

# Heat Flux and Entropy Produced by Thermal Fluctuations: Supplementary Informations

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## I. EXPERIMENTAL DETAILS

### A. Experimental set up

The electric systems and amplifiers are inside a Faraday cage and mounted on a floating optical table to reduce mechanical and acoustical noise. The resistance  $R_1$ , which is cooled by liquid Nitrogen vapors, changes of less than 0.1% in the whole temperature range. Its temperature is measured by a PT1000 which is inside the same shield of  $R_1$ . The signal  $V_1$  and  $V_2$  are amplified by two custom designed JFET amplifiers [1] with an input current of  $1pA$  and a noise of  $0.7nV\sqrt{Hz}$  at frequencies larger than  $1Hz$  and increases at  $8nV\sqrt{Hz}$  at  $0.1Hz$ . The resistances  $R_1$  and  $R_2$  have been used as input resistances of the amplifiers. The two signals  $V_1$  and  $V_2$  are amplified  $10^4$  times and the amplifier outputs are filtered (at  $4KHz$  to avoid aliasing) and acquired at  $8KHz$  by 24 bits-ADC. We used different sets of  $C_1$ ,  $C_2$  and  $C$ . The values of  $C_1$  and  $C_2$  are essentially set by the input capacitance of the amplifiers and by the cable length  $680pF < C_1 < 780pF$  and  $400pF < C_2 < 500pF$ . Instead  $C$  has been changed from  $100pF$  to  $1000pF$ . The system has always been calibrated in equilibrium at  $T_1 = T_2 = 296K$  using the FDT and estimating the spectrum using the values of the capacitances, see next sections.

### B. Noise spectrum of the amplifiers

The noise spectrum of the amplifiers  $A_1$  and  $A_2$  (Fig.1 of the main text), measured with a short circuit at the inputs, is plotted in fig.S.1a) and compared with the spectrum  $Sp_1$  of  $V_1$  at  $T_1 = 88K$ . We see that the useful signal is several order of magnitude larger than the amplifiers noise.

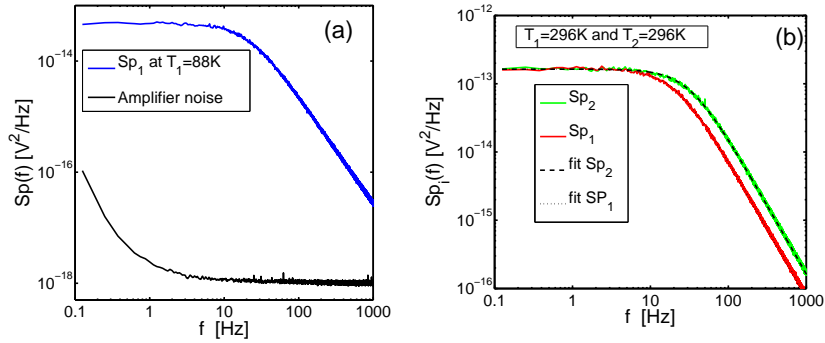


FIG. S.1: a) The power spectra  $Sp_1$  of  $V_1$  measured at  $T_1 = 88K$  (blue line) ( $C = 100pF$ ,  $C_1 = 680pF$ ,  $C_2 = 430pF$ ) is compared to the spectrum of the amplifier noise. b) The equilibrium spectra  $Sp_1$  (red line) and  $Sp_2$  (green line) measured at  $T_1 = T_2 = 296K$  are compared with prediction of eqs.S.1 and S.2 in order to check the values of the capacitances ( $C_1$ ,  $C_2$ ).

### C. Check of the calibration

The equilibrium spectra of  $V_1$  and  $V_2$  at  $T_1 = T_2$  used for calibration of the capacitances are:

$$S_1(\omega) = \frac{4k_B T_1 R_1 [1 + \omega^2 (C^2 R_1 R_2 + R_2^2 (C_2 + C)^2)]}{(1 - \omega^2 X R_1 R_2)^2 + \omega^2 Y^2} \quad (\text{S.1})$$

$$S_2(\omega) = \frac{4k_B T_2 R_2 [1 + \omega^2 (C^2 R_1 R_2 + R_1^2 (C_1 + C)^2)]}{(1 - \omega^2 X R_1 R_2)^2 + \omega^2 Y^2} \quad (\text{S.2})$$

where  $Y = [(C_1 + C)R_1 + (C_2 + C)R_2]$  and  $X = C_2 C_1 + C(C_1 + C_2)$ . This spectra can be easily obtained by applying FDT to the circuit of fig.1 in the main text.

