Mechanical vibration - System identification and modal analysis of 3-DOF linear system

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1 Dynamical system

1.1 The linear model

The chosen model is a linear plant consisting of 3 masses, 3 springs between them, and 3 dampers between each mass and the ground. The model is shown in Figure 1.

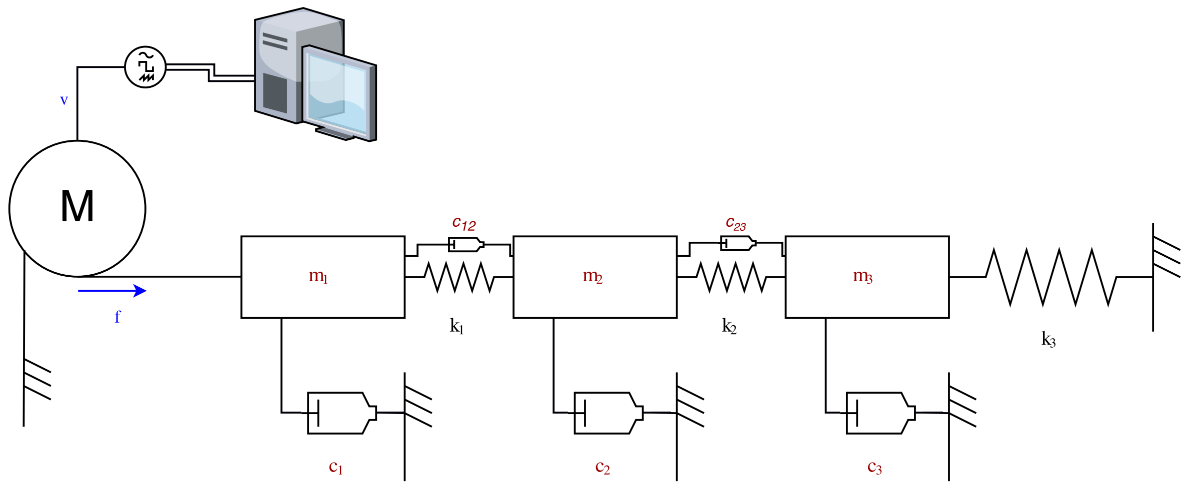

Figure 1: The chosen plant, in red the unknown parameters

1.2 equation of motion

$$\begin{cases} m_1\ddot{x}_1 = +k_1\left(x_2 - x_1\right) + c_{12}\left(\dot{x}_2 - \dot{x}_1\right) - c_1\dot{x}_1 + g_v v \\ m_2\ddot{x}_2 = +k_1\left(x_1 - x_2\right) + k_2\left(x_3 - x_2\right) + c_{12}\left(\dot{x}_1 - \dot{x}_2\right) + c_{23}\left(\dot{x}_3 - \dot{x}_2\right) - c_2\dot{x}_2 \\ m_3\ddot{x}_3 = +k_2\left(x_2 - x_3\right) + c_{23}\left(\dot{x}_2 - \dot{x}_3\right) - c_3\dot{x}_3 - k_3 x_3 \end{cases} \tag{1}$$

In the classical matrix form:

$$\boldsymbol{M}\ddot{x} + \boldsymbol{C}\dot{x} + \boldsymbol{K}x = \boldsymbol{b} \tag{2a}$$

where:

$$\boldsymbol{K} = \begin{bmatrix} +k_1 & -k_1 & 0 \\ -k_1 & +k_1 + k_2 & -k_2 \\ 0 & -k_2 & +k_3 \end{bmatrix} \tag{2b}$$

$$\boldsymbol{M} = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \tag{2c}$$

$$\boldsymbol{b} = \begin{bmatrix} K_v \\ 0 \\ 0 \end{bmatrix} \tag{2d}$$

$$\boldsymbol{C} = \begin{bmatrix} +c_1 + c_{12} & -c_{12} & 0 \\ -c_{12} & +c_2 + c_{12} + c_{23} & -c_{23} \\ 0 & -c_{23} & c_3 + c_{23} \end{bmatrix} \tag{2e}$$

