Q(1) Given that,

An AR(1) process (order 1), is a type of time series model where the current value of a variable is based on its immediate past value, along with some random noise.

Specifically, we have;

$$\varepsilon_i = \phi\, \varepsilon_{i-1} + a_i$$

where:

ε_i - is the current value of the process.

ϕ - is a parameter that measures the degree of auto correlation it can range from -1 & 1

ε_{i-1} - is the previous value of the process.

a_i - represents a white noise error term.

a) The AR(1) process is weakly stationary means, the mean, variance, and auto covariance donot change over time, they remain constant regardless of when we measure them.

for the above condition to hold, the parameter ϕ must satisfy $|\phi| < 1$. This ensures the process remain stable, so previous acts diminish rather than going away.

b) Assuming the white noise term a_i in the AR(1) model, it must meet the following conditions

(i) Zero Mean : The average of a_i is zero so it doesn't create a trend

$(E(a_i) = 0)$

(ii) Constant variance : The variability of a_i is constant over time.

$(var(a_i) = \sigma^2)$

(iii) No Autocorrelation : Each a_i is independent of others, means they don't affect each other.

$(cov(a_i, a_j) = 0 \text{ for } i \neq j)$

These assumptions ensure that the white noise term does not add any patterns or dependencies to the process.

Q (2) Yes, during the class we have shown that; using the assumptions in Question 1:

$$E(\varepsilon_i) = 0$$

$$var(\varepsilon_i) = \frac{\sigma^2}{1-\phi^2}$$

$$cov(\varepsilon_i, \varepsilon_{i-1}) = \frac{\phi\sigma^2}{1-\phi^2} = \phi \, var(\varepsilon_i)$$

In this problem, we will solve

a) To show that

$$cov(\varepsilon_i, \varepsilon_{i-2}) = \frac{\phi^2\sigma^2}{1-\phi^2} = \phi^2 var(\varepsilon_i)$$

given that in general,

$$cov(\varepsilon_i, \varepsilon_{i-s}) = \frac{\phi^s\sigma^2}{1-\phi^2} = \phi^s var(\varepsilon_i) \text{ for } s \geq 0$$

but here we don't have to show the general case, but just the case where $s=2$,

→ Generally, in the AR(1) process, the value at any time i is related to its previous value by a parameter ϕ.

This relationship allows us to calculate covariances between terms that are seperated by a certain number of lags.

→ we know that,

$$cov(\varepsilon_i, \varepsilon_{i-1}) = \frac{\phi \sigma^2}{1-\phi^2} = \phi \, var(\varepsilon_i) \quad — \text{(1)}$$

→ Since, each term in the AR(1) process depends linearly on the previous one, the covariance between ε_i, and ε_{i-2} can be obtained by using equation (1).

$$cov(\varepsilon_i, \varepsilon_{i-2}) = \phi \, cov(\varepsilon_{i-1})$$

Let's substitute equation 1, we get.

$$cov(\varepsilon_i, \varepsilon_{i-2}) = \phi \cdot \frac{\phi \sigma^2}{1-\phi^2} = \frac{\phi^2 \sigma^2}{1-\phi^2}$$

we know that, $\frac{\sigma^2}{1-\phi^2} = var(\varepsilon_i)$

Thus, we have shown that;

$$\boxed{cov(\varepsilon_i, \varepsilon_{i-2}) = \frac{\phi^2 \sigma^2}{1-\phi^2} = \phi^2 var(\varepsilon_i)}$$

b) to show,

$$\rho_2 = \text{corr}(\varepsilon_i, \varepsilon_{i+2}) = \phi^2$$

given that in general,

$$\rho_K = \text{corr}(\varepsilon_i, \varepsilon_{i+K}) = \phi^K \quad \text{for} \quad K = 1, 2, \ldots$$

but here we don't have to show the general case,

but just the case where $K = 2$,

→ The correlation $\rho_2 = \text{corr}(\varepsilon_i, \varepsilon_{i+2})$ is defined as:

$$\rho_2 = \frac{\text{cov}(\varepsilon_i, \varepsilon_{i+2})}{\sqrt{\text{var}(\varepsilon_i) \cdot \text{var}(\varepsilon_{i+2})}} \quad \text{— (2)}$$

→ For a stationary process the variance is same at all time points so, $\text{var}(\varepsilon_i) = \text{var}(\varepsilon_{i+2})$

So, we can simplify equation 2 as :

$$\rho_2 = \frac{\text{cov}(\varepsilon_i, \varepsilon_{i+2})}{\text{var}(\varepsilon_i)}$$

we have shown, $\text{cov}(\varepsilon_i, \varepsilon_{i-2}) = \phi^2 \text{var}(\varepsilon_i)$ in part(a)

Similar to that we can say that :

$$\text{cov}(\varepsilon_i, \varepsilon_{i+2}) = \phi^2 \text{var}(\varepsilon_i)$$

substituting this into the formula;

$$\rho_2 = \frac{\phi^2 var(\varepsilon_i)}{var(\varepsilon_i)}$$

Therefore, $\rho_2 = \phi^2$

Thus, shown that:

$$\boxed{corr(\varepsilon_i, \varepsilon_{i+2}) = \rho^2 = \phi^2}$$

Q(5) Given that,

using the same notation as in Problem 1, we can model the data in problem 3 and 4 as,

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \varepsilon_i \quad — ①$$

$$\varepsilon_i = \phi \varepsilon_{i-1} + a_i$$

Given change of variables;

$$y_i' = y_i - \phi y_{i-1}$$

$$x_{i1}' = x_{i1} - \phi x_{i-1,1}$$

$$x_{i2}' = x_{i2} - \phi x_{i-1,2}$$

a) To show that, with the new variables y_i', x_{i1}' and x_{i2}' we can write equation ① as

$$y_i' = \beta_0(1-\phi) + \beta_1 x_{i1}' + \beta_2 x_{i2}' + a_i$$

Let's first substitute the error term ε_i into equation 1.

we get,

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \phi \varepsilon_{i-1} + a_i$$

Now, let's substitute above equation to y_i', we get

$$y_i' = (\beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \phi \varepsilon_{i-1} + a_i) - \phi y_{i-1} \quad \text{— 2}$$

Expanding ϕy_{i-1},

$$y_{i-1} = \beta_0 + \beta_1 x_{i-1,1} + \beta_2 x_{i-1,2} + \varepsilon_{i-2}$$

So, $\phi y_{i-1} = \phi (\beta_0 + \beta_1 x_{i-1} + \beta_2 x_{i-1,2} + \varepsilon_{i-1})$

Substituting ϕy_{i-1} to equation 2, we get

$$y_i' = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \cancel{\phi \varepsilon_{i-1}} + a_i - \phi\beta_0 - \phi \beta_1 x_{i-1,1} - \phi \beta_2 x_{i-1,2} - \cancel{\phi \varepsilon_{i-1}}$$

$$y_i' = \beta_0(1-\phi) + \beta_1(x_{i1} - \phi x_{i-1,1}) + \beta_2 (x_{i2} - \phi x_{i-1,2}) + a_i$$

$$\Rightarrow \boxed{y_i' = \beta_0(1-\phi) + \beta_1 x_{i1}' + \beta_2 x_{i2}' + a_i}$$

Hence Shown