5. DISCRETE RANDOM VARIABLES

Introduction

- We introduced random events and probability in the previous chapter.
- Now we will formalize the random events mathematically.
- This chapter focuses on discrete random variables, while the next chapter will focus on continuous random variables.

Random variables

Random variables

- A random variable is a variable whose values are determined by chance.
  - In other words, random variable is the random outcome of an experiment.

- Discrete random variable – random variable that assumes countable values.
  - Example:
    - The number of students getting an A
    - The number of fish caught on a fishing trip

- We usually denote a random variable by X (uppercase letter), and x (lowercase letter) for a specific value of the random variable.

Probability distribution

- The probability distribution is a table or function, that lists all the possible values for a random variable and their corresponding probabilities.
- Also called probability mass function (pmf) for discrete random variables.
- Notation: $P(x)$ or $P(X = x)$ is the probability that the random variable X takes the value x.
- Two properties of probability distribution of discrete random variable:
  - $0 \leq P(X = x) \leq 1$
  - $\sum_x P(X = x) = 1$

Example

- Consider rolling a die.
  - Let X be the observed outcome when rolling a die.
  - The following table is the probability distribution of X:

x	1	2	3	4	5	6
$P(X=x)$	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$

  - $P(X=1)=1/6$, means that the probability of X takes value 1 is $1/6$
  - Also note that

$$\sum P(X=x)=P(1)+P(2)+\cdots+P(6)=1$$

Exercise

- Determine whether the following represents a probability distribution. If it does not, state why.

7.

X	15	16	20	25
$P(X)$	0.2	0.5	0.7	−0.8

8.

X	5	7	9
$P(X)$	0.6	0.8	−0.4

9.

X	−5	−3	0	2	4
$P(X)$	0.1	0.3	0.2	0.3	0.1

10.

X	20	30	40	50
$P(X)$	0.05	0.35	0.4	0.2

11.

X	3	6	9	1
$P(X)$	0.3	0.4	0.3	0.1

12.

X	3	7	9	12	14
$P(X)$	$\frac{4}{13}$	$\frac{1}{13}$	$\frac{3}{13}$	$\frac{1}{13}$	$\frac{2}{13}$

Two properties

1) $0 \leq P(X=x) \leq 1$

2) $\sum P(X=x) = 1$

7) $P(X=25) = -0.8 < 0$
Not a probability distribution

8) $P(X=9) = -0.4 < 0$
Not a probability distribution

9) $0 \leq P(X=x) \leq 1$

$$\Sigma P(X=x) = 0.1 + 0.3 + 0.2 + 0.3 + 0.1 = 1$$

Yes, it is a probability distribution

10) $0 \leq P(X=x) \leq 1$

$$\Sigma P(X=x) = 0.05 + 0.35 + 0.4 + 0.2 = 1$$

Yes, it is a probability distribution

11) $0 \leq P(X=x) \leq 1$

$$\Sigma P(X=x) = 0.3 + 0.4 + 0.3 + 0.1 = 1.1 \neq 1$$

This is not a probability distribution

12) $0 \leq P(X=x) \leq 1$

$$\Sigma P(X=x) = \frac{4}{13} + \frac{1}{13} + \frac{3}{13} + \frac{1}{13} + \frac{2}{13} = \frac{11}{13} \neq 1$$

Not a probability distribution

Exercise

- An industrial psychologist administered a personality inventory test for passive-aggressive traits to 150 employees. Individuals were given a score from 1 to 5, where 1 was extremely passive and 5 extremely aggressive. A score of 3 indicated neither trait. Construct a probability distribution for the random variable X.

Score, x	Frequency, f
1	24
2	33
3	42
4	30
5	21

$P(X=x)$

$24/150 = 0.16$

$33/150 = 0.22$

0.28

0.2

0.14

Exercise

- At a drop-in mathematics tutoring center, each teacher sees 4 to 8 students per hour. The probability that a tutor sees 4 students in an hour is 0.117; 5 students, 0.123; 6 students, 0.295; and 7 students, 0.328.
  a) Find the probability that a tutor sees 8 students in an hour
  b) Construct the probability distribution.
  c) Find the probability that a tutor sees 6 or less students in an hour.

a) $\sum P(X=x) = 1$

$P(4) + P(5) + P(6) + P(7) + P(8) = 1$

$P(8) = 1 - P(4) - P(5) - P(6) - P(7)$

$= 1 - 0.117 - 0.123 - 0.295 - 0.328$

$= 0.137$

b)

x	4	5	6	7	8
$P(X=x)$	0.117	0.123	0.295	0.328	0.137

c) $P(X \leq 6) = P(4) + P(5) + P(6)$

$= 0.117 + 0.123 + 0.295$

$= 0.535$

Exercise

- The number 1, 2, 3 and 4 are printed one each on one side of card. The cards are placed face down and mixed. Choose two cards at random; and let X be the sum of the two numbers. Construct the probability distribution for this random variable X.

Possible outcomes : 1 & 2   1 & 3   1 & 4   2 & 3   2 & 4   3 & 4

X : 3   4   5   5   6   7

x	3	4	5	6	7
$P(X=x)$	$1/6$	$1/6$	$2/6 = 1/3$	$1/6$	$1/6$

eg: $P(X < 5) = P(3) + P(4)$

$$= \frac{1}{6} + \frac{1}{6}$$

$$= \frac{1}{3}$$

Cumulative distribution function

- Another important concept in the distribution of random variable is the cumulative distribution function (also called distribution function).

- If X is a discrete random variable, the cumulative distribution function is given by

$$F(x) = P(X \leq x) = \sum_{t \leq x} P(X = t)$$

for $-\infty < x < \infty$.

- $F(x)$ is the probability of X taking value less than equal to x.

