

Mechanical vibration – System identification and modal analysis of 3-DOF linear system

Giammarco Valenti

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1 Dynamical system

1.1 The system and the experimental setup

The system consists of three different bodies on three carriers. The carriers are aligned and the bodies are constrained to slide along the common axis. Between the bodies two springs are located, and a third spring connects the frame and the last body. The first body is rigidly connected through a rack-pinion gearing with a motor which is controlled in voltage with a PC interface. The position of each body is provided by an encoder. The zeroes of the positions are at the springs rest position. The scheme of the model is depicted in Figure 1. The displacement of the masses in meters is available from the encoder with the following resolution:

$$\Delta x = \frac{2\pi r_e}{16000}. \quad (1)$$

1.2 The dynamical model

1.2.1 Assumptions

To define a model of the system, some assumptions were made:

Rectilinear motion All the bodies (masses), and the rack of the rack-pinion gearing are supposed to move and exert forces along the same axis, which is the motion axis. Consequently, all the quantities are meant to be projected on this axis.

Viscous friction Only viscous frictions are present

Instantaneous electrical dynamics The model of the electrical dynamics of the motor is only a gain from voltage to force, expressed by the voltage-to-force factor, which is exploited in Section 2.1.

Motor mechanics merged Inertia and damping of the motor are merged respectively into m_1 and c_1 (refer to Figure 1)

$$\begin{cases} m_1 = m_{block} + \frac{J_{motor}|_{zz}}{r^2} \\ c_2 = c_{block} + \frac{c_{motor}}{r^2} \end{cases} \quad (2)$$

where r is the radius of the gear-rack coupling (gear wheel), $J_{motor}|_{zz}$ is the inertia of the motor, c_{motor} the rotational damping and "block" quantities are the ones strictly related to the physical first mass.

Last viscosities merged The term c_3 (refer to Figure 1) contains the viscous friction with the ground and the one due to the spring. In the model those two contributions cannot be quantified separately.

1.2.2 The linear model

The chosen model is a linear plant consisting of 3 lumped masses, 3 lumped springs between them (the last to the frame), and 3 dampers between each mass and the ground. The model is shown in Figure 1.



Figure 1: The chosen plant, in red the unknown parameters

1.3 equation of motion

$$\begin{cases} m_1 \ddot{x}_1 = +k_1 (x_2 - x_1) + c_{12} (\dot{x}_2 - \dot{x}_1) - c_1 \dot{x}_1 + g_v v(t) \\ m_2 \ddot{x}_2 = +k_1 (x_1 - x_2) + k_2 (x_3 - x_2) + c_{12} (\dot{x}_1 - \dot{x}_2) + c_{23} (\dot{x}_3 - \dot{x}_2) - c_2 \dot{x}_2 \\ m_3 \ddot{x}_3 = +k_2 (x_2 - x_3) + c_{23} (\dot{x}_2 - \dot{x}_3) - c_3 \dot{x}_3 - k_3 x_3 \end{cases} \quad (3)$$

In the classical matrix form:

$$M\ddot{x} + C\dot{x} + Kx = b \quad (4a)$$

where:

$$K = \begin{bmatrix} +k_1 & -k_1 & 0 \\ -k_1 & +k_1 + k_2 & -k_2 \\ 0 & -k_2 & +k_3 \end{bmatrix} \quad (4b)$$

$$M = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \quad (4c)$$

$$b = \begin{bmatrix} g_v \\ 0 \\ 0 \end{bmatrix} \quad (4d)$$

$$C = \begin{bmatrix} +c_1 + c_{12} & -c_{12} & 0 \\ -c_{12} & +c_2 + c_{12} + c_{23} & -c_{23} \\ 0 & -c_{23} & c_3 + c_{23} \end{bmatrix} \quad (4e)$$

1.4 state-space model

The linear model of the plant, expressed by the equation 4, is a SIMO model. A state-space form was chosen to represent this model. The matrices are the following:

$$\begin{cases} \dot{x} = Ax + Bv \\ y = Cx + Dv \end{cases} \quad (5a) \quad x = [x_1 \ x_2 \ x_3 \ \dot{x}_1 \ \dot{x}_2 \ \dot{x}_3]^T \quad (5b)$$

$$A = \begin{bmatrix} Z_{3 \times 3} & I_{3 \times 3} \\ -M^{-1}K & -M^{-1}C \end{bmatrix} \quad (5c) \quad B = \begin{bmatrix} Z_{3 \times 1} \\ -M^{-1}b \end{bmatrix} \quad (5d)$$

$$C = [I_{3 \times 3} \ Z_{3 \times 3}] \quad (5e) \quad D = [Z_{3 \times 1}] \quad (5f)$$

where I is the identity matrix and Z is a matrix with all the entries equal to zero.

