Exercise 4

1. Based on its analysis of the future demand for its products, the financial department at Tipper Corporation has determined that there is a 0.17 probability that the company will lose $1.2 million during the next year, a 0.21 probability that it will lose $0.7 million, a 0.37 probability that it will make a profit of $0.9 million, and a 0.25 probability that it will make a profit of $2.3 million.

   a. Let X be a random variable that denotes the profit earned by this corporation during the next year. Write the probability distribution of X.

x	-1.2	-0.7	0.9	2.3
$P(X = x)$	0.17	0.21	0.37	0.25

   b. Find the mean and standard deviation of the probability distribution of part a. Give a brief interpretation of the value of the mean.

$$\mu = \sum xP(X = x) = \left((-1.2) \times 0.17\right) + \left((-0.7) \times 0.21\right) + (0.9 \times 0.37) + (2.3 \times 0.25)$$
$$= 0.557$$

$$\sigma^2 = \sum x^2 P(X = x) - \mu^2$$
$$= \left(1.2^2 \times 0.17\right) + \left(0.7^2 \times 0.21\right) + \left(0.9^2 \times 0.37\right) + \left(2.3^2 \times 0.25\right) - 0.557^2$$
$$= 1.6597$$

$$\sigma = \sqrt{1.6597} = 1.288$$

We expect that the company will make a profit of $0.557 million.

2. If X has the distribution function

$$F(x) = \begin{cases} 0 & \text{for } x < 1 \\ {}^{1}\!/_{8} & \text{for } 1 \le x < 4 \\ {}^{5}\!/_{8} & \text{for } 4 \le x < 6 \\ {}^{7}\!/_{8} & \text{for } 6 \le x < 10 \\ 1 & \text{for } x \ge 10 \end{cases}$$

   find

   a. $P(2 < X \le 6)$.

$$P(2 < X \le 6) = P(X \le 6) - P(X \le 2) = \frac{7}{8} - \frac{1}{8} = \frac{3}{4}$$

   b. $P(X = 4)$.

$$P(X = 4) = P(X \le 4) - P(X \le 3) = \frac{5}{8} - \frac{1}{8} = \frac{1}{2}$$

c. the probability distribution of X.

x	1	4	6	10
$P(X=x)$	$\frac{1}{8}$	$\frac{5}{8}-\frac{1}{8}=\frac{1}{2}$	$\frac{7}{8}-\frac{5}{8}=\frac{1}{4}$	$1-\frac{7}{8}=\frac{1}{8}$

3. The number of calls that come into a small mail-order company follows a Poisson distribution. Currently, these calls are serviced by a single operator. The manager knows from past experience that an additional operator will be needed if the rate of calls exceeds 20 per hour. The manager observes that 9 calls came into the mail-order company during a randomly selected 15-minute period.

a. If the rate of calls is actually 20 per hour, what is the probability that 9 or more calls will come in during a given 15-minute period?

X = number of calls in 15-minute period.

X has a Poisson distribution with mean $\lambda = \frac{20}{4} = 5$.

$$P(X \geq 9) = 1 - \big(P(X=0) + P(X=1) + \cdots + P(X=8)\big)$$

$$= 1 - \left(\frac{5^0 e^{-5}}{0!} + \cdots + \frac{5^8 e^{-5}}{8!}\right) = 1 - 0.9319 = 0.06809$$

b. If the rate of calls is really 30 per hour, what is the probability that 9 or more calls will come in during a given 15-minute period?

X has a Poisson distribution with mean $\lambda = \frac{30}{4} = 7.5$.

$$P(X \geq 9) = 1 - \big(P(X=0) + P(X=1) + \cdots + P(X=8)\big)$$

$$= 1 - \left(\frac{7.5^0 e^{-7.5}}{0!} + \cdots + \frac{7.5^9 e^{-7.5}}{9!}\right) = 1 - 0.66197 = 0.3380$$

c. Based on the calculations in parts a and b, do you think that the rate of incoming calls is more likely to be 20 or 30 per hour?

30 per hour is more likely, because the probability when the rate is 30 per hour is higher.

d. Would you advise the manager to hire a second operator? Explain.

Yes. The probability of getting 9 or more calls if the rate is 20 calls per hour is 0.06809, which is small. It is more likely that the rate is actually higher than 20 calls per hour.

4. Spoke Weaving Corporation has eight weaving machines of the same kind and of the same age. The probability that any weaving machine will break down at any time is 0.04.

   a. Let X be the number of weaving machines that will break down at any given time. What is the distribution of X and what are the parameters?

      X has a binomial distribution with $n = 8$, $p = 0.04$, $q = 0.96$

   b. Find the probability that at any given time

      i. all eight weaving machines will be broken down

         $P(X = 8) = {}^{8}C_{8}(0.04)^{8}(0.96)^{0} = 6.554 \times 10^{-12}$

      ii. at most two weaving machines will be broken down

         $$\begin{aligned} P(X \leq 2) &= P(X = 0) + P(X = 1) + P(X = 2) \\ &= {}^{8}C_{0}(0.04)^{0}(0.96)^{8} + {}^{8}C_{1}(0.04)^{1}(0.96)^{7} + {}^{8}C_{2}(0.04)^{2}(0.96)^{6} \\ &= 0.9969 \end{aligned}$$

      iii. at least seven of the weaving machines will be broken down

         $$\begin{aligned} P(X \geq 7) &= P(X = 7) + P(X = 8) = {}^{8}C_{7}(0.04)^{7}(0.96)^{1} + {}^{8}C_{8}(0.04)^{8}(0.96)^{0} \\ &= 1.265 \times 10^{-9} \end{aligned}$$

   c. What are the mean and variance for number of weaving machines that will break down at a given time?

      $\mu = np = 8 \times 0.04 = 0.32$

      $\sigma^2 = npq = 8 \times 0.04 \times 0.96 = 0.3072$

5. A player plays a roulette game in a casino by betting on a single number each time. Because the wheel has 38 numbers, the probability that the player will win in a single play is 1⁄38. Note that each play of the game is independent of all previous plays.

   a. What probability distribution would describe the number of plays it takes for the player to win for the first time? What are the mean and variance?

      X = number of plays until win first time. X has a geometric distribution with $p = \frac{1}{38}$.

      $$\mu = \frac{1}{p} = 38$$

      $$\sigma^2 = \frac{1-p}{p^2} = \frac{1 - \frac{1}{38}}{\left(\frac{1}{38}\right)^2} = 1406$$

   b. Find the probability that the player will win for the first time on the 10th play.

$$P(X=10)=(1-p)^9 p=\left(\frac{37}{38}\right)^9\left(\frac{1}{38}\right)=0.02070$$

c. It is given that for $p \neq 1$,

$$\sum_{t=1}^{x}(1-p)^{t-1}p=1-(1-p)^x.$$

Using the equation given above, find the probability that it takes the player more than 50 plays to win for the first time.

$$P(X>50)=1-P(X\leq 50)=1-\sum_{t=1}^{50}(1-p)^t p=1-\left(1-(1-p)^{50}\right)$$

$$=\left(1-\frac{1}{38}\right)^{50}=0.2636$$

d. A gambler claims that because he has 1 chance in 38 of winning each time he plays, he is certain to win at least once if he plays 38 times. Does this sound reasonable to you? Find the probability that he will win at least once in 38 plays.

Not reasonable. It is a random chance.

$$P(X\leq 38)=\sum_{t=1}^{38}(1-p)^t p=1-(1-p)^{38}=0.6370$$