Cumulative distribution function

- Properties of a discrete distribution function:
  - $0 \leq F(x) \leq 1$ for all x
  - If $a < b$, then $F(a) \leq F(b)$
  - $F(x)$ is a step function
  - $F(-\infty) = 0$ and $F(\infty) = 1$

- Additionally:
  - $P(a < X \leq b) = F(b) - F(a)$
  - $P(X > a) = 1 - P(X \leq a) = 1 - F(a)$
  - $P(X < a) = P(X \leq a - 1) = F(a - 1)$
  - $P(X = a) = P(X \leq a) - P(X \leq a - 1) = F(a) - F(a - 1)$

Example – CDF

- Using the following probability distribution, find its cumulative distribution function $F(x)$.

x	0	1	2	3
$P(X = x)$	$1/8$	$3/8$	$3/8$	$1/8$

- From the table we get:

$P(X \le 0)$

$$F(0) = \frac{1}{8}$$

$P(X \le 1)$

$$F(1) = \frac{1}{8} + \frac{3}{8} = \frac{1}{2}$$

$P(X \le 2)$

$$F(2) = \frac{1}{2} + \frac{3}{8} = \frac{7}{8}$$

$$F(3) = \frac{7}{8} + \frac{1}{8} = 1$$

Example – CDF

- Additionally, $F(x) = 0$ for $x < 0$, and $F(x) = 1$ for $x > 3$.

- Therefore:

$$F(x) = \begin{cases} 0, & x < 0 \\ {}^{1}/_{8}, & 0 \leq x < 1 \\ {}^{1}/_{2}, & 1 \leq x < 2 \\ {}^{7}/_{8}, & 2 \leq x < 3 \\ 1, & x \geq 3 \end{cases}$$

Exercise

- You are given the following table of probability distribution of X. Find the distribution function $F(x)$.

x	1	2	3	4	5
$P(X=x)$	0.16	0.22	0.28	0.20	0.14

$P(X \le x)$ 0.16   0.38   0.66   0.86   1

+0.22   +0.28

$$F(x) = \begin{cases} 0, & x < 1 \\ 0.16, & 1 \le x < 2 \\ 0.38, & 2 \le x < 3 \\ 0.66 & 3 \le x < 4 \\ 0.86 & 4 \le x < 5 \\ 1 & x \geq 5 \end{cases}$$

Exercise

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- Given the following cumulative distribution function for a discrete random variable X:

$$F(x) = \begin{cases} 0, & x < 1 \\ 1/6, & 1 \le x < 2 \\ 1/2, & 2 \le x < 3 \\ 5/6, & 3 \le x < 4 \\ 1, & x \ge 4 \end{cases}$$

$P(X=1) = \frac{1}{6} - 0 = \frac{1}{6}$

$P(X=2) = \frac{1}{2} - \frac{1}{6} = \frac{1}{3}$

$P(X=3) = \frac{5}{6} - \frac{1}{2} = \frac{1}{3}$

$P(X=4) = 1 - \frac{5}{6} = \frac{1}{6}$

- Find
  a) $P(1 < X < 3)$
  b) $P(X > 2)$
  c) $P(2 \le X \le 4)$
  d) The probability distribution of X

a) $P(1 < X < 3) = P(X \leq 2) - P(X \leq 1) = F(2) - F(1)$

$= P(1 < X \leq 2) \qquad = \frac{1}{2} - \frac{1}{6} = \frac{1}{3}$

$= P(X = 2)$

b) $P(X > 2) = 1 - P(X \leq 2) = 1 - F(2) = 1 - \frac{1}{2} = \frac{1}{2}$

c) $P(2 \leq X \leq 4) = P(X \leq 4) - P(X \leq 1) = F(4) - F(1)$

$= P(1 < X \leq 4) = 1 - \frac{1}{6} = \frac{5}{6}$

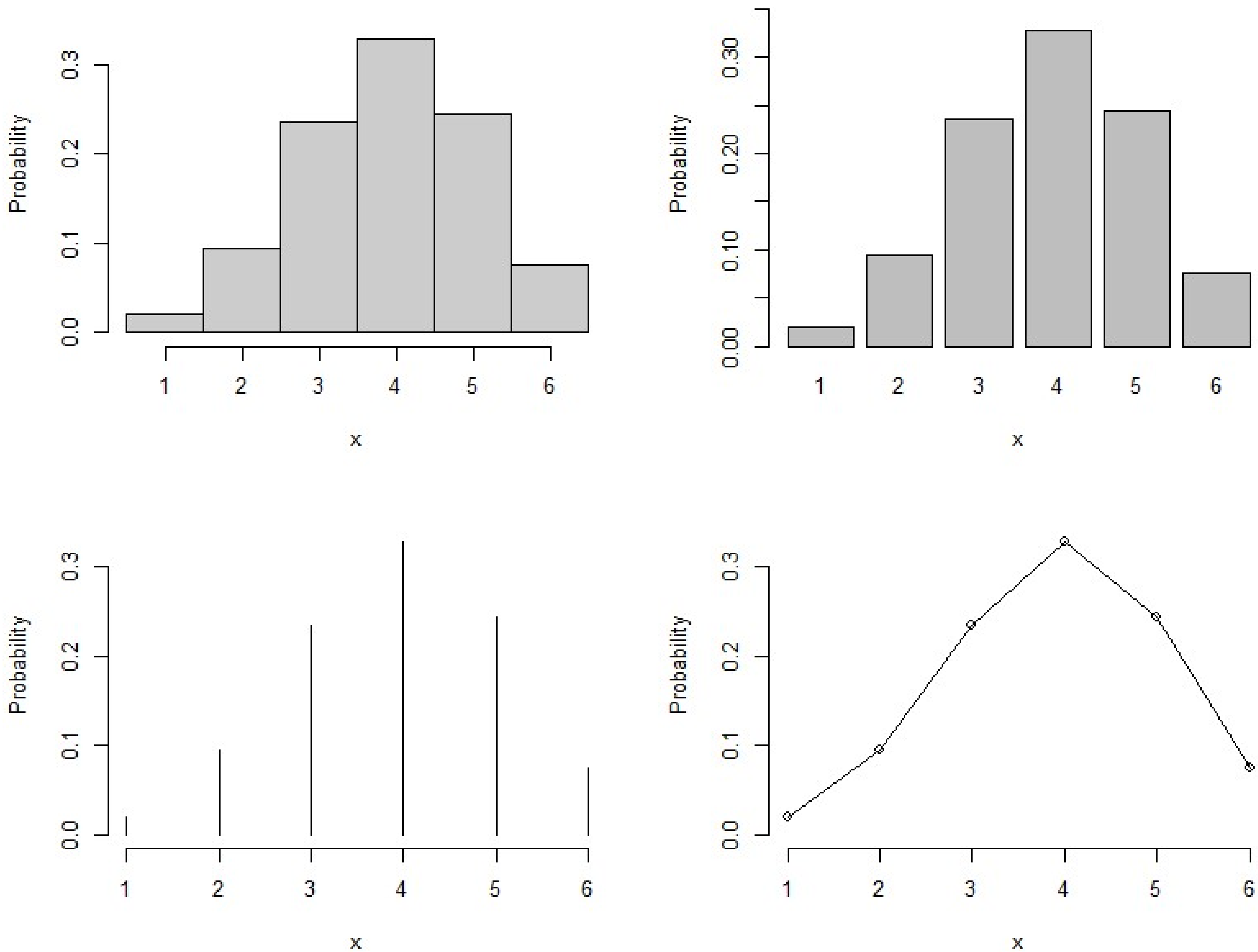
d)