1.5 experimental setup

1.5.1 data processing

Few operations on data must be performed in order to use them. At first, the data on displacements is provided in *encoder counts*. They are converted in meters with the following conversion factor (g_x):

$$g_x = \frac{\Delta x}{\Delta \text{counts}} = 2\pi r_e \cdot \frac{\Delta \text{counts}}{16000 \frac{\text{counts}}{\text{encoder revolution}}}. \quad (6)$$

1.6 parameters and data available

The linear system in Section 1.2.2 is fully represented by parameters, they are shown in Table 1.6. Only the stiffnesses of the springs (k_i) are known.

| Name | g_v | k_1 | k_2 | k_3 | m_1 | m_2 | m_3 | c_1 | c_2 | c_3 | c_{12} | c_{23} |
|-------|-------|-------|-------|-------|-------|-------|-------|---|-------|-------|----------|----------|
| Value | - | 774 | 770 | 396 | - | - | - | - | - | - | - | - |
| Units | N/V | N/m | | | K_g | | | $Ns\ m^{-1}$ | | | | |
| Notes | | | | | | | | c_a and c_b if proportional damping | | | | |

Table 1: Parameters ("-" stands for unavailable parameter)

The meaning of each parameter is depicted in Figure 1, except for the *voltage-to-force* coefficient. Such coefficient g_v is the factor which converts the voltage of the signal sent to the motor in the force exerted on the rack. It is the product of several physical gains as shown in Equation 7.

$$f = (k_a \cdot k_t \cdot k_{mp})v(t) = g_v v(t). \quad (7)$$

Where the physical meanings are:

$$k_a = \frac{1}{R_{motor}} \approx 2\ A V^{-1} \quad (8a)$$

Electrical conductance of the motor

$$k_t \approx 0.1\ Nm A^{-1} \quad (8b)$$

Motor torque constant

$$k_{mp} = \frac{1}{r_{pinion}} \approx 26.25\ m^{-1} \quad (8c)$$

Trasmission ratio of the gearing

2 System identification

2.1 steady state analysis

From the *step response analysis* the steady state values of input and output are considered. In this analysis the "static" coefficients can be studied, they are:

- voltage to force g_v
- springs' stiffness k_i with $i \in 1, 2, 3$

The goal of this section is to estimate g_v and to verify the ratios between the stiffnesses k_i , w.r.t the nomina values. It is not possible to verify directly the k_i values because the steady state values are 3 as many as the equations of the generated system (linear) while the variables would be 4.

2.1.1 Calculations

In order to generate the system of equations, the static gain vector g_{dc} of the system is computed. A procedure may be to apply the "CAB" formula from the state space formulation:

$$g_{dc} = CA^{-1}B. \quad (9)$$

which is the transfer function at $s = 0$. Since equation 9 implies the inverse of the 6×6 matrix A , another computation is performed. Using the formulation in Equation 4, the following limits are performed:

$$\begin{cases} \lim_{t \rightarrow +\infty} \dot{x} = 0 \\ \lim_{t \rightarrow +\infty} \ddot{x} = 0 \end{cases} \quad (10)$$

The substitution 10 in 4a yields:

$$Kx = b \quad (11)$$

The static gain is then:

$$g_{dc} = K^{-1}b \quad (12)$$

$$g_{dc} = \begin{bmatrix} g_v \frac{k_1 k_2 + k_3 k_2 + k_3 k_1}{k_1 k_2 k_3} & g_v \frac{k_2 + k_3}{k_2 k_3} & g_v \frac{1}{k_3} \end{bmatrix}^T \quad (13)$$

Comment 1. Apply 10 to equation 4a is equivalent to apply the Laplace tranform to the equation 4a and apply the final value theorem to it

Comment 2. Looking at the Equation 13 is evident the series connection of the springs.

The steady state value of the three output are available. Some other computations has to be made to make equation 13 suitable for the check on the stiffnesses ratios and the new estimation of g_v . The equation for the steady state values is:

$$Cx_\infty = g_{dc}v_\infty \quad (14)$$

Where the ∞ denotes the steady state value of the quantity ($\lim_{t \rightarrow \infty} f(t)$). The Equation 14 is now expressed in terms of stiffnesses ratio: k_3 is fixed to the nominal value and two ratio are defined as following.

$$R_{31} = \frac{k_3}{k_1} \quad (15a) \quad R_{32} = \frac{k_3}{k_2} \quad (15b)$$

This operations are made in order to make the system easy to solve, and uncoupling the nonlinear factor g_v . This allows to solve the system in a cascade fashion from g_v to R_{13} . The equation 14 is now expressed in the following form:

$$k_3 \begin{bmatrix} x_1(\infty) \\ x_2(\infty) \\ x_3(\infty) \end{bmatrix} = u(\infty) \begin{bmatrix} g_v + g_v R_{13} + g_v R_{23} \\ g_v R_{23} \\ g_v \end{bmatrix} \quad (16)$$

2.1.2 Steady state value picking

Now the next step is to use the measured value of steady state response (input and output), take g_v as new estimated value and verify that R_{12} and R_{32} matches the nominal values. The input is set on $u(\infty) = 0.5$.

