4. PROBABILITY

Introduction

- The first three chapters are on descriptive statistics – organizing data, plot the data, numerical descriptive measures.

- Now we enter the topic on probability and random variables.

- Why study probability?
  - Statistics also deal with uncertainty.
  - For example, we might guess that it will rain tomorrow. But how sure are we?

Experiment, outcome, sample space

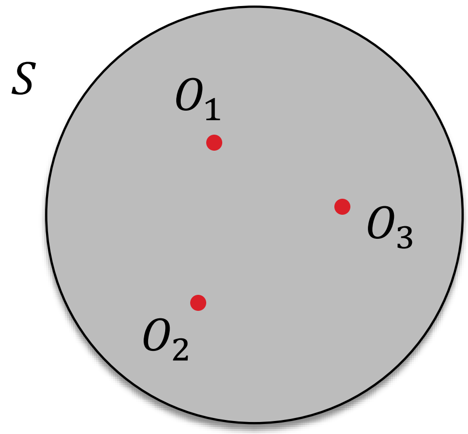
Experiment, outcome, sample space

- A probability experiment is a chance process that leads to well-defined results called outcomes.
- An outcome is the result of a single trial of a probability experiment.
- A sample space is the set of all possible outcomes of a probability experiment.

- Denote sample space by S.

Set notation

- We write the sample space as S, and the individual outcomes as O_i.
- When we write $S = \{O_1, O_2, O_3\}$, that means the outcomes O_1, O_2, O_3 belongs to the set S.
- The set $\{O_1, O_2, O_3\}$ is the same as $\{O_2, O_3, O_1\}$. Ordering does not matter in the set.



Example of experiments, outcomes and sample spaces

Experiment	Outcomes	Sample Space
Toss a coin once	Head, Tail	$S = \{\text{Head}, \text{Tail}\}$
Roll a die once	1, 2, 3, 4, 5, 6	$S = \{1,2,3,4,5,6\}$
Toss a coin twice	HH, HT, TH, TT	$S = \{HH, HT, TH, TT\}$
Play lottery	Win, Lose	$S = \{\text{Win}, \text{Lose}\}$
Take a test	Pass, Fail	$S = \{\text{Pass}, \text{Fail}\}$
Select a student	Male, Female	$S = \{\text{Male}, \text{Female}\}$

Venn and tree diagrams

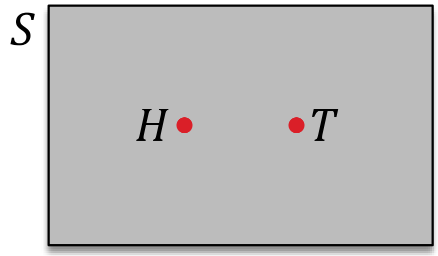
- It may be useful to use Venn diagram or tree diagram to illustrate the sample space.

- Venn diagram:
  - Illustrates all the possible outcomes.
  - Draw a rectangle, and label it as S. The rectangle represents all the possible outcomes.
  - Draw the individual outcomes inside the rectangle.

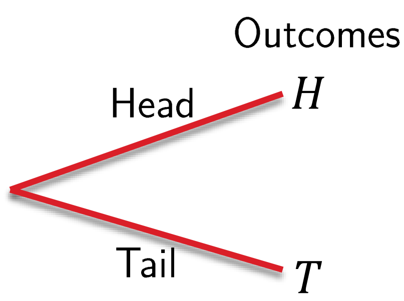
- Tree diagram:
  - Illustrates the flow of events.
  - Draw the branches starting from the same point.
  - Each branch represents a possible outcome.

Example (tossing coin once)

- Consider the experiment of tossing a coin once
  - Two possible outcome – head (H) or tail (T).
  - Sample space: $S = \{H, T\}$
  - Venn diagram:



  - Tree diagram:

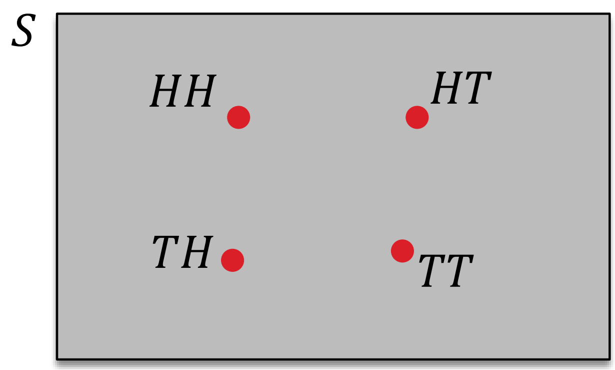


Example (tossing coin twice)

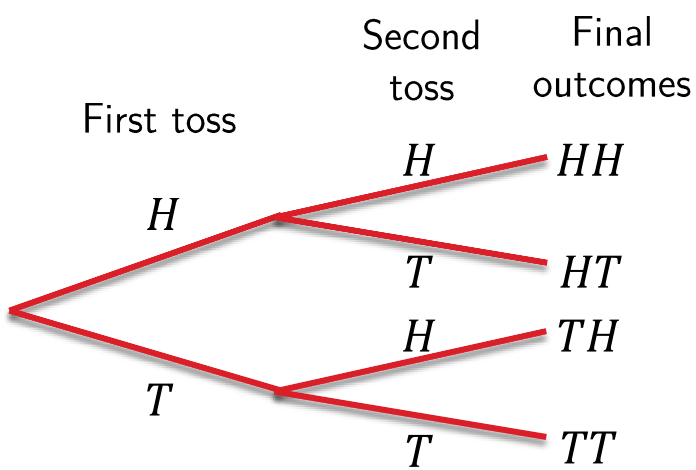
- For the experiment of tossing a coin twice.
  - Split the experiment into two parts: the first toss and second toss.
  - Denote
    - HH as the outcome when both tosses are head
    - HT as the outcome when first toss is head and second toss is tail
    - TH as the outcome when first toss is tail and second toss is head
    - TT as the outcome when both tosses are tail
  - Therefore there are four possible outcome, with the sample space $S = \{HH, HT, TH, TT\}$

Example (tossing coin twice)

- Venn diagram:



- Tree diagram:



Events

- An event is a collection of one or more outcomes of an experiment.
- Simple event – event with a single outcome
- Compound event – event with more than one outcome

Example

- Consider the experiment of tossing coin twice.
  - $S = \{HH, HT, TH, TT\}$

  - The event of getting both heads:
    - $E = \{HH\}$
    - Simple event

  - The event of getting at least one head:
    - $E = \{HH, HT, TH\}$
    - Compound event

Another example

EXAMPLE 4–6 Preference for Ice Tea

In a group of college students, some like ice tea and others do not. There is no student in this group who is indifferent or has no opinion. Two students are randomly selected from this group.

(a) How many outcomes are possible? List all the possible outcomes.