The two computed spectra are compared to the measured ones in fig. S.1a). This comparison allows us to check the values of the capacitances  $C_1$  and  $C_2$  which depend on the cable length. We see that the agreement between the prediction and the measured power spectra is excellent and the global error on calibration is of the order of 1%. This corresponds exactly to the case discussed by Nyquist in which the two resistances at the same temperature are exchanging energy via an electric circuit ( $C$  in our case).

### D. The power spectra of $V_1$ and $V_2$ out-of-equilibrium

When  $T_1 \neq T_2$  the power spectra of  $V_1$  and  $V_2$  are:

$$S_1(\omega) = \frac{4k_B T_1 R_1 [1 + \omega^2 (C^2 R_1 R_2 + R_2^2 (C_2 + C)^2)]}{(1 - \omega^2 X R_1 R_2)^2 + \omega^2 Y^2} + \frac{4k_B (T_2 - T_1) \omega^2 C^2 R_1^2 R_2}{(1 - \omega^2 X R_1 R_2)^2 + \omega^2 Y^2} \quad (\text{S.3})$$

$$S_2(\omega) = \frac{4k_B T_2 R_2 [1 + \omega^2 (C^2 R_1 R_2 + R_1^2 (C_1 + C)^2)]}{(1 - \omega^2 X R_1 R_2)^2 + \omega^2 Y^2} + \frac{4k_B (T_1 - T_2) \omega^2 C^2 R_2^2 R_1}{(1 - \omega^2 X R_1 R_2)^2 + \omega^2 Y^2} \quad (\text{S.4})$$

These equations have been obtained by Fourier transforming eqs. S.7,S.8, solving for  $\tilde{V}_1(\omega)$  and  $\tilde{V}_2(\omega)$  and computing the modula. The integral of eqs. S.3 and S.4 gives the variances eq. S.24 directly computed from the distributions.

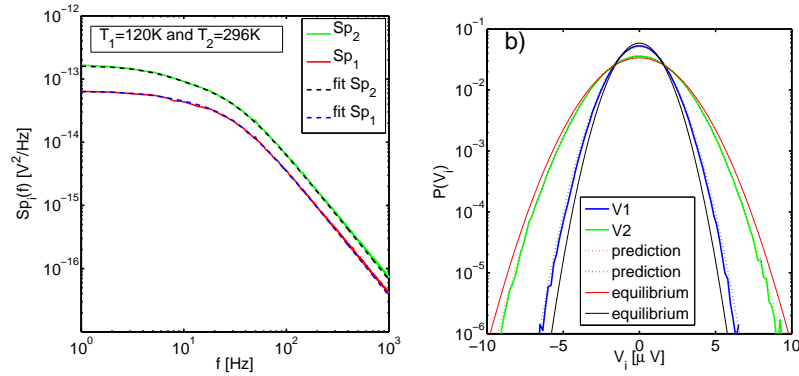


FIG. S.2: a) The power spectra  $Sp_1$  of  $V_1$  and  $Sp_2$  of  $V_2$  measured at  $T_1 = 120K$  and  $T_2 = 296K$  ( $C = 100pF$ ,  $C_1 = 680pF$ ,  $C_2 = 430pF$ ) are compared with the prediction of eq.S.3 and S.4 (dashed lines) b) The corresponding Probability Density Function  $P(V_1)$  of  $V_1$  (green line) and  $P(V_2)$  of  $V_2$  (blue line) measured at  $T_1 = 120K$  and  $T_2 = 296K$ . Dotted lines are the out-of-equilibrium PDF, whose variance is estimated from the measure of the heat flux (see fig.3 in the main text) and eq.S.24 in the following section. The continuous red line is the equilibrium  $P(V_2)$  at  $T_1 = 296K$  and the black continuous line corresponds to the equilibrium  $P(V_1)$  at  $T_2 = 120K$ .