The value of steady state response are picked from the data directly, as shown in Figure 2. The input is shown in Figure 3.

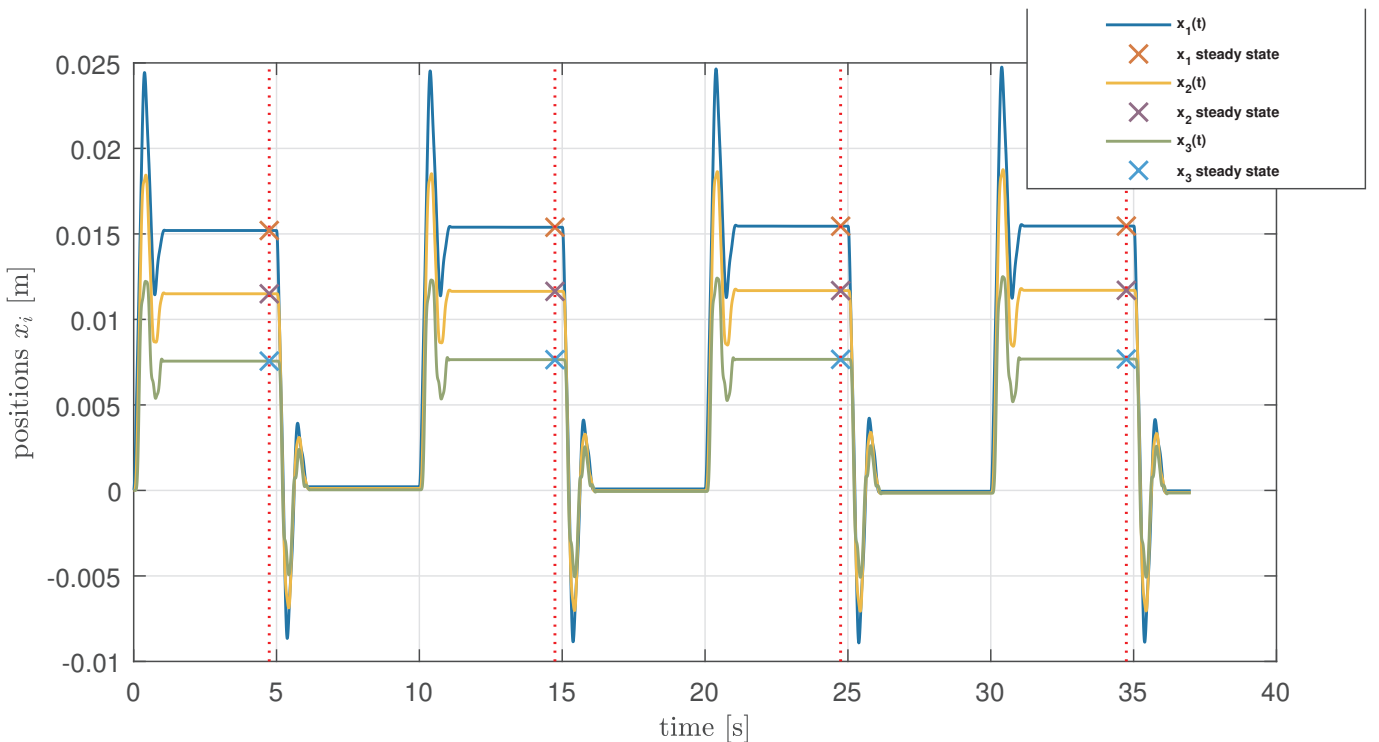


Figure 2: Steady state value picking

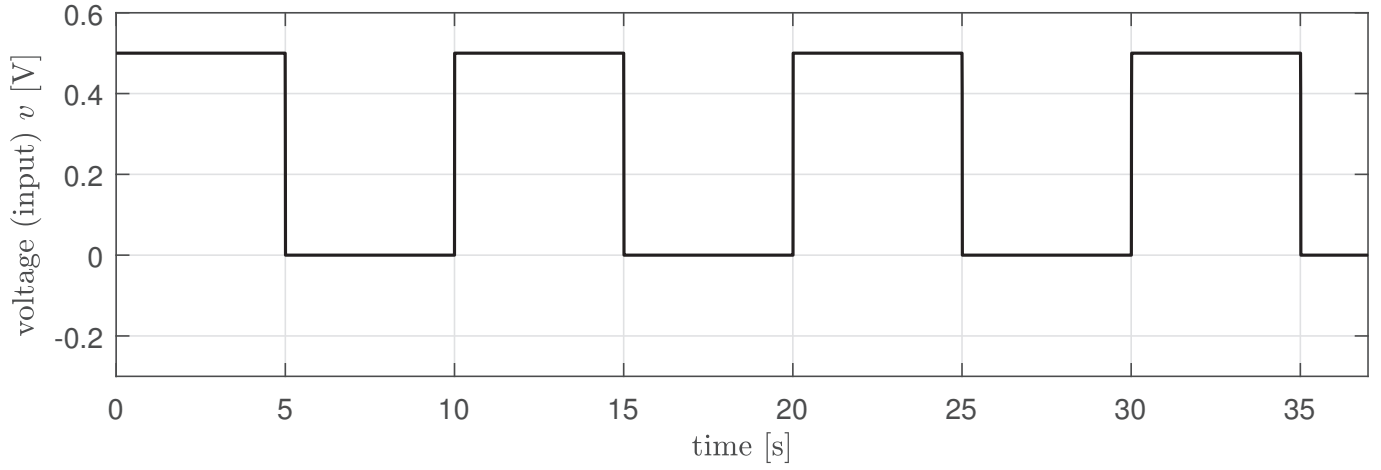


Figure 3: Step input

The obtained values for each output (x_∞) are 4. The mean value of the ratios ($R31, R32$) is estimated by the **arithmetic average**. Given (x_∞) averaged from the data, Equation 16 is solved and the results are shown in Table 2.1.2. The error is defined as $\text{error}\% = 100 \frac{\text{data value}}{\text{nominal}}$

| | from data | nominal | error % |
|-----------|-----------|---------|---------|
| k_3/k_2 | 0.5227 | 0.5143 | 6.7 % |
| k_3/k_1 | 0.4900 | 0.5116 | 4.4 % |
| | from data | initial | error % |
| g_v | 6.0497 | 5.2500 | 15.2 % |

Table 2: Stiffnesses ratios and voltage-to-force coefficients results from steady state analysis

The *voltage-to-force* estimation is shown in table 2.1.2.

2.2 Parameters estimation

To estimate the parameters the impulse response is used. The *voltage-to-force* coefficient g_v is one of the parameters to be estimate in order to include the possibility to cross-check the result with the step response result. The parameters to estimate are shown in table 1.6.

