Université Grenoble Alpes

Tsunami modelling

Numerical modelling workshop

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Contents

The modelling of Tsunami waves can be done both using the shallow water [8] or Boussinesq [13] equations. The shallow water equations do not take into account dispersion phenomena unlike Boussinesq's.

In our work we will use the shallow water equations.

1 Shallow water equations

The shallow water equations are widely used for tsunami modeling. Indeed, the main assumption is that the wavelength is much larger than the water depth which is the case for tsunamis ($\lambda \sim 10^2 km$) unlike wind related waves ($\lambda \sim 10^2 m$). Consequently, the propagation is assumed horizontal, the vertical acceleration is negligible.

In the following section, we derive the shallow water equations from the conservative Navier Stokes equations [14].

1.1 Conservative Navier Stokes equations

These equations describe the motion of a viscous fluid with a density ρ, velocity $\overrightarrow{V}$ that undergoes a set of given forces.
The Navier stokes system of equations is composed of 1 continuity equation derived from mass conservation law and 1 momentum equation derived from momentum conservation law.

Continuity equation

$$\frac{\partial \rho}{\partial t} + \overrightarrow{\nabla}(\rho \overrightarrow{V}) = 0 \tag{1}$$

Momentum equation

$$\rho(\frac{\partial \overrightarrow{V}}{\partial t} + (\overrightarrow{V}.\overrightarrow{\nabla})\overrightarrow{V}) = \rho \overrightarrow{g} - \overrightarrow{\nabla} P + \overrightarrow{\nabla}.\Sigma \tag{2}$$

With,
$\overrightarrow{V}$: Velocity
$\overrightarrow{g}$: Gravity acceleration
P : Pressure
ρ : Density
Σ : Viscous stress tensor

The forces in the system are:

Pressure gradient

$$\overrightarrow{\nabla} P = \begin{pmatrix} \frac{\partial P}{\partial x} \\ \frac{\partial P}{\partial y} \\ \frac{\partial P}{\partial z} \end{pmatrix} \tag{3}$$

Weight

$$\rho \overrightarrow{g} = \begin{pmatrix} \rho g_x \\ \rho g_y \\ \rho g_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -\rho g \end{pmatrix} \tag{4}$$

Divergence stress tensor

$$\overrightarrow{\nabla}.\Sigma = \begin{pmatrix} \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} \\ \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} \\ \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} \end{pmatrix} \tag{5}$$

We can project the vectorial momentum equation onto the X Y Z components.

This gives the following general expressions,

$$\begin{cases} \frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} = 0 \\ \frac{\partial \rho u}{\partial t} + \frac{\partial \rho u^2}{\partial x} + \frac{\partial \rho uv}{\partial y} + \frac{\partial \rho uw}{\partial z} = 0 - \frac{\partial P}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} \\ \frac{\partial \rho v}{\partial t} + \frac{\partial \rho uv}{\partial x} + \frac{\partial \rho v^2}{\partial y} + \frac{\partial \rho vw}{\partial z} = 0 - \frac{\partial P}{\partial y} + \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} \\ \frac{\partial \rho w}{\partial t} + \frac{\partial \rho uw}{\partial x} + \frac{\partial \rho vw}{\partial y} + \frac{\partial \rho w^2}{\partial z} = -\rho g - \frac{\partial P}{\partial z} + \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} \end{cases}$$

We consider, the fluid vertical acceleration (z component) to be negligible.

Thus we get a relation between pressure and gravity, by integrating we find an expression of the pressure which is hydrostatic.

So the z momentum equation becomes:

$$\frac{\partial P}{\partial z} = -\rho g \tag{6}$$

$$P(z) = -\int_{\eta}^{z} \rho g dz = -\left[\rho g z\right]_{\eta}^{z} = \rho g(\eta - z) \tag{7}$$

Now, we write the boundary conditions in $z = \eta$ and $z = -h$ (figure 1),

At $z = \eta$ (surface),

$$P(0) = 0 \tag{8}$$

$$w(\eta) = \frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} + v\frac{\partial \eta}{\partial y} \tag{9}$$

At $z = -h$ (bottom),

$$w(-h) = -u\frac{\partial h}{\partial x} - v\frac{\partial h}{\partial y} \tag{10}$$

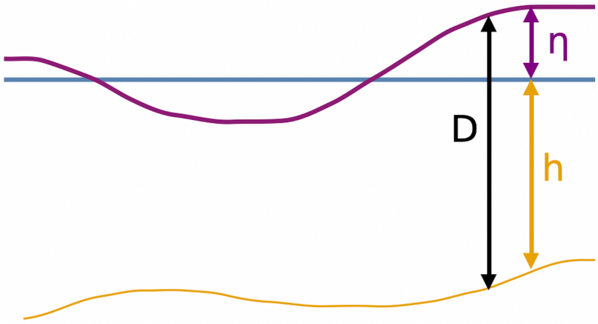


Figure 1: Scheme of the model parameters

1.2 Vertical integration over the water column

Now, we integrate the Navier-Stokes equations from the bottom to the surface using the Liebnitz rule [9].

Thus, we get the shallow water equations ,

$$\begin{cases} \frac{\partial \eta}{\partial t} + \frac{\partial(\eta+h)u}{\partial x} + \frac{\partial(\eta+h)v}{\partial y} = 0 \\ \frac{\partial(\eta+h)u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{((\eta+h)u)^2}{\eta+h}\right) + \frac{\partial}{\partial y}\left(\frac{(\eta+h)u(\eta+h)v}{\eta+h}\right) = -g(\eta+h)\frac{\partial \eta}{\partial x} - \frac{\tau_x}{\rho} \\ \frac{\partial(\eta+h)v}{\partial t} + \frac{\partial}{\partial x}\left(\frac{(\eta+h)u(\eta+h)v}{\eta+h}\right) + \frac{\partial}{\partial y}\left(\frac{((\eta+h)v)^2}{\eta+h}\right) = -g(\eta+h)\frac{\partial \eta}{\partial y} - \frac{\tau_y}{\rho} \end{cases}$$

We can write [3],
u = depth averaged speed in x
v = depth averaged speed in y
M = Du, x direction discharge flux
N = Dv, y direction discharge flux
D= $\eta + h$ the total water colum length
g= gravity acceleration
τ_x = bottom friction in x
τ_y = bottom friction in y

Thus we get the following system of equations,

$$\begin{cases} \frac{\partial \eta}{\partial t} + \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} = 0 \\ \frac{\partial M}{\partial t} + \frac{\partial}{\partial x}\left(\frac{M^2}{D}\right) + \frac{\partial}{\partial y}\left(\frac{MN}{D}\right) = -gD\frac{\partial \eta}{\partial x} - \frac{\tau_x}{\rho} \\ \frac{\partial N}{\partial t} + \frac{\partial}{\partial x}\left(\frac{MN}{D}\right) + \frac{\partial}{\partial y}\left(\frac{N^2}{D}\right) = -gD\frac{\partial \eta}{\partial y} - \frac{\tau_y}{\rho} \end{cases}$$

2 1D linear case

2.1 1D linear equations

We don't consider the bottom friction.

We have this general system of equations,

$$\begin{cases} \frac{\partial \eta}{\partial t} + \frac{\partial M}{\partial x} = 0 \\ \frac{\partial M}{\partial t} + \frac{\partial}{\partial x}\left(\frac{M^2}{D}\right) = -gD\frac{\partial \eta}{\partial x} \end{cases}$$

Assuming that u and v are small, h= Constante (flat bathymetry) and $h >> \eta$ gives,

$$\begin{cases} \frac{\partial \eta}{\partial t} + h\frac{\partial u}{\partial x} = 0 \\ \frac{\partial u}{\partial t} = -g\frac{\partial \eta}{\partial x} \end{cases}$$

Taking the x derivative of the first equation and the time derivative of the second gives,

$$\begin{cases} \frac{\partial^2 \eta}{\partial t^2} + h\frac{\partial^2 u}{\partial xt} = 0 \\ \frac{\partial^2 u}{\partial tx} = -g\frac{\partial^2 \eta}{\partial x^2} \end{cases}$$

Combining the 2 equations,

$$\frac{\partial^2 \eta}{\partial t^2} - C_0^2 \frac{\partial^2 \eta}{\partial x^2} = 0 \tag{11}$$

With,

$$C_0^2 = gh \tag{12}$$

This is the basic second order linear wave equation with only one unknown.

2.2 Explicit Finite difference scheme

We use an explicit finite difference methode and use a second order accuracy central scheme in both space and time. The main advantage is the low computation time, the main drawback is the poor stability.

Using Taylor's expansion as an approximation of the second order derivatives.

$$\frac{\partial^2 \eta}{\partial t^2} = \frac{\eta_j^{n+1} - 2\eta_j^n + \eta_j^{n-1}}{\Delta t^2} \tag{13}$$

With an associated truncation error $O(\Delta t^2)$

$$\frac{\partial^2 \eta}{\partial x^2} = \frac{\eta_{j+1}^n - 2\eta_j^n + \eta_{j-1}^n}{\Delta x^2} \tag{14}$$

With an associated truncation error $O(\Delta x^2)$

Thus, replacing the terms in equation (11) by the expressions in (13) (14) we get,

$$\frac{\eta_j^{n+1} - 2\eta_j^n + \eta_j^{n-1}}{\Delta t^2} - C_0^2 \frac{\eta_{j+1}^n - 2\eta_j^n + \eta_{j-1}^n}{\Delta x^2} = 0 \tag{15}$$

Isolating , η_j^{n+1} gives,

$$\eta_j^{n+1} = C_0^2 \Delta t^2 \frac{\eta_{j+1}^n - 2\eta_j^n + \eta_{j-1}^n}{\Delta x^2} + 2\eta_j^n - \eta_j^{n-1} \tag{16}$$

With an associated truncation error $O(\Delta x^2) + O(\Delta t^2)$

Free displacement boundary condition,

$\eta(\Omega) = \eta$

2.2.1 Von Neumann stability analysis

Von Neumann stability analysis [12] is based on Fourier decomposition and can only be used for linear partial differential equations (PDE).

(See the full calculations in the A ANNEX)

We get the stability condition of the scheme,

$$C_0 \leq \frac{\Delta x}{\Delta t} \tag{17}$$

The scheme is conditionally stable.

2.2.2 Numerical Dispersion analysis

We calculate the numerical dispersion equation buy considering a plane wave solution [4]:

$P = P_0 e^{-i(\omega t + kx)}$.

We inject this solution into the equation (16).

