Mechanical vibration 2015/16 - System identification and modal analysis of 3-DOF linear system

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1 The system

1.1 The system and the experimental setup

The system consists of three different bodies on three carriers. The carriers are aligned and the bodies are constrained to slide along the common axis. Between the bodies two springs are located, and a third spring connects the frame and the last body. The first body is rigidly connected trough a rack-pinion gearing with a motor which is controlled in voltage with a PC interface. The position of each body is provided by an encoder. The zeroes of the positions are at the springs equilibrium positions. The scheme of the model is depicted in Figure 1. The displacement of the masses in meters is available from the encoder with the following resolution:

$$\Delta x = \frac{2\pi r_e}{16000}. \tag{1}$$

1.2 The dynamical model

1.2.1 Assumptions

To define a linear model of the system, some assumptions were made:

Rectilinear motion All the bodies (masses), and the rack of the rack-pinion gearing are supposed to move and exert forces along the same axis, which is the motion axis. Consequently, all the quantities are meant to be projected on this axis.

Viscous friction Only viscous frictions are present in the model.

Instantaneous electrical dynamics The model of the electrical dynamics of the motor is only a gain from voltage to force, expressed by the voltage-to-force factor, which is exploited in Section 2.1.

Motor mechanics merged Inertia and damping of the motor are merged respectively into m_1 and c_1 (refer to Figure 1)

$$\begin{cases} m_1 = m_{body} + \frac{J_{motor}|_{zz}}{r^2} \\ c_1 = c_{body} + \frac{c_{motor}}{r^2} \end{cases} \tag{2}$$

where r is the radius of the gear-rack coupling (gear wheel), $J_{motor}|_{zz}$ is the inertia of the motor , c_{motor} the rotational damping and "body" quantities are the ones stricly related to the physical first mass.

Last viscosities merged The term c_3 (refer to Figure 1) contains the viscous friction with the ground and the one due to the spring. In the model those two contributions cannot be quantified separately.

1.2.2 The linear model

The chosen model is a linear plant consisting of 3 lumped masses, 3 lumped springs between them (the last to the frame), and 5 dampers between each mass and the ground. The model is shown in Figure 1.

