Homework - Intro to Probability

1. A unbiased die is thrown once. Compute the probability of the following events.
   a). The number of spots shown is odd.
   b). The number of spots shown is less than 3.
   c). The number of spots shown is a prime number.

2. A prisoner in a Kafkaesque prison is put in the following situation. A regular deck of 52 cards is placed in front of him. He must choose cards one at a time, and once a card is chosen, the card is replaced in the deck and the deck is shuffled. If the prisoner happens to select three consecutive red cards, he is executed. If he happens to selects 6 cards before three consecutive red cards appear he is granted freedom. What is the probability that the prisoner is executed.

3. Three marksmen fire simultaneously and independently at a target. What is the probability of the target being hit at least once, given that marksman one hits a target 9 times out of 10, marksman two hits a target 8 times out of 10 while marksman three only hits a target 1 out of every 2 times.

4. Calculate the expected value and variance of a random variable that follows the exponential distribution with parameter λ.

5. Calculate the expected value and variance of a random variable that follows the Poisson distribution with parameter λ.

6. Jobs to be performed on a machine arrive according to a Poisson distribution with a rate of two per hour. Suppose that the machine breaks down from time to time and it takes 1 hour to be repaired. What is the probability that a) zero, b) two, and c) five new jobs will arrive during this time?

7. SSI chips, essential to the running of a computer system, fail in accordance with a Poisson distribution at the rate of one chip per five weeks. If there are two spare chips on hand and if a new supply will arrive in eight weeks, what is the probability that during the next eight weeks the system will be down for a week or more owing to lack of chips?

8. Jobs arrive to a computer system have been found to require a CPU time that can be modelled by an exponential distribution with mean 140 msec. The CPU scheduling algorithm is quantum-oriented so that a job not completing within a quantum of 100 msec will be routed back to the tail of the queue of waiting jobs.
   a). What is the probability that an arriving job will be forced to wait for a second quantum?
   b). Of the 800 jobs coming during a day, how many are expected to finish within the first quantum?

Solutions

1

The sample space for this experiment is:

$$S = \{1, 2, 3, 4, 5, 6\}$$

All outcomes are equally possible to appear so:

$$P[\{1\}] = P[\{2\}] = P[\{3\}] = P[\{4\}] = P[\{5\}] = P[\{6\}] = 1/6$$

1. The odd outcomes are: $O = \{2, 3, 5\}$. Since all outcomes are independent
$$P[\{2, 3, 5\}] = P[\{2\}] + P[\{3\}] + P[\{5\}] = 1/2$$
2. The outcomes less than 3 are: $L = \{1, 2\}$. Since all outcomes are independent:
$$P[\{1, 2\}] = P[\{1\}] + P[\{2\}] = 1/3$$
3. The outcomes that are prime are: $L = \{1, 2, 3, 5\}$. Since all outcomes are independent:
$$P[\{1, 2, 3, 5\}] = P[\{1\}] + P[\{2\}] + P[\{3\}] + P[\{5\}] = 2/3$$

2

The following events lead to the prisoner's execution:

1. The first three cards he draws are all red. This event occurs with probability $1/8$.

2. The first card drawn is black, but the next three are red. This event occurs with probability 1/16.

3.
   - The first card is red, the second black and the next three red, or
   - The first card is black, the second black and the next three red.

   Both these events occur with probability 1/32, so their sum is 1/16.

4. The third card is black and the last three are red. The first two cards may be either black or red. This gives four possibilities each with probability 1/64. The sum in this case is 1/16.

Therefore, the probability that the prisoner is executed is the sum of the probabilities of these four events, which is equal to 5/16.

In terms of elementary events, we can take an elementary event to be any combination of red and black cards from rrrrrr through bbbbbb, where r indicates a red card and b, a black card. Each event is equiprobable with probability 1/64. There are 8 elementary events of the form rrrxxx, where x indicates a don't care; 4 elementary events of the form brrrxx; 4 elementary events of the form xbrrrx and 4 of the form xxbrrr. Each of these elementary events lead to the prisoners execution. Since there are a total of 20 of them, the probability that the prisoner is executed is given by $20/64 = 5/16$.

