E\nabla(f(x)) = E[\nabla f(x)]$$
$$= E[f(x) - f(x-h)]$$
$$= Ef(x) - E(f(x-h))$$
$$= f(x+h) - f(x-h+h)$$
$$= f(x+h) - f(x)$$
$$= \Delta f(x)$$

$\Delta = \nabla E$

$$\nabla E y_1 = (\nabla y_1) E$$
$$= (y_1 - y_0) E$$
$$= E y_1 - E y_0$$
$$= y_2 - y_1$$
$$\nabla E y_1 = \Delta y_1$$
$$\nabla E = \Delta$$

$$E\nabla y_1 = E[\nabla y_1]$$
$$= E(y_1 - y_0)$$
$$= E y_1 - E y_0$$
$$= y_2 - y_1$$
$$= \Delta y_1 \Rightarrow \boxed{E\nabla = \Delta}$$

② P.T $\delta E^{1/2} = \Delta$.

we have $\delta = E^{1/2} - E^{-1/2}$

$$\delta E^{1/2} = (E^{1/2} - E^{-1/2}) E^{1/2}$$
$$= E - 1$$
$$= \Delta$$

③ PT $1 + \mu^2 \delta^2 = (1 + \frac{1}{2}\delta^2)^2$.

$$LHS = 1 + \mu^2 \delta^2$$
$$= 1 + (\tfrac{1}{2}(E^{1/2} + E^{-1/2})^2)(E^{1/2} - E^{-1/2})^2$$
$$= 1 + \tfrac{1}{4}[(E^{1/2})^2 - (E^{-1/2})^2]^2$$

$$= 1 + \frac{1}{4}[E - E^{-1}]^2.$$

$$= \frac{4 + [E - E^{-1}]^2}{4}.$$

$$= \frac{(E + E^{-1})^2}{4}.$$

$$RHS = \left(1 + \frac{1}{2}\delta^2\right)^2.$$

$$= \left[1 + \frac{1}{2}\left(E^{1/2} - E^{-1/2}\right)^2\right].$$

$$= \left[1 + \frac{1}{2}\left(E + E^{-1} - 2\right)\right]^2$$

$$= \left[\frac{\cancel{2} + E + E^{-1} - \cancel{2}}{2}\right]^2$$

$$= \frac{(E + E^{-1})^2}{4}$$

(4) : P.T $\mu\delta = \frac{1}{2}\Delta E^{-1} + \frac{1}{2}\Delta.$

$$RHS = \frac{1}{2}\Delta\left(E^{-1} + 1\right)$$

$$= \frac{1}{2}[E - 1][E^{-1} + 1]$$

$$= \frac{1}{2}[\cancel{1} + E - E^{-1} - \cancel{1}]$$

$$= \frac{1}{2}[E - E^{-1}] \longrightarrow (1).$$

$$LHS = \mu\delta = \frac{1}{2}\left(E^{1/2} + E^{-1/2}\right)\left(E^{1/2} - E^{-1/2}\right)$$

$$= \frac{1}{2}[E - E^{-1}] \quad -(2).$$

LHS = RHS.

Hence proved

⑤ P.T $\Delta = \frac{1}{2}\delta^2 + \delta\sqrt{1+\frac{\delta^2}{4}}$.

sol $\quad \frac{1}{2}\delta^2 + \delta\sqrt{1+\frac{\delta^2}{4}}$.

$= \frac{1}{2}\delta^2 + \frac{1}{2}\delta\sqrt{4+\delta^2}$.

$= \frac{\delta}{2}\left[\delta + \sqrt{4+\delta^2}\right]$

$= \frac{\delta}{2}\left[E^{1/2} - E^{-1/2} + \sqrt{4+(E^{1/2}-E^{-1/2})^2}\right]$

$= \frac{\delta}{2}\left[E^{1/2} - E^{-1/2} + \sqrt{(E^{1/2}+E^{-1/2})^{\cancel{2}}}\right]$

$= \frac{\delta}{2}\left[E^{1/2} - \cancel{E^{-1/2}} + E^{1/2} + \cancel{E^{-1/2}}\right]$

$= \frac{\delta}{\cancel{2}}\left[\cancel{2}\cdot E^{1/2}\right]$

$= (E^{1/2} - E^{-1/2})E^{1/2}$.

$= E - 1 = \Delta$.

⑥ The following table gives a set of values of x. and corresponding values of $y = f(x)$

x	10	15	20	25	30	35
y	19.97	21.51	22.47	23.52	24.65	25.89

form the forward difference table and write down. the values of $\Delta f(10)$, $\Delta^2 f(10)$, $\Delta^3 f(15)$, $\Delta^4 f(15)$.

Sol.

x	f(x)					
10	19.97	1.54	-0.58	0.67	-0.68	0.72
15	21.51	0.96	0.09	-0.01	0.04	
20	22.47	1.05	0.08	0.03		
25	23.52	1.13	0.11			
30	24.65	1.24				
35	25.89					

Q. construct a forward difference table from the following data:

x	0	1	2	3	4
y	1	1.5	2.2	3.1	4.6

Evaluate $\Delta^3 y_1$.

x	f(x)	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
x_0 0	y_0 1	Δy_0 0.5	$\Delta^2 y_0$ 0.2	$\Delta^3 y_0$ 0	$\Delta^4 y_0$ 0.4
x_1 1	y_1 1.5	Δy_1 0.7	$\Delta^2 y_1$ 0.2	$\Delta^3 y_1$ 0.4	
x_2 2	y_2 2.2	Δy_2 0.9	$\Delta^2 y_2$ 0.6		
x_3 3	y_3 3.1	Δy_3 1.5			
x_4 4	y_4 4.6				

$\Delta^3 y_1 = 0.4$

⑧ Find the missing term in the following data.

x	0	1	2	3	4
y	1	3	9	-	81

Sol: Here 4 values are known.

$\Rightarrow$ Third differences are constant.

$\Rightarrow$ fourth differences are zero.

consider $\Delta^4 y_0 = 0$.

$$(E-1)^4 y_0 = 0$$

$$[{}^4C_0 E^4 - {}^4C_1 E^3 + {}^4C_2 E^2 - {}^4C_3 E + {}^4C_4 E^0] y_0 = 0$$

$$\Rightarrow [E^4 - 4E^3 + 6E^2 - 4E + 1] y_0 = 0$$

$$E^4 y_0 - 4E^3 y_0 + 6 y_0 E^2 - 4 y_0 E + y_0 = 0$$

$$y_4 - 4y_3 + 6 \cdot y_2 - 4y_1 + y_0 = 0$$

$$81 - 4y_3 + 6(9) - 4(3) + 1 = 0$$

$$\boxed{y_3 = 31}$$

⑨ Find the missing term from following data.

x	1	2	3	4	5
y	2	5	7	-	32

Here 4 values are known

$\Rightarrow$ Third differences are constant.