### E. Measure of the equilibrium variance of $V_1$ as a function of $T_1$

This measure is necessary to estimate  $\langle Q_1 \rangle$  starting from the measurement of the variances as explained in fig.4 of the main text. We first measure  $\sigma_{m,eq}^2(T_1)$  at  $T_1 = T_2 = 296K$ . Indeed in equilibrium the variance must be proportional to

$T_1 = T_2$ , i.e.  $\sigma_{m,\text{eq}}^2(T_1) = \alpha_m T_1$ , and from the equilibrium measurements at  $T_1 = T_2$  one gets the proportionality constant  $\alpha_m = \sigma_{m,\text{eq}}^2(T_1)/T_1$ . Thus when  $T_1 < T_2$  one can estimate the values of the equilibrium variances  $\sigma_{m,\text{eq}}^2(T_1) = \alpha_m T_1$ . As explained in the main text  $\langle \dot{Q}_1 \rangle = (\sigma_1^2(T_1) - \sigma_{m,\text{eq}}^2(T_1))/R_1$ . In fig. S.2b) we compare the measured PDF of  $V_1$  and  $V_2$  with the equilibrium and the out-of-equilibrium distributions as computed by using the theoretical predictions eq.S.24 for the variance.

## II. DYNAMICAL EQUATIONS FOR $V_m$ AND $Q_m$ .

We want to describe, with a set of coupled Langevin equations, the dynamical evolution of both the electric and thermodynamic variables for the circuit in fig. 1 of the main text. For this purpose we write the Langevin equations governing the dynamical evolutions for the voltages across the circuit:

$$(C_1 + C)\dot{V}_1 = C\dot{V}_2 + \frac{1}{R_1}(\eta_1 - V_1) \quad (\text{S.5})$$

$$(C_2 + C)\dot{V}_2 = C\dot{V}_1 + \frac{1}{R_2}(\eta_2 - V_2) \quad (\text{S.6})$$

where we have substituted eqs. (5)-(6) into eqs. (3)-(4) in the main text. We rearrange these equations in a standard form, and obtain

$$\dot{V}_1 = f_1(V_1, V_2) + \sigma_{11}\eta_1 + \sigma_{12}\eta_2 = f_1(V_1, V_2) + \xi_1 \quad (\text{S.7})$$

$$\dot{V}_2 = f_2(V_1, V_2) + \sigma_{21}\eta_1 + \sigma_{22}\eta_2 = f_2(V_1, V_2) + \xi_2 \quad (\text{S.8})$$

where the “forces” acting on the circuits read

$$f_1(V_1, V_2) = \alpha_1 V_1 + \alpha_2 V_2 = -\frac{C_2 R_2 V_1 + C(R_2 V_1 + R_1 V_2)}{[C_2 C + C_1(C_2 + C)]R_1 R_2}, \quad (\text{S.9})$$

$$f_2(V_1, V_2) = \gamma_1 V_1 + \gamma_2 V_2 = -\frac{C_1 R_1 V_2 + C(R_2 V_1 + R_1 V_2)}{[C_2 C + C_1(C_2 + C)]R_1 R_2}, \quad (\text{S.10})$$

the coefficients  $\sigma_{ij}$  read

$$\begin{aligned} \sigma_{11} &= \frac{C_2 + C}{X R_1} \\ R_2 \sigma_{12} &= R_1 \sigma_{21} = \frac{C}{X} \\ \sigma_{22} &= \frac{C_1 + C}{X R_2}, \end{aligned}$$

and the noises  $\xi_i$  introduced in eqs. (S.7)-(S.8) are now correlated  $\langle \xi_i \xi_j' \rangle = 2\theta_{ij}\delta(t - t')$ , where

$$\theta_{11} = \frac{T_1(C_2 + C)^2}{R_1(C_2 C + C_1(C_2 + C))^2} + \frac{T_2 C^2}{R_2(C_2 C + C_1(C_2 + C))^2}, \quad (\text{S.11})$$

$$\theta_{12} = \frac{T_1(C(C_2 + C))}{R_1(C_2 C + C_1(C_2 + C))^2} + \frac{T_2(C(C_1 + C))}{R_2(C_2 C + C_1(C_2 + C))^2}, \quad (\text{S.12})$$

$$\theta_{22} = \frac{T_1 C^2}{R_1(C_2 C + C_1(C_2 + C))^2} + \frac{T_2(C_1 + C)^2}{R_2(C_2 C + C_1(C_2 + C))^2}, \quad (\text{S.13})$$

and  $\theta_{12} = \theta_{21}$ . We now notice that the rate of the dissipated heat in circuit  $m$  reads

$$\dot{Q}_m = V_m i_m = \frac{V_m}{R_m}(V_m - \eta_m) = V_m \left[ (C_m + C)\dot{V}_m - C\dot{V}_{m'} \right], \quad (\text{S.14})$$

where  $m' = 2$  if  $m = 1$ , and  $m' = 1$  if  $m = 2$ . The rightmost equality in eq. (S.14) follows immediately from eqs. (S.5)-(S.6). So one has a formalism where both the voltages and the dissipated heats are described as stochastic processes, driven by the thermal noises  $\eta_m$ .