(b) Consider the following events. List all the outcomes included in each of these events. Mention whether each of these events is a simple or a compound event.

- **(i)** Both students like ice tea.
- **(ii)** At most one student likes ice tea.
- **(iii)** At least one student likes ice tea.
- **(iv)** Neither student likes ice tea.

Another example

- Let L denotes the event that a student likes ice tea, and let N denotes the event that a student does not like ice tea.

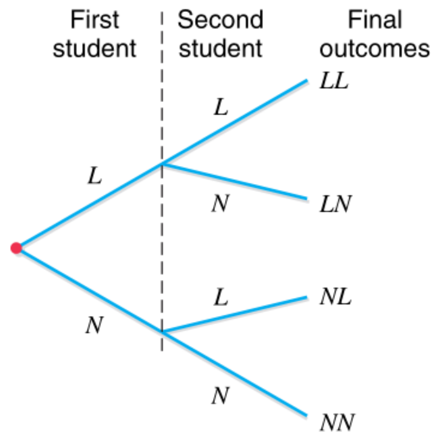
a) There are four outcomes:
   - LL = both students like ice tea
   - LN = first student likes ice tea, while the second student does not
   - NL = first student does not like ice tea, while the second does
   - NN = both students does not like ice tea

b)

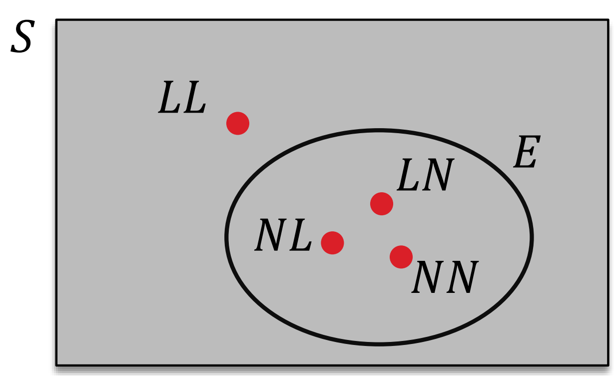
   i. Both students like ice tea = $\{LL\}$ (simple)
   ii. At most one student likes ice tea = $\{LN, NL, NN\}$ (compound)
   iii. At least one student likes ice tea = $\{LL, LN, NL\}$ (compound)
   iv. Neither student likes ice tea = $\{NN\}$ (simple)

Another example

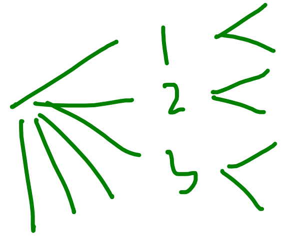
- Tree diagram:



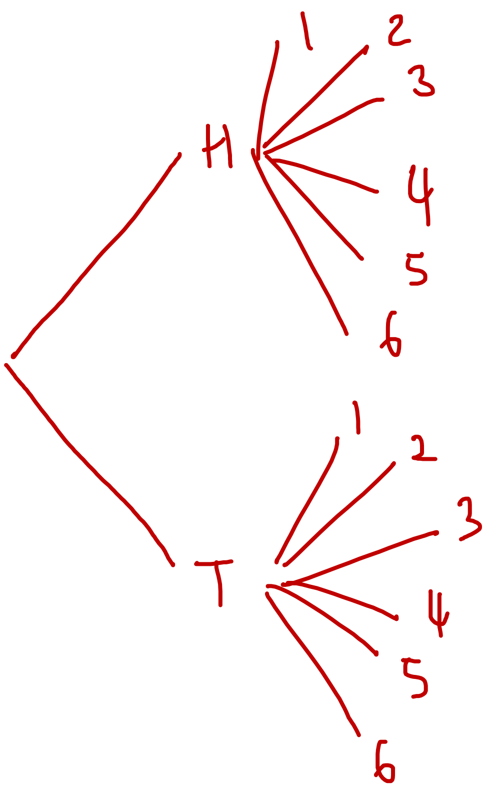
- Venn diagram (for (ii)):



Exercise



- In an experiment, a die and a coin are tossed and the results are recorded.
  - Draw a tree diagram on the outcomes of this experiment.
  - How many possible outcomes are there? Write down the sample space.
  - List down the event in which head is observed.



There are 12 outcomes

$S = \{H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6\}$

$E = \{H1, H2, H3, H4, H5, H6\}$

Calculating probability

What is probability

- Probability is the numerical measure of the likelihood that a specific event will occur.

- Properties of probability:
  - $0 \leq P(E) \leq 1$
  - $\sum P(O_i) = 1$ where O_i is the individual outcomes in the sample space

- If probability of an event is 1, then the event is certain to happen.
- If probability of an event is 0, then the event is impossible.

What is probability

- Example: toss a coin
  - $P(\text{get a head or tail}) = 1$
  - $P(\text{coin standing on its edge}) = 0$
  - $P(\text{Head}) + P(\text{Tail}) = 1$

- Three approaches to probability:
  - Classical probability
  - Empirical probability
  - Subjective probability

Classical probability

- In classical probability, we assume that all outcomes in the sample space are equally likely to occur.
- The probability of event E with sample space S:

$$P(E) = \frac{\text{Number of outcomes in } E}{\text{Total number of outcomes in } S}$$

- Example: Tossing a coin once
  - $S = \{\text{Head}, \text{Tail}\}$
  - $P(\text{Head}) = \frac{1}{2}$
  - $P(\text{Tail}) = \frac{1}{2}$

Example

- Find the probability of obtaining an even number in one roll of a die.
  - Sample space, $S = \{1,2,3,4,5,6\}$
  - Let E be the event that an even number is observed.
  - $E = \{2,4,6\}$

$$P(E) = \frac{\text{Number of outcomes included in } E}{\text{Total number of outcomes}} = \frac{3}{6} = \frac{1}{2}$$

Example

- Consider the experiment of tossing a coin twice. What is the probability of getting exactly one head?
  - Sample space, $S = \{HH, HT, TH, TT\}$
  - Let E be the event of getting exactly one head
  - $E = \{HT, TH\}$

$$P(E) = \frac{\text{Number of outcomes in } E}{\text{Total number outcomes}} = \frac{2}{4} = \frac{1}{2}$$

Empirical probability

- Unlike classical probability, the events are not assumed to be equally likely to occur.
- Empirical probability uses relative frequency as the probability.
- Probability of event E:

$$P(E) = \frac{\text{Frequency of } E}{\text{Total frequency}} = \frac{f}{n}$$

Example

- 500 randomly selected new cars are checked whether they are in good or bad condition:

Condition	Frequency
Good	490
Bad	10

- Then,

$$P(\text{Good}) = \frac{490}{500} = 0.98$$

$$P(\text{Bad}) = \frac{10}{500} = 0.02$$

Law of large numbers

- The relative frequencies are not exact probabilities, but approximate probabilities unless from the whole population.
- But if the sample size increases, the relative frequencies will approach the actual probability of outcome.
- This phenomenon is called the law of large numbers.