$\Rightarrow$ fourth differences are zero.

$$\Delta^4 y_0 = 0.$$

$$(E-1)^4 y_0 = 0.$$

$$\left[{}^4C_0 E^4 - {}^4C_1 E^3 + {}^4C_2 E^2 - {}^4C_3 E + {}^4C_4 E^0\right] y_0 = 0.$$

$$[E^4 - 4E^3 + 6E^2 - 4E + 1] y_0 = 0.$$

$$E^4 y_0 - 4E^3 y_0 + 6E^2 y_0 - 4E y_0 + y_0 = 0.$$

$$y_4 - 4y_3 + 6y_2 - 4y_1 + y_0 = 0.$$

$$32 - 4y_3 + 6(7) - 4(5) + 2 = 0.$$

$$32 - 4y_3 + 42 - 20 + 2 = 0.$$

$$\cancel{4}y_3 = \cancel{56}\ 14,$$

$$\boxed{y_3 = 14}$$

(10) Find the missing terms in table in following table.

x	1	2	3	4	5	6	7	8.
y	1	8	—	64	—	216	343	512.
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7.

$$\Delta^6 y_0 = 0 \ \& \ \Delta^6 y_1 = 0.$$

$$\Delta^6 y_0 = 0.$$

$$(E-1)^6 y_0 = 0.$$

$$\left({}^6C_0 E^6 - {}^6C_1 E^5 + {}^6C_2 E^4 - {}^6C_3 E^3 + {}^6C_4 E^2 - {}^6C_5 E + {}^6C_6 E^0\right) y_0 = 0$$

$$(E^6 - 6E^5 + 15E^4 - 20E^3 + 15E^2 - 6E + 1) y_0 = 0.$$

$$y_6 - 6y_5 + 15y_4 - 20y_3 + 15y_2 - 6y_1 + y_0 = 0. \qquad 152$$

$$343 - 6(216) + 15y_4 - 20(64) + 15y_2 - 6(8) + 1 = 0 \Rightarrow 15(y_4 + y_2) = 228$$

$$\boxed{y_4 + y_2 = 152.} \quad \text{--- (1).}$$

$\Delta^6 y_1 = 0.$ $\qquad\qquad\qquad\qquad E^n y_r = y_{r+n}.$

$(E-1)^6 y_1 = 0.$

$\Rightarrow [E^6 - 6E^5 + 15E^4 - 20E^3 + 15E^2 - 6E + 1] y_1 = 0.$

$y_7 - 6y_6 + 15y_5 - 20y_4 + 15y_3 - 6y_2 + y_1 = 0.$

$512 - 6(343) + 15(216) - 20y_4 + 15(64) - 6y_2 + 8 = 0.$

$-20y_4 - 6y_2 = -2662.$

$20y_4 + 6y_2 = 2662$ --- (2).

$y_4 + y_2 = 152$ --- (1).

<u>solve</u> (1) & (2).

$$\boxed{y_4 = 125}$$

$$\boxed{y_2 = 27.}$$

(11) - If $f(x) = x^3 + 5x - 7$ form a table of forward difference taking $x = -1, 0, 1, 2, 3, 4, 5$. show that third differences are constant.

x	-1	0	1	2	3	4	5
y	-13	-7	-1	11	35	77	143

x	$f(x)$ y			
-1	-13	6	0	6
0	-7	6	6	6
1	-1	12	12	6
2	11	24	18	6
3	35	42	24	
4	77	66		
5	143			

$$\cos C - \cos D = -2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$$
$$= 2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{D-C}{2}\right)$$

Hence 3rd differences are constant.

⑫ Evaluate ① $\Delta \cos x$ ② $\Delta \tan^{-1}x$ ③ $\Delta^n e^{ax+b}$.

① $\Delta \cos x = \cos(x+h) - \cos x$.
$$= 2\sin\left(\frac{x+h+x}{2}\right)\sin\left(\frac{x+h-\cancel{x}}{2}\right)$$
$$= 2\sin\left(\frac{2x+h}{2}\right)\sin\left(\frac{h}{2}\right)$$

② $\Delta \tan^{-1}x = \tan^{-1}\frac{x}{x+h} - \tan^{-1}\frac{y}{x}$.
$$= \tan^{-1}\left(\frac{x+h-x}{1+x(x+h)}\right)$$
$$= \tan^{-1}\left(\frac{h}{1+x(x+h)}\right).$$

$$\boxed{\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)}$$

③ $\Delta^n e^{ax+b}$.
$$\Delta e^{ax+b} = e^{a(x+h)+b} - e^{ax+b}.$$
$$= e^{ax+ah+b} - e^{ax+b}.$$
$$= e^{ax+b}\cdot e^{ah} - e^{ax+b}.$$
$$= e^{ax+b}(e^{ah}-1).$$

$$\Delta^2 e^{ax+b} = \Delta(\Delta e^{ax+b})$$

$$= \Delta(e^{ah}-1)(e^{ax+b})$$

$$= e^{ah}-1(\Delta e^{ax+b})$$

$$= (e^{ah}-1)\left[(e^{ah}-1)(e^{ax+b})\right].$$

$$= (e^{ah}-1)^2 e^{ax+b}.$$

$$\vdots$$

$$\Delta^n e^{ax+b} = (e^{ah}-1)^n e^{ax+b}.$$

(4) If interval of differencing is unity prove.

$$\Delta x(x+1)(x+2)(x+3) = 4(x+1)(x+2)(x+3) \quad \boxed{h=1}$$

$$\Delta x(x+1)(x+2)(x+3) = f(x+1) - f(x)$$

$$= (x+1)(x+2)(x+3)(x+4) - x(x+1)(x+2)(x+3)$$

$$= (x+1)(x+2)(x+3)\left[\cancel{x}+4-\cancel{x}\right]$$

$$= 4(x+1)(x+2)(x+3)$$

(14) Find the first difference of the polynomial $x^4 - 12x^3 + 42x^2 - 30x + 9$ with interval of differencing is 2.

Given $h = 2$.

$f(x) = x^4 - 12x^3 + 42x^2 - 30x + 9$.

$\Delta f(x) = f(x+h) - f(x)$.

$= f(x+2) - f(x)$.

$\Delta f(x) = (x+2)^4 - 12(x+2)^3 + 42(x+2)^2 - 30(x+2) + 9$.

$- (x^4 - 12x^3 + 42x^2 - 30x + 9)$.

$$= (x+2)(x^3 + 8 + 6x^2 + 12x \underbrace{- 12(x^2+4+4x)}_{-12x^2 - 48 - 48x} + 42x + 84 - 30) + \cancel{9}$$

$- x^4 + 12x^3 - 42x^2 + 30x \cancel{9}$.

$= (x+2)(x^3 - 6x^2 + 6x + 44) - x^4 + 12x^3 - 42x^2 + 30x$.

$= x^4 - 6x^3 + 6x^2 + 44x + 2x^3 - 12x^2 + 12x + 88 - \cancel{x^4} + 12x^3 - 42x^2 + 30x$.

$= 8x^3 - 48x^2 + 86x - 28$ //

Newton's Forward interpolation (equal interval)

Let $y = f(x)$ be a polynomial of degree n, then Newton's forward interpolation formula is given by.

$$y = f(x) = f(x_0 + ph) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0$$

$$\cdots \frac{p(p-1)\cdots(p-(n-1))}{n!}\Delta^n y_0.$$

Here $x = x_0 + ph$. h = interval of difference.

$p = \frac{x - x_0}{h}$.

Note:- Newton's forward interpolation is used to interpolate at the begining of the given data.

Newton's backward interpolation.

let $y=f(x)$ be a polynomial of degree n, then Newton's backward interpolation formula is given by

$$y=f(x) = f(x_n+ph) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n$$

$$+ \cdots + \frac{p(p+1)(p+2)\cdots(p+(n-1))}{n!}\nabla^n y_n.$$

Note:- Newton's backward interpolation is useful for the end of the tabulated values.