x	1	2	3	4
$P(X = x)$	$1/6$	$1/3$	$1/3$	$1/6$

Graphing probability distribution

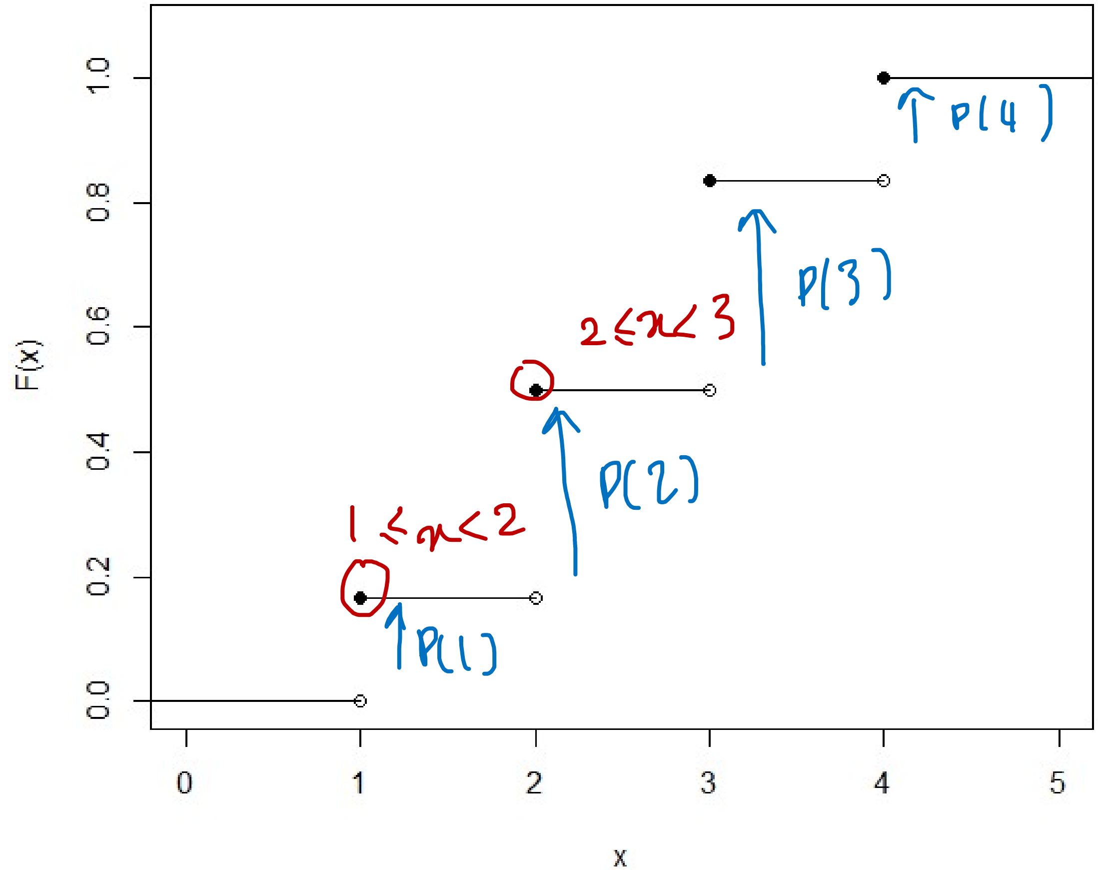
- The probability distribution can be shown visually using for example:
  - Histogram
  - Line plot
- The distribution function can be shown visually using:
  - Line plot
- Presenting the probability distribution visually helps with describing the distribution.

Example – graphing probability distribution



Example – graphing distribution function

$$F(x) = \begin{cases} 0, & x < 1 \\ {}^{1}/_{6}, & 1 \le x < 2 \\ {}^{1}/_{2}, & 2 \le x < 3 \\ {}^{5}/_{6}, & 3 \le x < 4 \\ 1, & x \ge 4 \end{cases}$$



Mean, variance, standard deviation of discrete random variables

Mean of discrete random variables

- The mean, variance and standard deviation for random variable are calculated in a similar way to grouped data.
- We use the probability $P(X = x)$ to replace f/N (frequency over total population).
- The mean of random discrete variable is defined as:

$$\mu = \sum xP(X = x)$$

Variance and standard deviation of discrete random variables

- Similarly, the variance and standard deviation can be calculated using:

$$\sigma^2 = \sum (x-\mu)^2 P(X=x), \qquad \sigma = \sqrt{\sum (x-\mu)^2 P(X=x)}$$