2.2.1 Estimation strategy

The strategy to estimate the parameters consists in use the **linear model** described in 1.2.2, use the same input as the real model, and compare the output with the real output tuning the parameters to minimize the difference.

Box 1 Estimation strategy keypoints

1. Choose the linear model
2. Provide the same input as the real system (an impulse-like signal)
3. Simulate the output (time domain)
4. Compare the outputs with the real one.
5. Computing the sum of squares of the residuals
6. Tune the parameters iteratively to minimize this quantity

The implementation of the strategy in Box 1 is possible creating a MATLAB function which presents in input the parameters and in output the sum of the squares of the residuals. This is performed with a simulation of the system executed inside the function, the data of the real system are available as a global variable, and the only output is the sum of square to minimize. The strategy and the function to minimize (orange box) is shown in Figure 4.

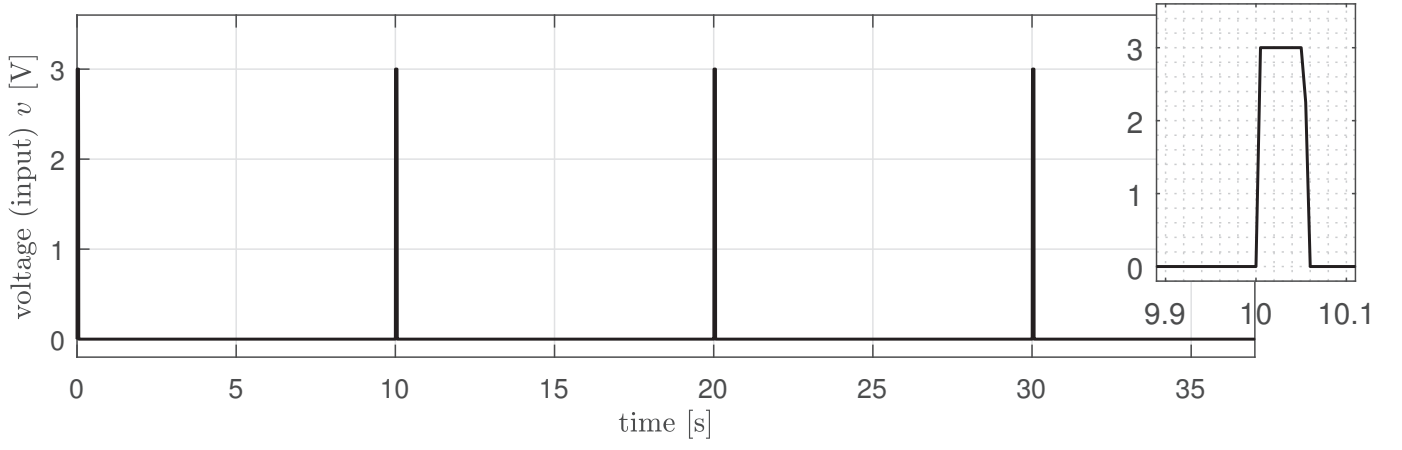


Figure 5: The pulse-like input (zoom of the single pulse on the right)