Problems

① The population of a town in the decimal census was given below. Estimate the population for the year 1895, 1925

year x.	1891	1901	1911	1921	1931
population y (Thousands)	46	66	81	93	101

$h=10 \quad x=1895 \quad x_0=1891.$

$x=x_0+ph.$

$$p=\frac{x-x_0}{h} = \frac{1895-1891}{10} = \frac{4}{10} = 0.4$$

x	y	Δy_0	$\Delta^2 y_0$	$\Delta^3 y_0$	$\Delta^4 y_0$
1891	46 (y_0)				
		20			
1901	66		-5		
		15		2	
1911	81		-3		-3
		12		-1	($\Delta^4 y_4$)
1921	93		-4	($\Delta^3 y_4$)	
		8	($\Delta^2 y_4$)		
1931	101 (y_4)	(Δy_4)			

Newton's forward interpolation.

$$y = f(x) = f(x_0 + ph)$$

$$= y_0 + p\Delta y_0 + \frac{p(p-1)}{2!} \cdot \Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 - - -$$

$$= y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0$$

$$= 46 + (0.4)(20) + \frac{(0.4)(0.4-1)}{2}(-5) + \frac{(0.4)(0.4-1)(0.4-2)}{6}(2)$$

$$+ \frac{(0.4)(0.4-1)(0.4-2)(0.4-3)}{24}(-3)$$

$$= 46 + 8 + \frac{(0.4)(-0.6)(-5)}{2} + \frac{(0.4)(-0.6)(-1.6)}{6}(2)$$

$$+ \frac{(0.4)(-0.6)(-1.6)(-2.6)}{24}(-3)$$

$$= \underline{\underline{54.8528}}.$$

Newton's backward interpolation.

$$y = f(x) = f(x_n + ph)$$

$$= y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_n$$

$h = 10$ $\quad x = 1925$ $\quad x_0 = 1931 \Rightarrow x = x_0 + ph.$

$$p = \frac{x - x_0}{h} = \frac{1925 - 1931}{10} = -0.6$$

$$y = 101 + (-0.6)(8) + \frac{(-0.6)(-0.6+1)}{2}(-4) + \frac{(-0.6)(-0.6+1)(-0.6+2)}{6}(-1)$$

$$+ \frac{(-0.6)(-0.6+1)(-0.6+2)(-0.6+3)}{24}(-3)$$

$$y = 101 + (-0.6)(8) + \frac{(-0.6)(0.4)}{2}(-4) + \frac{(-0.6)(0.4)(1.4)}{6}(-1)$$

$$+ \frac{(-0.6)(0.4)(1.4)(2.4)}{24}(-3).$$

$$= 96.8368.$$

(2) Applying Newton's forward formula, compute the value of $\sqrt{5.5}$, given that $\sqrt{5} = 2.236$, $\sqrt{6} = 2.449$, $\sqrt{7} = 2.646$, $\sqrt{8} = 2.828$.

x	$f(x) = y = \sqrt{x}$			
5	2.236			
		0.213		
6	2.449		−0.016	
		0.197		0.001
7	2.646		−0.015	
		0.182		
8	2.828			

Newton's law of forward interpolation.

$$y = f(x) = f(x_0 + ph)$$

$$= y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 \ldots$$

$h = 1$

$x = \cancel{2.2030} \cdot 5.5$

$x_0 = 5$

$p = \dfrac{x - x_0}{h} = \dfrac{5.5 - 5}{10} = 0.05$

$p = 0.5$

$$y = 2.236 + 0.213 + \frac{(0.213)(-0.787)(-0.016)}{2!} + \frac{(0.213)(-0.787)(-1.787)(-0.016)}{6}$$

$= 2.3445$

③ Given $\sin 45 = 0.7071$, $\sin 50 = 0.7660$, $\sin 55 = 0.8192$, $\sin 60 = 0.8660$, find $\sin 52°$ using forward

x	45	50	55	60
y	0.7071	0.766	0.8192	0.866

$x = 52 \quad x_0 = 45$

$h = \dfrac{52 - 45}{5} \Rightarrow \dfrac{7}{5} = 0.14$

x	y			
45	0.7071	0.0589	-5.7×10^{-3}	-0.7×10^{-3}
50	0.766	0.0532	-6.4×10^{-3}	
55	0.8192	0.0468		
60	0.866			

$$y = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 \cdots$$

$$y = 0.7071 + 0.14 \times 0.0589 + \frac{\overset{0.07}{\cancel{0.14}}(-0.86)(-5.7 \times 10^{-3})}{2} + \frac{\overset{0.07}{\cancel{0.14}}(-0.86)(-1.86)(-0.7 \times 10^{-3})}{3}$$

③ $y = 707.0\times10^{-5} + 824.6\times10^{-5} + 34.314\times10^{-5} + 2.61268\times10^{-5}$

$y = 71571.5268\times10^{-5}$

④ 96 $f(1.15) = 1.0723$, $f(1.20) = 1.0954$, $f(1.25) = 1.1180$, $f(1.30) = 1.140$

find $f(1.28)$

x	y			
1.15	1.0723			
		0.0231		
1.20	1.0954		-5×10^{-4}	
		0.226		0
1.25	1.1180		-5×10^{-4}	
		0.0221		
1.30	1.1401			

$x = 1.28 \qquad x_0 = 1.30$

$$p = \frac{x - x_n}{h} = \frac{-0.02}{5} = -4\times10^{-3}$$

Newtons backward interpolation.

$$y = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{6!}\nabla^3 y_n$$

$$y = 1.1401 + (-4\times10^{-3})(0.0221) + \frac{(-4\times10^{-3})(0.996)}{2}(-5\times10^{-4})$$

$y = 1.1401 - 0.0884\times10^{-3} + 4.96\times10^{-7}$

$y = 1.1400116 + 9.46\times10^{-7}$

$y = 1.140012596$

⑤ The following table gives corresponding values of x and y. Construct difference table and then express y as function of x.

x	0	1	2	3	4
y	3	6	11	18	27

Sol

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	3				
		3			
1	6		2		
		5		0	
2	11		2		0
		7		0	
3	18		2		
		9			
4	27				

$h=1 \quad x=x \quad x_0=0$

$$P=\frac{x-x_0}{h}=\frac{x-0}{1}=x$$

$P=x$

Newton's forward interpolation

$$y=f(x)=y_0+p\Delta y_0+\frac{P(P-1)}{2!}\Delta^2 y_0+\frac{P(P-1)(P-2)}{3!}\Delta^3 y_0+\cdots$$

$$y=3+x(3)+\frac{x(x-1)}{\cancel{2}}\cancel{(2)}$$

$$=x^2-x+3x+3$$

$$y=x^2+2x+3$$

⑥ Find the newton's forward difference interpolating polynomial for the data.

x	0	1	2	3
y	1	3	7	13

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	1			
		2		
1	3		2	
		4		0
2	7		2	
		6		
3	13			