- As before, for easier computation:

$$\boxed{\sigma^2 = \sum x^2 P(X=x) - \mu^2}, \qquad \boxed{\sigma = \sqrt{\sum x^2 P(X=x) - \mu^2}}$$

Example

- Three coins are tossed and let X be the number of heads that occur.

x	0	1	2	3
$P(X=x)$	$1/8$	$3/8$	$3/8$	$1/8$

$$\begin{aligned}\mu &= \sum xP(X=x)\\ &= \left(0\times\frac{1}{8}\right)+\left(1\times\frac{3}{8}\right)+\left(2\times\frac{3}{8}\right)+\left(3\times\frac{1}{8}\right)\\ &= 1.5\end{aligned}$$

$$\begin{aligned}\sigma^2 &= \sum x^2P(X=x)-\mu^2\\ &= \left(0^2\times\frac{1}{8}\right)+\left(1^2\times\frac{3}{8}\right)+\left(2^2\times\frac{3}{8}\right)+\left(3^2\times\frac{1}{8}\right)-1.5^2\\ &= 3-2.25=0.75\end{aligned}$$

$$\sigma=\sqrt{0.75}=0.8660$$

Another example

- The probability distribution for the number of batteries sold over the weekend at a convenience store is given below:

x	2	4	6	8
$P(x)$	0.20	0.40	0.32	0.08

$$\begin{aligned}\mu &= \sum xP(X=x)\\ &= 2(0.2)+4(0.4)+6(0.32)+8(0.08)\\ &= 4.56\end{aligned}$$

$$\begin{aligned}\sigma^2 &= \sum x^2P(X=x)-\mu^2\\ &= 2^2(0.2)+4^2(0.4)+6^2(0.32)+8^2(0.08)-4.56^2\\ &= 23.84-20.7936=3.046\end{aligned}$$

$$\sigma = \sqrt{3.046} = 1.745$$

Exercise

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5.29 Let x be the number of heads obtained in two tosses of a coin. The following table lists the probability distribution of x.

x	0	1	2
$P(x)$	.25	.50	.25

Calculate the mean and standard deviation of x. Give a brief interpretation of the value of the mean.

$$\mu = \sum x P(X=x)$$

$$= (0 \times 0.25) + (1 \times 0.5) + (2 \times 0.25)$$

$$= 1$$

$$\sigma^2 = \sum x^2 P(X=x) - \mu^2$$

$$= (0^2 \times 0.25) + (1^2 \times 0.5) + (2^2 \times 0.25)$$

$$- 1^2$$

$$= 0.5$$

$$\sigma = \sqrt{0.5} = 0.7071$$

If the experiment is repeated many times, the average number of heads in two tosses is 1

Expectation

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- Another concept related to the mean for a probability distribution is that of expected value or expectation

- The expected value of a random variable is the theoretical average of the variable.

- For discrete random variable, the formula is

$$\boxed{E[X] = \sum xP(X = x) = \mu}$$

- Also note that the variance

$$\sigma^2 = E[(X-\mu)^2] = E[X^2] - E[X]^2$$

$\Sigma x^2 P(X=x)$ μ^2

Exercise

- A person pays \$2 to play a certain game by rolling a single die once. If a 1 or a 2 comes up, the person wins nothing. If, however, the player rolls a 3, 4, 5, or 6, he or she wins the difference between the number rolled and \$2. Find the expectation and variance for this game. Is the game fair?

Die outcome	1	2	3	4	5	6
Amount win	0	0	1	2	3	4

Let X = amount win

x	0	1	2	3	4
$P(X=x)$	$2/6 = 1/3$	$1/6$	$1/6$	$1/6$	$1/6$

$$E[X] = \mu = \sum x P(X=x) = \left(0 \times \frac{1}{3}\right) + \left(1 \times \frac{1}{6}\right) + \left(2 \times \frac{1}{6}\right)$$
$$+ \left(3 \times \frac{1}{6}\right) + \left(4 \times \frac{1}{6}\right)$$
$$= \frac{5}{3} = 1.66\bar{7}$$

$$\sigma^2 = \sum x^2 P(X=x) - \mu^2$$
$$= \left(0^2 \times \frac{1}{3}\right) + \left(1^2 \times \frac{1}{6}\right) + \left(2^2 \times \frac{1}{6}\right) + \left(3^2 \times \frac{1}{6}\right)$$
$$+ \left(4^2 \times \frac{1}{6}\right) - \left(\frac{5}{3}\right)^2$$
$$= \frac{20}{9} = 2.222$$

The game is not fair because the expected amount of winnings (\$1.67) is less than the cost (\$2)

Exercise

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- The probability distribution for the number of batteries sold over the weekend at a convenience store is given below:

x	2	4	6	8
$P(X = x)$	0.20	0.40	0.32	0.08

- Calculate
  a) $E[X]$
  b) $E[2X]$
  c) $E[X^2]$

a) $E[x] = \sum x P(X = x)$

$= (2 \times 0.2) + (4 \times 0.4) + (6 \times 0.32)$

$+ (8 \times 0.08)$

$= 4.56$

b) $E[\underline{2X}] = \sum \underline{2x}\, P(X=x)$

$$= (2\times 2\times 0.2)+(2\times 4\times 0.4)+(2\times 6\times 0.32)$$
$$+(2\times 8\times 0.08)$$
$$= 9.12$$

$$\left(\begin{array}{l} E[2X] = 2E[X] \\ E[aX+b] = aE[X]+b \end{array} \right)$$

c) $E[\underline{X^2}] = \sum \underline{x^2}\, P(X=x)$

$$= (2^2\times 0.2)+(4^2\times 0.4)+(6^2\times 0.32)$$
$$+(8^2\times 0.08)$$
$$= 23.84 \qquad \left(E[X^2] \neq E[X]^2 \right)$$

Binomial distribution

Binomial experiment

- Consider a probability experiment that only has two outcomes – "success" or "failure".
  - Eg: Toss a coin. If head is observed, then it is a success.