### III. PROBABILITY DISTRIBUTION FUNCTION FOR THE VOLTAGES

We now study the joint probability distribution function (PDF)  $P(V_1, V_2, t)$ , that the system at time  $t$  has a voltage drop  $V_1$  across the resistor  $R_1$  and a voltage drop  $V_2$  across the resistor  $R_2$ . As the time evolution of  $V_1$  and  $V_2$  is described by the Langevin equations (S.7)-(S.8), it can be proved that the time evolution of  $P(V_1, V_2, t)$  is governed by the Fokker-Planck equation [2]

$$\begin{aligned} \partial_t P(V_1, V_2, Q_1, t) = L_0 P(V_1, V_2, t) = & -\frac{\partial}{\partial V_1} (f_1 P) - \frac{\partial}{\partial V_2} (f_2 P) + \theta_{11} \frac{\partial^2}{\partial V_1^2} P + \theta_{22} \frac{\partial^2}{\partial V_2^2} P \\ & + 2\theta_{12} \frac{\partial^2}{\partial V_1 \partial V_2} P \end{aligned} \quad (\text{S.15})$$

We are interested in the long time steady state solution of eq. (S.15), which is time independent  $P(V_1, V_2, t \rightarrow \infty) = P_{ss}(V_1, V_2)$ . As the deterministic forces in eqs. (S.7)-(S.8) are linear in the variables  $V_1$  and  $V_2$ , such a steady state solution reads

$$P_{ss}(V_1, V_2) = \frac{2\pi e^{-(aV_1^2 + bV_1V_2 + cV_2^2)}}{\sqrt{-b^2 + 4ca}} \quad (\text{S.16})$$

where the coefficients

$$\begin{aligned} a &= \frac{X \{C_1 T_2 Y + C[CR_2 T_1 + T_2(C_1 R_1 + CR_1 + C_2 R_2)]\}}{2[Y^2 T_1 T_2 + C^2 R_1 R_2 (T_1 - T_2)^2]}, \\ b &= -\frac{XC[(C_2 + C)R_2 T_1 + (C_1 + C)R_1 T_2]}{[Y^2 T_1 T_2 + C^2 R_1 R_2 (T_1 - T_2)^2]}, \\ c &= \frac{X \{C_2 T_1 Y + C[CR_1 T_2 + T_1(C_1 R_1 + CR_2 + C_2 R_2)]\}}{2[Y^2 T_1 T_2 + C^2 R_1 R_2 (T_1 - T_2)^2]}, \end{aligned}$$

can be obtained by replacing eq. (S.16) into eq. (S.15), and by imposing the steady state condition  $\partial_t P = 0$ . We are furthermore interested in the unconstrained steady state probabilities  $P_{1,ss}(V_1)$ , and  $P_{2,ss}(V_2)$ , which are obtained as follows

$$P_{1,ss}(V_1) = \int dV_2 P_{ss}(V_1, V_2) = \frac{e^{-\frac{V_1^2}{2\sigma_1^2}}}{\sqrt{2\pi\sigma_1^2}} \quad (\text{S.17})$$

$$P_{2,ss}(V_2) = \int dV_1 P_{ss}(V_1, V_2) = \frac{e^{-\frac{V_2^2}{2\sigma_2^2}}}{\sqrt{2\pi\sigma_2^2}} \quad (\text{S.18})$$

where the variances read

$$\sigma_1^2 = \frac{T_1(C + C_2)Y + (T_2 - T_1)C^2 R_1}{XY} \quad (\text{S.19})$$

$$\sigma_2^2 = \frac{T_2(C + C_1)Y - (T_2 - T_1)C^2 R_2}{XY} \quad (\text{S.20})$$