1.3 state-space model

The linear model of the plant, expressed by the equation 2, is a SIMO model. A state-space form was chosen to represent this model. The matrices are the follwing:

$$A = \begin{bmatrix} \boldsymbol{Z}_{3\times3} & \boldsymbol{I}_{3\times3} \\ -\boldsymbol{M}^{-1}\boldsymbol{K} & -\boldsymbol{M}^{-1}\boldsymbol{C} \end{bmatrix} \tag{3a}$$

$$\boldsymbol{B} = \begin{bmatrix} \boldsymbol{Z}_{3\times1} \\ -\boldsymbol{M}^{-1}\boldsymbol{b} \end{bmatrix} \tag{3b}$$

$$\boldsymbol{C} = \begin{bmatrix} \boldsymbol{I}_{3\times3} & \boldsymbol{Z}_{3\times3} \end{bmatrix} \tag{3c}$$

$$\boldsymbol{D} = \begin{bmatrix} \boldsymbol{Z}_{3\times1} \end{bmatrix} \tag{3d}$$

where $\boldsymbol{I}$ is the identity matrix and $\boldsymbol{Z}$ is a matrix with all the entries equal to zero.

1.4 experimental setup

1.5 parameters and data available

1.6 initial hypothesis and approximations

- neglected motor electrical dynamics (instantaneous transmission of torque)
- rectilinear motion (all perfect aligned)
- inertia and damping of the motor are merged respectively into m_1 and c_1.

$$\begin{cases} m_1 = m_{block} + \frac{J_{motor}|_{zz}}{r^2} \\ c_2 = c_{block} + \frac{c_{motor}}{r^2} \end{cases} \tag{4}$$

where r is the radius of the gear-rack coupling (gear wheel), $J_{motor}|_{zz}$ is the inertia of the motor , c_{motor} the rotational damping and "block" quantities are the ones stricly related to the physical first mass.

2 System identification

2.1 step response analysis

First of all, the step response analysis can be performed. In this analysis the "static" coefficients can be estimated, they are:

- voltage to force g_v
- springs' stiffness k_i with $i \in 1, 2, 3$

The coefficient to be estimated is the *voltage-to-force* coefficient

$$f = (\boldsymbol{k_a} \cdot \boldsymbol{k_t} \cdot \boldsymbol{k_{mp}})v = g_v v \tag{5}$$

. In order to estimate the parameters the static gain vector g_{dc} of the system has to be computed. The standard procedure is to apply the "CAB" formula from the state space formulation:

$$g_{dc} = \boldsymbol{C}\boldsymbol{A}^{-1}\boldsymbol{B} \tag{6}$$

which is the transfer function at $s = 0$. Since this computation implies the inverse of the 6×6 matrix $\boldsymbol{A}$, another computation is performed. Using the formulation in Equation 2, we can take the following limits:

$$\begin{cases} \lim_{t\to+\infty} \dot{x} = 0 \\ \lim_{t\to+\infty} \ddot{x} = 0 \end{cases} \tag{7}$$

. The substitution 7 in 2a yields:

$$\boldsymbol{K}x = \boldsymbol{b} \tag{8}$$

. The static gain is then:

$$g_{dc} = \boldsymbol{K}^{-1}b \tag{9}$$

$$g_{dc} = \begin{bmatrix} \frac{k_1k_2 + k_3k_2 + k_3k_1}{k_1k_2k_3} & \frac{k_2 + k_3}{k_2k_3} & \frac{1}{k_3} \end{bmatrix}^\top \tag{10}$$

Note that in 10 is evident from the expression the parallele between the stiffnesses (at steady state inertia and damping are iinvisible).

2.2 Parameters estimation

$$ciao \tag{11a}$$

$$ciaone \tag{11b}$$