- This experiment is called a Bernoulli trial.

- On the other hand, suppose we repeat the trial a few times and count how many success we observed.

- These are called binomial experiments.

Binomial experiment

- A binomial experiment is a probability experiment that satisfies the following four requirements:
  - There must be a fixed number of trials.
  - Each trial can have only two outcomes. These outcomes can be considered as either "success" or "failure".
  - The outcomes of each trial must be independent of one another.
  - The probability of a success must remain the same for each trial.

- "Success" does not imply that something good or positive has occurred.

Binomial distribution

- Let X be the number of successes observed in a binomial experiment with n number of trials.

- Then the probability distribution of X is called the binomial distribution.

- Example:
  - Toss a coin 10 times, and let X be the number of times head is observed.
  - Roll a die 5 times, and let X be the number of times odd numbers are observed.

Notation for binomial distribution

- We usually use these notations:
  - n: Total number of trials.
  - p: The probability of success.
  - q: The probability of failure. $q = 1 - p$.
  - X: Number of successes in n trials.

- Note that $0 \leq X \leq n$. X can take the values 0, 1, 2, …, n

Probability distribution for binomial

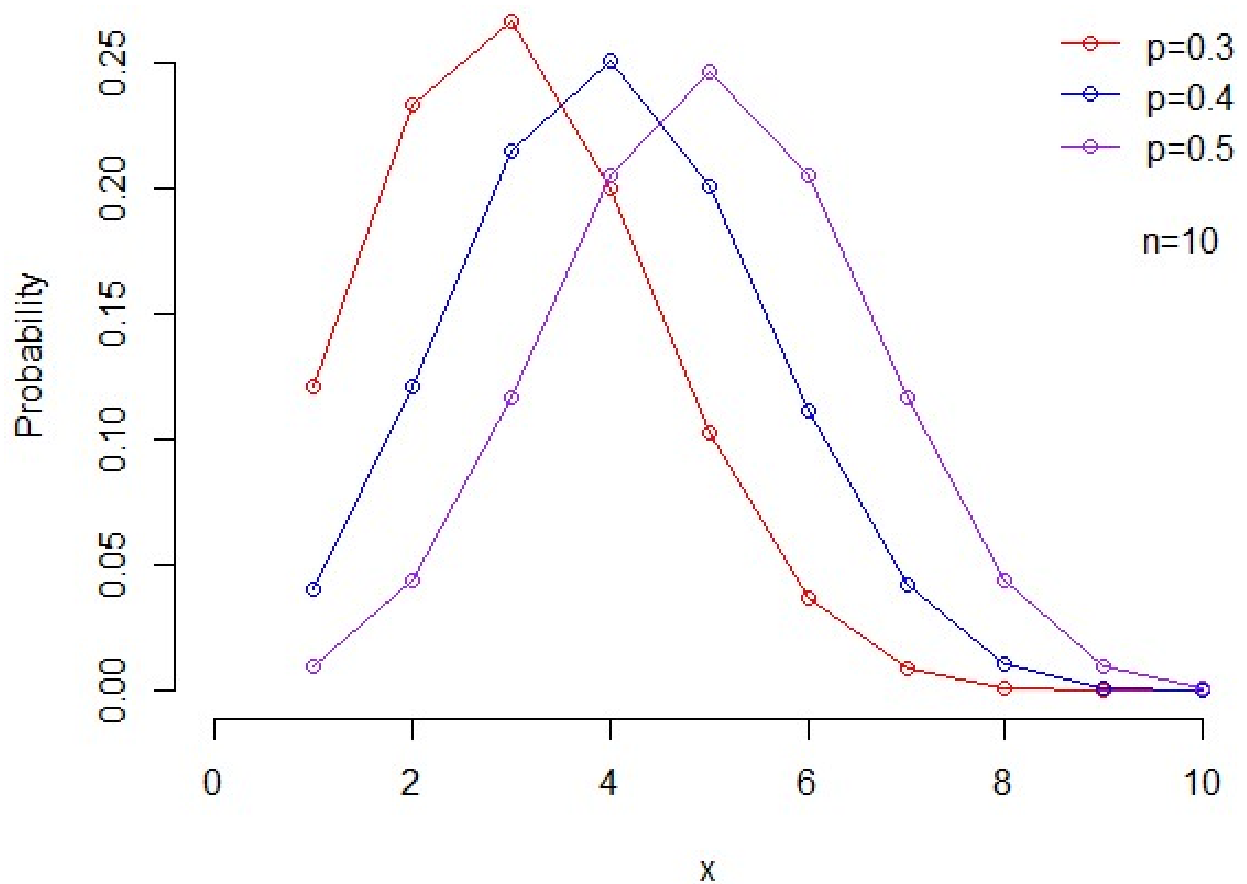
- The probability distribution of X:

$$P(X = x) = {}^{n}C_{x} p^{x} q^{n-x}$$

  for $x = 0,1,\ldots,n$

- How do we get this probability?
  - Out of the n trials, x number of them are successes.
    - Total possible way to get x successes out of n trials is ${}^{n}C_{x}$
  - There are x number of successes, each with probability p.
    - The probability for this is p^{x}
  - There are $n - x$ number of failures, each with probability q.
    - The probability for this is q^{n-x}
  - Since they are independent, we can multiply their probabilities.

Probability distribution for binomial



Mean and variance for binomial distribution

- Mean:

$$\mu = np$$

- Variance:

$$\sigma^2 = npq = np(1-p)$$

- Standard deviation:

$$\sigma = \sqrt{npq} = \sqrt{np(1-p)}$$

Example

- A coin is tossed 3 times.
  - Find the probability that we observe 2 heads.
  - Find the mean, variance, and standard deviation of the number of heads that will be obtained.