$h=1 \quad x=x \quad x_0=0$

$$P=\frac{x}{1}=x$$

$$y=f(x)=y_0+p\Delta y_0+\frac{p(p-1)}{2!}\Delta^2 y_0+\frac{p(p-1)(p-2)}{3!}\Delta^3 y_0$$

$$\Rightarrow y=1+x(2)+\frac{x(x-1)}{2}(2)$$

$$=1+2x+x^2-x$$

$$\boxed{y=x^2+x+1}$$

Gauss forward interpolation formula is given by

$$y = f(x) = f(x_0 + ph)$$

$$= y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{p(p-1)(p+1)}{3!}\Delta^3 y_{-1} + \frac{p(p-1)(p+1)(p-2)}{4!}\Delta^4 y_{-2}$$

$$+ \frac{(p+2)(p+1)p(p-1)(p-2)}{5!}\Delta^5 y_{-2}.$$

Gauss backward interpolation is given by

$$y = f(x) = f(x_0 + ph)$$

$$= y_0 + p\Delta y_{-1} + \frac{p(p+1)}{2!}\Delta^2 y_{-1} + \frac{p(p+1)(p-1)}{3!}\Delta^3 y_{-2} + \frac{p(p+1)(p+2)(p-1)}{4!}\Delta^4 y_{-2}.$$

$$+ \frac{p(p+1)(p+2)(p-1)(p-2)}{5!}\Delta^5 y_{-3} + \cdots$$

<u>Central difference</u>

$$\begin{array}{cccccc}
x & y & & & & \\
x_{-2} & y_{-2} & & & & \\
 & & \Delta y_{-2} & & & \\
x_{-1} & y_{-1} & & \Delta^2 y_{-2} & & \\
 & & \Delta y_{-1} & & \Delta^3 y_{-2} & \\
x_0 & y_0 & & \Delta^2 y_{-1} & & \Delta^4 y_{-2} \\
 & & \Delta y_0 & & \Delta^3 y_{-1} & \\
x_1 & y_1 & & \Delta^2 y_0 & & \\
 & & \Delta y_1 & & & \\
x_2 & y_2 & & & &
\end{array}$$

① **Problem**: From the following table values of x & $y = e^x$ interpolate values of y when $x = 1.91$.

x	1.7	1.8	1.9	2	2.1	2.2
y	5.4739	6.0496	6.6859	7.3891	8.1662	9.0250

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
$1.7\ x_{-2}$	$5.4739\ (y_{-2})$					
		$0.5757\ (\Delta y_{-2})$				
$1.8\ x_{-1}$	$6.0496\ (y_{-1})$		$0.0606\ (\Delta^2 y_{-2})$			
		$0.6363\ (\Delta y_{-1})$		$0.0063\ (\Delta^3 y_{-2})$		
$1.9\ x_0$	$6.6859\ (y_0)$		$0.0669\ (\Delta^2 y_{-1})$		$0.0007\ (\Delta^4 y_{-2})$	
		$0.7032\ (\Delta y_0)$		$0.007\ (\Delta^3 y_{-1})$		$0.0001\ (\Delta^5 y_{-2})$
$2\ x_1$	$7.3891\ (y_1)$		$0.0739\ (\Delta^2 y_0)$		$0.0008\ (\Delta^4 y_{-1})$	
		$0.777\ (\Delta y_1)$		$0.0078\ (\Delta^3 y_0)$		
$2.1\ x_2$	$8.1662\ (y_2)$		$0.0817\ (\Delta^2 y_1)$			
		$0.8588\ (\Delta y_2)$				
$2.2\ x_3$	$9.0250\ (y_3)$					

Gauss forward interpolation is

$$y = f(x) = f(x_0 + ph)$$

$$= y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{p(p-1)(p+1)}{3!}\Delta^3 y_{-1} + \frac{p(p-1)(p+1)(p-2)}{4!}\Delta^4 y_{-2}$$

$$+ \frac{(p+2)(p+1)p(p-1)(p-2)}{5!}\Delta^5 y_{-2}$$

$x = 1.91$

$$p = \frac{x - x_0}{h} = \frac{1.91 - 1.9}{0.1} = 0.1$$

$$= 6.6859 + (0.1)(0.7032) + \frac{(0.1)(0.1-1)}{2}(0.0669) + \frac{(0.1)(0.1-1)(0.1+1)}{3!}(0.007)$$

$$+ (0.1)(0.1-1)(0.1+1)$$

② Use gauss forward interpolation formula to find $f(3.3)$ from following table.

x	1	2	3	4	5
$y = f(x)$	15.30	15.10	15.00	14.50	14.00

Difference table-

x	$y = f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1	y_{-2} 15.30				
		Δy_{-2} −0.20			
2	y_{-1} 15.10		$\Delta^2 y_{-2}$ 0.10		
		Δy_{-1} −0.10		$\Delta^3 y_{-2}$ −0.50	
3	y_0 15.00		$\Delta^2 y_{-1}$ −0.40		$\Delta^4 y_{-2}$ 0.90
		Δy_0 −0.50		$\Delta^3 y_{-1}$ 0.40	
4	y_1 14.50		$\Delta^2 y_0$ 0		
		Δy_1 −0.50			
5	y_2 14.00				

$x = x_0 + ph$ $\quad h = 1,\ x_0 = 3,\ x = 3.3$

$p = \dfrac{x - x_0}{h} = \dfrac{3.3 - 3}{1} = 0.3$

Gauss forward interpolation

$$y = f(x) = f(x_0 + ph) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{p(p+1)(p-1)}{3!}\Delta^3 y_{-1}$$

$$+ \frac{p(p+1)(p-1)(p-2)}{4!}\Delta^4 y_{-2} - - -$$

$$= 15 + (0.3)(-0.50) + \frac{(0.3)(0.3-1)}{2}(-0.40) + \frac{(0.3+1)(0.3)(0.3-1)}{6}(0.40)$$

$$+ \frac{(0.3+1)(0.3)(0.3-1)(0.3-2)}{24}(0.90) = 14.89$$

$f(3.3) = 14.89$

③ find $y(25)$ given that $y_{20} = 24, y_{24} = 32, y_{28} = 35,$
$y_{32} = 40$. using gauss forward difference formula

x	y			
x_{-1} 20	24			
		Δy_{-1} 8		
x_0 24	32		$\Delta^2 y_{-1}$ -5	
		Δy_0 3		$\Delta^3 y_{-1}$ 7
x_1 28	35		2	
		5		
x_2 32	40			

$h = 4, \; x_0 = 24, \; x = 25$

$p = \frac{x - x_0}{h} = \frac{25 - 24}{4} = 0.25$

Gauss forward interpolation.