- Eg: Toss a fair coin.
  - Toss 10 times: 3 heads, 7 tails (prob: 0.3 and 0.7)
  - Toss 100 times: 43 heads, 57 tails (prob: 0.43 and 0.57)
  - Toss 1000 times: 513 heads, 487 tails (prob: 0.513 and 0.487)

Subjective probability

- Another approach to probability.
- Probability based on individual's subjective belief and judgement.
- Probabilities are simply guess or estimate.

- Example:
  - A student believe that his/her probability of getting an A is 70%.
  - A doctor believe that his/her patient has a 50% chance of surviving an operation.

Exercise

classical probability

13 13 13 13 (52 total)

EXAMPLE 4–7 **Drawing Cards**

A card is drawn from an ordinary deck. Find the probability of getting

a. A heart

b. A black card

c. The 8 of diamonds

d. A queen

e. A face card

a) $P(\text{Heart}) = \dfrac{\#\text{ heart}}{\text{total}}$

$= \dfrac{13}{52} = \dfrac{1}{4}$

b) $P(\text{Black}) = \dfrac{26}{52} = \dfrac{1}{2}$

c) $P(8\diamondsuit) = \dfrac{1}{52}$

d) $P(\text{Queen}) = \dfrac{4}{52} = \dfrac{1}{13}$

e) $P(\text{face}) = \dfrac{12}{52} = \dfrac{3}{13}$

Exercise

classical probability

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14. Rolling a Die If a die is rolled one time, find these probabilities:

a. Getting a number less than 7.

b. Getting a number greater than or equal to 3

c. Getting a number greater than 2 and an even number

d. Getting a number less than 1

$S = \{1, 2, 3, 4, 5, 6\}$

a) $E = \{1, 2, 3, 4, 5, 6\}$

$P(E) = \frac{6}{6} = 1$

b) $E = \{3, 4, 5, 6\}$

$P(E) = \frac{4}{6} = \frac{2}{3}$

c) $E = \{4, 6\}$

$P(E) = \frac{2}{6} = \frac{1}{3}$

d) $E = \{\}$

$= \emptyset$

$P(E) = \frac{0}{6}$

$= 0$

Exercise

empirical probability

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21. Human Blood Types Human blood is grouped into four types. The percentages of Americans with each type are listed below.

O 43% A 40% B 12% AB 5%

Choose one American at random. Find the probability that this person

a. Has type B blood

b. Has type AB or O blood

c. Does not have type O blood

a) $P(\text{Type B}) = 0.12$

b) $P(\text{AB or O}) = \underset{AB}{0.05} + \underset{O}{0.43} = 0.48$

c) $P(\text{not O}) = \underset{A}{0.4} + \underset{B}{0.12} + \underset{AB}{0.05} = 0.57$

$= 1 - \underset{O}{0.43} = 0.57$

Exercise

empirical probability

28. Computers in Elementary Schools Elementary and secondary schools were classified by the number of computers they had.

Computers	1–10	11–20	21–50	51–100	100+
Schools	3170	4590	16,741	23,753	34,803

Choose one school at random. Find the probability that it has

a. 50 or fewer computers

b. More than 100 computers

c. No more than 20 computers

Source: World Almanac.

Total freq = 83 057

a) $P(50 \text{ or fewer}) = \dfrac{3170+4590+16741}{83057}$

$= 0.2950$

b) $P(\text{more than } 100)$

$= \dfrac{34803}{83057} = 0.4190$

c) $P(\text{no more than } 20)$

$= \dfrac{3170+4590}{83057}$

$= 0.09343$

Marginal probability, conditional probability, and complementary events

Marginal and joint probabilities

- Now consider the outcome consists of <u>multiple events</u>.
- Marginal probability is the probability of a single event without consideration of any other event
- Joint probability is the probability of two (or more) events occurring at the same time.

Example

- Suppose all 100 employees of a company were asked whether they agree or disagree in paying high salaries to CEOs of U.S. companies.

	Agree	Disagree	Total
Male	15	45	60
Female	4	36	40
Total	19	81	100

- Then these are the marginal probabilities:
  - $P(\text{Male}) = \frac{60}{100} = 0.6$
  - $P(\text{Female}) = \frac{40}{100} = 0.4$
  - $P(\text{Agree}) = \frac{19}{100} = 0.19$
  - $P(\text{Disagree}) = \frac{81}{100} = 0.81$

Example

	Agree	Disagree	Total
Male	15	45	60
Female	4	36	40
Total	19	81	100

- And these are the joint probabilities
  - $P(\text{Male and Agree}) = \frac{15}{100} = 0.15$
  - $P(\text{Male and Disagree}) = \frac{45}{100} = 0.45$
  - $P(\text{Female and Agree}) = \frac{4}{100} = 0.04$
  - $P(\text{Female and Disagree}) = \frac{36}{100} = 0.36$

Conditional probability

- Conditional probability on the other hand is the probability of an event occurring given that another event has already occurred.

- Suppose using the previous example, a male employee is selected, and we would like to know if he agrees or disagrees in paying high salaries to CEOs.

- In this case:
  - We already know the gender of the employee.
  - And interested in knowing an employees opinion given that the employee is a male.

Conditional probability

- We write $P(A|B)$ as the probability of event A given event B has occurred.
- Suppose $n(A \text{ and } B)$ is the number of outcome in which A and B occur, and $n(B)$ is the number of outcome in which event B occur.

$$P(A|B) = \frac{n(A \text{ and } B)}{n(B)} = \frac{P(A \text{ and } B)}{P(B)}$$

Example

	Agree	Disagree	Total
Male	15	45	60
Female	4	36	40
Total	19	81	100

- $P(\text{Agree}|\text{Male}) = \frac{15}{60} = \frac{1}{4}$
- $P(\text{Male}|\text{Agree}) = \frac{15}{19}$
- $P(\text{Disagree}|\text{Female}) = \frac{36}{40} = \frac{9}{10}$
- $P(\text{Female}|\text{Disagree}) = \frac{36}{81} = \frac{4}{9}$

Complementary events

- The complement of event E, denoted by $\bar{E}$ and is the event that includes all the outcomes for an experiment that are not in E.
- Read as "E bar" or "E complement".