- Let X be the number of times we observe heads. X has a binomial distribution.
- In this case, "success" is defined as "observing head".

$$p = P(\text{head}) = 0.5, \qquad q = P(\text{tail}) = 0.5$$

- The number of trials, n is 3.
- Probability of observing 2 heads:

$$P(X = \cancel{3}) = {}^3C_2(0.5)^2(0.5)^1 = 0.375$$

2

$${}^nC_x p^x q^{n-x}$$

Example

- Mean:

$$\mu = np = 3(0.5) = 1.5$$

- Variance:

$$\sigma^2 = npq = 3(0.5)(0.5) = 0.75$$

- Standard deviation:

$$\sigma = \sqrt{npq} = \sqrt{0.75} = 0.8660$$

Another example

- A die is rolled 3 times.
  - Find the probability that we observe numbers 1 or 2 two times.
  - Find the mean and variance of the number of times 1 or 2 are observed.

- Let X be the number of times 1 or 2 are observed. X has a binomial distribution.
- Number of trials: $n = 3$.
- "Success" is observing 1 or 2.
- Probability of success and failure:

$$p = \frac{2}{6} = \frac{1}{3}, \qquad q = 1 - p = \frac{2}{3}$$

- Probability $X = 2$:

$$P(2) = {}^{3}C_{2}\left(\frac{1}{3}\right)^{2}\left(\frac{2}{3}\right)^{1} = 0.2222$$

Another example

- Mean:

$$\mu = np = 3\left(\frac{1}{3}\right) = 1$$

- Variance:

$$\sigma^2 = npq = 3\left(\frac{1}{3}\right)\left(\frac{2}{3}\right) = \frac{2}{3}$$

Exercise

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- Forty percent of prison inmates were unemployed when they entered prison. If 5 inmates are randomly selected, find these probabilities:
  a) Exactly 3 were unemployed.
  b) At most 4 were unemployed.
  c) At least 3 were unemployed.
  d) Fewer than 2 were unemployed.

Let X be the number of inmates that were unemployed out of the five inmates.

X has a binomial distribution with

$n=5$, $p=0.4$, $q=0.6$

X can take values $0, 1, 2, 3, 4, 5$

a) $P(X=3) = {}^{n}C_{x}\, p^{x} q^{n-x} = {}^{5}C_{3}(0\cdot 4)^{3}(0\cdot 6)^{2}$

$= 0\cdot 2304$

b) $P(X \leqslant 4) = 1 - P(5) = 1 - {}^{5}C_{5}(0\cdot 4)^{5}(0\cdot 6)^{0}$

$= 0\cdot 9898$ (or $P(0)+P(1)+P(2)+P(3)+P(4)$)

c) $P(X \geqslant 3) = P(3)+P(4)+P(5)$

$= {}^{5}C_{3}(0\cdot 4)^{3}(0\cdot 6)^{2} + {}^{5}C_{4}(0\cdot 4)^{4}(0\cdot 6)^{1}$

$+ {}^{5}C_{5}(0\cdot 4)^{5}(0\cdot 6)^{0}$

$= 0\cdot 3174$

d) $P(X<2) = P(0)+P(1) = {}^{5}C_{0}(0\cdot 4)^{0}(0\cdot 6)^{5} + {}^{5}C_{1}(0\cdot 4)^{1}(0\cdot 6)^{4}$

$= 0\cdot 3370$

Exercise

- Thirty-two percent of adult Internet users have purchased groceries online. For a random sample of 200 adult Internet users, find the mean, variance, and standard deviation for the number who have purchased groceries online.

X = number of adults who have purchased groceries online, out of 200 internet users.

X has a binomial distribution with

$n = 200$, $p = 0.32$, $q = 0.68$

$$\mu = np = 200 \times 0.32 = 64$$

$$\sigma^2 = npq = 200 \times 0.32 \times 0.68 = 43.52$$

$$\sigma = \sqrt{43.52} = 6.597$$

Poisson distribution

Poisson experiment

- A Poisson experiment is a probability experiment that satisfies the following requirements:
  - The random variable X is the number of occurrences of an event over some interval (i.e., length, area, volume, period of time, etc.).
  - The occurrences occur randomly.
  - The occurrences are independent of one another.
  - The average number of occurrences over an interval is known.

- Note that $X \geq 0$ and can take the values 0, 1, 2, …

Poisson distribution

- Let X be the outcome of a Poisson experiment.
- Then the distribution of X is a Poisson distribution.
- Example:
  - Number of patients admitted in a hospital in a day.
  - Number of customers in 5 hours interval.
  - Number of typographical error in a page.

Probability distribution for Poisson

- Denote λ be the mean number of occurrences in that interval.
- Then probability distribution of X:

