Mechanical vibration - System identification and modal analysis of 3-DOF linear system

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1 Dynamical system

1.1 The linear model

The chosen model is a linear plant consisting of 3 masses, 3 springs between them, and 3 dampers between each mass and the ground. The model is shown in Figure 1.

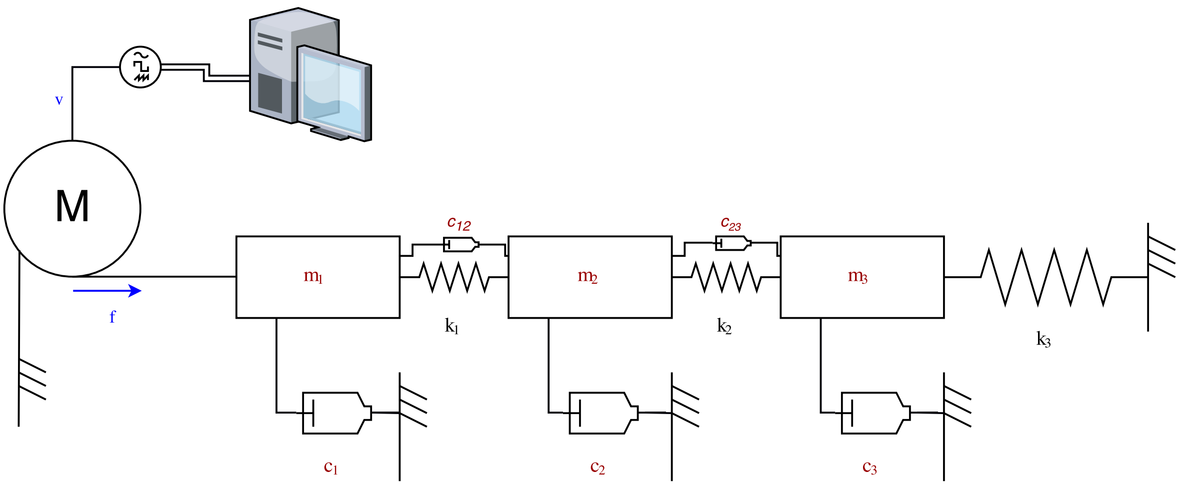

Figure 1: The chosen plant, in red the unknown parameters

1.2 equation of motion

$$\begin{cases} m_1\ddot{x}_1 = +k_1\left(x_2 - x_1\right) + c_{12}\left(\dot{x}_2 - \dot{x}_1\right) - c_1\dot{x}_1 + g_v v \\ m_2\ddot{x}_2 = +k_1\left(x_1 - x_2\right) + k_2\left(x_3 - x_2\right) + c_{12}\left(\dot{x}_1 - \dot{x}_2\right) + c_{23}\left(\dot{x}_3 - \dot{x}_2\right) - c_2\dot{x}_2 \\ m_3\ddot{x}_3 = +k_2\left(x_2 - x_3\right) + c_{23}\left(\dot{x}_2 - \dot{x}_3\right) - c_3\dot{x}_3 - k_3 x_3 \end{cases} \tag{1}$$

In the classical matrix form:

$$\boldsymbol{M}\ddot{x} + \boldsymbol{C}\dot{x} + \boldsymbol{K}x = \boldsymbol{b} \tag{2a}$$

where:

$$\boldsymbol{K} = \begin{bmatrix} +k_1 & -k_1 & 0 \\ -k_1 & +k_1 + k_2 & -k_2 \\ 0 & -k_2 & +k_3 \end{bmatrix} \tag{2b}$$

$$\boldsymbol{M} = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \tag{2c}$$

$$\boldsymbol{b} = \begin{bmatrix} g_v \\ 0 \\ 0 \end{bmatrix} \tag{2d}$$

$$\boldsymbol{C} = \begin{bmatrix} +c_1 + c_{12} & -c_{12} & 0 \\ -c_{12} & +c_2 + c_{12} + c_{23} & -c_{23} \\ 0 & -c_{23} & c_3 + c_{23} \end{bmatrix} \tag{2e}$$

1.3 state-space model

The linear model of the plant, expressed by the equation 2, is a SIMO model. A state-space form was chosen to represent this model. The matrices are the follwing:

$$A = \begin{bmatrix} \boldsymbol{Z}_{3\times3} & \boldsymbol{I}_{3\times3} \\ -\boldsymbol{M}^{-1}\boldsymbol{K} & -\boldsymbol{M}^{-1}\boldsymbol{C} \end{bmatrix} \tag{3a}$$

$$\boldsymbol{B} = \begin{bmatrix} \boldsymbol{Z}_{3\times1} \\ -\boldsymbol{M}^{-1}\boldsymbol{b} \end{bmatrix} \tag{3b}$$

$$\boldsymbol{C} = \begin{bmatrix} \boldsymbol{I}_{3\times3} & \boldsymbol{Z}_{3\times3} \end{bmatrix} \tag{3c}$$

$$\boldsymbol{D} = \begin{bmatrix} \boldsymbol{Z}_{3\times1} \end{bmatrix} \tag{3d}$$

where $\boldsymbol{I}$ is the identity matrix and $\boldsymbol{Z}$ is a matrix with all the entries equal to zero.