$y = f(x) = f(x_0 + ph)$

$= y_0 + p\Delta y_0 + \frac{p(p-1)}{2!} \Delta^2 y_{-1} + \frac{p(p+1)(p-1)}{3!} \Delta^3 y_{-1}$

$= 32 + (0.25)(3) + \frac{(0.25)(0.25-1)}{2}(-5) + \frac{0.25(0.25+1)(0.25-1)}{6}(7)$

$f(25) = 32.945$

④ Given $\sqrt{6500} = 80.6223$, $\sqrt{6510} = 80.6846$, $\sqrt{6520} = 80.7456$
$\sqrt{6530} = 80.8084$ find $\sqrt{6526}$ using gauss backward interpolation

x	y			
x_{-3} 6500	y_{-3} 80.6223			
		0.0623		
x_{-2} 6510	y_{-2} 80.6846		-0.0013	
		0.061		0.0031
x_{-1} 6520	y_{-1} 80.7456		0.0018	
		0.0628		
x_0 6530	y_0 80.8084			

$x = 6526$, $h = 10$

$x_0 = 6530$

$p = \frac{x - x_0}{h} = \frac{6526 - 6530}{10} = -0.4$

$y = f(x) = f(x_0 + ph)$

$= y_0 + p\Delta y_{-1} + \frac{p(p+1)}{2!}\Delta^2 y_{-1} + \frac{p(p-1)(p+1)}{3!}\Delta^3 y_{-2}$

$= 80.8084 + (-0.4)(0.0628) + \frac{(0.4)(-0.4+1)}{2}(0.0018)$

$+ \frac{(-0.4)(-0.4+1)(-0.4-1)}{6}(0.0031)$

$f(2526) = 80.783$

⑤ Using gauss backward difference formula find $y(8)$ from the table:

x	0	5	10	15	20	25
$f(x) = y$	7	11	14	18	24	32

	x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
y_{-2}	0	7					
			4 (Δy_{-2})				
y_{-1}	5	11		-1 ($\Delta^2 y_{-2}$)			
			3		2 ($\Delta^3 y_{-2}$)		
y_0	10	14		1		-1 ($\Delta^4 y_{-2}$)	
			4		1		0 ($\Delta^5 y_{-2}$)
y_1	15	18		2		-1	
			6		0 ($\Delta^3 y_0$)	($\Delta^3 y_1$)	
y_2	20	24		2 ($\Delta^2 y_1$)			
			8 (Δy_2)				
y_3	25	32					

$y = f(x) = f(x_0 + ph)$

$= y_0 + p\Delta y_{-1} + \frac{p(p+1)}{2!}\Delta^2 y_{-1} + \frac{p(p+1)(p-1)}{3!}\Delta^3 y_{-2} + \frac{p(p+1)(p+2)(p-1)}{4!}\Delta^4 y_{-2}$

$+ \frac{p(p+1)(p-1)(p+2)(p-2)}{5!}\Delta^5 y_{-3} + \cdots$

$x = 8 \qquad x_0 = 10 \qquad h = 5$

$$p = \frac{x - x_0}{h} = \frac{8 - 10}{5} = \frac{-2}{5} = -0.4$$

$$y(8) = 14 + (-0.4)(3) + \frac{(-0.4)(-0.4+1)}{2}(1) + \frac{(-0.4)(-0.4+1)(-0.4-1)}{6}(2)$$

$$+ \frac{(-0.4)(-0.4+1)(-0.4+2)(-0.4-1)}{24}(-1)$$

$$+ \frac{(-0.4)(-0.4+1)(-0.4-1)(-0.4+2)(-0.4-2)}{120}(0)$$

$$\underline{y(8) = 12.76}$$

⑧ find $f(2.36)$ from the following table

x	1.6	1.8	2.0	2.2	2.4	2.6
y	4.95	6.05	7.39	9.03	11.02	13.48

	x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
x_{-4}	1.6	4.95	Δy_{-4} 1.1	$\Delta^2 y_{-4}$ 0.24	$\Delta^3 y_{-4}$ 0.06	$\Delta^4 y_{-4}$ -0.01	$\Delta^5 y_{-4}$ 0.06
x_{-3}	1.8	6.05	Δy_{-3} 1.34	$\Delta^2 y_{-3}$ 0.3	$\Delta^3 y_{-3}$ 0.05	$\Delta^4 y_{-3}$ 0.05	
x_{-2}	2.0	7.39	Δy_{-2} 1.64	$\Delta^2 y_{-2}$ 0.35	$\Delta^3 y_{-2}$ 0.1		
x_{-1}	2.2	9.03	Δy_{-1} 1.99	$\Delta^2 y_{-1}$ 0.45			
x_0	2.4	11.02	Δy_0 2.44				
x_1	2.6	13.48					

$h = 0.2$. $\qquad p = \dfrac{2.36 - 2.4}{0.2} = -0.2$

$x = 2.36$.

$x_0 = 2.4$.

Gauss backward interpolation.

$y = f(x) = f(x_0 + ph)$

$$= y_0 + p\Delta y_{-1} + \frac{p(p+1)}{2!}\Delta^2 y_{-1} + \frac{p(p+1)(p-1)}{3!}\Delta^3 y_{-2} + \frac{p(p+1)(p+2)(p-1)}{4!}\Delta^4 y_{-2}$$

$$+ \frac{p(p+1)(p+2)(p-1)(p-2)}{5!}\Delta^5 y_{-3}.$$

$$= 11.02 + (-0.2)(1.99) + \frac{(-0.2)(0.2+1)}{2}(0.45) + \frac{(-0.2)(-0.2+1)(-0.2-1)}{6}(0.1)$$

$$+ \frac{(0.2)(-0.2+1)(-0.2+2)(-0.2-1)}{24}(0) + 0.$$

$y = 10.5892.$

Lagranges interpolation (unequal intervals)

Let $y = f(x)$ be a polynomial of n^{th} degree, then Lagranges interpolation formula is given by.

$$y = f(x) = \frac{(x-x_1)(x-x_2) \cdots (x-x_n)}{(x_0-x_1)(x_0-x_2) \cdots (x_0-x_n)} f(x_0) +$$

$$\frac{(x-x_0)(x-x_2)(x-x_3) \cdots (x-x_n)}{(x_1-x_0)(x_1-x_2)(x_1-x_3) \cdots (x-x_n)} f(x_1) + \cdots$$

$$+ \frac{(x-x_0)(x-x_1) \cdots (x-x_{n-1})}{(x_n-x_0)(x_n-x_1) \cdots (x_n-x_{n-1})} f(x_n)$$

Problems

① Evaluate $f(10)$ given $f(x) = 168, 192, 336$ at $x = 1, 7, 15$ respectively.

Given values can be written as.

	x_0	x_1	x_2
x	1	7	15
$y = f(x)$	168	192	336
	y_0	y_1	y_2

By Lagrange's interpolation formula.

$f(10) =$

$x = 10$.

$$y = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1)$$

$$+ \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2)$$

Given $x = 10$.

y at $x = 10$

$$y(10) = \frac{(10-7)(10-15)}{(1-7)(1-15)}(168) + \frac{(10-1)(10-15)}{(7-1)(7-15)}(192) +$$

$$\frac{(10-1)(10-7)}{(15-1)(15-7)}(336)$$

$$= \frac{3(-5)}{(-6)(-14)}(168) + \frac{9(-5)}{6(-8)}(192) + \frac{(9)(3)}{(14)(8)}(336)$$

$$= 231.$$

② A curve passes through points $(0,18)$, $(1,10)$, $(3,-18)$ and $(6,90)$ find slope of curve at $x=2$.

x	0	1	3	6
y	18	10	-18	90

Lagranges interpolation formula.