(See the full calculations in the B ANNEX)

After some calculations we get the numerical dispersion equation of the scheme (figure 2).

$$C_k = \frac{2}{k\Delta t} sin^{-1}\left(\frac{C_0 \Delta t}{\Delta x} sin\left(\frac{k\Delta x}{2}\right)\right) \tag{18}$$

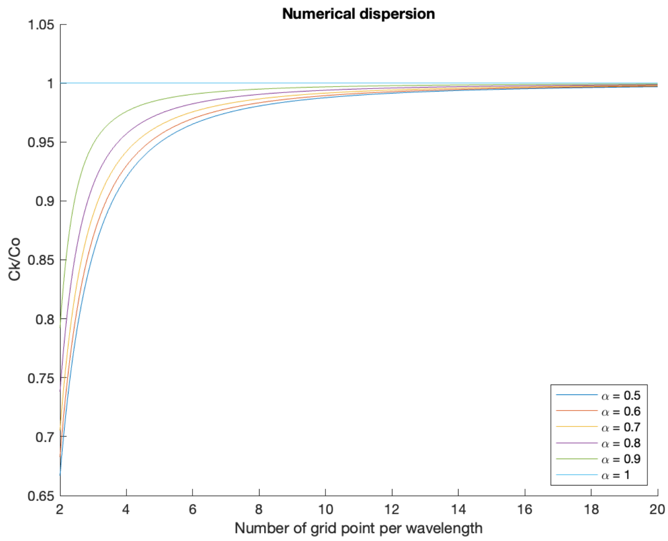


Figure 2: Numerical dispersion 1D explicit scheme

The parameters C_0, Δt, Δx have a direct impact on the numerical dispersion since $\alpha = \frac{C_0 \Delta t}{\Delta x}$.

We notice that, if $\alpha = \frac{C_0 \Delta t}{\Delta x} = 1$, then equation (18) writes,

$$C_k = \frac{\Delta x}{\Delta t} \tag{19}$$

In this particular case, there is no numerical dispersion (C_k constant).

2.3 Implicite finite difference scheme

We use the implicite Crank-Nicolson scheme [10]. This scheme is implicite, so unconditionnaly stable. However, it requires to solve a linear system of equations at each time step which is computationaly expensive.
This scheme can be seen as a weighted average of the spatial derivative approximations evaluated at different time steps with the same coefficients as in the time derivative approximation [7].

It writes:

$$\begin{aligned} &\frac{\eta_j^{n+1} - 2\eta_j^n + \eta_j^{n-1}}{\Delta t^2} - \frac{C_0^2}{4}(\frac{\eta_{j+1}^{n-1} - 2\eta_j^{n-1} + \eta_{j-1}^{n-1}}{\Delta x^2} \\ &+2\frac{\eta_{j+1}^n - 2\eta_j^n + \eta_{j-1}^n}{\Delta x^2} + \frac{\eta_{j+1}^{n+1} - 2\eta_j^{n+1} + \eta_{j-1}^{n+1}}{\Delta x^2}) = 0 \end{aligned} \tag{20}$$

Writing, $\alpha = \frac{C_0^2 \Delta t^2}{4 \Delta x^2}$ gives,

$$\begin{aligned} &\eta_j^{n+1} - 2\eta_j^n + \eta_j^{n-1} - \alpha(\eta_{j+1}^{n-1} - 2\eta_j^{n-1} + \eta_{j-1}^{n-1} \\ &+2\eta_{j+1}^n - 4\eta_j^n + 2\eta_{j-1}^n + \eta_{j+1}^{n+1} - 2\eta_j^{n+1} + \eta_{j-1}^{n+1}) = 0 \end{aligned} \tag{21}$$

we isolate all the n+1 terms in the left and side,

$$\begin{aligned} -\alpha\eta_{j+1}^{n+1} + (2\alpha + 1)\eta_j^{n+1} - \alpha\eta_{j-1}^{n+1} = \alpha\eta_{j+1}^{n-1} - (1 + 2\alpha) \\ \eta_j^{n-1} + \alpha\eta_{j-1}^{n-1} + 2\alpha\eta_{j+1}^n + (2 - 4\alpha)\eta_j^n + 2\alpha\eta_{j-1}^n \end{aligned} \tag{22}$$

This is a linear system of equations and can be written as,

$$R\eta^{n+1} = V \tag{23}$$

With,

$$\eta^{n+1} = \begin{pmatrix} \eta_1^{n+1} \\ \eta_2^{n+1} \\ ... \\ \eta_{N-1}^{n+1} \\ \eta_N^{n+1} \end{pmatrix} \tag{24}$$

$$V = \alpha\eta_{j+1}^{n-1} - (1+2\alpha)\eta_j^{n-1} + \alpha\eta_{j-1}^{n-1} + 2\alpha\eta_{j+1}^{n} + (2-4\alpha)\eta_j^{n} + 2\alpha\eta_{j-1}^{n} \tag{25}$$

Free displacement boundary condition,

$\eta(\Omega) = \eta$

$$R = \begin{pmatrix} 2\alpha+1 & -2\alpha & 0 & 0 & 0 & ... & ... & ... \\ -\alpha & 2\alpha+1 & -\alpha & 0 & 0 & ... & ... & ... \\ 0 & -\alpha & 2\alpha+1 & -\alpha & 0 & ... & ... & ... \\ ... & ... & ... & ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... & ... & ... & ... \\ ... & ... & ... & 0 & -\alpha & 2\alpha+1 & -\alpha & 0 \\ ... & ... & ... & 0 & 0 & -\alpha & 2\alpha+1 & -\alpha \\ ... & ... & ... & 0 & 0 & 0 & -2\alpha & 2\alpha+1 \end{pmatrix} \tag{26}$$

With an associated truncation error $O(\Delta x^2) + O(\Delta t^2)$

2.3.1 Von Neumann stability analysis

The stability condition writes,

$$1 \geq cos(k\Delta x) \tag{27}$$

Which is true for any Δx and k.
Therefore the scheme is unconditionally stable,
(See the full calculations in the C ANNEX)

2.3.2 Numerical Dispersion analysis

We have the following numerical dispersion equation of the implicit scheme (figure 3).

$$C_k = cos^{-1}(\frac{2-\beta^2+\beta^2 cos(k\Delta x)}{2+\beta^2-\beta^2 cos(k\Delta x)}) \times \frac{1}{k\Delta t} \tag{28}$$

With $\beta = \frac{C_0\Delta t}{\Delta x}$.
(See the full calculations in the D ANNEX)

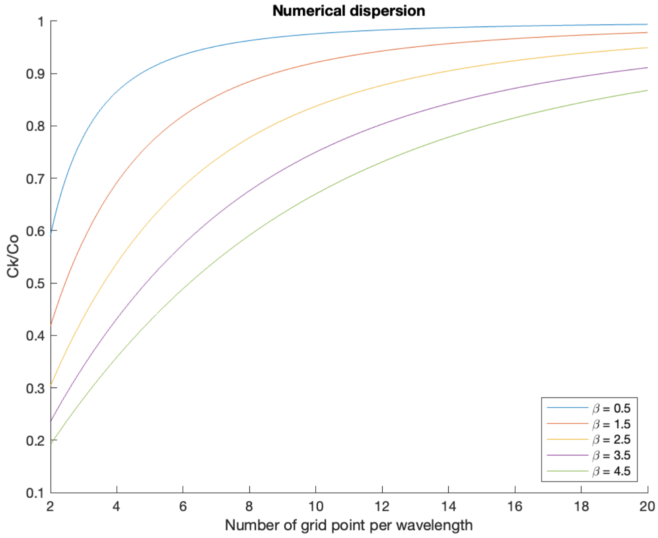


Figure 3: Numerical dispersion 1D implicit scheme

The smaller the β, the smaller the numerical dispersion.

2.4 Model

Parameters:
L=1000 km : Model length
H = 1000 m : Water depth
g = 9.81 N/kg
$C_0 = \sqrt{gH}$: Propagation speed

Initial condition (t=0):

$\eta(x, t = 0) = e^{-\frac{(x-\mu)^2}{2*\sigma^2}}$: Gaussian source

We consider the Gaussian width (W) as 6σ and set W at 100 km.

Thus, we have,
W = 100km
σ = W / 6 = 16667 m
μ = 200 km

$f_{max} = 4.10^{-5} m^{-1}$(Amplitude spectrum reading)
$\lambda_{min} = \frac{1}{f_{max}}$
Spatial sampling condition (20 samples per λ_{min}) : $\Delta x \leq \frac{\lambda_{min}}{20}$
Stability condition : $C_0 \leq \frac{\Delta x}{\Delta t} \rightarrow \Delta t \leq \frac{\Delta x}{C_0}$

We take: $\Delta x = 1250m$, $\Delta t = 10s$

Free displacement boundary conditions (BC) : $\eta(\Omega) = \eta$

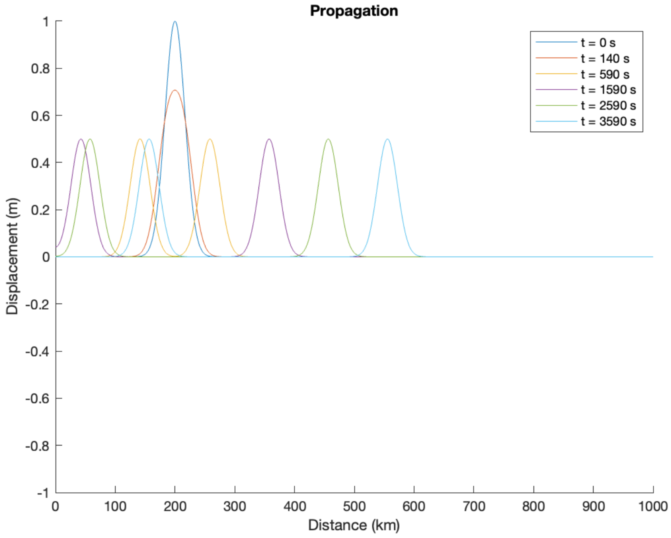


Figure 4: 1D Linear propagation

2.5 Local Truncation Error analysis

The Local Error represents the error introduced in one iteration of a scheme.

This error has 2 distinct origines :

-Local Truncation error from Taylor's series approximation (often dominant)
-Round off error due to the number of digits precision (often negligeable).

We use the analytical solution as reference to derive the errors.

$$\eta(x,t) = \frac{1}{2}(e^{-\frac{(x+C_0t-\mu)^2}{2*\sigma^2}} + e^{-\frac{(x-C_0t-\mu)^2}{2*\sigma^2}}) \tag{29}$$

To get the spatial local truncation error of the scheme we vary the spatial step.
To get the time local truncation error of the scheme we vary the time step.

Loglog slope Error(dx) : accuracy order of the spatial Local Truncation error (figure 5).
Loglog slope Error(dt) : accuracy order of the time Local Truncation error (figure 6).