1.4 experimental setup

1.4.1 data processing

Few operations on data must be performed in order to use them. At first, the data on diplacements is provided in *encoder counts*. They are converted in meters with the following convertion factor (g_x):

$$g_x = \frac{\Delta x}{\Delta \texttt{counts}} = 2\pi r_e \cdot \frac{\Delta \texttt{counts}}{16000 \frac{\texttt{counts}}{\texttt{encoder revolution}}} \tag{4}$$

1.5 parameters and data available

1.6 initial hypothesis and approximations

- neglected motor electrical dynamics (instantaneous transmission of torque)
- rectilinear motion (all perfect aligned)
- inertia and damping of the motor are merged respectively into m_1 and c_1.

$$\begin{cases} m_1 = m_{block} + \frac{J_{motor}|_{zz}}{r^2} \\ c_2 = c_{block} + \frac{c_{motor}}{r^2} \end{cases} \tag{5}$$

  where r is the radius of the gear-rack coupling (gear wheel), $J_{motor}|_{zz}$ is the inertia of the motor , c_{motor} the rotational damping and "block" quantities are the ones stricly related to the physical first mass.

- The term c_3 contains the viscous friction with the ground and the one due to the spring. In the model those two contributions cannot be quantified separately.

2 System identification

2.1 step response analysis

First of all, the step response analysis can be performed. In this analysis the "static" coefficients can be estimated, they are:

- voltage to force g_v
- springs' stiffness k_i with $i \in 1, 2, 3$

The coefficient to be estimated is the *voltage-to-force* coefficient

$$f = (\boldsymbol{k_a} \cdot \boldsymbol{k_t} \cdot \boldsymbol{k_{mp}})v = g_v v \tag{6}$$

. In order to estimate the parameters the static gain vector g_{dc} of the system has to be computed. The standard procedure is to apply the "CAB" formula from the state space formulation:

$$\boldsymbol{g_{dc}} = \boldsymbol{C}\boldsymbol{A}^{-1}\boldsymbol{B} \tag{7}$$

which is the transfer function at $s = 0$. Since this computation implies the inverse of the 6×6 matrix $\boldsymbol{A}$, another computation is performed. Using the formulation in Equation 2, we can perform the following limits:

$$\begin{cases} \lim_{t\to+\infty} \dot{x} = 0 \\ \lim_{t\to+\infty} \ddot{x} = 0 \end{cases} \tag{8}$$

	data	nominal	error %
k_3/k_2	0.523	0.390	6.7 %
k_3/k_1	0.490	0.388	20.9 %

Table 1: Stiffnesses ratios results

The substitution 8 in 2a yields:

$$\boldsymbol{K}x = \boldsymbol{b} \tag{9}$$

The static gain is then:

$$\boldsymbol{g_{dc}} = \boldsymbol{K^{-1}b} \tag{10}$$

Note that it is equivalent to apply the Laplace tranform to the equation 2a and apply the final value theorem to it.

$$\boldsymbol{g_{dc}} = \left[\begin{array}{ccc} g_v \dfrac{k_1k_2 + k_3k_2 + k_3k_1}{k_1k_2k_3} & g_v \dfrac{k_2 + k_3}{k_2k_3} & g_v \dfrac{1}{k_3} \end{array} \right]^\top \tag{11}$$

Note that in equation 11 is evident from the expression the parallel between the stiffnesses (at steady state inertia and damping are invisible). The steady state value of the three output are available. Some other computations has to be made to make equation 11 suitable for the check on the stiffnesses ratios and the new estimation of g_v. The equation for the steady state values is:

$$\boldsymbol{Cx}_\infty = \boldsymbol{g_{dc}}u_\infty \tag{12}$$

Where the ∞ denotes the steady state value of the quantity. The Equation 12 is now expressed in terms of stiffnesses ratio: k_3 is fixed to the nominal value and two ratio are defined:

- fix k_3 on nominal value
- multiplied both sides times k_3
- replace $R_{32} = \dfrac{k_3}{k_2}$
- replace $R_{31} = \dfrac{k_3}{k_1}$

This operations are made in order to make the system easy to solve, and uncoupling the nonlinear factor g_v. This allow to make to solve the system in a cascade fashion from g_v to R_{13}. The equation 12 is now expressed in the following form:

$$k_3 \left[\begin{array}{c} x_1(\infty) \\ x_2(\infty) \\ x_3(\infty) \end{array} \right] = u(\infty) \left[\begin{array}{c} g_v + g_vR_{13} + g_vR_{23} \\ g_vR_{23} \\ g_v \end{array} \right] \tag{13}$$

Now the next step is to use the measured value of steady state response (input and output), take g_v as new estimated value and verify that R_{12} and R_{32} matches the nominal values. As set before $u(\infty) = 0.5$. Results on the ratios are shown in Table 2.1

2.2 Parameters estimation

$$ciao \tag{14a}$$

$$ciaone \tag{14b}$$