$$y = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}(y_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1 +$$

$$\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3$$

$$y = \frac{(x-1)(x-3)(x-6)}{(0-1)(0-3)(0-6)}(18) + \frac{(x-0)(x-3)(x-6)}{(1-0)(1-3)(1-6)}(10) +$$

$$\frac{(x-0)(x-1)(x-6)}{(3-0)(3-1)(3-6)}(-18) + \frac{(x-0)(x-1)(x-3)}{(6-0)(6-1)(6-3)}90.$$

$$= \frac{(x-1)(x-3)(x-6)}{(-1)(-3)(-6)}(18) + \frac{x(x-3)(x-6)}{(1)(-2)(-5)}(10) +$$

$$\frac{x(x-1)(x-6)}{(3)(2)(-3)}(-18) + \frac{x(x-1)(x-3)}{(6)(5)(3)} 90.$$

$$= -\left[(x^2-3x-x+3)(x-6) + (x^2-3x)(x-6) + (x^2-x)(x-6) + (x^2-x)(x-3)\right]$$

$$= -\left[(x^2-4x+3)(x-6) + x^3-6x^2-3x^2 \cancel{+6x^2} -18x + x^3-6x^2 - x^2+6x+x^3-3x^2-x^2+3x\right]$$

$$= -[x^3-6x^2-4x^2+24x+3x-18] + 3x^3-20x^2+27x$$

$$= -x^3+10x^2-\cancel{27x}+18+3x^3-20x^2+\cancel{27x}$$

$$y = 2x^3-10x^2+18.$$

Slope of curve.

$$y = 2x^3-10x^2+18.$$

$$y' = 6x^2-20x.$$

at $(x=2)$ $y' = 6(4)-20(2)$

$$= 24-40$$

$$= -16.$$

(3) find parabola passing through points (0,1), (1,3) and (3, 55) using lagrange's interpolation.

	x_0	x_1	x_2
x	0	1	3
y	1	3	55
	y_0	y_1	y_2

$$y = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) + \dots$$

$$\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2).$$

$$= \frac{(x-1)(x-3)}{(0-1)(0-3)}(1) + \frac{(x-0)(x-3)}{(1-0)(1-3)}(3) +$$

$$\frac{(x-0)(x-1)}{(3-0)(3-1)}(55)$$

$$= \frac{x^2-4x+3}{3} + \left(\frac{-3x^2+9x}{+2}\right) + \frac{55x^2-55x}{6}.$$

$$= \frac{2x^2-8x+6-9x^2+27x+55x^2-55x}{6}$$

$$= \frac{48x^2-36x+6}{6} = 8x^2-6x+1$$

$$y = 8x^2-6x+1.$$

$$y = 8\left(x^2-\frac{3}{4}x\right)+1$$

$a=8 \quad b=-6 \quad \pm\left(\frac{b}{2a}\right)^2$

$$y = 8\left(x^2-\frac{3}{4}x+\left(\frac{9}{64}\right)-\left(\frac{9}{64}\right)\right)+1$$

$$y = 8\left(\left(x-\frac{3}{8}\right)^2 - \frac{9}{64}\right) + 1$$

$$y = 8\left(x-\frac{3}{8}\right)^2 - \frac{9}{8} + 1$$

$$\boxed{y = 8\left(x-\frac{3}{8}\right)^2 - \frac{1}{8}}$$

(4) Using lagranges interpolation formula, find $y(10)$ from following table.

	x_0	x_1	x_2	x_3
x	5	6	9	11
y	12	13	14	16
	y_0	y_1	y_2	y_3

Lagrange's interpolation formula.

$$y = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}(y_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3$$

$$= \frac{(x-6)(x-9)(x-11)}{(5-6)(5-9)(5-11)}(12) + \frac{(x-5)(x-9)(x-11)}{(6-5)(6-9)(6-11)}(13)$$

$$+ \frac{(x-5)(x-6)(x-11)}{(9-5)(9-6)(9-11)}(14) + \frac{(x-5)(x-6)(x-9)}{(11-5)(11-6)(11-9)}(16)$$

$$= \frac{(x^3-26x^2+219x-594)}{-2} + \frac{(x^3-25x^2+209x-495)13}{15}$$

$$+ \frac{(x^3-22x^2+151x-330)7}{12} + \frac{(x^3-20x^2+129x-270)4}{15}$$

$$y = \frac{\cancel{4}^{2} \times 1 \times 1}{\cancel{2}} - \frac{\cancel{5}}{\cancel{15}\,\cancel{3}}(\cancel{13})^{4.33} + \frac{5 \times \cancel{4} \times 1 \times 7}{12\;3} + \frac{5 \times 4 \times 1 \times 4}{15}$$

$$y = 2 - 4.33 + 11.66 + 5.33$$

$$\boxed{y = 14.66}$$

Curve fitting

Method of least squares.

① Fitting a straight line.

Let $y = a + bx$ is a straight line to be fitted to the given data, then the normal equations are.

$$\Sigma y = ma + b\Sigma x \quad — ②$$

$$\Sigma xy = a\Sigma x + b\Sigma x^2 \quad — ③$$

Solving 2 & 3 equations for a, b we get required straight line of best fit.

② Fitting a parabola :- (Second polynomial)

Let $y = a + bx + cx^2$ — ① is a parabola to be fitted to the given data, The normal equations are.

$$\Sigma y = ma + b\Sigma x + c\Sigma x^2 \quad — ②$$

$$\Sigma xy = a\Sigma x + b\Sigma x^2 + c\Sigma x^3 \quad — ③$$

$$\Sigma x^2 y = a\Sigma x^2 + b\Sigma x^3 + c\Sigma x^4 \quad — ④$$

Solving ②, ③ & ④ equations for a, b, c, we get required parabola of best fit.

Problems

1. By the method of least squares, find the straight line that best fit the following data.

x	1	2	3	4	5
y	14	27	40	55	68

Sol

x	y	xy	x^2
1	14	14	1
2	27	54	4
3	40	120	9
4	55	220	16
5	68	340	25

$\Sigma x = 15$ $\Sigma y = 204$ $\Sigma xy = 748$ $\Sigma x^2 = 55$.

let $y = a + bx$. —— ①

Normal equations are

$\Sigma y = ma + b\Sigma x$ —— ②

$\Sigma xy = a\Sigma x + b\Sigma x^2$ —— ③

$204 = 5a + b(15)$ —— ④

$748 = 15a + 55b$ —— ⑤

Solving ④ & ⑤.

$a = 0,\ b = 13.6$.

Substitute a, b values in ①

$\boxed{y = 13.6x}$

② Fit a st line in following data.

x	6	7	7	8	8	8	9	9	10
y	5	5	4	5	4	3	4	3	3

x	y	xy	x^2
6	5	30	36
7	5	35	49
7	4	28	49
8	5	40	64
8	4	32	64
8	3	24	64
9	4	36	81
9	3	27	81
10	3	30	100
$\Sigma x=72$	$\Sigma y=36$	$\Sigma xy=282$	$\Sigma x^2=588$

let $y = a+bx$ —①

Normal equations are

$\Sigma y = na + b\Sigma x$ —②

$\Sigma xy = a\Sigma x + b\Sigma x^2$ —③

$36 = 9(a) + b(72)$ —④

$282 = a(72) + b(588)$ —⑤

solving ④ & ⑤.

$a = 8 \quad b = -0.5.$

$$\boxed{y = -0.5x + 8}$$

③ A chemical company wishing to study the effect of extraction time on the efficiency of an extraction operation obtained the data shown in following data.

Extraction time in min. x	27	45	41	19	3	39	19	49	15	31
efficiency y	57	64	80	46	62	72	52	77	57	68

fit a st line

Let $y = a + bx$ — ①.