### IV. PROBABILITY DISTRIBUTION FOR THE DISSIPATED HEAT AND AVERAGE RATE

We start by noticing that the heat injected from the bath 1 is then dissipated in the bath 2 (and vice-versa), and so we expect the probability distribution of  $Q_1$  and  $Q_2$  to be symmetric. Thus in the following, we will only study the probability distribution of  $Q_1$ . We now proceed by introducing the joint probability distribution function of the variables  $V_1$ ,  $V_2$ , and  $Q_1$ ,  $\Phi(V_1, V_2, Q_1, t)$ . As each of these three variables evolves according to a Langevin equation, the time evolution of their PDF is described by the Fokker-Planck equation [3, 4]

$$\begin{aligned} \partial_t \Phi(V_1, V_2, Q_1, t) = & -\frac{\partial}{\partial V_1} (f_1 \Phi) - \frac{\partial}{\partial V_2} (f_2 \Phi) + \theta_{11} \frac{\partial^2}{\partial V_1^2} \Phi + \theta_{22} \frac{\partial^2}{\partial V_2^2} \Phi + 2\theta_{12} \frac{\partial^2}{\partial V_1 \partial V_2} \Phi \\ & - \frac{\partial}{\partial Q_1} \left\{ r_{11} \left[ \frac{\partial}{\partial V_1} (V_1 \Phi) + \left( V_1 \frac{\partial}{\partial V_1} \Phi \right) \right] + 2r_{12} \frac{\partial}{\partial V_2} (V_1 \Phi) + \frac{V_1^2}{R_1} \Phi \right\} \\ & + V_1^2 r_{22} \frac{\partial^2}{\partial Q_1^2} \Phi \end{aligned} \quad (\text{S.21})$$

with

$$\begin{aligned} r_{11} &= k_1\theta_{11} + k_2\theta_{12}, \\ r_{12} &= k_1\theta_{12} + k_2\theta_{22}, \\ r_{22} &= k_1^2\theta_{11} + k_2^2\theta_{22} + 2k_1k_2\theta_{12}, \end{aligned} \quad (\text{S.22})$$

and  $k_1 = (C_1 + C)$ ,  $k_2 = -C$ . It is worth noting that the first part of the right hand side of eq. (S.21) is identical to the rhs of eq. (S.15).

We proceed by proving eq. (7) in the main text, expressing the dissipated heat rate as a function of the system parameters. We have

$$\begin{aligned} \partial_t \langle Q_1 \rangle_t &= \partial_t \int dV_1 dV_2 dQ_1 Q_1 \Phi(V_1, V_2, Q_1, t) = \int dV_1 dV_2 dQ_1 Q_1 \partial_t \Phi(V_1, V_2, Q, t) = -r_{11} + \frac{1}{R_1} \langle V_1^2 \rangle \\ &= \frac{C^2 \Delta T}{XY}, \end{aligned} \quad (\text{S.23})$$

where we have replaced the time derivative  $\partial_t \Phi(V_1, V_2, Q, t)$  with the rhs of eq. (S.21) and used the equality  $\langle V_1^2 \rangle = \sigma_1^2$ , with  $\sigma_1$  as given by eq. (S.19). This equation corresponds to the one given in the main text.

We can now obtain the expressions for the variance of  $V_1$  and  $V_2$ , as introduced in the main text. Using eq. (S.23) we can express eq. (S.19) and eq. (S.20) in the following way:

$$\sigma_m^2 = \sigma_{m,\text{eq}}^2 + \langle \dot{Q}_m \rangle R_m \quad (\text{S.24})$$

where  $\sigma_{m,\text{eq}}^2 = \frac{T_m(C+C_{m'})}{X}$  is the equilibrium value of  $\sigma_m^2$  at  $\langle \dot{Q}_m \rangle = 0$ .

## V. CONSERVATION LAW

We now turn our attention to eq. (2), in the main text, and provide a formal proof for it. In order to do this, we derive a relation between the reservoir entropy change  $\Delta S_{r,\tau}$  and the system dynamics. For simplicity, in the following we divide the time into small intervals  $\Delta t$ : let us assume that the system (the circuit in our case) is in the state  $\mathbf{V} = (V_1, V_2)$  at time  $t$ , and let's denote by  $\mathbf{V}' = (V_1 + \Delta V_1, V_2 + \Delta V_2)$  its state at time  $t + \Delta t$ . Let  $\mathcal{P}_F(\mathbf{V} \rightarrow \mathbf{V}' | \mathbf{V}, t)$  be the probability that the system undergoes a transition from  $\mathbf{V}$  to  $\mathbf{V}'$  provided that its state at time  $t$  is  $\mathbf{V}$ , and let  $\mathcal{P}_R(\mathbf{V}' \rightarrow \mathbf{V} | \mathbf{V}', t + \Delta t)$  be the probability of the time-reverse transition. We have