We fulfill the Local Truncation Error analysis for 3 schemes:

-Explicit scheme second order wave equation
-Implicit scheme second order wave equation
-Implicit scheme first order wave equation (not detailed in the report)

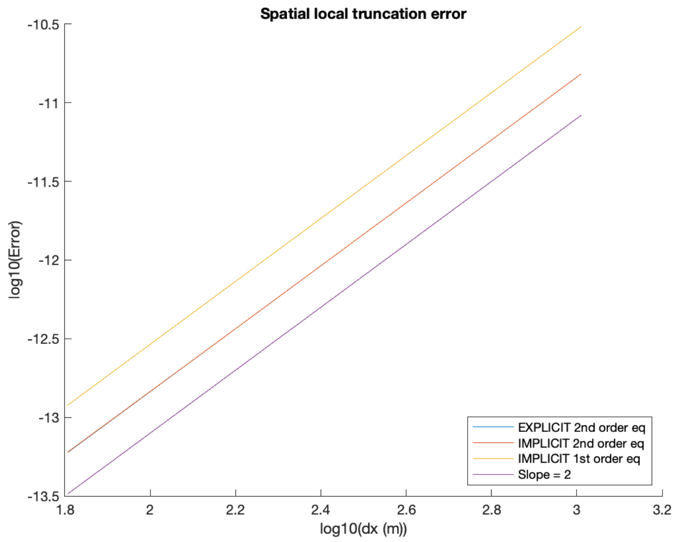


Figure 5: Spatial Local Truncation Error

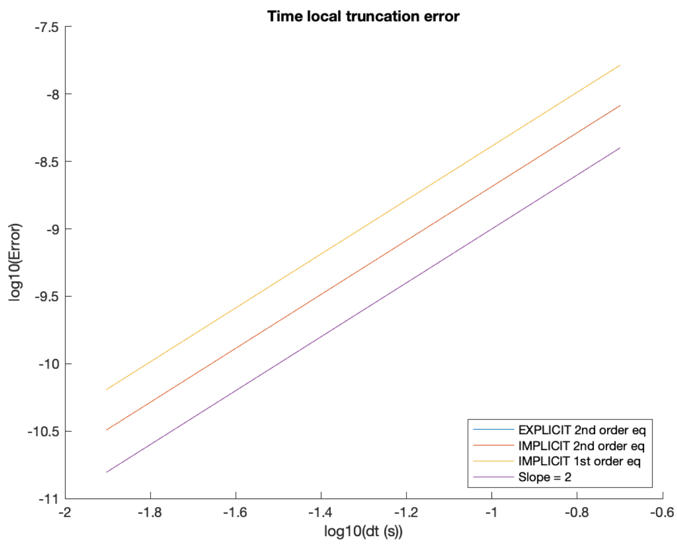


Figure 6: Time Local Truncation Error

We observe that the 3 schemes are second order accuracy local truncation error in both space and time.

3 2D linear case

Since it's similar to the previous section, we wont demonstrate all the equations and will only compute an explicit finite difference scheme.

We have,

$$\frac{\partial^2 \eta}{\partial t^2} - C_0^2 \frac{\partial^2 \eta}{\partial x^2} - C_0^2 \frac{\partial^2 \eta}{\partial y^2} = 0 \tag{30}$$

With,

$$C_0^2 = gh \tag{31}$$

3.1 Explicit finite difference scheme

We use an explicit finite difference methode and use a second order accuracy central scheme in both space and time.

We get the recursive expression,

$$\begin{aligned} \eta_{i,j}^{n+1} = C_0^2 \Delta t^2 (\frac{\eta_{i,j+1}^n - 2\eta_{i,j}^n + \eta_{i,j-1}^n}{\Delta x^2} \\ + \frac{\eta_{i+1,j}^n - 2\eta_{i,j}^n + \eta_{i-1,j}^n}{\Delta y^2}) + 2\eta_{i,j}^n - \eta_{i,j}^{n-1} \end{aligned} \tag{32}$$

With an associated truncation error $O(\Delta x^2) + O(\Delta y^2) + O(\Delta t^2)$

3.1.1 Von Neumann stability analysis

The stability condition can be easily derived by a Von Neumann stability analysis as before,

$$C_0 \sqrt{2} \leq \frac{\Delta x}{\Delta t} \tag{33}$$

3.2 Model

The parameters are similar as for the 1D case except $\Delta y = \Delta x = 1250$m and $\Delta t = 5$ s.

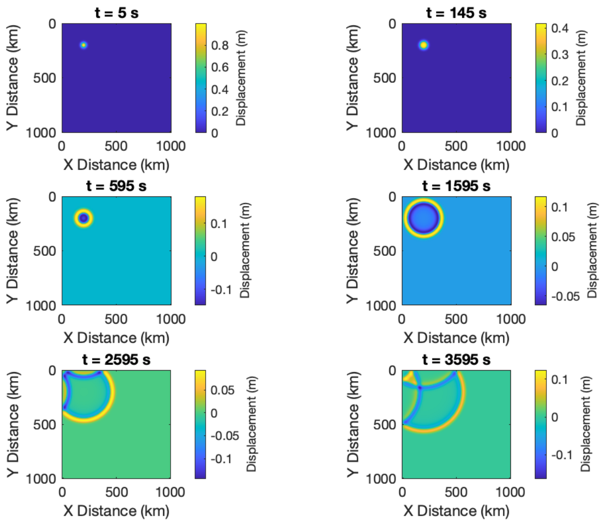


Figure 7: 2D Linear propagation

4 1D nonlinear case

Now we consider the 1D nonlinear shallow water system which takes into account a variable bathymetry as well as bottom friction.

4.1 1D nonlinear equations

The 1D nonlinear system writes $\begin{cases} \frac{\partial \eta}{\partial t} + \frac{\partial M}{\partial x} = 0 \\ \frac{\partial M}{\partial t} + \frac{\partial}{\partial x}\left(\frac{M^2}{D}\right) = -gD\frac{\partial \eta}{\partial x} - \frac{\tau_x}{\rho} \end{cases}$

The bottom friction term (τ_x) is a nonlinear term that can be written,

$$\tau_x = \frac{\rho m^2 M^2}{D^{7/3}} \tag{34}$$

With m : Manning's roughness

4.2 Explicit Two steps Lax-Wendroff scheme

The Two steps Lax-Wendroff scheme [11] [1] is often used in fluid dynamics for nonlinear modeling as it little oscilates and is pretty stable.

As the name mention it, this scheme is composed of two steps, the 1D pattern is pretty simple (figure 8):

First step, we calculate the orange points at time n+1/2 from the yellow points at time n.

Second step, we calculate the red point at time n+1 from the orange points at time n+1/2 and the central yellow point at time n.

(See the full calculations in the E ANNEX)

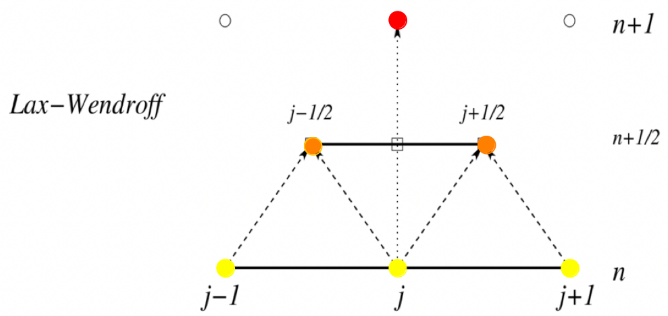


Figure 8: Two steps Lax-Wendroff 1D Pattern [6]

With an associated truncation error $O(\Delta x^2) + O(\Delta t^2)$

Free displacement null velocity boundary condition,

$\eta(\Omega) = \eta$

$M(\Omega) = 0$

4.3 Implicit Crank-Nicolson scheme

We use the implicit Crank-Nicolson scheme.
We average the Forward and Backward Euler schemes.
This gives,

$$\begin{cases} \frac{\eta_j^{n+1}-\eta_j^n}{\Delta t} + \frac{1}{4\Delta x}(M_{j+1}^{n+1} - M_{j-1}^{n+1} + M_{j+1}^n - M_{j-1}^n) = 0 \\ \dfrac{M_j^{n+1} - M_j^n}{\Delta t} + \dfrac{1}{4\Delta x}(\dfrac{{M_{j+1}^{n+1}}^2}{D_{j+1}^{n+1}} - \dfrac{{M_{j-1}^{n+1}}^2}{D_{j-1}^{n+1}} + \dfrac{{M_{j+1}^n}^2}{D_{j+1}^n} - \dfrac{{M_{j-1}^n}^2}{D_{j-1}^n}) \\ \quad + \dfrac{g}{4\Delta x}(D_j^{n+1}(\eta_{j+1}^{n+1} - \eta_{j-1}^{n+1}) + D_j^n(\eta_{j+1}^n - \eta_{j-1}^n)) + \dfrac{gm^2}{2}(\dfrac{{M_j^{n+1}}^2}{{D_j^{n+1}}^{7/3}} + \dfrac{{M_j^n}^2}{{D_j^n}^{7/3}}) = 0 \end{cases}$$

We write, $\alpha = \frac{\Delta t}{4\Delta x}$

$$\begin{cases} (F_1) \quad \eta_j^{n+1} - \eta_j^n + \alpha(M_{j+1}^{n+1} - M_{j-1}^{n+1} + M_{j+1}^n - M_{j-1}^n) = 0 \\ (F_2) \quad M_j^{n+1} - M_j^n + \alpha(\dfrac{{M_{j+1}^{n+1}}^2}{D_{j+1}^{n+1}} - \dfrac{{M_{j-1}^{n+1}}^2}{D_{j-1}^{n+1}} + \dfrac{{M_{j+1}^n}^2}{D_{j+1}^n} - \dfrac{{M_{j-1}^n}^2}{D_{j-1}^n}) \\ \quad + g\alpha(D_j^{n+1}(\eta_{j+1}^{n+1} - \eta_{j-1}^{n+1}) + D_j^n(\eta_{j+1}^n - \eta_{j-1}^n)) + \dfrac{gm^2\Delta t}{2}(\dfrac{{M_j^{n+1}}^2}{{D_j^{n+1}}^{7/3}} + \dfrac{{M_j^n}^2}{{D_j^n}^{7/3}}) = 0 \end{cases}$$

We apply the iterative newton's methode to solve the nonlinear system of equations [5],

$$J \times \begin{pmatrix} \delta\eta \\ \delta M \end{pmatrix} = - \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} \tag{35}$$

Jacobian matrix,

$$J = \begin{pmatrix} J_{F1}^{\eta} & J_{F1}^{M} \\ J_{F2}^{\eta} & J_{F2}^{M} \end{pmatrix} \tag{36}$$

(See the full Jacobian calculations in the F ANNEX)
We start with an initial guess η_0 and M_0. Then we solve, the linear system and find $\delta\eta$ and δM. We compute $\eta^{n+1} = \eta^n + \delta\eta$ and $M^{n+1} = M^n + \delta M$, and iterate until convergence.
We shall take as initial guess η_0 and M_0, the values of the previous time step to reduce the number of iterations, since the function is slowly varying.