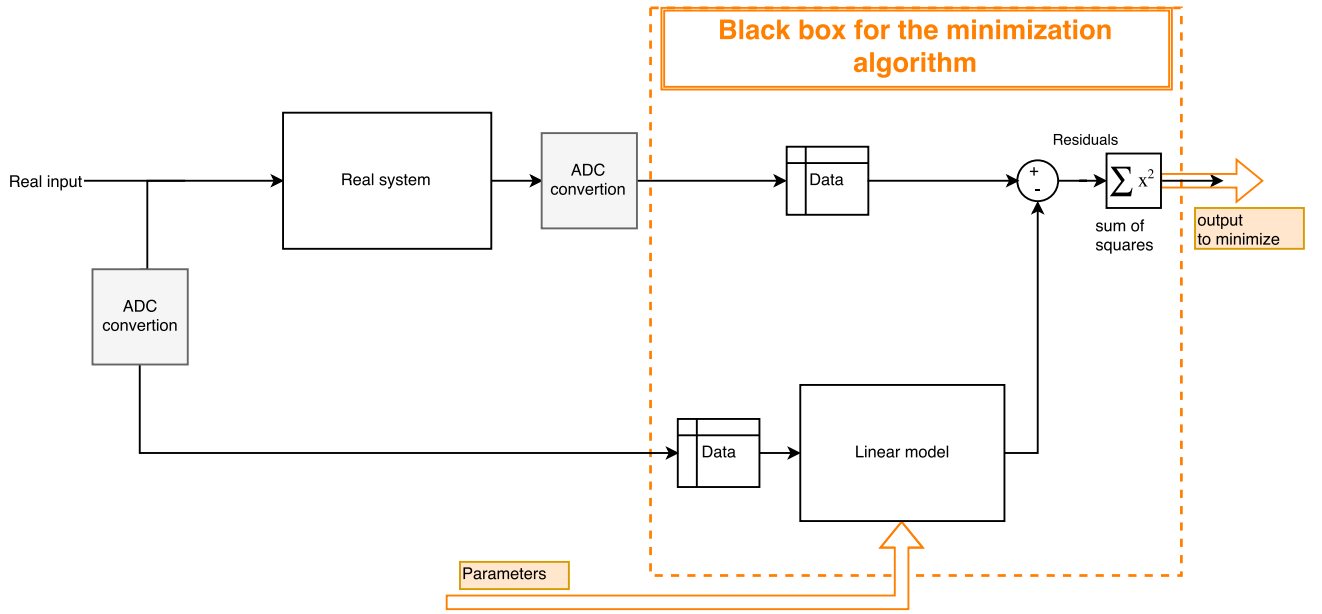


Figure 4: estimation strategy scheme, Function to minimize in orange

Comment 3. A single evaluation of the function corresponds to a simulation of the linear system, the parameters are updated by the algorithm after the evaluation of the function

The implementation of the strategy is performed with MATLAB, the function `lsqnonlin` is used. The input used is a pulse-like waveform which excite the system four times, the signal is shown in Figure 5.

2.2.2 Free damping case

The first estimation is performed using the model in Equations 5.