Normal eqns are:

$\Sigma y = am + b\Sigma x$ — ②

$\Sigma xy = a\Sigma x + b\Sigma x^2$ — ③

$635 = 10a + 288b$ — ④

$19991 = 288a + 10274b$ — ⑤

solving ④ & ⑤

$a = 48.90 \quad b = 0.5066.$

$y = 48.90 + (0.5066)x.$

x	y	xy	x^2
27	57	1539	729
45	64	2880	2025
41	80	3280	1681
19	46	874	361
3	62	186	9
39	72	2808	1521
19	52	988	361
49	77	3773	2401
15	57	855	225
31	68	2108	961
$\Sigma x = 288$	$\Sigma y = 635$	$\Sigma xy = 19991$	$\Sigma x^2 = 10274$.

Q) fit st line to form $y = a + bx$. for following.

x	0	5	10	15	20	25
y	12	15	17	22	24	30

x	y	xy	x^2
0	12	0	0
5	15	75	25
10	17	170	100
15	22	330	225
20	24	480	400
25	30	750	625

$\Sigma x = 75$ $\quad \Sigma xy = 1805$

$\Sigma y = 120$ $\quad \Sigma x^2 = 1375$

let $y = a + bx$

Normal equations are

$$\Sigma y = na + b\Sigma x$$

$$\Sigma xy = a\Sigma x + b\Sigma x^2$$

$$120 = 6a + b(75) \quad \text{—①}$$

$$1805 = a(75) + b(1375) \quad \text{—②}$$

Solving ① & ②.

$a = 11.28$ $\quad b = 0.69$

$$y = 11.28 + (0.69)x$$

(5) Fit a second degree polynomial for following data by the method of least squares.

x	0	1	2	3	4
y	1	1.8	1.3	2.5	6.3

second degree polynomial is

$$y = a + bx + cx^2 \quad (2)$$

Normal eq are.

$\Sigma y = na + b\Sigma x + c\Sigma x^2$

$\Sigma xy = a\Sigma x + b\Sigma x^2 + c\Sigma x^3$

$\Sigma x^2 y = a\Sigma x^2 + b\Sigma x^3 + c\Sigma x^4$

x	y	xy	x^2y	x^2	x^3	x^4
0	1	0	0	0	0	0
1	1.8	1.8	1.8	1	1	1
2	1.3	2.6	5.2	4	8	16
3	2.5	7.5	22.5	9	27	81
4	6.3	25.2	100.8	16	64	256

$\Sigma x = 10$ $\Sigma y = 12.9$ $\Sigma xy = 37.1$ $\Sigma x^2 y = 130.3$ $\Sigma x^2 = 30$ $\Sigma x^3 = 100$ $\Sigma x^4 = 354$

$12.9 = 5(a) + b(10) + c(30)$

$37.1 = a(10) + b(30) + c(100)$

$130.3 = a(30) + b(100) + c(354)$

$a = 1.42$ $b = -1.07$ $c = 0.55$

$0.55x^2 - 1.07x + 1.42$

(6) By the method of least squares fit a parabola, of the form.

x	2	4	6	8	10
y	3.07	17.85	31.47	57.38	91.29

x	y	xy	x^2y	x^2	x^3	x^4
2	3.07	6.14	12.28	4	8	16
4	12.85	51.4	205.6	16	64	256
6	31.47	188.82	1132.92	36	216	1296
8	57.38	459.04	3672.32	64	512	4096
10	91.29	912.9	9129	100	1000	10000
30	196.06	1618.3	14152.12	220	1800	15664

$$196.06 = 5a + 30b + 220c$$

$$1618.3 = 30a + 220b + 1800c$$

$$14152.12 = 220a + 1800b + 15664c$$

$$a = 0.696$$

$$b = -0.855$$

$$c = 0.991$$

eqn of parabola.

$$y = 0.696 - 0.855x + 0.991x^2$$

Fit a parabola of the form $y = a + bx + cx^2$ to the following data

x	1	2	3	4	5	6	7
y	2.3	5.2	9.7	16.5	29.4	35.5	54.4

x	y	xy	x^2	x^2y	x^3	x^4
1	2.3	2.3	1	2.3	1	1
2	5.2	10.4	4	20.8	8	16
3	9.7	29.1	9	87.3	27	81
4	16.5	66.0	16	264.0	64	256
5	29.4	147.0	25	735.0	125	625
6	35.5	213.0	36	1278	216	1296
7	54.4	380.8	49	2665.6	343	2401
28	153.0	848.6	140	5053	784	4676

$$153 = 7a + 28b + 140c$$

$$848.6 = 28a + 140b + 784$$

$$5053 = 140a + 784b + 4676c$$

$a = 2.3714 \quad b = -1.0928 \quad c = 1.1928$

$$y = 1.1928x^2 - (1.0928)x + 2.3714$$

Non linear curves.

power curve.

1. let $y = a \cdot e^{bx}$ — ①.

Taking $\log_e$ on both sides.

$$\log_e y = \log_e a e^{bx}.$$

$$\log_e y = \log_e a + \log_e e^{bx}.$$

$$\log_e y = \log_e a + bx.$$

$$y = A + bx.$$

Now the normal equations are

$$\Sigma y = mA + b\Sigma x \quad \text{—②}$$

$$\Sigma xy = A\Sigma x + b\Sigma x^2 \quad \text{—③}.$$

Solving ②, ③ equations to get A & b.

since $A = \log_e a$

$$a = e^A.$$

problem:- Find the curve of best fit of the type $y = a \cdot e^{bx}$ to the following data by the method of least squares.

x	1	5	7	9	12.
y	10.	15	12	15	21.

Given $y = ae^{bx}$ — ① [$\because \log_e = \ln$]

take $\log_e$ on both sides.

$$\log_e y = \log_e a + bx.$$

$$y = A + bx.$$

Normal equations are.

$\Sigma y = nA + b\Sigma x$ — ①

$\Sigma xy = A\Sigma x + b\Sigma x^2$ — ③.

x.	y	Y = ln y.	xY	x^2.
1	10	2.3025.	2.3025.	1
5	15.	2.7080.	13.54.	25.
7	12.	2.4849.	17.3943.	49.
9	15.	2.7080	24.372.	81
12.	21.	3.0445.	36.534.	144.
34	73	13.249	94.1428.	300

sub $\Sigma x, \Sigma x^2, \Sigma Y, \Sigma xY$ values in ② & ③.

$13.249 = 5A + 34b$ — ④.

$94.1428 = 34A + 300b$ — ⑤.

solving ④ & ⑤.

$A = 2.2495 \quad b = 0.0588.$

$a = e^A = e^{2.2495} = 9.4829.$

substitute a & b in ①.

Hence $y = 9.4829\, e^{(0.0588)x}$.

② find the curve of best fit of type $y = a \cdot e^{bx}$ to the following data by the method of least squares.

x	0.0	0.5	1.0	1.5	2.0	2.5
y	0.10	0.45	2.15	9.15	40.35	180.75

x	y	$Y = \ln y$	~~xy~~	xY	x^2
0.0	0.10	−2.3025	~~0~~	0	0
0.5	0.45	−0.7985	~~0.2185~~	−0.39925	0.25
1.0	2.15	0.7654	~~2.15~~	0.7654	1
1.5	9.15	2.2137	~~20.5875~~	3.3205	2.25
2.0	40.35	3.6975		7.395	4
2.5	180.75	5.1971		12.9927	6.25
7.5	232.85	8.7727		24.0744	13.75

$$8.7727 = 6A + 7.5b$$

$$24.0744 = 7.5A + 13.75b$$

$$A = -2.2831$$
$$b = 2.9962$$

$$a = e^A = 0.1019$$

$$y = A + bx$$
$$= 0.1019 + e^{(2.9962)x}$$

(3) Fit the curve $y = a \cdot e^{bx}$.

x	0	1	2	3	4	5	6	7	8.
y	20	30	52	77	135	211	326	550	1052.