Associated truncation error $O(\Delta x^2) + O(\Delta t^2)$

Free displacement null velocity boundary condition,

$\eta(\Omega) = \eta$

$M(\Omega) = 0$

4.4 Model

Real bathymetry data [2] smoothed using a gaussian filter with $\sigma_g = 20$

$\Delta x = 100$ m and $\Delta t = 0.5$ s

Initial condition (t=0):

$\eta(x, t = 0) = e^{-\frac{(x-\mu)^2}{2*\sigma^2}}$: Gaussian source

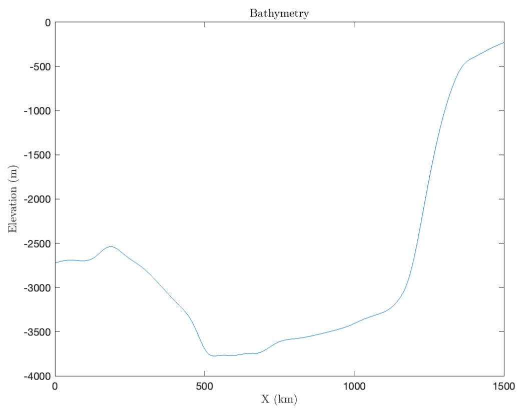
$\sigma = 16667\text{m}$

$\mu = 750$ km

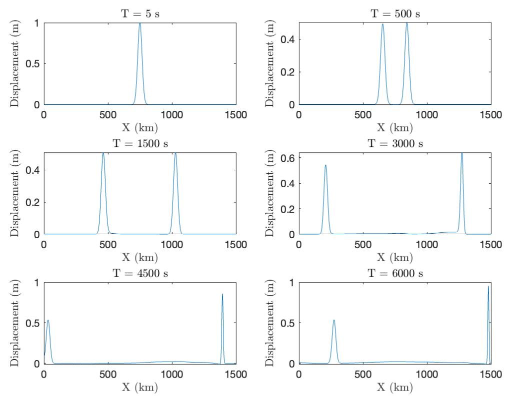
Free displacement null velocity boundary condition

$\eta(\Omega) = \eta$

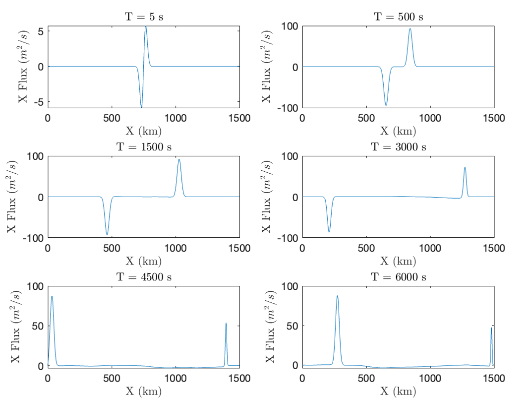
$M(\Omega) = 0$



(a) Bathymetry



(b) Displacement



(c) X Flux

Figure 9: 1D nonlinear propagation

5 2D nonlinear case

Now we consider the 2D nonlinear shallow water system.

5.1 2D nonlinear equations

The 2D nonlinear system writes,

$$\begin{cases} \frac{\partial \eta}{\partial t} + \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} = 0 \\ \frac{\partial M}{\partial t} + \frac{\partial}{\partial x}\left(\frac{M^2}{D}\right) + \frac{\partial}{\partial y}\left(\frac{MN}{D}\right) = -gD\frac{\partial \eta}{\partial x} - \frac{\tau_x}{\rho} \\ \frac{\partial N}{\partial t} + \frac{\partial}{\partial x}\left(\frac{MN}{D}\right) + \frac{\partial}{\partial y}\left(\frac{N^2}{D}\right) = -gD\frac{\partial \eta}{\partial y} - \frac{\tau_y}{\rho} \end{cases}$$

The bottom friction terms (τ_x, τ_y) are nonlinear terms that can be written,

$$\begin{cases} \tau_x = \frac{\rho m^2}{D^{7/3}} M\sqrt{M^2 + N^2} \\ \tau_y = \frac{\rho m^2}{D^{7/3}} N\sqrt{M^2 + N^2} \end{cases}$$

With m : Manning's roughness

5.2 Explicit Two steps Lax-Wendroff scheme

This explicit scheme is composed of two steps, the pattern is more complex in 2D[1] (Figure 10).

First step, we calculate the orange points at time n+1/2 from the yellow points at time n.

Second step, we calculate the red point at time n+1 from the orange points at time n+1/2 and the central yellow point at time n.

(See the full calculations in the G ANNEX)

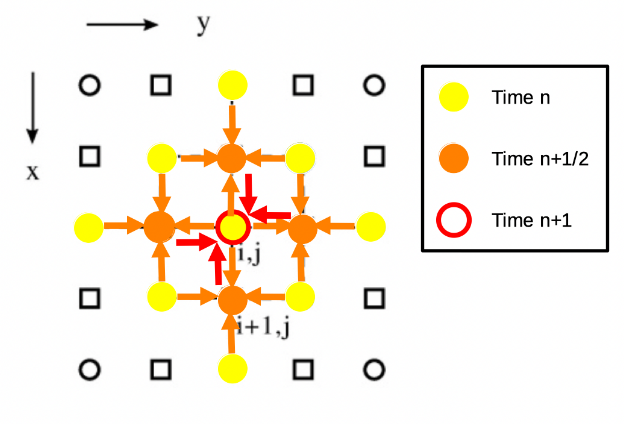


Figure 10: Two steps Lax-Wendroff 2D Pattern

With an associated truncation error $O(\Delta x^2) + O(\Delta y^2) + O(\Delta t^2)$
Free displacement null velocity boundary condition,

$\eta(\Omega) = \eta$

$M(\Omega) = 0$

5.3 Model

Real bathymetry data [2] smoothed using a gaussian filter with $\sigma_g = 20$

$\Delta x = \Delta y = 1000$ m and $\Delta t = 3$ s

Initial condition (t=0):

$$\eta(x, y, t = 0) = e^{-(\frac{(x-\mu_x)^2}{2*\sigma_x^2} + \frac{(y-\mu_y)^2}{2*\sigma_y^2})} \text{ : Gaussian source}$$

$$\sigma_x = \sigma_y = 16667\text{m}$$

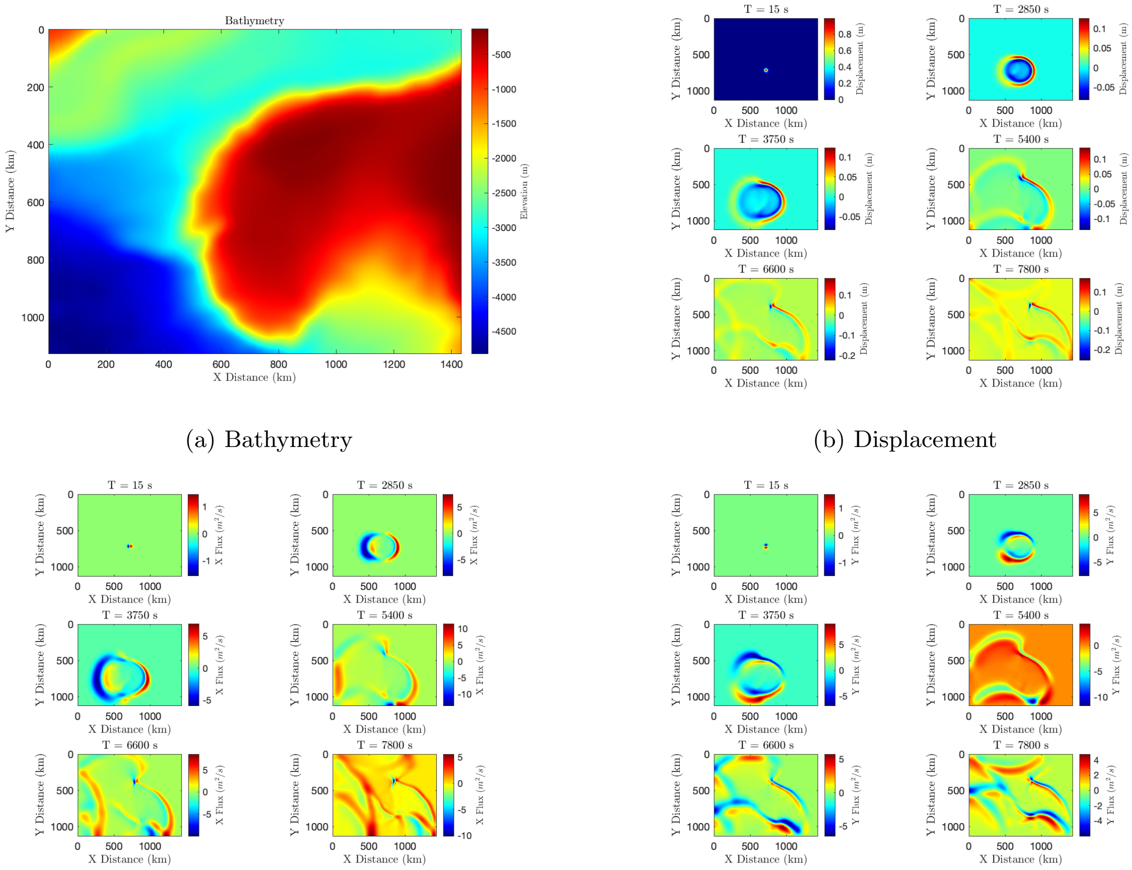
$$\mu_x = \mu_y = 716{,}5 \text{ km}$$

Free displacement null velocity boundary condition

$$\eta(\Omega) = \eta$$

$$M(\Omega) = 0$$

$$N(\Omega) = 0$$



(a) Bathymetry

(b) Displacement

(c) X Flux

(d) Y Flux

Figure 11: 2D nonlinear propagation

The modelling provides sensible results as we observe low, large and fast waves on deep regions and high, narrow and slow waves on shallow regions.

6 Personal experience

This project gave me a full spectrum of the theory, use, advantages, drawbacks and limitations of the Finite difference methode.

Explicit, implicit, 1D and 2D schemes have been used in this project.

I had already computed once a 1D explicite finite difference scheme but never an implicit or 2D scheme, so it was obscure to me.

I also learned the way to work on a model, which is: going step by step slowly and pay attention to all the parameters, make stability analysis as well as dispersion analysis to see how the model behaves, and above all, never complexify a model until it works perfectly otherwise we lose in efficiency.

That was one of the main issues i encountered in this project: i wanted to go too fast without respecting all the successive steps.

The nonlinear aspect in the shallow water equations, made me observe the unstabilities and difficulties that are linked to such nonlinear cases (unstability for very shallow bathymetry).

This project is the first one that made me really focus on the computational cost, as i saw the limit of my personal laptop in term of computational power. That was visible in the 2D cases with explicit scheme but also i saw the difference in computational effort between an explicit scheme and an implicit scheme.

7 ANNEX

7.1 A : 1D linear explicit Von Neumann stability analysis

Von Neumann stability analysis [12] is based on Fourier decomposition and can only be used for linear partial differential equations (PDE).
It writes that the error satisfies the discretized equation and can be written as,

$$\epsilon(x,t) = A(t)e^{ikx} \tag{37}$$

Inserting the error in (16) gives,

$$\begin{aligned} \epsilon(x,t+\Delta t) =& C_0^2 \Delta t^2 \frac{\epsilon(x+\Delta x,t) - 2\epsilon(x,t) + \epsilon(x-\Delta x,t)}{\Delta x^2} \\ &+ 2\epsilon(x,t) - \epsilon(x,t-\Delta t) \end{aligned} \tag{38}$$

With,

$\epsilon(x+\Delta x,t) = A(t)e^{ik(x+\Delta x)}$
$\epsilon(x-\Delta x,t) = A(t)e^{ik(x-\Delta x)}$
$\epsilon(x,t+\Delta t) = A(t+\Delta t)e^{ikx}$
$\epsilon(x,t-\Delta t) = A(t-\Delta t)e^{ikx}$

$A(t)$ gives the amplitude of the error at each time step, thus we call $G = \frac{A(t+\Delta t)}{A(t)}$ the amplification factor and its absolute value must be below or equal to 1 to ensure numerical stability of the scheme.

The stability analysis provide a stability condition that gives us the couples Δx, Δt that make the model stable ($G \leq 1$).