$$P(X = x) = \frac{\lambda^x e^{-\lambda}}{x!}$$

for $x = 0,1,2, \ldots$

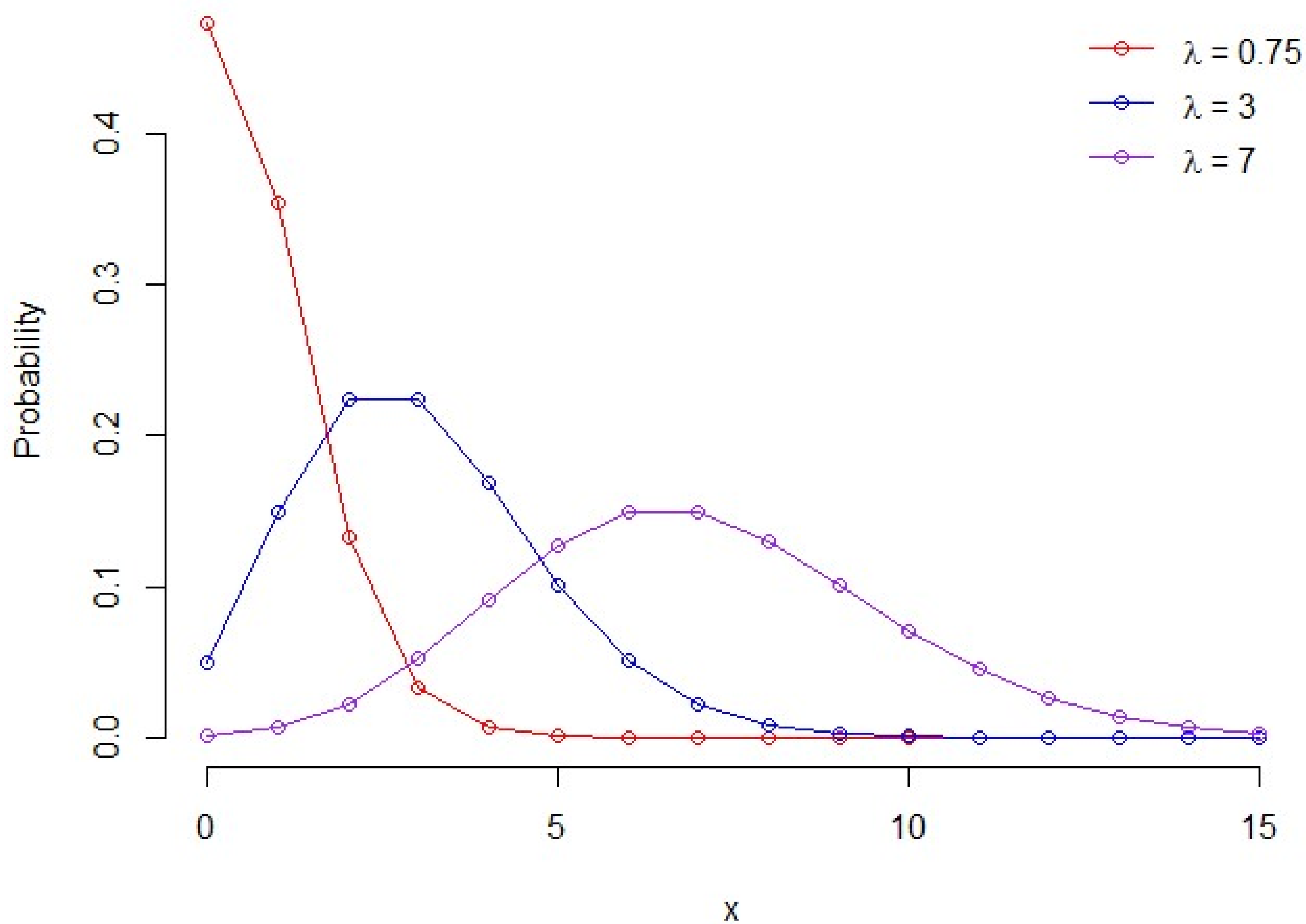
- Mean:

$$\mu = \lambda$$

- Variance and standard deviation:

$$\sigma^2 = \lambda, \qquad \sigma = \sqrt{\lambda}$$

Probability distribution for Poisson



Example

- A sales firm receives, on average, 3 calls per hour on its toll-free number. For any given hour, find the probability that it receives exactly 5 calls.

- Let X be the number calls received in an hour. X has a Poisson distribution.
- Mean number of calls per hour, $\lambda = 3$.
- Probability receiving exactly 5 calls:

$$P(5) = \frac{3^5 e^{-3}}{5!} = 0.1008$$

$$\frac{\lambda^x e^{-\lambda}}{x!}$$

Another example

- A sales firm receives, on average, 3 calls per hour on its toll-free number. Find the probability that it receives exactly 5 calls in a two-hour period.

- Let X be the number calls received in a two-hour period. X has a Poisson distribution.
- Mean number of calls in a two-hour period:

$$\lambda = \text{mean number per hour} \times 2 = 3 \times 2 = 6$$

- Probability receiving exactly 5 calls:

$$P(5) = \frac{6^5 e^{-6}}{5!} = 0.1606$$

Exercise

- A recent study of robberies for a certain geographic region showed an average of 1 robbery per 20,000 people. In a city of 80,000 people, find the probability of the following.
  a) 0 robberies
  b) 1 robbery
  c) 2 robberies

X = number of robberies in a city of 80 000 people

X has a Poisson distribution

$$\lambda = \frac{1}{20000 \text{ people}} \times 80000 \text{ people} = 1 \times 4 = 4$$

a) $P(0) = \dfrac{\lambda^{x} e^{-\lambda}}{x!} = \dfrac{4^{0} e^{-4}}{0!} = 0.01832$

b) $P(1) = \dfrac{4^{1} e^{-4}}{1!} = 0.07326$

c) $P(2) = \dfrac{4^{2} e^{-4}}{2!} = 0.1465$

Exercise

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- The mean number of accidents per month at a certain intersection is three. Find the probability that in any given month,
  a) No accidents will occur.
  b) At least one accident will occur.
  c) Less than three accident will occur.

X = number of accidents in a month

X has a Poisson distribution with

$\lambda = 3$

a) $P(X=0) = \dfrac{\lambda^{x} e^{-\lambda}}{x!} = \dfrac{3^{0} e^{-3}}{0!} = 0.04979$

b) $P(X \geq 1) = 1 - P(X=0) = 1 - 0.04979$

$= 0.9502$

c) $P(X<3) = P(0) + P(1) + P(2)$

$= \dfrac{3^{0} e^{-3}}{0!} + \dfrac{3^{1} e^{-3}}{1!} + \dfrac{3^{2} e^{-3}}{2!}$

$= 0.4232$

Geometric distribution

Geometric distribution

- A geometric experiment is a probability experiment such that:
  - Each trial has two outcomes, either success or failure
  - The outcomes are independent of each other
  - The probability of a success is the same for each trial
  - The experiment continues until a success is obtained.

- Geometric distribution is related to the binomial.
  - Binomial random variable counts number of successes in n trials.
  - Geometric random variable counts number of trials until success.