$$\begin{aligned} P_F(\mathbf{V} \rightarrow \mathbf{V}' | \mathbf{V}, t) &= \int d\eta_1 d\eta_2 \delta(\Delta V_1 - \Delta t \cdot (f_1(V_1, V_2) + \sigma_{11}\eta_1 + \sigma_{12}\eta_2)) \\ &\quad \times \delta(\Delta V_2 - \Delta t \cdot (f_2(V_1, V_2) + \sigma_{21}\eta_1 + \sigma_{22}\eta_2)) p_1(\eta_1) p_2(\eta_2), \end{aligned} \quad (\text{S.25})$$

where  $\delta(x)$  is the Dirac delta function. Given that the noises are Gaussian distributed, their probability distributions read

$$p_m(\eta_m) = \exp \left[ -\frac{\eta_m^2 \Delta t}{4R_m k_B T_m} \right] \sqrt{\frac{\Delta t}{4\pi R_m k_B T_m}} \quad (\text{S.26})$$

and expressing the Dirac delta in Fourier space  $\delta(x) = 1/(2\pi) \int dq \exp(iqx)$ , eq. (S.25) becomes

$$P_F(\mathbf{V} \rightarrow \mathbf{V}' | \mathbf{V}, t) = \frac{1}{(2\pi)^2} \int dq_1 dq_2 \exp[i(q_1 \Delta V_1 + q_2 \Delta V_2)] \int \prod_m d\eta_m e^{\Delta t \left[ i q_m (f_m + \sigma_{m1}\eta_1 + \sigma_{m2}\eta_2) - \frac{\eta_m^2}{4R_m k_B T_m} \right]} \quad (\text{S.27})$$

$$\begin{aligned} &= \exp \left\{ -\frac{\Delta t}{4k_B T_1 T_2} \left[ C_1^2 R_1 T_2 (\dot{V}_1 - f_1)^2 + C_2^2 R_2 T_1 (\dot{V}_2 - f_2)^2 \right. \right. \\ &\quad \left. \left. + 2C(\dot{V}_1 - f_1 - \dot{V}_2 + f_2)(C_1 R_1 T_2 (\dot{V}_1 - f_1) - C_2 R_2 T_1 (\dot{V}_2 - f_2)) \right. \right. \\ &\quad \left. \left. + C^2 (R_2 T_1 + R_1 T_2) (\dot{V}_1 - f_1 - \dot{V}_2 + f_2)^2 \right] \right\} \frac{X}{4\pi k_B \Delta t} \sqrt{\frac{R_1 R_2}{T_1 T_2}}; \end{aligned} \quad (\text{S.28})$$

where we have taken  $\Delta V_m/\Delta t \simeq \dot{V}_m$ , and exploited the fact that all the integrals in eq. (S.27) are Gaussian integrals. Similarly, for the reverse transition we obtain

$$P_R(\mathbf{V}' \rightarrow \mathbf{V}|\mathbf{V}', t + \Delta t) = \int d\eta_1 d\eta_2 \delta(\Delta V_1 + \Delta t(f_1(V'_1, V'_2) + \sigma_{11}\eta_1 + \sigma_{12}\eta_2)) \times \delta(\Delta V_2 + \Delta t(f_2(V'_1, V'_2) + \sigma_{21}\eta_1 + \sigma_{22}\eta_2)) p_1(\eta_1) p_2(\eta_2) \quad (\text{S.29})$$

$$= \exp \left\{ -\frac{\Delta t}{4k_B T_1 T_2} \left[ C_1^2 R_1 T_2 (\dot{V}_1 + f_1)^2 + C_2^2 R_2 T_1 (\dot{V}_2 + f_2)^2 + 2C(\dot{V}_1 + f_1 - \dot{V}_2 - f_2)(C_1 R_1 T_2 (\dot{V}_1 + f_1) - C_2 R_2 T_1 (\dot{V}_2 + f_2)) + C^2 (R_2 T_1 + R_1 T_2) (\dot{V}_1 + f_1 - \dot{V}_2 - f_2)^2 \right] \right\} \frac{X}{4\pi k_B \Delta t} \sqrt{\frac{R_1 R_2}{T_1 T_2}}. \quad (\text{S.30})$$