We write for more convenience, $\alpha = C_0 \frac{\Delta t}{\Delta x}$.

$$\begin{aligned} A(t+\Delta t)e^{ikx} =& \alpha^2 (A(t)e^{ik(x+\Delta x)} - 2A(t)e^{ikx} + A(t)e^{ik(x-\Delta x)}) \\ &+ 2A(t)e^{ikx} - A(t-\Delta t)e^{ikx} \end{aligned} \tag{39}$$

We divide equation (39) by $A(t-\Delta t)$ and the e^{ikx} terms cancel out.

$$\begin{aligned} \frac{A(t+\Delta t)}{A(t-\Delta t)} =& \alpha^2 (\frac{A(t)}{A(t-\Delta t)} e^{ik\Delta x} - 2\frac{A(t)}{A(t-\Delta t)} + \frac{A(t)}{A(t-\Delta t)} e^{ik\Delta x}) \\ &+ 2\frac{A(t)}{A(t-\Delta t)} - 1 \end{aligned} \tag{40}$$

Writing $G = \frac{A(t)}{A(t-\Delta t)}$ and $G^2 = \frac{A(t+\Delta t)}{A(t-\Delta t)}$ we get,

$$G^2 = \alpha^2 (Ge^{ik\Delta x} - 2G + Ge^{ik\Delta x}) + 2G - 1 \tag{41}$$

We recognize the cosine function through euler formula $cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2}$.

$$G^2 = \alpha^2 (2Gcos(k\Delta x) - 2G) + 2G - 1 \tag{42}$$

This can be written as,

$$G^2 - 2\beta G + 1 = 0 \tag{43}$$

With, $\beta = 1 - 2\alpha^2 sin^2(\frac{k\Delta x}{2})$

This is a quadratic equation, with :
$a = 1$
$b = -2\beta$
$c = 1$

The discriminant writes,

$$\Delta = b^2 - 4ac = 4\beta^2 - 4 \tag{44}$$

$$\begin{cases} \Delta \leq 0, -1 \leq \beta \leq 1 \\ \Delta > 0, \beta < -1 \cup \beta > 1 \end{cases}$$

For $\beta < -1 \cup \beta > 1$, the roots are real and write,

$$R1 = \frac{-b - \sqrt{\Delta}}{2a} = \frac{2\beta - \sqrt{4\beta^2 - 4}}{2} = \beta - \sqrt{\beta^2 - 1} \tag{45}$$

$$R2 = \frac{-b + \sqrt{\Delta}}{2a} = \frac{2\beta + \sqrt{4\beta^2 - 4}}{2} = \beta + \sqrt{\beta^2 - 1} \tag{46}$$

One of the roots is always greater than 1, so it's unstable for $\beta < -1 \cup \beta > 1$

For $-1 \leq \beta \leq 1$, the roots are complex and write,

$$C1 = \frac{-b - i\sqrt{-\Delta}}{2a} = \frac{2\beta - i\sqrt{-4\beta^2 + 4}}{2} = \beta - i\sqrt{-\beta^2 + 1} \tag{47}$$

$$C2 = \frac{-b + i\sqrt{-\Delta}}{2a} = \frac{2\beta + i\sqrt{-4\beta^2 + 4}}{2} = \beta + i\sqrt{-\beta^2 + 1} \tag{48}$$

The module of the complex root C1 and C2 writes,

$$\begin{aligned} \|C1\| = \|C2\| &= \sqrt{\Re(C1)^2 + \Im(C1)^2} \\ &= \sqrt{\beta^2 - \beta^2 + 1} \\ &= 1 \end{aligned} \tag{49}$$

The norm of the two complex roots is equal to 1 for $-1 \leq \beta \leq 1$, so it's stable on this interval (figure 12).

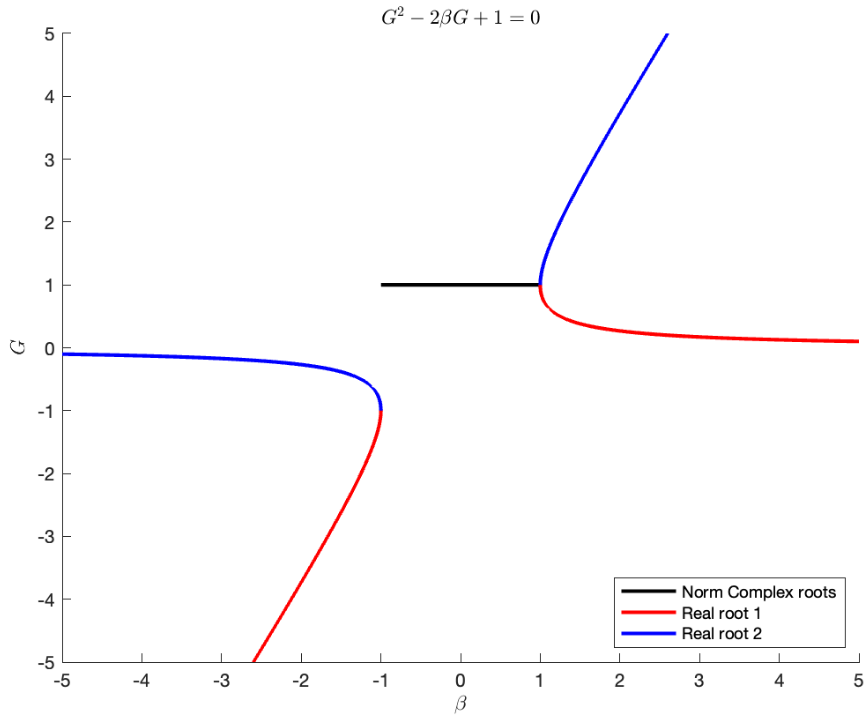


Figure 12: Roots of $G^2 - 2\beta G + 1 = 0$

Since $\beta = 1 - 2\alpha^2 sin^2(\frac{k\Delta x}{2})$, the stability condition is,

$$-1 \leq 1 - 2\alpha^2 sin^2(\frac{k\Delta x}{2}) \leq 1 \tag{50}$$

Therefore the scheme is conditionally stable,

$$C_0 \leq \frac{\Delta x}{\Delta t} \tag{51}$$

7.2 B : 1D linear explicit Numerical dispersion

We consider a plane wave solution [4] : $P = P_0 e^{-i(\omega t + kx)}$.

We inject this solution into the equation (15).

It writes,

$$\begin{aligned} &\frac{P_0 e^{-i(\omega(t+\Delta t)+kx)} - 2P_0 e^{-i(\omega t+kx)} + P_0 e^{-i(\omega(t-\Delta t)+kx)}}{\Delta t^2} \\ -C_0^2 &\frac{P_0 e^{-i(\omega t+k(x+\Delta x))} - 2P_0 e^{-i(\omega t+kx)} + P_0 e^{-i(\omega t+k(x-\Delta x))}}{\Delta x^2} = 0 \end{aligned} \tag{52}$$

We simplify the equation by $P_0 e^{-i(\omega t+kx)}$,

$$e^{-i\omega\Delta t} - 2 + e^{i\omega\Delta t} - \frac{C_0^2\Delta t^2}{\Delta x^2}(e^{-ik\Delta x} - 2 + e^{ik\Delta x}) = 0 \tag{53}$$

Using Euler's formula this can be written as,

$$cos(\omega\Delta t) - 1 = \frac{C_0^2\Delta t^2}{\Delta x^2}(cos(k\Delta x) - 1) \tag{54}$$

We use the following trigonometry property,

$$sin(\frac{x}{2}) = \sqrt{\frac{1 - cos(x)}{2}} \tag{55}$$

We get,

$$sin(\frac{\omega\Delta t}{2}) = \frac{C_0\Delta t}{\Delta x} sin(\frac{k\Delta x}{2}) \tag{56}$$

We isolate ω,

$$\omega = \frac{2}{\Delta t} sin^{-1}(\frac{C_0\Delta t}{\Delta x} sin(\frac{k\Delta x}{2})) \tag{57}$$

Since the phase velocity $C_k = \frac{\omega}{k}$ we write,

$$C_k = \frac{2}{k\Delta t} sin^{-1}(\frac{C_0\Delta t}{\Delta x} sin(\frac{k\Delta x}{2})) \tag{58}$$

This provides the Numerical dispersion relation of the numerical scheme .

7.3 C : 1D linear implicit Von Neumann stability analysis

Von Neumann stability analysis [12] is based on Fourier decomposition and can only be used for linear partial differential equations (PDE).
It writes that the error satisfies the discretized equation and can be written as,

$$\epsilon(x, t) = A(t)e^{ikx} \tag{59}$$

Injecting in the eq (22) gives,

$$\begin{aligned}
&-\alpha\epsilon(x + \Delta x, t + \Delta t) + (2\alpha + 1)\epsilon(x, t + \Delta t) - \alpha\epsilon(x - \Delta x, t + \Delta t) = \\
&\alpha\epsilon(x + \Delta x, t - \Delta t) - (1 + 2\alpha)\epsilon(x, t - \Delta t) + \alpha\epsilon(x - \Delta x, t - \Delta t) \\
&+ 2\alpha\epsilon(x + \Delta x, t) + (2 - 4\alpha)\epsilon(x, t) + 2\alpha\epsilon(x - \Delta x, t)
\end{aligned} \tag{60}$$

With, $\alpha = \frac{C_0^2\Delta t^2}{4\Delta x^2}$.

$A(t)$ gives the amplitude of the error at each time step, thus we call $G = \frac{A(t+\Delta t)}{A(t)}$ the amplification factor and its absolute value must be below or equal to 1 to ensure numerical stability of the scheme.

The stability analysis provide a stability condition that gives us the couples Δx, Δt (the α values) that make the model stable ($G \leq 1$).

$$\begin{aligned}
&-\alpha A(t+\Delta t)e^{ik(x+\Delta x)} + (2\alpha+1)A(t+\Delta t)e^{ikx} - \alpha A(t+\Delta t)e^{ik(x-\Delta x)} = \\
&\alpha A(t-\Delta t)e^{ik(x+\Delta x)} - (1+2\alpha)A(t-\Delta t)e^{ikx} + \alpha A(t-\Delta t)e^{ik(x-\Delta x)} \\
&+ 2\alpha A(t)e^{ik(x+\Delta x)} + (2-4\alpha)A(t)e^{ikx} + 2\alpha A(t)e^{ik(x-\Delta x)}
\end{aligned} \tag{61}$$

We divide equation (61) by $A(t-\Delta t)$ and the e^{ikx} terms cancel out.

$$\begin{aligned}
&-\alpha\frac{A(t+\Delta t)}{A(t-\Delta t)}e^{ik\Delta x} + (2\alpha+1)\frac{A(t+\Delta t)}{A(t-\Delta t)} - \alpha\frac{A(t+\Delta t)}{A(t-\Delta t)}e^{-ik\Delta x} = \\
&\alpha e^{ik\Delta x} - (1+2\alpha) + \alpha e^{-ik\Delta x} \\
&+ 2\alpha\frac{A(t)}{A(t-\Delta t)}e^{ik\Delta x} + (2-4\alpha)\frac{A(t)}{A(t-\Delta t)} + 2\alpha\frac{A(t)}{A(t-\Delta t)}e^{-ik\Delta x}
\end{aligned} \tag{62}$$

Writing $G = \frac{A(t)}{A(t-\Delta t)}$ and $G^2 = \frac{A(t+\Delta t)}{A(t-\Delta t)}$ we get,

$$\begin{aligned}
&-\alpha G^2 e^{ik\Delta x} + (2\alpha+1)G^2 - \alpha G^2 e^{-ik\Delta x} = \\
&\alpha e^{ik\Delta x} - (1+2\alpha) + \alpha e^{-ik\Delta x} + 2\alpha G e^{ik\Delta x} + (2-4\alpha)G + 2\alpha G e^{-ik\Delta x}
\end{aligned} \tag{63}$$

We recognize the cosine function through euler formula $cos(\theta) = \frac{e^{i\theta}+e^{-i\theta}}{2}$.

$$G^2(2\alpha+1-2\alpha cos(k\Delta x)) + G(4\alpha-2-4\alpha cos(k\Delta x)) + 2\alpha+1-2\alpha cos(k\Delta x) = 0 \tag{64}$$

We write, $\beta = 2\alpha + 1 - 2\alpha cos(k\Delta x)$.