- Note: in some books, the geometric r.v. may refer to the number of failures until success

Probability distribution for geometric

- Probability distribution:

$$P(X = x) = (1 - p)^{x-1}p$$

for $x = 1,2,3, \ldots$

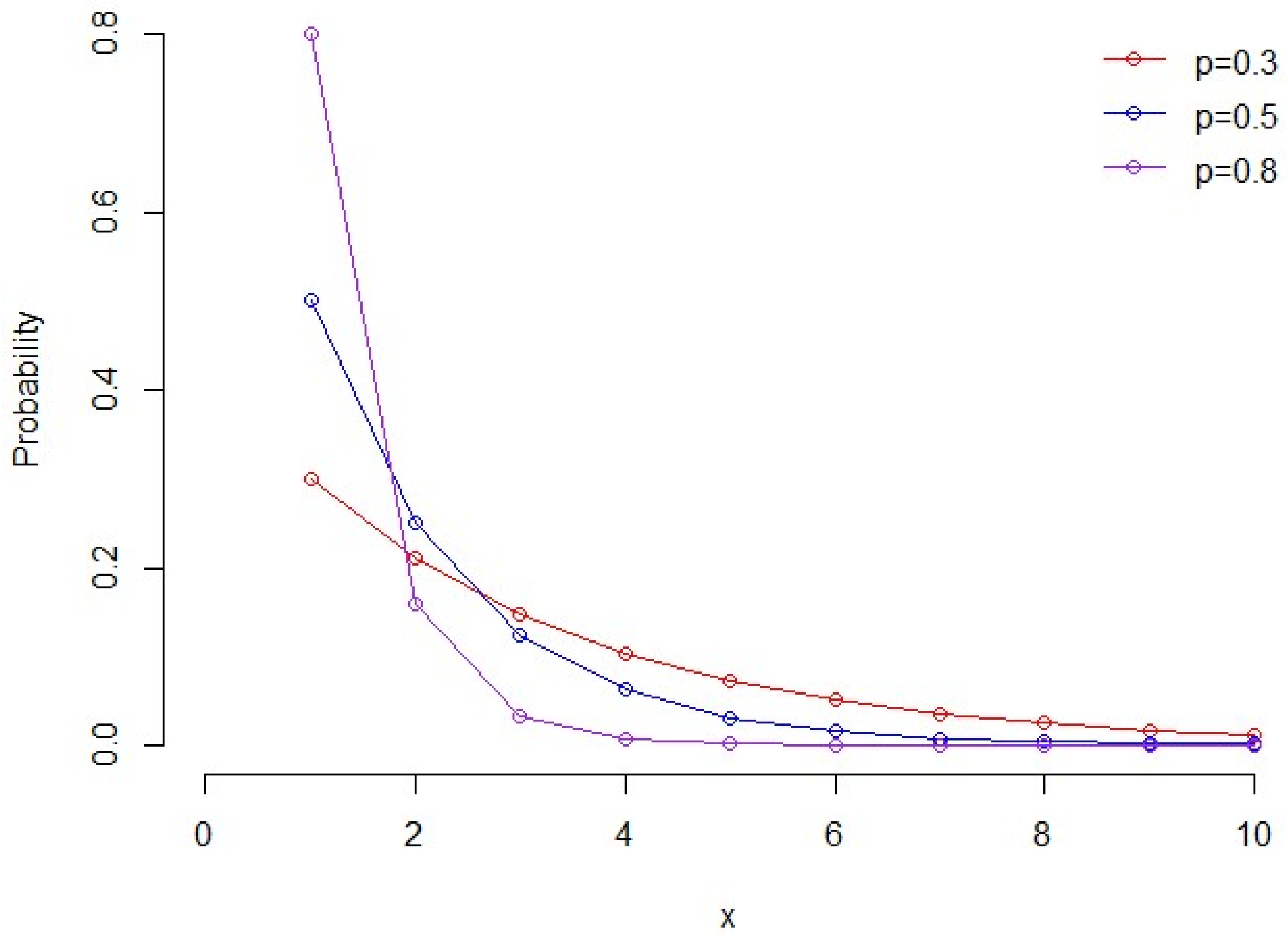
- Mean:

$$\mu = \frac{1}{p}$$

- Variance:

$$\sigma^2 = \frac{1 - p}{p^2}$$

Probability distribution for geometric



Example

- Suppose that the probability that an applicant for a driver's license will pass the road test on any given try is 0.75, and assume the tests are independent.

- Let X be the number of road test taken by an applicant until they pass the test.

- In this case, X can take values 1, 2, 3, … and X follows the geometric distribution with $p = 0.75$.

- For example, the probability that it will take an applicant passing the test for the first time at the 4th attempt is

$$P(X = 4) = (0.25)^3 0.75 = 0.01171$$

- Additionally, the mean number of attempts it will take for an applicant to pass the test is $\frac{1}{p} = 1.333$

Exercise

- The probability that you will make a sale on any given telephone call is 0.19. Find the probability that you:
  a) make your first sale on the fifth call.
  b) make your first sale on the first, second, or third call.
  c) do not make a sale on the first three calls.

X = number of call until the first sale

X has a geometric distribution with $p = 0.19$

a) $P(X=5) = (1-p)^{x-1} p = (1-0.19)^4 (0.19)$

$= (0.81)^4 (0.19)$

$= 0.08179$

b) $P(1 \le X \le 3) = P(1) + P(2) + P(3)$

$$= (0.81)^0(0.19) + (0.81)^1(0.19)$$

$$+ (0.81)^2(0.19)$$

$$= 0.4686$$

c) $P(X > 3) = 1 - P(X \le 3)$

$$= 1 - P(1) - P(2) - P(3)$$

$$= 1 - 0.4686$$

$$= 0.5314$$

Summary

- This chapter introduces discrete random variables.
- We first introduced what random variables are.
- Then we focused on discrete random variables including
  - Probability distribution
  - Distribution function
  - Mean
  - Variance
- We looked at three special case of discrete random variables:
  - Binomial distribution – number of successes in n trials.
  - Poisson distribution – number of events in an interval.
  - Geometric distribution – number of trials until success