- Example: Rolling a die.
  - Let E = an even number is observed = $\{2, 4, 6\}$
  - Then $\bar{E} = \{1, 3, 5\}$

- Probability of complementary event:

$$P(\bar{E}) = 1 - P(E)$$

Complementary events

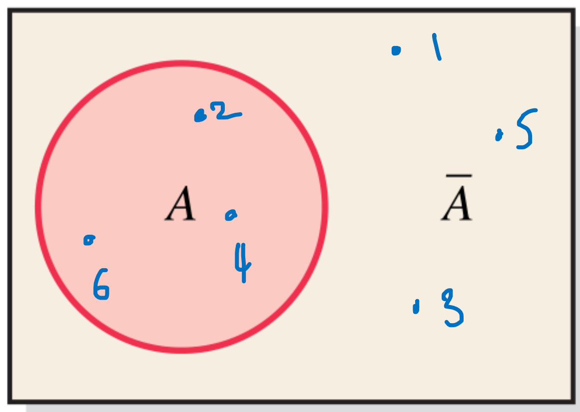
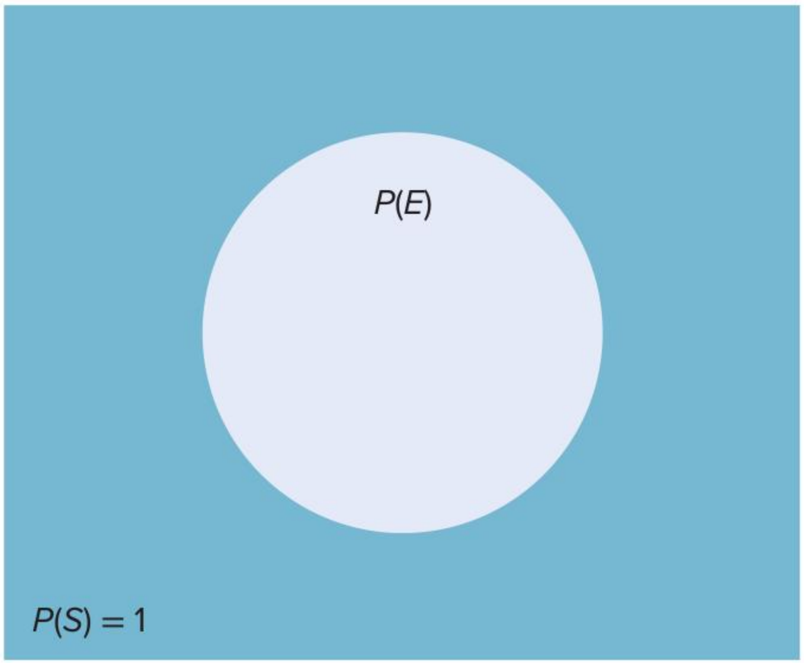


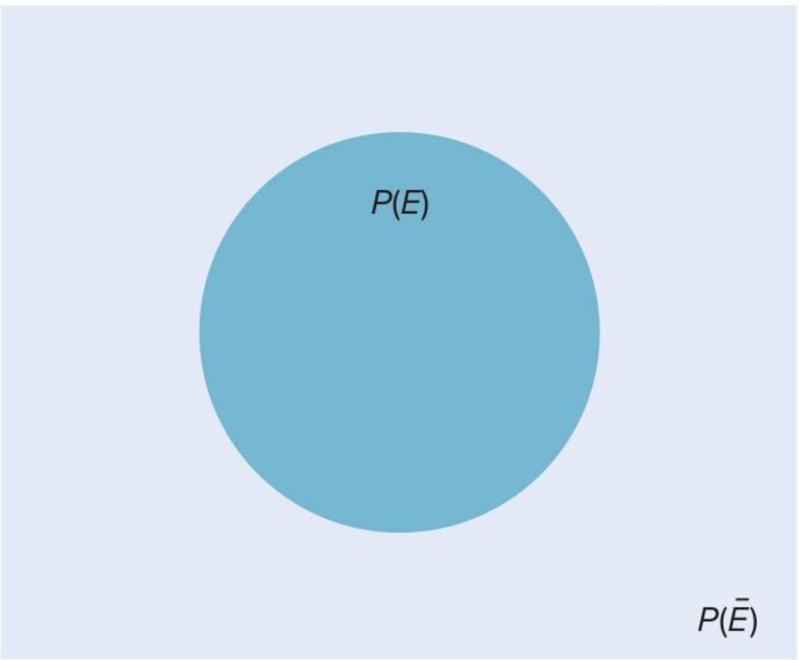
Figure 4.9 Two complementary events.

- Any element that is outside of A is inside $\bar{A}$.

Complementary events



(a) Simple probability



(b) $P(\bar{E}) = 1 - P(E)$

Example

	Agree	Disagree	Total
Male	15	45	60
Female	4	36	40
Total	19	81	100

- $P(\overline{\text{Male}}) = 1 - P(\text{Male}) = 1 - \frac{60}{100} = 0.4 \;(= P(\text{Female}))$
- Let $A = \{\text{Male and Agree}\}$
  - $P(A) = \frac{15}{100} = 0.15$
  - $P(\bar{A}) = 1 - P(A) = 0.85$

$= \frac{4+45+36}{100}$

Exercise

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4.53 Two thousand randomly selected adults were asked whether or not they have ever shopped on the Internet. The following table gives a two-way classification of the responses.

	Have Shopped	Have Never Shopped	Total
Male	500	700	1200
Female	300	500	800
Total	800	1200	2000

If one adult is selected at random from these 2000 adults, find the probability that this adult

i. has never shopped on the Internet
ii. is a male
iii. has shopped on the Internet given that this adult is a female
iv. is a male given that this adult has never shopped on the Internet

i) $P(\text{never shopped}) = \frac{n(\text{never shopped})}{n(\text{Total})} = \frac{1200}{2000} = 0.6$

ii) $P(\text{male}) = \frac{n(\text{male})}{n(\text{Total})} = \frac{1200}{2000} = 0.6$

iii) $P(\text{has shopped} \mid \text{female}) = \dfrac{n(\text{has shopped and female})}{n(\text{female})}$

$= \dfrac{300}{800}$

$= 0.375$

iv) $P(\text{male} \mid \text{never shopped}) = \dfrac{700}{1200} = 0.5833$

$$\boxed{P(A \mid B) = \frac{n(A \text{ and } B)}{n(B)}}$$

Exercise

- The probability that a randomly selected college student attended at least one major league baseball game last year is 0.12.
  - What is the complementary event?
  - What is the probability of this complementary event?

Let E = college student attended at least one major league baseball game

$\bar{E}$ = college student has never attended major league baseball game

$P(\bar{E}) = 1 - P(E) = 1 - 0.12 = 0.88$

Intersection of events

Mutually exclusive events

- Events that cannot occur together are said mutually exclusive events.

- Example: Gender of employee.
  - Let A be the event that the randomly selected employee is male.
  - Let B be the event that the randomly selected employee is female.
  - Then A and B are mutually exclusive.

Example

- Consider rolling a die and observing these events:
  - A = an even number is observed = $\{2, 4, 6\}$
  - B = an odd number is observed = $\{1, 3, 5\}$
  - C = a number less than 5 is observed = $\{1, 2, 3, 4\}$

- Then events A and B are mutually exclusive events. They do not share any similar outcome.
- Events A and C are not mutually exclusive events. Outcome 2 and 4 are included in both events.