Then eq (61) writes,

$$\beta G^2 + (2\beta - 4)G + \beta = 0 \tag{65}$$

This is a quadratic equation, with :
$a = \beta$
$b = 2\beta - 4$
$c = \beta$

The discriminant writes,

$$\Delta = b^2 - 4ac = 4\beta^2 - 16\beta + 16 - 4\beta^2 = -16\beta + 16 \tag{66}$$

$$\begin{cases} \Delta \leq 0, \beta \geq 1 \\ \Delta > 0, \beta < 1 \end{cases}$$

For $\beta < 1$, the roots are real and write,

$$R1 = \frac{-b - \sqrt{\Delta})}{2a} = \frac{4 - 2\beta - \sqrt{-16\beta + 16})}{2\beta} \tag{67}$$

$$R2 = \frac{-b + \sqrt{\Delta})}{2a} = \frac{4 - 2\beta + \sqrt{-16\beta + 16})}{2\beta} \tag{68}$$

One of the roots is always greater than 1, so it's unstable for $\beta < 1$

For $\beta \geq 1$, the roots are complex and write

$$C1 = \frac{-b - i\sqrt{-\Delta})}{2a} = \frac{4 - 2\beta - i\sqrt{16\beta - 16})}{2\beta} \tag{69}$$

$$C2 = \frac{-b + i\sqrt{-\Delta})}{2a} = \frac{4 - 2\beta + i\sqrt{16\beta - 16})}{2\beta} \tag{70}$$

The module of the complex root C1 and C2 writes,

$$\begin{aligned} \|C1\| = \|C2\| &= \sqrt{\Re(C1)^2 + \Im(C1)^2} \\ &= \sqrt{(\frac{4 - 2\beta}{2\beta})^2 + (\frac{\sqrt{16\beta - 16}}{2\beta})^2} \\ &= \sqrt{\frac{16 - 16\beta + 4\beta^2}{4\beta^2} + \frac{16\beta - 16}{4\beta^2}} \\ &= 1 \end{aligned} \tag{71}$$

The norm of the two complex roots is equal to 1 for $\beta \geq 1$, so it's stable on this interval (figure 13).

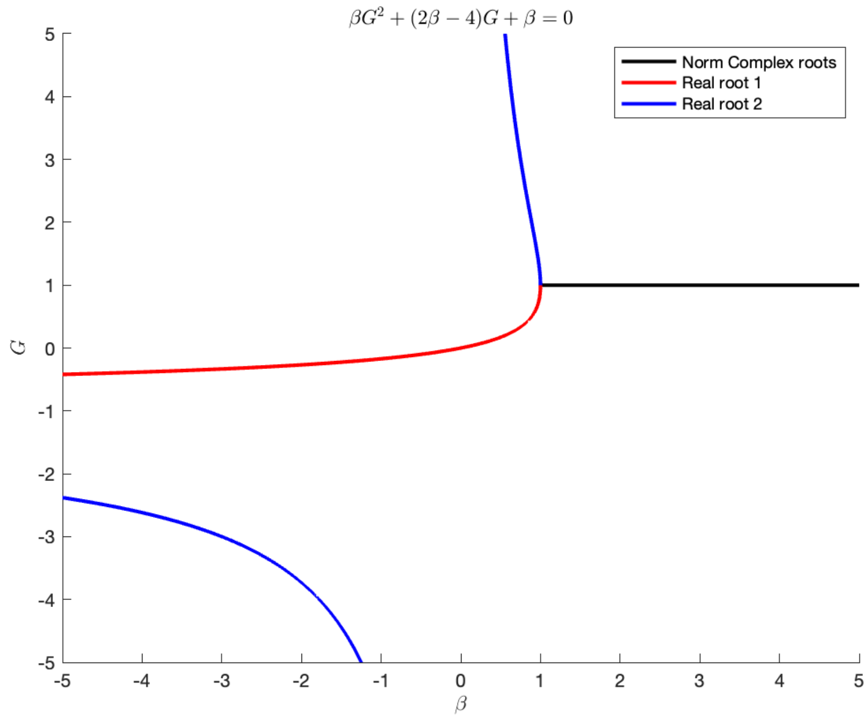


Figure 13: Roots of $\beta G^2 + (2\beta - 4)G + \beta = 0$

Since $\beta = 2\alpha + 1 - 2\alpha cos(k\Delta x)$, the stability condition is,

$$2\alpha + 1 - 2\alpha cos(k\Delta x) \geq 1 \tag{72}$$

$$1 \geq cos(k\Delta x) \tag{73}$$

Which is true for any Δx and k.
Therefore the scheme is unconditionally stable.

7.4 D : 1D linear implicit Numerical dispersion

We consider a plane wave solution [4] : $P = P_0 e^{-i(\omega t + kx)}$.

We inject this solution into the equation (20).

It writes,

$$\begin{aligned}
&\frac{P_0 e^{-i(\omega(t+\Delta t)+kx)} - 2P_0 e^{-i(\omega t+kx)} + P_0 e^{-i(\omega(t-\Delta t)+kx)}}{\Delta t^2} \\
-\frac{C_0^2}{4}(&\frac{P_0 e^{-i(\omega(t-\Delta t)+k(x+\Delta x))} - 2P_0 e^{-i(\omega(t-\Delta t)+kx)} + P_0 e^{-i(\omega(t-\Delta t)+k(x-\Delta x))}}{\Delta x^2} \\
&+ 2\frac{P_0 e^{-i(\omega t+k(x+\Delta x))} - 2P_0 e^{-i(\omega t+kx)} + P_0 e^{-i(\omega t+k(x-\Delta x))}}{\Delta x^2} \\
+&\frac{P_0 e^{-i(\omega(t-\Delta t)+k(x+\Delta x))} - 2P_0 e^{-i(\omega(t-\Delta t)+kx)} + P_0 e^{-i(\omega(t-\Delta t)+k(x-\Delta x))}}{\Delta x^2}) = 0
\end{aligned} \tag{74}$$

We simplify the equation by $P_0 e^{-i(\omega t + kx)}$,

$$\begin{aligned}&\frac{e^{-i\omega\Delta t}-2+e^{i\omega\Delta t}}{\Delta t^2}-\frac{C_0^2}{4}(\frac{e^{-i(-\omega\Delta t+k\Delta x)}-2e^{i\omega\Delta t}+e^{i(\omega\Delta t+k\Delta x)}}{\Delta x^2}\\&+\frac{2e^{-ik\Delta x}-4+2e^{ik\Delta x}}{\Delta x^2}+\frac{e^{-i(\omega\Delta t+k\Delta x)}-2e^{-i\omega\Delta t}+e^{-i(\omega\Delta t-k\Delta x)}}{\Delta x^2})=0\end{aligned} \tag{75}$$

Writing $\beta = \frac{C_0 \Delta t}{\Delta x}$,

$$\begin{aligned}&e^{-i\omega\Delta t}-2+e^{i\omega\Delta t}-\frac{\beta^2}{4}e^{-i(-\omega\Delta t+k\Delta x)}+\frac{\beta^2}{2}e^{i\omega\Delta t}-\frac{\beta^2}{4}e^{i(\omega\Delta t+k\Delta x)}-\frac{\beta^2}{2}e^{-ik\Delta x}\\&+\beta^2-\frac{\beta^2}{2}e^{ik\Delta x}-\frac{\beta^2}{4}e^{-i(\omega\Delta t+k\Delta x)}+\frac{\beta^2}{2}e^{-i\omega\Delta t}-\frac{\beta^2}{4}e^{-i(\omega\Delta t-k\Delta x)}=0\end{aligned} \tag{76}$$

$$\begin{aligned}&2cos(\omega\Delta t)+\beta^2cos(\omega\Delta t)-\beta^2cos(k\Delta x)\\&-\frac{\beta^2}{2}cos(\omega\Delta t+k\Delta x)-\frac{\beta^2}{2}cos(\omega\Delta t-k\Delta x)=2-\beta^2\end{aligned} \tag{77}$$

$$\begin{aligned}&2cos(\omega\Delta t)+\beta^2cos(\omega\Delta t)-\beta^2cos(k\Delta x)-\frac{\beta^2}{2}(cos(\omega\Delta t)cos(k\Delta x)-sin(\omega\Delta t)sin(k\Delta x))\\&-\frac{\beta^2}{2}(cos(\omega\Delta t)cos(k\Delta x)+sin(\omega\Delta t)sin(k\Delta x))=2-\beta^2\end{aligned} \tag{78}$$

$$2cos(\omega\Delta t)+\beta^2cos(\omega\Delta t)-\beta^2cos(\omega\Delta t)cos(k\Delta x)=2-\beta^2+\beta^2cos(k\Delta x) \tag{79}$$

$$cos(\omega\Delta t)(2+\beta^2-\beta^2cos(k\Delta x))=2-\beta^2+\beta^2cos(k\Delta x) \tag{80}$$

$$cos(\omega\Delta t)=\frac{2-\beta^2+\beta^2cos(k\Delta x)}{2+\beta^2-\beta^2cos(k\Delta x)} \tag{81}$$

$$\omega=cos^{-1}(\frac{2-\beta^2+\beta^2cos(k\Delta x)}{2+\beta^2-\beta^2cos(k\Delta x)})\times\frac{1}{\Delta t} \tag{82}$$

Since the phase velocity $C_k = \frac{\omega}{k}$ we write,

$$C_k=cos^{-1}(\frac{2-\beta^2+\beta^2cos(k\Delta x)}{2+\beta^2-\beta^2cos(k\Delta x)})\times\frac{1}{k\Delta t} \tag{83}$$

This is the dispersion relation of the implicit scheme.,

7.5 E : 1D nonlinear explicit scheme

We apply the two-steps Lax-Wendroff scheme on the 1D nonlinear shallow water system.