We now consider the ratio between the probability of the forward and backward trajectories, and by substituting the explicit definitions of  $f_1(V_1, V_2)$  and  $f_2(V_1, V_2)$ , as given by eqs. (S.9)-(S.10), into eqs. (S.28) and (S.30), we finally obtain

$$\log \frac{P_F(\mathbf{V} \rightarrow \mathbf{V}'|\mathbf{V}, t)}{P_R(\mathbf{V}' \rightarrow \mathbf{V}|\mathbf{V}', t + \Delta t)} = -\Delta t \left( V_1 \frac{(C_1 + C)\dot{V}_1 - C\dot{V}_2}{k_B T_1} + V_2 \frac{(C_2 + C)\dot{V}_2 - C\dot{V}_1}{k_B T_2} \right) = \Delta t \left( \frac{\dot{Q}_1}{k_B T_1} + \frac{\dot{Q}_2}{k_B T_2} \right), \quad (\text{S.31})$$

where we have exploited eq. (S.14) in order to obtain the rightmost equality. Thus, by taking a trajectory  $\mathbf{V} \rightarrow \mathbf{V}'$  over an arbitrary time interval  $[t, t + \tau]$ , and by integrating the right hand side of eq. (S.31) over such time interval, we finally obtain

$$k_B \log \frac{P_F(\mathbf{V} \rightarrow \mathbf{V}'|\mathbf{V}, t)}{P_R(\mathbf{V}' \rightarrow \mathbf{V}|\mathbf{V}', t + \tau)} = \left( \frac{Q_1}{T_1} + \frac{Q_2}{T_2} \right) = \Delta S_{r,\tau} \quad (\text{S.32})$$

We now note that the system is in an out-of-equilibrium steady state characterized by a PDF  $P_{ss}(V_1, V_2)$ , and so, along any trajectory connecting two points in the phase space  $\mathbf{V}$  and  $\mathbf{V}'$  the following equality holds

$$\exp[\Delta S_{tot}/k_B] = \exp[(\Delta S_{r,\tau} + \Delta S_{s,\tau})/k_B] = \frac{\mathcal{P}_F(\mathbf{V} \rightarrow \mathbf{V}'|\mathbf{V}, t) P_{ss}(\mathbf{V})}{\mathcal{P}_R(\mathbf{V}' \rightarrow \mathbf{V}|\mathbf{V}', t + \tau) P_{ss}(\mathbf{V}')}, \quad (\text{S.33})$$

where we have exploited eq. (S.32), and the definition of  $\Delta S_{s,\tau}$  in eq. (1) in the main text. Thus we finally obtain

$$\mathcal{P}_F(\mathbf{V} \rightarrow \mathbf{V}'|\mathbf{V}, t) P_{ss}(\mathbf{V}) \exp[-\Delta S_{tot}/k_B] = \mathcal{P}_R(\mathbf{V}' \rightarrow \mathbf{V}|\mathbf{V}', t + \tau) P_{ss}(\mathbf{V}') \quad (\text{S.34})$$

and summing up both sides over all the possible trajectories connecting any two points  $\mathbf{V}, \mathbf{V}'$  in the phase space, and exploiting the normalization condition of the backward probability, namely

$$\sum_{\mathbf{V}', \mathbf{V}} \mathcal{P}_R(\mathbf{V}' \rightarrow \mathbf{V}|\mathbf{V}', t + \tau) P_{ss}(\mathbf{V}') = 1, \quad (\text{S.35})$$

one obtains eq. (2). It is worth noting that the explicit knowledge of  $P_{ss}(\mathbf{V})$  is not required, in order to prove eq. (2).

Finally, we note that, from a general perspective, eqs. (S.7)-(S.8) correspond to the Langevin equations of a stochastic system, whose variables  $V_1$  and  $V_2$  interact through non-conservative forces, and where the white noise is correlated. Therefore our proof of eq. (S.35), and thus of eq. (2) in the main text, holds in general for systems with such characteristics.

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- [1] G. Cannatà, G. Scandurra, C. Ciofin, *Rev. Scie. Instrum.* **80**, 114702 (2009).
  - [2] R. Zwanzig, *Nonequilibrium Statistical Mechanics*, Oxford University Press, Oxford, 2001.
  - [3] A. Imparato, L. Peliti, G. Pesce, G. Rusciano, A. Sasso, *Phys. Rev. E*, **76**: 050101R (2007).
  - [4] H. C. Fogedby, A. Imparato, *J. Stat. Mech.* P04005 (2012).