Example

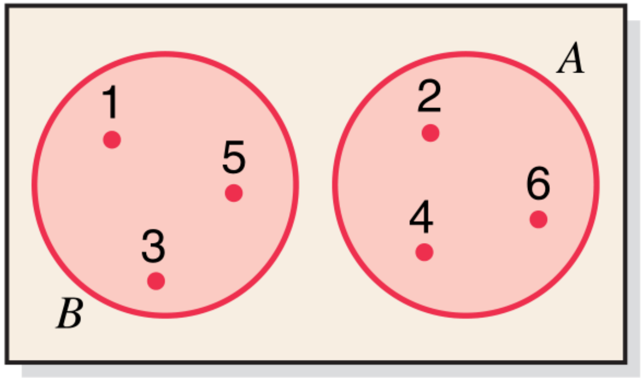


Figure 4.6 Mutually exclusive events A and B.

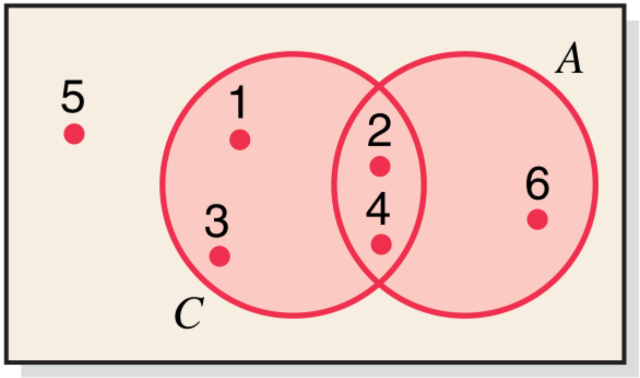


Figure 4.7 Mutually nonexclusive events A and C.

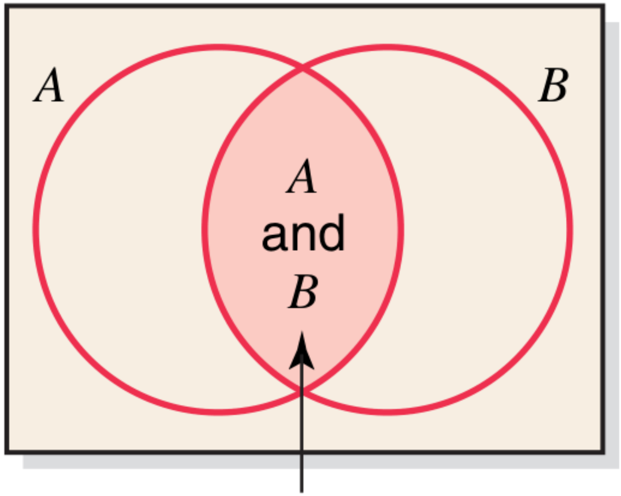
Intersection of events

- The intersection of two events is given by the outcomes that are common to both events.
- It can be denoted as $(A \text{ and } B)$ or $(A \cap B)$.

- Example: Roll a die.
  - Let $A = \{\text{even number}\} = \{2,4,6\}$
  - Let $B = \{\text{less than 3}\} = \{1,2\}$
  - Then the outcome that is even number and less than 3 is only 2. $A \cap B = \{2\}$

- If the two events are mutually exclusives, then $A \cap B = \phi$ is the empty set. (ie. no intersection at all) .

Intersection of events



Intersection of A and B

Independent events

- Two events are independent if the outcome for one event does not affect the other event.
- When two events are independent:

$$P(A \text{ and } B) = P(A) \times P(B)$$

- On the other hand, two events are dependent if the outcome for one event affects the outcome for the other event.
- When two events are dependent:

$$P(A \text{ and } B) = P(A) \times P(B|A)$$

$$= P(B) \times P(A|B)$$

Example

- Tossing a coin twice.
  - The two tosses do not affect each other.
  - They are independent events.

- Draw a card from the deck. Then throw the card away and draw another card.
  - The outcome of second draw depends on the outcome of the first draw.
  - They are dependent events.

Example

- The probability that a college graduate will find a full-time job within three months after graduation from college is 0.27. Three college students, who will be graduating soon, are randomly selected. What is the probability that all three of them will find full-time jobs within three months after graduation from college?

- Let:

$A = \{\text{first student find full}-\text{time job within 3 months after graduation}\}$

$B = \{\text{second student find full}-\text{time job within 3 months after graduation}\}$

$C = \{\text{third student find full}-\text{time job within 3 months after graduation}\}$

- The three events are independent. Therefore

$$P(A \text{ and } B \text{ and } C) = 0.27 \times 0.27 \times 0.27 = 0.0197$$

Example

- In a group of 20 college students, 7 like iced tea and others do not. Two students are randomly selected from this group.
  - Find the probability that both of the selected students like iced tea.
  - Find the probability that the first student selected likes iced tea and the second does not.

- Let
  - $L_1 = \{\text{first student likes iced tea}\}$
  - $N_1 = \{\text{first student does not like iced tea}\}$
  - $L_2 = \{\text{second student likes iced tea}\}$
  - $N_2 = \{\text{second student does not like iced tea}\}$

Example

- $P(L_1) = \frac{7}{20}$.
- If first student likes iced tea, there are only 6 students left out of 19 students who like iced tea. So $P(L_2|L_1) = \frac{6}{19}$.