First step, we calculate the orange points at time n+1/2 from the yellow points at time n (figure 8).

$$\begin{cases} \eta_{j+1/2}^{n+1/2} = \frac{1}{2}(\eta_{j+1}^n + \eta_j^n) - \frac{\Delta t}{2\Delta x}(M_{j+1}^n - M_j^n) \\ \eta_{j-1/2}^{n+1/2} = \frac{1}{2}(\eta_j^n + \eta_{j-1}^n) - \frac{\Delta t}{2\Delta x}(M_j^n - M_{j-1}^n) \\ M_{j+1/2}^{n+1/2} = \dfrac{1}{2}(M_{j+1}^n + M_j^n) - \dfrac{\Delta t}{2\Delta x}(\dfrac{{M_{j+1}^n}^2}{D_{j+1}^n} - \dfrac{M_j^{n2}}{D_j^n}) \\ \qquad - \dfrac{g\Delta t}{4\Delta x}(D_{j+1}^n + D_j^n)(\eta_{j+1}^n - \eta_j^n) - \dfrac{gm^2\Delta t}{4}(\dfrac{{M_{j+1}^n}^2}{{D_{j+1}^n}^{7/3}} + \dfrac{M_j^{n2}}{D_j^{n7/3}}) \\ M_{j-1/2}^{n+1/2} = \dfrac{1}{2}(M_j^n + M_{j-1}^n) - \dfrac{\Delta t}{2\Delta x}(\dfrac{M_j^{n2}}{D_j^n} - \dfrac{{M_{j-1}^n}^2}{D_{j-1}^n}) \\ \qquad - \dfrac{g\Delta t}{4\Delta x}(D_j^n + D_{j-1}^n)(\eta_j^n - \eta_{j-1}^n) - \dfrac{gm^2\Delta t}{4}(\dfrac{M_j^{n2}}{D_j^{n7/3}} + \dfrac{{M_{j-1}^n}^2}{{D_{j-1}^n}^{7/3}}) \end{cases}$$

Second step, we calculate the red point at time n+1 from the orange points at time n+1/2 and the central yellow point at time n (figure 8).

$$\begin{cases} \eta_j^{n+1} = \eta_j^n - \frac{\Delta t}{\Delta x}(M_{j+1/2}^{n+1/2} - M_{j-1/2}^{n+1/2}) \\ M_j^{n+1} = M_j^n - \dfrac{\Delta t}{\Delta x}(\dfrac{{M_{j+1/2}^{n+1/2}}^2}{D_{j+1/2}^{n+1/2}} - \dfrac{{M_{j-1/2}^{n+1/2}}^2}{D_{j-1/2}^{n+1/2}}) \\ \qquad - \dfrac{g\Delta t}{2\Delta x}(D_{j+1/2}^{n+1/2} + D_{j-1/2}^{n+1/2})(\eta_{j+1/2}^{n+1/2} - \eta_{j-1/2}^{n+1/2}) - \dfrac{gm^2\Delta t}{2}(\dfrac{{M_{j+1/2}^{n+1/2}}^2}{{D_{j+1/2}^{n+1/2}}^{7/3}} + \dfrac{{M_{j-1/2}^{n+1/2}}^2}{{D_{j-1/2}^{n+1/2}}^{7/3}}) \end{cases}$$

7.6 F : Full Jacobian description

Jacobian matrix for free displacement null velocity boundary condition,

$$J = \begin{pmatrix} J_{F1}^{\eta} & J_{F1}^{M} \\ J_{F2}^{\eta} & J_{F2}^{M} \end{pmatrix} \tag{84}$$

With,

$$J_{F1}^{\eta} = \begin{pmatrix} \frac{\partial F_1}{\partial \eta_j} & 2\frac{\partial F_1}{\partial \eta_{j+1}} & 0 & 0 & ... & ... \\ \frac{\partial F_1}{\partial \eta_{j-1}} & \frac{\partial F_1}{\partial \eta_j} & \frac{\partial F_1}{\partial \eta_{j+1}} & 0 & ... & ... \\ 0 & \frac{\partial F_1}{\partial \eta_{j-1}} & \frac{\partial F_1}{\partial \eta_j} & \frac{\partial F_1}{\partial \eta_{j+1}} & ... & ... \\ ... & ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... & ... \\ ... & ... & 0 & \frac{\partial F_1}{\partial \eta_{j-1}} & \frac{\partial F_1}{\partial \eta_j} & \frac{\partial F_1}{\partial \eta_{j+1}} \\ ... & ... & 0 & 0 & 2\frac{\partial F_1}{\partial \eta_{j-1}} & \frac{\partial F_1}{\partial \eta_j} \end{pmatrix} \tag{85}$$

$$\begin{cases} \frac{\partial F_1}{\partial \eta_{j-1}} = 0 \\ \frac{\partial F_1}{\partial \eta_j} = 1 \\ \frac{\partial F_1}{\partial \eta_{j+1}} = 0 \end{cases}$$

$$J_{F1}^{M}=\begin{pmatrix} \frac{\partial F_1}{\partial M_j} & \frac{\partial F_1}{\partial M_{j+1}} & 0 & 0 & ... & ... \\ \frac{\partial F_1}{\partial M_{j-1}} & \frac{\partial F_1}{\partial M_j} & \frac{\partial F_1}{\partial M_{j+1}} & 0 & ... & ... \\ 0 & \frac{\partial F_1}{\partial M_{j-1}} & \frac{\partial F_1}{\partial M_j} & \frac{\partial F_1}{\partial M_{j+1}} & ... & ... \\ ... & ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... & ... \\ ... & ... & 0 & \frac{\partial F_1}{\partial M_{j-1}} & \frac{\partial F_1}{\partial M_j} & \frac{\partial F_1}{\partial M_{j+1}} \\ ... & ... & 0 & 0 & \frac{\partial F_1}{\partial M_{j-1}} & \frac{\partial F_1}{\partial M_j} \end{pmatrix} \tag{86}$$

$$\begin{cases} \frac{\partial F_1}{\partial M_{j-1}} = -\alpha \\ \frac{\partial F_1}{\partial M_j} = 0 \\ \frac{\partial F_1}{\partial M_{j+1}} = \alpha \end{cases}$$

$$J_{F2}^{\eta}=\begin{pmatrix} \frac{\partial F_2}{\partial \eta_j} & 2\frac{\partial F_2}{\partial \eta_{j+1}} & 0 & 0 & ... & ... \\ \frac{\partial F_2}{\partial \eta_{j-1}} & \frac{\partial F_2}{\partial \eta_j} & \frac{\partial F_2}{\partial \eta_{j+1}} & 0 & ... & ... \\ 0 & \frac{\partial F_2}{\partial \eta_{j-1}} & \frac{\partial F_2}{\partial \eta_j} & \frac{\partial F_2}{\partial \eta_{j+1}} & ... & ... \\ ... & ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... & ... \\ ... & ... & 0 & \frac{\partial F_2}{\partial \eta_{j-1}} & \frac{\partial F_2}{\partial \eta_j} & \frac{\partial F_2}{\partial \eta_{j+1}} \\ ... & ... & 0 & 0 & 2\frac{\partial F_2}{\partial \eta_{j-1}} & \frac{\partial F_2}{\partial \eta_j} \end{pmatrix} \tag{87}$$

$$\begin{cases} \frac{\partial F_2}{\partial \eta_{j-1}} = \alpha {M_{j-1}^{n+1}}^2 (H_{j-1}+\eta_{j-1}^{n+1})^{-2} - g\alpha(H_j+\eta_j^{n+1}) \\ \frac{\partial F_2}{\partial \eta_j} = g\alpha(\eta_{j+1}^{n+1}-\eta_{j-1}^{n+1}) - \frac{7gm^2\Delta t}{6}\frac{{M_j^{n+1}}^2}{{D_j^{n+1}}^{10/3}} \\ \frac{\partial F_2}{\partial \eta_{j+1}} = -\alpha {M_{j+1}^{n+1}}^2 (H_{j+1}+\eta_{j+1}^{n+1})^{-2} + g\alpha(H_j+\eta_j^{n+1}) \end{cases}$$

$$J_{F2}^{M}=\begin{pmatrix} \frac{\partial F_2}{\partial M_j} & \frac{\partial F_2}{\partial M_{j+1}} & 0 & 0 & ... & ... \\ \frac{\partial F_2}{\partial M_{j-1}} & \frac{\partial F_2}{\partial M_j} & \frac{\partial F_2}{\partial M_{j+1}} & 0 & ... & ... \\ 0 & \frac{\partial F_2}{\partial M_{j-1}} & \frac{\partial F_2}{\partial M_j} & \frac{\partial F_2}{\partial M_{j+1}} & ... & ... \\ ... & ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... & ... \\ ... & ... & 0 & \frac{\partial F_2}{\partial M_{j-1}} & \frac{\partial F_2}{\partial M_j} & \frac{\partial F_2}{\partial M_{j+1}} \\ ... & ... & 0 & 0 & \frac{\partial F_2}{\partial M_{j-1}} & \frac{\partial F_2}{\partial M_j} \end{pmatrix} \tag{88}$$

$$\begin{cases} \frac{\partial F_2}{\partial M_{j-1}} = -2\alpha \frac{M_{j-1}^{n+1}}{H_{j-1}+\eta_{j-1}^{n+1}} \\ \frac{\partial F_2}{\partial M_j} = 1 + gm^2\Delta t \frac{M_j^{n+1}}{{D_j^{n+1}}^{7/3}} \\ \frac{\partial F_2}{\partial M_{j+1}} = 2\alpha \frac{M_{j+1}^{n+1}}{H_{j+1}+\eta_{j+1}^{n+1}} \end{cases}$$

7.7 G : 2D nonlinear explicit scheme

We apply the two-steps Lax-Wendroff scheme on the 2D nonlinear shallow water equations.

First step, we calculate the orange points at time n+1/2 from the yellow points at time n (Figure 10).

$$\begin{cases} \eta_{i+1,j}^{n+1/2} = \frac{1}{4}(\eta_{i,j}^n + \eta_{i+2,j}^n + \eta_{i+1,j+1}^n + \eta_{i+1,j-1}^n) - \Delta t \frac{M_{i+1,j+1}^n - M_{i+1,j-1}^n}{4\Delta x} - \Delta t \frac{N_{i+2,j}^n - N_{i,j}^n}{4\Delta y} \\ \eta_{i-1,j}^{n+1/2} = \frac{1}{4}(\eta_{i,j}^n + \eta_{i-2,j}^n + \eta_{i-1,j+1}^n + \eta_{i-1,j-1}^n) - \Delta t \frac{M_{i-1,j+1}^n - M_{i-1,j-1}^n}{4\Delta x} - \Delta t \frac{N_{i,j}^n - N_{i-2,j}^n}{4\Delta y} \\ \eta_{i,j+1}^{n+1/2} = \frac{1}{4}(\eta_{i,j}^n + \eta_{i,j+2}^n + \eta_{i+1,j+1}^n + \eta_{i-1,j+1}^n) - \Delta t \frac{M_{i,j+2}^n - M_{i,j}^n}{4\Delta x} - \Delta t \frac{N_{i+1,j+1}^n - N_{i-1,j+1}^n}{4\Delta y} \\ \eta_{i,j-1}^{n+1/2} = \frac{1}{4}(\eta_{i,j}^n + \eta_{i,j-2}^n + \eta_{i+1,j-1}^n + \eta_{i-1,j-1}^n) - \Delta t \frac{M_{i,j}^n - M_{i,j-2}^n}{4\Delta x} - \Delta t \frac{N_{i+1,j-1}^n - N_{i-1,j-1}^n}{4\Delta y} \end{cases}$$