$y = 18.95 \cdot e^{0.486x}$.

x.	y	Y = ln y.	xY	x^2.
0	20	2.9957.	0.	0
1	30	3.4011	3.4011.	1
2	52.	3.9512.	7.9024.	4
3	77.	4.3438.	13.0314.	9
4	135.	4.9052.	19.6208.	16
5	211	5.3518.	26.759.	25
6	326.	5.7868.	34.7208.	36
7	550.	6.3099.	44.1693.	49
8.	1052.	6.9584.	55.6672.	64.
36.		44.0039.	205.272	204.

Normal equations are.

$\Sigma Y = mA + b\Sigma x$.

$\Sigma xY = A\Sigma x + b\Sigma x^2$.

$44.0039 = 9A + b(36)$

$205.272 = A(36) + b(204)$

$A = 2.9388 \qquad b = 0.4876$.

$a = e^A = 18.8931$.

$y = a \cdot e^{bx} = (18.8931)e^{(0.4876)x}$.

② $y = ab^x$.

Taking $\log_{10}$ on both sides.

$\log_{10} y = \log_{10} (ab^x)$

$\log_{10} y = \log_{10} a + x \log_{10} b$

$\log y = \log a + x \log b$.

$Y = A + Bx$ — ②

Now normal equations of ②

$\Sigma Y = mA + B\Sigma x$ — ③

$\Sigma xY = A\Sigma x + B\Sigma x^2$ — ④.

Solving ③ & ④ for A & B.

$A = \log_{10} a \Rightarrow a = 10^A$.

$B = \log_{10} b \Rightarrow b = 10^B$.

Sub A & B in ① to get required curve $y = ab^x$.

Problems:- ① Fit $y = ab^x$ by method of least squares to following data.

x	0	1	2	3	4	5	6	7.
y	10	21	35	59	92	200	400	610.

Sol To fit a curve $y = a.b^x$. — ①.

Taking $\log_{10}$ on both sides

$\log_{10} y = \log_{10} a + x \log_{10} b$.

$Y = A + Bx$. — ②

Normal equations are.

$\Sigma Y = mA + B\Sigma x$ — ③

$\Sigma xY = A\Sigma x + B\Sigma x^2$ — ④.

x	y	$Y=\log_{10}y$	xY	x^2
0	10	1	0	0
1	21	1.3222	1.3222	1
2	35	1.5440	3.088	4
3	59	1.7708	5.3124	9
4	92	1.9637	7.8548	16
5	200	2.3010	11.505	25
6	400	2.6020	15.612	36
7	610	2.7853	19.4971	49
28	1427	15.289	64.1915	140

$$15.289 = 8A + B(28)$$

$$64.1915 = A(28) + B(140)$$

$$A = 1.0211 \Rightarrow a = 10^{1.0211} = 10.4978$$

$$B = 0.2542 \Rightarrow b = 10^{0.2542} = 1.7955$$

$$y = (10.4978)(1.7955)^x$$

② Fit curve of form

$y = ab^x$.

x	77	100	185	239	285
y	2.4	3.4	7.0	11.1	19.6

To fit a curve $y = ab^x$.

log on both sides.

$$\log_{10} y = \log_{10} a + x \log_{10} b$$

$$Y = A + Bx$$

Normal equations are

$$\Sigma Y = mA + B(\Sigma x)$$

$$\Sigma xY = A \cdot \Sigma x + B(\Sigma x^2)$$

x	y	$Y = \log_{10} y$	xY	x^2
77	2.4	0.3802	29.2754	5929
100	3.4	0.5314	53.14	10000
185	7.0	0.8450	156.325	34225
239	11.1	1.0453	249.8267	57121
285	19.6	1.2922	368.277	81225
886		4.0941	856.7941	188500

$$4.0941 = 5A + B(886)$$

$$856.7941 = 886A + B(188500)$$

$A = 0.0801$, $B = 0.00416$

$a = 10^{0.0801}$ $b = 10^{0.00416}$

$a = 1.2025$ $b = 1.0096$

$$y = (1.2025)(1.0096)^x$$

* ③ Fitting a curve $y = ax^b$ — ①

Taking $\log_{10}$ on both sides

$$\log_{10} y = \log_{10} ax^b$$

$$\log_{10} y = \log_{10} a + b \log_{10} x$$

$$Y = A + bX \quad \text{— ②}$$

write normal eqn of ②

$$\Sigma Y = mA + b\Sigma X \quad \text{— ③}$$

$$\Sigma XY = A\Sigma X + b\Sigma X^2 \quad \text{— ④}$$

solving ③ & ④ $A = \log_{10} a$ $a = 10^A$

subs a, b in ① to get required curve

① Fit a power curve of form $y = ax^b$ for following

x	1	2	4	6
y	6	4	2	2

Fit curve of $y = ax^b$ — ①

$$\log_{10} y = \log_{10} a + b \log_{10} x$$

$$Y = A + bX \quad \text{—} ②$$

Normal eqns are-

$$\Sigma Y = mA + b\Sigma X \quad \text{—} ③$$

$$\Sigma YX = A\Sigma X + b\Sigma X^2 \quad \text{—} ④$$

x	y	$Y = \log_{10} y$	$X = \log_{10} x$	XY	X^2
1	6	0.7781	0	0	0
2	4	0.6020	0.3010	0.1812	0.0906
4	2	0.3010	0.6020	0.1812	0.3624
6	2	0.3010	0.7781	0.2342	0.6054
		$\Sigma Y = 1.9821$	$\Sigma X = 1.6811$	$\Sigma XY = 0.5966$	$\Sigma X^2 = 1.0584$

substitute values in eq ③ & ④.

$$1.9821 = 4A + b(1.6811) \quad \text{—} ③$$

$$0.5966 = A(1.6811) + b(1.0584) \quad \text{—} ④$$

Solving ③ & ④

$A = 0.7779 \qquad b = -0.6719.$

$a = 10^A = 10^{0.7779} = 5.9965.$

substitute a, b values in ①.

$$y = (5.9965)\,x^{-0.6719}.$$

② Fit a curve $y = a x^b$ to following data

x	1	2	3	4	5	6.
y	2.98	4.26	5.21	6.10	6.80	7.50.

x	y	$Y = \log_{10} y$	$X = \log_{10} x$	XY	X^2
1	2.98	0.4742	0	0	0
2	4.26	0.6294	0.3010	0.1894	0.0906
3	5.21	0.7168	0.4771	0.3419	0.2276
4	6.10	0.7853	0.6020	0.4727	0.3624
5	6.80	0.8325	0.6989	0.5818	0.4884
6	7.50	0.8750	0.7781	0.6808	0.6054
		4.3132	2.8571	2.256	1.7744

$$4.3132 = 6A + 2.8571b$$

$$2.266 = 2.8571A + 1.7744b$$

$$A = 0.4748$$

$$b = 0.5125$$

$$a = 10^A = 10^{0.4748} = 2.9908$$

$$y = 2.9908\, x^{0.5125}$$