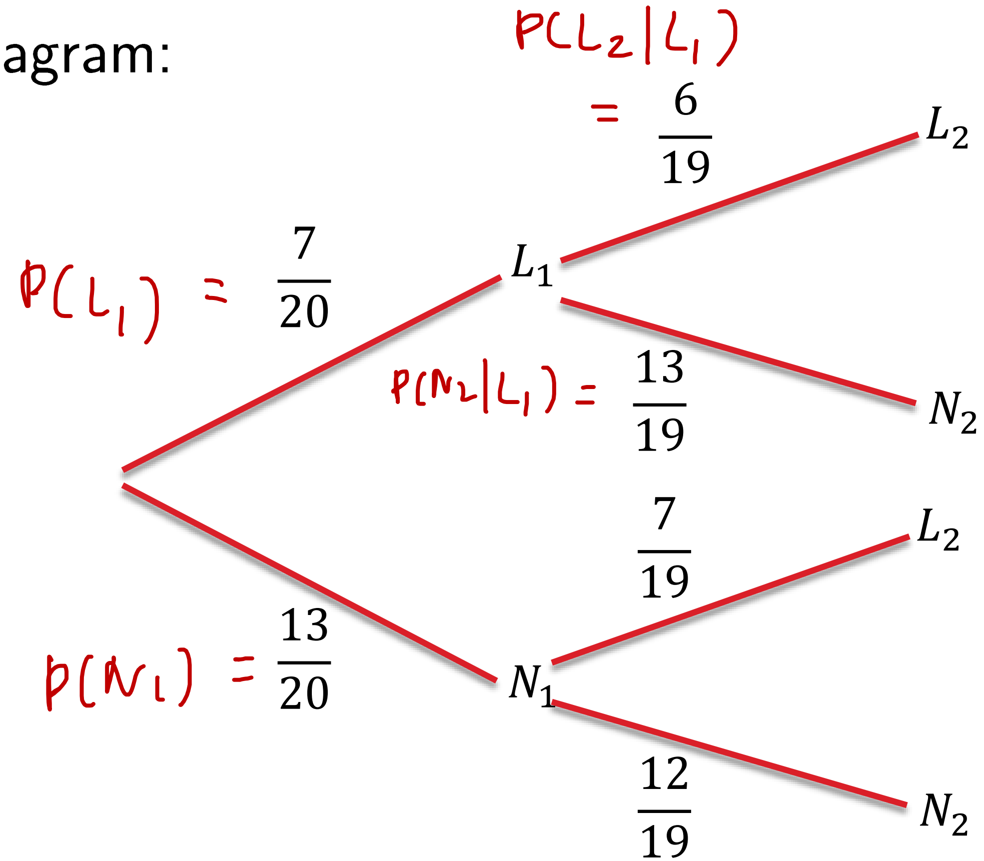
$$P(L_1 \text{ and } L_2) = P(L_1) \times P(L_2|L_1) = \frac{7}{20} \times \frac{6}{19} = 0.1105$$

- If first student likes iced tea, there are 13 more students out of 19 students who does not like iced tea. So $P(N_2|L_1) = \frac{13}{19}$.

$$P(L_1 \text{ and } N_2) = P(L_1) \times P(N_2|L_1) = \frac{7}{20} \times \frac{13}{19} = 0.2395$$

Example

- Tree diagram:



Exercise

4.92 A contractor has submitted bids for two state construction projects. The probability of winning each contract is .25, and it is the same for both contracts.

a. What is the probability that he will win both contracts?

b. What is the probability that he will win neither contract?

Draw a tree diagram for this problem.

a) Let W_1 = wins first contract

L_1 = loses " "

W_2 = wins second contract

L_2 = loses " "

The two contracts are independent events

$P(W_1 \cap W_2) = P(W_1) \times P(W_2) = 0.25 \times 0.25$

$= 0.0625$

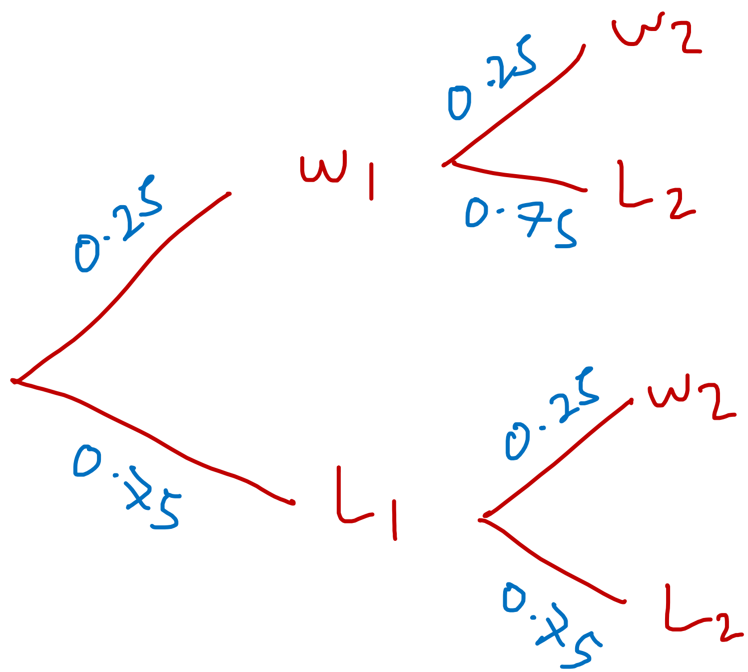
b) $P(\text{loses both}) = P(L_1 \cap L_2) = P(L_1) \times P(L_2)$

$= 0.75 \times 0.75$

$= 0.5625$

$P(L_1) = 1 - P(W_1)$

complementary event



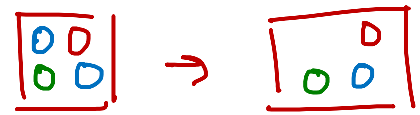
Exercise

10. Selecting Marbles A bag contains 9 red marbles, 8 white marbles, and 6 blue marbles. Randomly choose two marbles, one at a time, and without replacement. Find the following.

a. The probability that the first marble is red and the second is white

b. The probability that both are the same color

c. The probability that the second marble is blue



pick o

* dependent events

a) let R_1 = first is red
W_1 = " " white
B_1 = " " blue

similarly for R_2, W_2, B_2

Total $= 9 + 8 + 6 = 23$

$P(R_1 \cap W_2)$

$= P(R_1) \times P(W_2 \mid R_1)$

$= \dfrac{9}{23} \times \dfrac{8}{22} = 0.1423$

22 after throwing one out

b) $P(\text{same colour}) = P(R_1 \cap R_2) + P(W_1 \cap W_2) + P(B_1 \cap B_2)$

$$= \left(\frac{9}{23} \times \frac{8}{22}\right) + \left(\frac{8}{23} \times \frac{7}{22}\right) + \left(\frac{6}{23} \times \frac{5}{22}\right)$$

$$= 0.3123$$

c) $P(\text{second is blue}) = P(R_1 \cap B_2) + P(W_1 \cap B_2) + P(B_1 \cap B_2)$

$$= \left(\frac{9}{23} \times \frac{6}{22}\right) + \left(\frac{8}{23} \times \frac{6}{22}\right) + \left(\frac{6}{23} \times \frac{5}{22}\right)$$