$$\begin{cases} M_{i+1,j}^{n+1/2} = \dfrac{1}{4}(M_{i,j}^n + M_{i+2,j}^n + M_{i+1,j+1}^n + M_{i+1,j-1}^n) - \dfrac{\Delta t}{4\Delta x}(\dfrac{{M_{i+1,j+1}^n}^2}{D_{i+1,j+1}^n} - \dfrac{{M_{i+1,j-1}^n}^2}{D_{i+1,j-1}^n}) \\ - \dfrac{\Delta t}{4\Delta y}(\dfrac{M_{i+2,j}^n N_{i+2,j}^n}{D_{i+2,j}^n} - \dfrac{M_{i,j}^n N_{i,j}^n}{D_{i,j}^n}) - \dfrac{g\Delta t D_{i+1,j}^n}{4\Delta x}(\eta_{i+1,j+1}^n - \eta_{i+1,j-1}^n) \\ - \dfrac{gm^2\Delta t}{2{D_{i+1,j}^n}^{7/3}} M_{i+1,j}^n \sqrt{{M_{i+1,j}^n}^2 + {N_{i+1,j}^n}^2} \\ \\ M_{i-1,j}^{n+1/2} = \dfrac{1}{4}(M_{i,j}^n + M_{i-2,j}^n + M_{i-1,j+1}^n + M_{i-1,j-1}^n) - \dfrac{\Delta t}{4\Delta x}(\dfrac{{M_{i-1,j+1}^n}^2}{D_{i-1,j+1}^n} - \dfrac{{M_{i-1,j-1}^n}^2}{D_{i-1,j-1}^n}) \\ - \dfrac{\Delta t}{4\Delta y}(\dfrac{M_{i,j}^n N_{i,j}^n}{D_{i,j}^n} - \dfrac{M_{i-2,j}^n N_{i-2,j}^n}{D_{i-2,j}^n}) - \dfrac{g\Delta t D_{i-1,j}^n}{4\Delta x}(\eta_{i-1,j+1}^n - \eta_{i-1,j-1}^n) \\ - \dfrac{gm^2\Delta t}{2{D_{i-1,j}^n}^{7/3}} M_{i-1,j}^n \sqrt{{M_{i-1,j}^n}^2 + {N_{i-1,j}^n}^2} \\ \\ M_{i,j+1}^{n+1/2} = \dfrac{1}{4}(M_{i,j}^n + M_{i,j+2}^n + M_{i+1,j+1}^n + M_{i-1,j+1}^n) - \dfrac{\Delta t}{4\Delta x}(\dfrac{{M_{i,j+2}^n}^2}{D_{i,j+2}^n} - \dfrac{{M_{i,j}^n}^2}{D_{i,j}^n}) \\ - \dfrac{\Delta t}{4\Delta y}(\dfrac{M_{i+1,j+1}^n N_{i+1,j+1}^n}{D_{i+1,j+1}^n} - \dfrac{M_{i-1,j+1}^n N_{i-1,j+1}^n}{D_{i-1,j+1}^n}) - \dfrac{g\Delta t D_{i,j+1}^n}{4\Delta x}(\eta_{i,j+2}^n - \eta_{i,j}^n) \\ - \dfrac{gm^2\Delta t}{2{D_{i,j+1}^n}^{7/3}} M_{i,j+1}^n \sqrt{{M_{i,j+1}^n}^2 + {N_{i,j+1}^n}^2} \\ \\ M_{i,j-1}^{n+1/2} = \dfrac{1}{4}(M_{i,j}^n + M_{i,j-2}^n + M_{i+1,j-1}^n + M_{i-1,j-1}^n) - \dfrac{\Delta t}{4\Delta x}(\dfrac{{M_{i,j}^n}^2}{D_{i,j}^n} - \dfrac{{M_{i,j-2}^n}^2}{D_{i,j-2}^n}) \\ - \dfrac{\Delta t}{4\Delta y}(\dfrac{M_{i+1,j-1}^n N_{i+1,j-1}^n}{D_{i+1,j-1}^n} - \dfrac{M_{i-1,j-1}^n N_{i-1,j-1}^n}{D_{i-1,j-1}^n}) - \dfrac{g\Delta t D_{i,j-1}^n}{4\Delta x}(\eta_{i,j}^n - \eta_{i,j-2}^n) \\ - \dfrac{gm^2\Delta t}{2{D_{i,j-1}^n}^{7/3}} M_{i,j-1}^n \sqrt{{M_{i,j-1}^n}^2 + {N_{i,j-1}^n}^2} \end{cases}$$

$$
\begin{cases}
N^{n+1/2}_{i+1,j} = \frac{1}{4}(N^n_{i,j} + N^n_{i+2,j} + N^n_{i+1,j+1} + N^n_{i+1,j-1}) - \frac{\Delta t}{4\Delta x}(\frac{M^n_{i+1,j+1}N^n_{i+1,j+1}}{D^n_{i+1,j+1}} - \frac{M^n_{i+1,j-1}N^n_{i+1,j-1}}{D^n_{i+1,j-1}}) \\
-\frac{\Delta t}{4\Delta y}(\frac{{N^n_{i+2,j}}^2}{D^n_{i+2,j}} - \frac{{N^n_{i,j}}^2}{D^n_{i,j}})\frac{-g\Delta t D^n_{i+1,j}}{4\Delta y}(\eta^n_{i+2,j} - \eta^n_{i,j}) - \frac{gm^2\Delta t}{2{D^n_{i+1,j}}^{7/3}}N^n_{i+1,j}\sqrt{{M^n_{i+1,j}}^2 + {N^n_{i+1,j}}^2} \\
\\
N^{n+1/2}_{i-1,j} = \frac{1}{4}(N^n_{i,j} + N^n_{i-2,j} + N^n_{i-1,j+1} + N^n_{i-1,j-1}) - \frac{\Delta t}{4\Delta x}(\frac{M^n_{i-1,j+1}N^n_{i-1,j+1}}{D^n_{i-1,j+1}} - \frac{M^n_{i-1,j-1}N^n_{i-1,j-1}}{D^n_{i-1,j-1}}) \\
-\frac{\Delta t}{4\Delta y}(\frac{{N^n_{i,j}}^2}{D^n_{i,j}} - \frac{{N^n_{i-2,j}}^2}{D^n_{i-2,j}}) - \frac{g\Delta t D^n_{i-1,j}}{4\Delta y}(\eta^n_{i,j} - \eta^n_{i-2,j}) - \frac{gm^2\Delta t}{2{D^n_{i-1,j}}^{7/3}}N^n_{i-1,j}\sqrt{{M^n_{i-1,j}}^2 + {N^n_{i-1,j}}^2} \\
\\
N^{n+1/2}_{i,j+1} = \frac{1}{4}(N^n_{i,j} + N^n_{i,j+2} + N^n_{i+1,j+1} + N^n_{i-1,j+1}) - \frac{\Delta t}{4\Delta x}(\frac{M^n_{i,j+2}N^n_{i,j+2}}{D^n_{i,j+2}} - \frac{M^n_{i,j}N^n_{i,j}}{D^n_{i,j}}) \\
-\frac{\Delta t}{4\Delta y}(\frac{{N^n_{i+1,j+1}}^2}{D^n_{i+1,j+1}} - \frac{{N^n_{i-1,j+1}}^2}{D^n_{i-1,j+1}}) - \frac{g\Delta t D^n_{i,j+1}}{4\Delta y}(\eta^n_{i+1,j+1} - \eta^n_{i-1,j+1}) - \frac{gm^2\Delta t}{2{D^n_{i,j+1}}^{7/3}}N^n_{i,j+1}\sqrt{{M^n_{i,j+1}}^2 + {N^n_{i,j+1}}^2} \\
\\
N^{n+1/2}_{i,j-1} = \frac{1}{4}(N^n_{i,j} + N^n_{i,j-2} + N^n_{i+1,j-1} + N^n_{i-1,j-1}) - \frac{\Delta t}{4\Delta x}(\frac{M^n_{i,j}N^n_{i,j}}{D^n_{i,j}} - \frac{M^n_{i,j-2}N^n_{i,j-2}}{D^n_{i,j-2}}) \\
-\frac{\Delta t}{4\Delta y}(\frac{{N^n_{i+1,j-1}}^2}{D^n_{i+1,j-1}} - \frac{{N^n_{i-1,j-1}}^2}{D^n_{i-1,j-1}}) - \frac{g\Delta t D^n_{i,j-1}}{4\Delta y}(\eta^n_{i+1,j-1} - \eta^n_{i-1,j-1}) - \frac{g\Delta t m^2}{2{D^n_{i,j-1}}^{7/3}}N^n_{i,j-1}\sqrt{{M^n_{i,j-1}}^2 + {N^n_{i,j-1}}^2}
\end{cases}
$$

Second step, we calculate the red point at time n+1 from the orange points at time n+1/2 and the central yellow point at time n (Figure 10).

$$
\begin{cases}
\eta^{n+1}_{i,j} = \eta^n_{i,j} - \Delta t\frac{M^{n+1/2}_{i,j+1} - M^{n+1/2}_{i,j-1}}{2\Delta x} - \Delta t\frac{N^{n+1/2}_{i+1,j} - N^{n+1/2}_{i-1,j}}{2\Delta y} \\
M^{n+1}_{i,j} = M^n_{i,j} - \frac{\Delta t}{2\Delta x}(\frac{{M^{n+1/2}_{i,j+1}}^2}{D^{n+1/2}_{i,j+1}} - \frac{{M^{n+1/2}_{i,j-1}}^2}{D^{n+1/2}_{i,j-1}}) - \frac{\Delta t}{2\Delta y}(\frac{M^{n+1/2}_{i+1,j}N^{n+1/2}_{i+1,j}}{D^{n+1/2}_{i+1,j}} - \frac{M^{n+1/2}_{i-1,j}N^{n+1/2}_{i-1,j}}{D^{n+1/2}_{i-1,j}}) \\
\qquad - g\Delta t D^{n+1/2}_{i,j}(\frac{\eta^{n+1/2}_{i,j+1} - \eta^{n+1/2}_{i,j-1}}{2\Delta x}) - \frac{gm^2\Delta t}{{D^{n+1/2}_{i,j}}^{7/3}}M^{n+1/2}_{i,j}\sqrt{{M^{n+1/2}_{i,j}}^2 + {N^{n+1/2}_{i,j}}^2} \\
N^{n+1}_{i,j} = N^n_{i,j} - \frac{\Delta t}{2\Delta x}(\frac{M^{n+1/2}_{i,j+1}N^{n+1/2}_{i,j+1}}{D^{n+1/2}_{i,j+1}} - \frac{M^{n+1/2}_{i,j-1}N^{n+1/2}_{i,j-1}}{D^{n+1/2}_{i,j-1}}) - \frac{\Delta t}{2\Delta y}(\frac{{N^{n+1/2}_{i+1,j}}^2}{D^{n+1/2}_{i+1,j}} - \frac{{N^{n+1/2}_{i-1,j}}^2}{D^{n+1/2}_{i-1,j}}) \\
\qquad - g\Delta t D^{n+1/2}_{i,j}(\frac{\eta^{n+1/2}_{i+1,j} - \eta^{n+1/2}_{i-1,j}}{2\Delta y}) - \frac{gm^2\Delta t}{{D^{n+1/2}_{i,j}}^{7/3}}N^{n+1/2}_{i,j}\sqrt{{M^{n+1/2}_{i,j}}^2 + {N^{n+1/2}_{i,j}}^2}
\end{cases}
$$

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