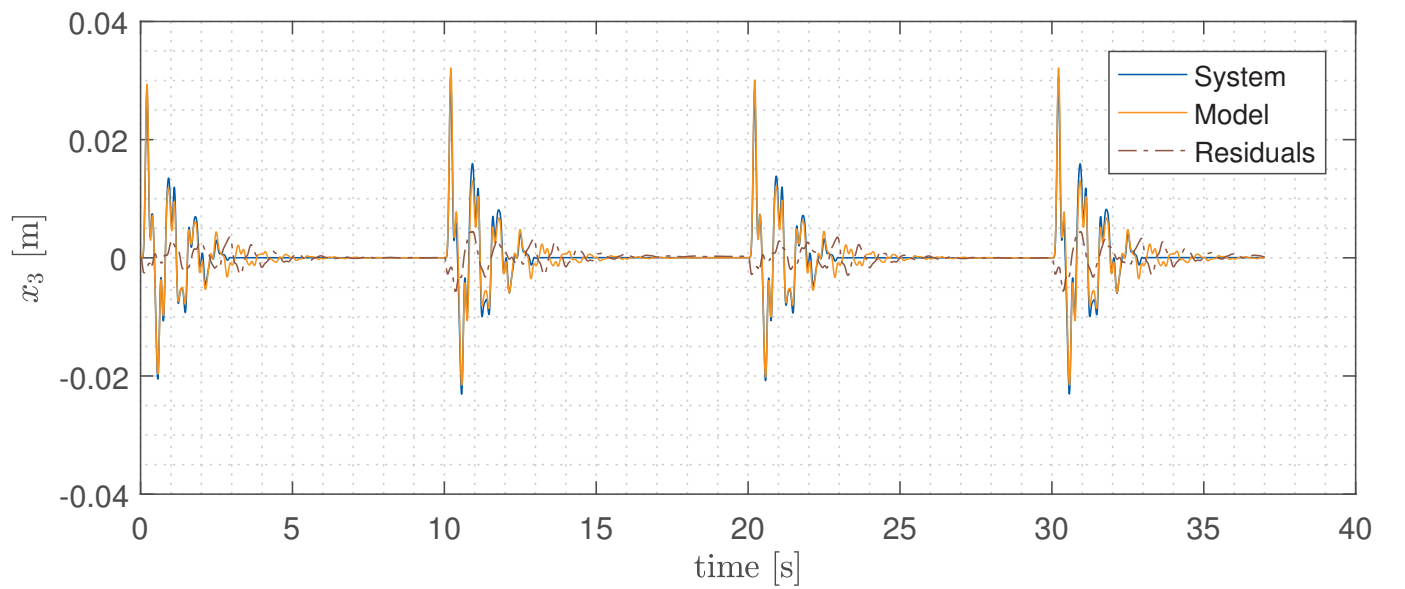
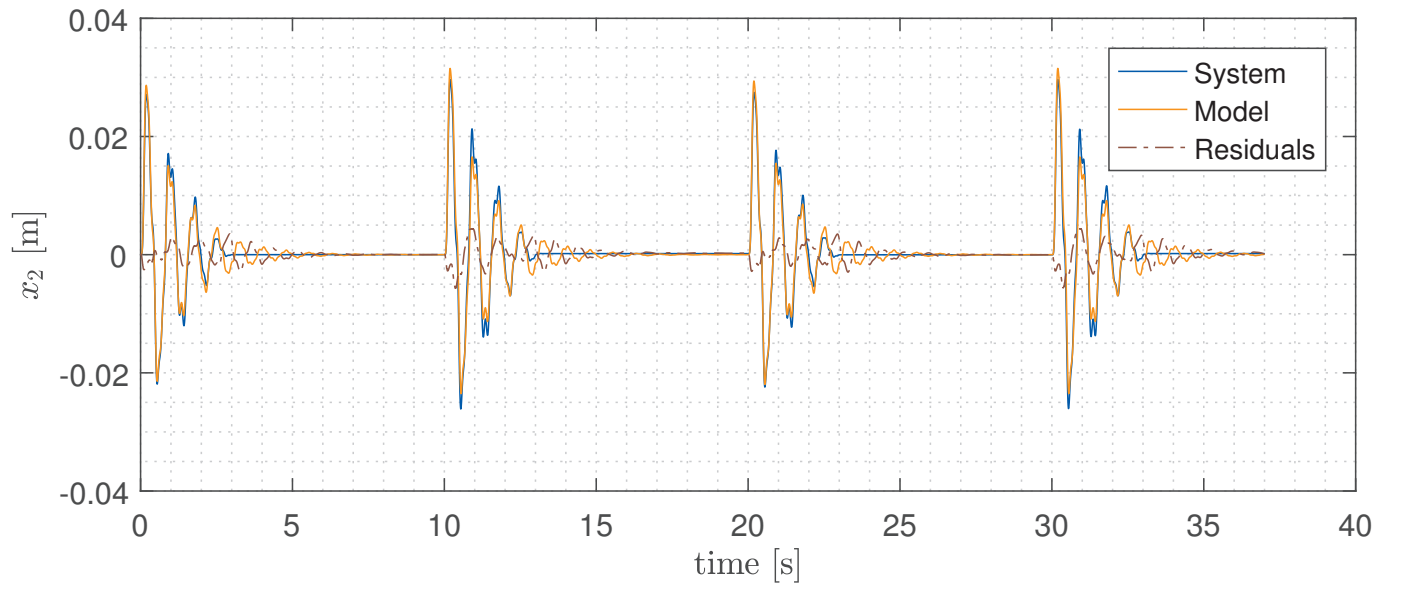
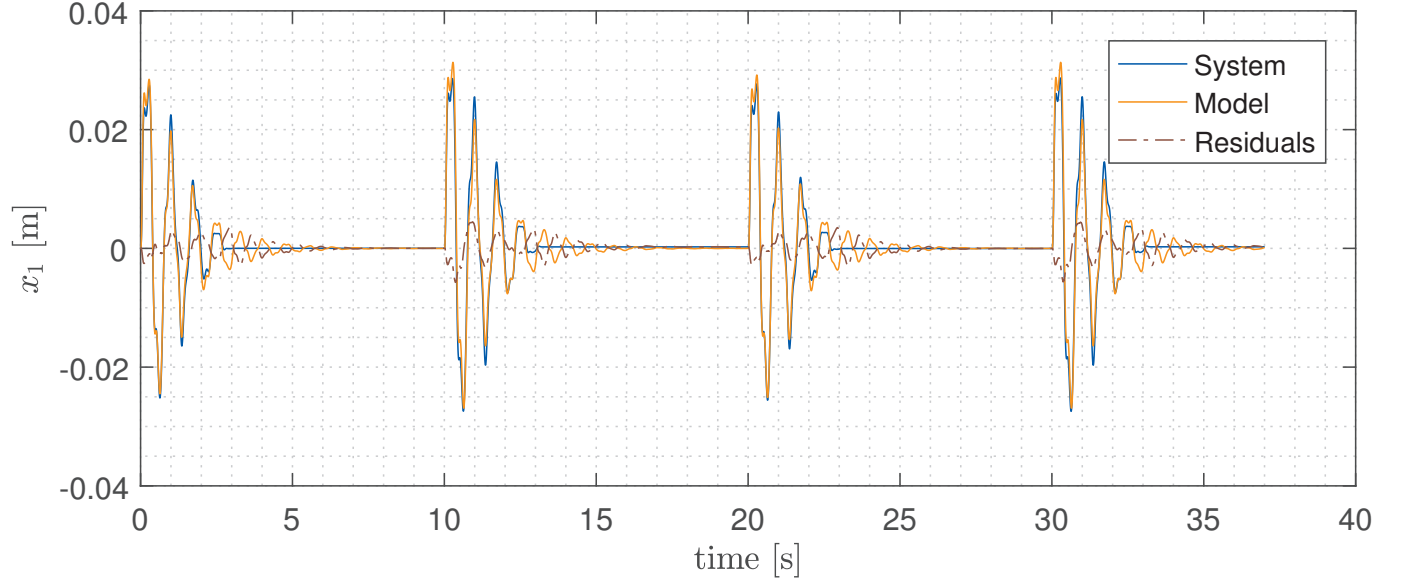
The results of this optimization are shown in table 2.3