$$= 0.2609$$

$$P(A \cap B) = P(A) \times P(B|A)$$

Union of events

Union of events

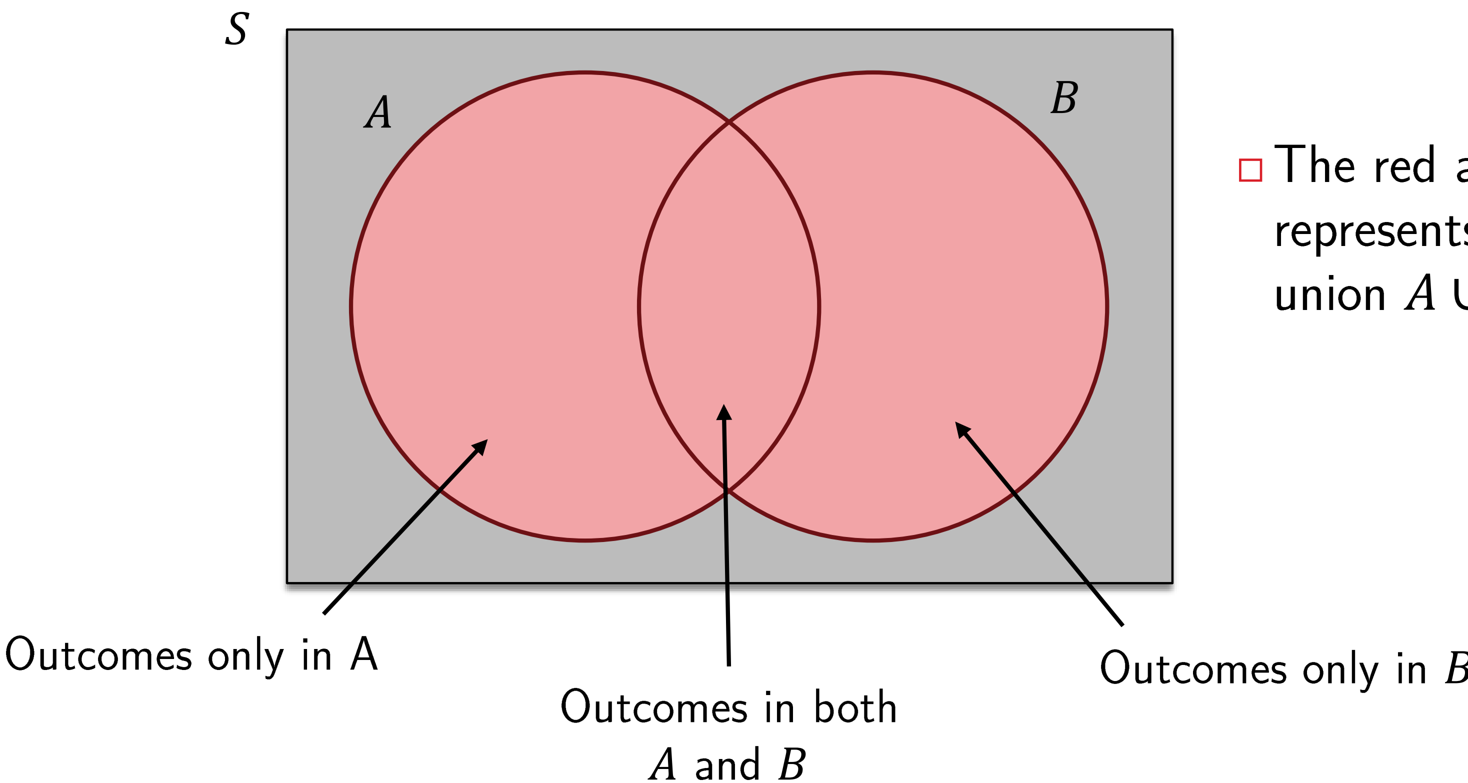
- The union of events A and B is given by the outcomes that are either in A, B or both.

- Denote the union as "A or B" or "$A \cup B$".

- Example: Suppose we are looking for a student who is a female or a first-year student. Are these outcomes inside the union?
  - Second-year female student. (Yes)
  - First-year male student. (Yes)
  - Second-year male student. (No)

Union of events

- In Venn diagram:



- The red area represents the union $A \cup B$.

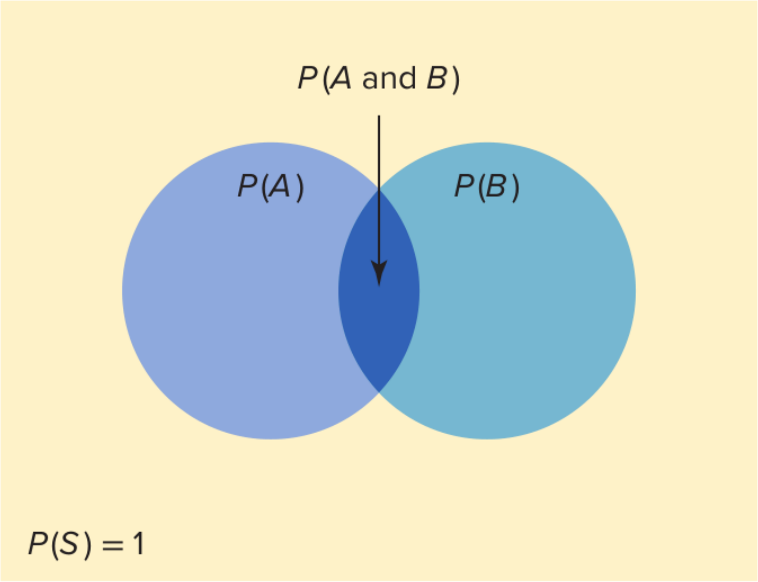
Probability for union of events

- The probability for union of A and B:

$$P(A \text{ or } B) = P(A) + P(B) - P(A \cap B)$$

- If A and B are mutually exclusive events, then $P(A \cap B) = 0$ and

$$P(A \text{ or } B) = P(A) + P(B)$$



- To get the area, we add $P(A)$ and $P(B)$ and have to remove $P(A \cap B)$ since it is counted twice.

Example

- A university president proposed that all students must take a course in ethics as a requirement for graduation. Three hundred staff members and students from this university were asked about their opinions on this issue.

	Favor	Oppose	Neutral	Total
Staff	45	15	10	70
Student	90	110	30	230
Total	135	125	40	300

- Find the probability that if one person is selected at random from these 300 persons, this person is a staff member or is in favor of this proposal.

Example

- Let T = staff is selected, F = is in favour
- Then $P(T) = \frac{70}{300}$, $P(F) = \frac{135}{300}$, and $P(T \text{ and } F) = \frac{45}{300}$
- Therefore

$$
\begin{aligned}
P(T \text{ or } F) &= P(T) + P(F) - P(T \text{ and } F) \\
&= \frac{70}{300} + \frac{135}{300} - \frac{45}{300} \\
&= \frac{8}{15} = 0.5333
\end{aligned}
$$

- This is equivalent to finding all outcomes with staff and in favour which is

$$
\frac{45 + 15 + 10 + 90}{300} = \frac{8}{15}
$$

Another example

- In the United States there are 59 different species of mammals that are endangered, 75 different species of birds that are endangered, and 68 species of fish that are endangered. If one animal is selected at random, find the probability that it is either a mammal or a fish.