3

In order to calculate the probability the target is hit only once, we can calculate the probability none hits the target. Thus:

P[targetis hit at least once] = 1 - P[none hits the target]

Since the marksmen hit the target simultaneously and independently, we have:

P[none hits the target] = P[(1^{st} marksman doesn't hit the target) $\cap$ (2^{nd} marksman doesn't hit the target) $\cap$ (3^{rd} marksman doesn't hit the target)] =

1 -P[1^{st} marksman doesn't hit the target] $\cdot$ P[2^{nd} marksman doesn't hit the target]$\cdot$ P[3^{rd} marksman doesn't hit the target]=

$$1 - (1 - 9/10) \cdot (1 - 8/10) \cdot (1 - 1/2) = 0.99$$

4

We know that the probability mass function for the exponential disrtibution is:

$$f_X(x) = \lambda e^{-\lambda x}$$

Thus using the definition for the expected value of a continuous R.V. we have:

$$E[X] = \int_{-\infty}^{\infty} x f_X(x) dx$$

So we have:

$$E[X] = \int_{-\infty}^{\infty} x e^{-\lambda x} dx$$

Using integration by parts we get:

$$E[X] = -xe^{\lambda x}|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} e^{-\lambda x} dx = 1/\lambda$$

In order to calculate the variance we use the definition below:

$$var[X] = E[X^2] - (E[X])^2$$

We already computed $E[X]$ above so we use the definition of $E[X]$ and set $X = U^2$ in order to calculate $E[U^2]$. We have then:

$$E[X^2] = \int_{-\infty}^{\infty} x^2 e^{-\lambda x} dx = \{using\ integration\ by\ parts\} = ... = 2/\lambda^2$$

So finally:

$$var[X] = 2/\lambda^2 - 1/\lambda^2 = 1/\lambda^2$$

5

In this case we use the definition of the expectation of a discrete R.V.:

$$E[X] = \sum_{k=0}^{\infty} k \frac{\lambda^k}{k!} e^{-\lambda} = e^{-\lambda} \sum_{k=0}^{\infty} k \frac{\lambda^k}{(k-1)!k} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^{(k+1)}}{k!} =$$

$$e^{-\lambda}\lambda\sum_{k=0}^{\infty}\frac{\lambda^k}{k!} = e^{-\lambda}\lambda e^{\lambda} = \lambda$$

Again by the definition of variance:

$$var[X] = E[X^2] - (E[X])^2 \qquad (1)$$

We have to calculate $E[X^2]$. We can do this by using the following observation:

$$E[X(X-1)] = E[X^2 - X] = E[X^2] - E[X] \Rightarrow E[X^2] = E[X(X-1)] + E[X]$$

Thus if we substitute $E[X^2]$ in (1) we have:

$$var[X] = E[X(X-1)] + E[X] - (E[X])^2 \qquad (2)$$

Now we have to calculate $E[X(X-1)]$. By definitili we have:

$$E[X(X-1)] = \sum_{k=0}^{\infty} k(k-1)\frac{\lambda^k}{k!}e^{-\lambda} = e^{-\lambda}\sum_{k=0}^{\infty} k(k-1)\frac{\lambda^k}{(k-2)!(k-1)k} =$$

$$e^{-\lambda}\sum_{k=0}^{\infty}\frac{\lambda^{(k+2)}}{k!} = e^{-\lambda}\lambda^2\sum_{k=0}^{\infty}\frac{\lambda^k}{k!} = e^{-\lambda}\lambda^2 e^{\lambda} = \lambda^2$$

Now we can calculate $var[X]$ using (2):

$$var[X] = \lambda^2 + \lambda - \lambda^2 = \lambda$$

6

1. The arrival of jobs is a Poisson distribution with rate $\lambda = 2/hr$. The probability to have k = 0 jobs arriving within the interval t = 1 hr that the machine is down is:

$$P[k=0] = e^{-2\cdot 1}\cdot(2\cdot 1)^0/0! = e^{-2}$$

2. Similarly to the above we get the probability to have k = 2 jobs arriving within the interval t = 1 hr:

$$P[k=0] = e^{-2\cdot 1}\cdot(2\cdot 1)^2/2! = 2e^{-2}$$

3. Finally the probability to have k = 5 jobs arriving within the interval t = 1 hr:

$$P[k=5] = e^{-2\cdot 1}\cdot(2\cdot 1)^5/5! = (4/15)e^{-2}$$

7

Since we have two spare SSI chips, the probability to have the system down for at least one week, is the probability to have at least three chip failures within at most 7 weeks. Thus:

P[the system is down for at least 1 week] = 1 - P[0 chips fail within at most 7 weeks] - P[1 chips fail within at most 7 weeks] - P[2 chips fail within at most 7 weeks] =

$1 - e^{-7/5}(1/5 \cdot 7)^0/0! - e^{-7/5}(1/5 \cdot 7)^1/1! - e^{-7/5}(1/5 \cdot 7)^2/2!$

8

1. A job will have to wait for a second quantum is the time it remais at the CPU is more than 100 msec. Since the service time is exponentially distributed we have:

$$P[X \geq 100] = \int_{100}^{\infty} f_X(x)dx = \int_{100}^{\infty} 1/140e^{-1/140 \cdot x}dx = e^{-10/14}$$

2. The probability that a job will finish within the first quantum is: $1 - P[X \geq 100]$. Thus the number of jobs that will finish within the first quantum is the sum of the independent probabilities that every job of the 800 will finish in the first quantum. This is:

$$k = 800 \cdot (1 - P[X \geq 100])$$