| Name | g_v | m_1 | m_2 | m_3 | c_1 | c_2 | c_3 | c_{12} | c_{23} |
|-------|-------|-------|-------|-------|--------------|-------|-------|----------|----------|
| Value | 6.223 | 1.565 | 1.461 | 1.146 | 2.923 | 1.806 | 2.002 | 0.00221 | 0.00001 |
| Unit | N/V | K_g | | | $Ns\ m_{-1}$ | | | | |

Table 3: Estimation results (model with free damping)

A plot with the comparison of the models is provided in Figure 6. The normalized root means square errors are provided for each DOF in table 5

Figure 6: Comparison between the response of the model and the response of the system



2.3 Proportional damping case

The procedure is exactly the same as the case of the free damping. The parameters to estimate now are 6 instead of 9. The damping now is represented by only 2 parameters, as shown in Equation 17.

$$\mathbf{C} = c_a \mathbf{M} + c_b \mathbf{K} \quad (17)$$

| | | | | | | |
|-------|-------|-------|-------|-------|--------------|-----------|
| Name | g_v | m_1 | m_2 | m_3 | c_a | c_b |
| Value | 6.223 | 1.565 | 1.461 | 1.146 | 1.528 | 8.583e-05 |
| Unit | N/V | K_g | | | $Ns\ m_{-1}$ | |

Table 4: Estimation results (model with proportional damping)

Figure 7: Comparison between the response of the model and the response of the system - proportional damping case