  - Let M = mammal is selected, and F = fish is selected.
  - Total animals = 59 + 75 + 68 = 202
  - Since M and F are mutually exclusive events, $P(M \text{ or } F) = P(M) + P(F)$

    $P(M \text{ and } F) = 0$

$$P(M \text{ or } F) = P(M) + P(F) = \frac{59}{202} + \frac{68}{202} = 0.6287$$

Exercise

12. Selecting a Book At a used-book sale, 100 books are adult books and 160 are children's books. Of the adult books, 70 are nonfiction while 60 of the children's books are nonfiction. If a book is selected at random, find the probability that it is

a. Fiction

b. Not a children's nonfiction book

c. An adult book or a children's nonfiction book

	Fiction	Nonfiction	Total
Adult	30	70	100
Children	100	60	160
Total	130	130	260

a) $P(\text{Fiction}) = \frac{130}{260} = \frac{1}{2}$

b) $P(\text{not a children's nonfiction})$

$= 1 - P(\text{children and nonfiction})$

$= 1 - \frac{60}{260}$

$= \frac{10}{13}$

mutually exclusive

c) $P(\text{adult or children's nonfiction})$

$= P(\text{adult}) + P(\text{children's nonfiction})$

$= \frac{100}{260} + \frac{60}{260} = \frac{8}{13}$

Exercise

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22. Medical Patients A recent study of 300 patients found that of 100 alcoholic patients, 87 had elevated cholesterol levels, and of 200 nonalcoholic patients, 43 had elevated cholesterol levels. If a patient is selected at random, find the probability that the patient is the following.

a. An alcoholic with elevated cholesterol level

b. A nonalcoholic

c. A nonalcoholic ~~with~~ or nonelevated cholesterol level

	Elevated	Nonelevated	total
Alcoholic	87	13	100
Nonalcoholic	43	157	200
Total	130	170	300

a) $P(\text{alcoholic and elevated}) = \frac{87}{300} = 0.29$

b) $P(\text{nonalcoholic}) = \frac{200}{300} = \frac{2}{3}$

c) $P(\text{nonalcoholic or nonelevated})$

$= P(\text{nonalcoholic}) + P(\text{nonelevated}) - P(\text{nonalcoholic and nonelevated})$

$= \frac{200}{300} + \frac{170}{300} - \frac{157}{300}$

$= 0.71$

$= \frac{43+157+13}{300}$

$= 0.71$

Counting rules

Counting rule

- If an experiment is a sequence of n events in which the first one has k_1 possible outcomes, the second one has k_2 possible outcomes, and the third has k_3 and so on, then the total possible outcome is

$$\text{Total outcomes} = k_1 \times k_2 \times k_3 \times \cdots \times k_n$$

- Example: Tossing a coin twice
  - Each toss can result in two outcomes – head or tail.
  - Total number of outcome $= 2 \times 2 = 4$

Example

- A coin is tossed and a die is rolled. Find the number of outcomes for the sequence of events.
  - When a coin is tossed, there are 2 possible outcomes – head or tail.
  - When a die is rolled, there are 6 possible outcomes.
  - Total number of outcomes $= 2 \times 6 = 12$

Factorial notation

- For any non-negative integer n,

$$n! = n(n-1)(n-2)\cdots(2)(1)$$

- Also by definition,

$$0! = 1$$

- Example
  - $3! = 3\times2\times1 = 6$
  - $4! = 4\times3\times2\times1 = 24$
  - $5! = 5\times4\times3\times2\times1 = 120$

Combination

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- Suppose we want to know how many possible combinations can be created.

- Example: There are 10 students and we want to select 2 of them. How many total combinations we can make?

- Total combinations of r objects selected from n objects:

$$^{n}C_{r} = \frac{n!}{(n-r)!\,r!}$$

$$\binom{n}{r} = {}^{n}C_{r}$$

- Read $^{n}C_{r}$ as "from n choose r"

Example

- There are four books labelled A, B, C and D, and I want to select two of them. How many possible combinations are there?
  - If we were to list all the possible combinations:

    AB, AC, AD, BC, BD, CD

  - Using formula:

$$^{4}C_{2} = \frac{4!}{(4-2)!\,2!} = \frac{24}{2 \times 2} = 6$$

Permutation

- A permutation is an arrangement of n objects in a specific order.
- Suppose we have a list of object and we want to arrange some them. How many arrangements can we make?
- Example: There are 10 digits. How many 4 digits password can a person make if no digits are repeated?
- Total permutation of r objects from n objects:

$$^{n}P_{r} = \frac{n!}{(n-r)!}$$

- Read $^{n}P_{r}$ as "from n permute r"

Example

- There are four books labeled A, B, C and D. Suppose I want to choose 2 of them but this time the order is important. That is AB is not the same as BA. How many arrangements can I make?
  - If we were to list all possible arrangements:

    AB, AC, AD, BC, BD, CD

    BA, CA, DA, CB, DB, DC

  - Using formula:

$$^{4}P_{2} = \frac{4!}{2!} = \frac{24}{2} = 12$$

$\underline{4} \times \underline{3} = 12$

First book, 4 possible books

second book, 3 possibility

Exercise

- There are 5 male and 4 female students in a class. 2 students from each gender is selected to form a group for a competition.
  - How many total combinations can be formed?
  - Suppose Ahmad is a male student in the class. How many combinations can be formed with Ahmad being one of the selected students?
  - If all combinations of students are equally likely, what is the probability of Ahmad being selected?

a) Total $= {}^{5}C_{2} \times {}^{4}C_{2}$

From 5 males, choose 2

From 4 females, choose 2.

$= 10 \times 6$

$= 60$

b) Total combinations with Ahmad

$$= {}^{4}C_{1} \times {}^{4}C_{2}$$

From 4 other males, choose 1

$$= 4 \times 6$$

$$= 24$$

c) $P(\text{Ahmad selected}) = \dfrac{\text{number with Ahmad}}{\text{Total combinations}}$

$$= \frac{24}{60}$$

$$= \frac{2}{5}$$

Exercise

$$\underset{\text{chair}}{\underline{17}} \times \underset{\text{secretary}}{\underline{16}} \times \underset{\text{webmaster}}{\underline{15}}$$

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- A student advisory board consists of 17 members. Three members serve as the board's chair, secretary, and webmaster. Each member is equally likely to serve in any of the positions.
  - How many ways can the three positions be filled?
  - What is the probability of selecting at random the three members who currently hold the three positions?

a) Total way $= {}^{17}P_3 = \frac{17!}{14!} = 4080$

$$= 17 \times 16 \times 15 = 4080$$

b) P(selecting the 3 members, in any position)

$$= \frac{\text{number of permutations with the 3 members}}{\text{Total}}$$

$$= \frac{{}^{3}P_{3}}{{}^{17}P_{3}} = \frac{3 \times 2 \times 1}{4080}$$

$$= 0.001471$$

Summary

- Experiment, sample space, outcome, event.
- Venn and tree diagrams.
- Calculating probability
  - Classical probability
  - Empirical probability
- Marginal probability, conditional probability, and complementary events.
- Intersection of events.
- Union of events.
- And lastly, how to count.