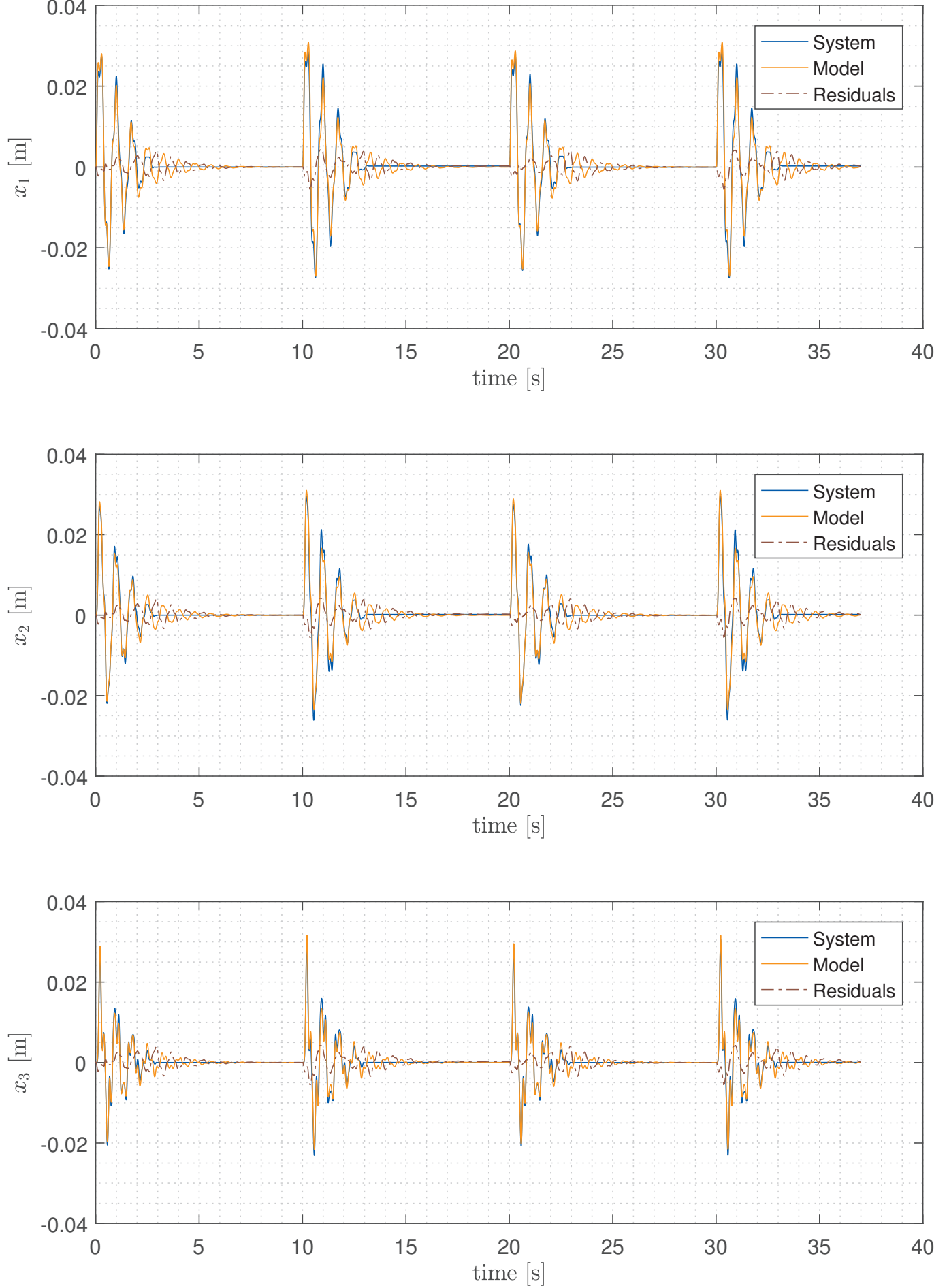


Table 5: NRMSE for the outputs

| | Free damping | Proportional damping |
|-------------|--------------|----------------------|
| x_1 NRMSE | 81.52 % | 81.22 % |
| x_2 NRMSE | 81.35 % | 81.05 % |
| x_3 NRMSE | 81.87 % | 81.35 % |

2.4 Conclusion

3 modal analysis

3.1 Eigenvalue problem

3.2 Undamped case

3.2.1 Raileight quotient

The Rayleigh quotient is

$$R_q(x) = \frac{x^T K x}{x^T M x} \quad (18)$$

where x is the vector of the positions, R_q as a function of the ladder, presents stationary points in the neighborhood of the modal shapes, and the value corresponds to the respective eigenfrequency. The first stationary point it's easy to find, but the other two are not. In order to find the other stationary points, the property of orthogonality of the modal shapes vector is used. The basic idea is to reduce a degree of freedom every frequency using this property. The procedure is the following:

- Provide an initial guess, the convergence to the first minima is quite robust to the first guess. Vector $[1 \ 1 \ 1]^T$ is used.
- Use Matlab function `fminunc` to find the minima and the value of x , varying only two parameters of the vector ($x = [1 \ \alpha \ \beta]^T$)
- Check if the minimum is the lowest frequency with a contour plot or performing the next steps. Define the first modal shape vector u_{1R} .
- Compute the null space of the first modal shape vector $Ker(u_{1R}^T)$
- find the minimum of

$$R_{q2} = \frac{[1 \ \alpha] B_k^T K B_k [1 \ \alpha]^T}{[1 \ \alpha] B_k^T M B_k [1 \ \alpha]^T} \quad (19)$$

where $img(B_k) = Ker(u_{1R}^T)$ is a base of the Kernel of the first modal shape vector, the value of the function is another resonance frequency.

- The last modal shape vector and the last resonance frequency can be computed respectively as a base of the Kernel of the two other modal shape vectors and evaluating R_q at the last modal shape vector.

$$u_{3R} \in Ker([u_{1R} \ u_{2R}]^T) \quad (20)$$

The results are shown in Table ?? and the contour plot of the Raileight coefficient is shown in Figure ??

3.3 Proportional damping case

(BIBLIO RAO PAG 914).

given the modal shape vectors matrix U obtained from the eigenvalue problem 3.1 with the masses from the proportional damping estimation, the modal coordinates w.r.t the previous coordinates are the following:

$$\tilde{x} = U^T x \quad (21)$$

Which is basically a projection of the coordinate on the modal shapes vector (scalar products). The matrices M , K and C becomes...