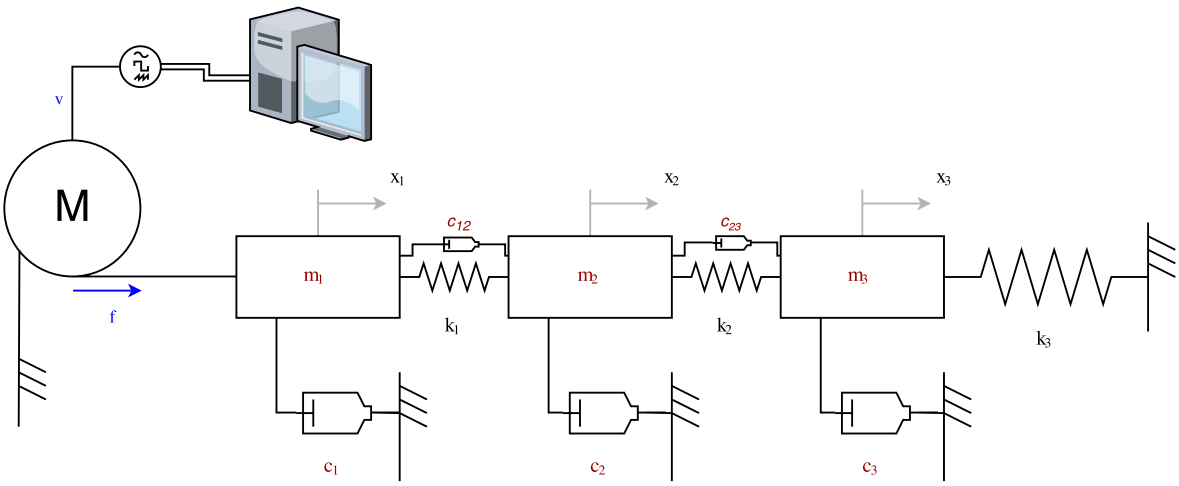


Figure 1: The chosen plant, in red the unknown parameters

1.3 equation of motion

$$\begin{cases} m_1\ddot{x}_1 = +k_1\left(x_2 - x_1\right) + c_{12}\left(\dot{x}_2 - \dot{x}_1\right) - c_1\dot{x}_1 + g_v v(t) \\ m_2\ddot{x}_2 = +k_1\left(x_1 - x_2\right) + k_2\left(x_3 - x_2\right) + c_{12}\left(\dot{x}_1 - \dot{x}_2\right) + c_{23}\left(\dot{x}_3 - \dot{x}_2\right) - c_2\dot{x}_2 \\ m_3\ddot{x}_3 = +k_2\left(x_2 - x_3\right) + c_{23}\left(\dot{x}_2 - \dot{x}_3\right) - c_3\dot{x}_3 - k_3 x_3 \end{cases} \tag{3}$$

In the classical matrix form:

$$\boldsymbol{M}\ddot{x} + \boldsymbol{C}\dot{x} + \boldsymbol{K}x = \boldsymbol{b}v \tag{4a}$$

where:

$$\boldsymbol{K} = \begin{bmatrix} +k_1 & -k_1 & 0 \\ -k_1 & +k_1 + k_2 & -k_2 \\ 0 & -k_2 & +k_3 \end{bmatrix} \tag{4b}$$

$$\boldsymbol{M} = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \tag{4c}$$

$$\boldsymbol{b} = \begin{bmatrix} g_v \\ 0 \\ 0 \end{bmatrix} \tag{4d}$$

$$\boldsymbol{C} = \begin{bmatrix} +c_1 + c_{12} & -c_{12} & 0 \\ -c_{12} & +c_2 + c_{12} + c_{23} & -c_{23} \\ 0 & -c_{23} & c_3 + c_{23} \end{bmatrix} \tag{4e}$$

1.4 State-space model

The linear model of the plant, expressed by the equation (4), is a SIMO model. A state-space form was chosen to represent this model. The matrices are the follwing:

$$\begin{cases} \dot{x} = \boldsymbol{A}x + \boldsymbol{B}v \\ y = \boldsymbol{C}x + \boldsymbol{D}v \end{cases} \tag{5a}$$

$$x = \begin{bmatrix} x_1 & x_2 & x_3 & \dot{x}_1 & \dot{x}_2 & \dot{x}_3 \end{bmatrix}^\top \tag{5b}$$

$$A = \begin{bmatrix} \boldsymbol{Z}_{3\times3} & \boldsymbol{I}_{3\times3} \\ -\boldsymbol{M}^{-1}\boldsymbol{K} & -\boldsymbol{M}^{-1}\boldsymbol{C} \end{bmatrix} \tag{5c}$$

$$\boldsymbol{B} = \begin{bmatrix} \boldsymbol{Z}_{3\times1} \\ -\boldsymbol{M}^{-1}\boldsymbol{b} \end{bmatrix} \tag{5d}$$

$$\boldsymbol{C} = \begin{bmatrix} \boldsymbol{I}_{3\times3} & \boldsymbol{Z}_{3\times3} \end{bmatrix} \tag{5e}$$

$$\boldsymbol{D} = \begin{bmatrix} \boldsymbol{Z}_{3\times1} \end{bmatrix} \tag{5f}$$

where $\boldsymbol{I}$ is the identity matrix and $\boldsymbol{Z}$ is a matrix with all the entries equal to zero.

1.5 Experimental setup

1.5.1 Data processing

An operation on data must be performed in order to use them. The data on diplacements is provided in *encoder counts*. They are converted in meters with the following convertion factor (g_x):

$$g_x = \frac{\Delta x}{\Delta \texttt{counts}} = 2\pi r_e \cdot \frac{\Delta \texttt{counts}}{16000 \frac{\texttt{counts}}{\texttt{encoder revolution}}}. \tag{6}$$

1.6 parameters and data available

The linear system in Section 1.2.2 is fully represented by 12 parameters, they are shown in Table 1. Only the stiffnesses of the springs (k_i) are known.

Name	g_v	k_1	k_2	k_3	m_1	m_2	m_3	c_1	c_2	c_3	c_{12}	c_{23}
Value	-	774	770	396	-	-	-	-	-	-	-	-
Units	N/V	N/m			K_g			$N s\, m^{-1}$				
Notes								c_a and c_b if proportional damping				

Table 1: Parameters (”-” stands for unavailable parameter)

The meaning of each parameter is depicted in Figure 1, except for the *voltage-to-force* coefficient. Such coefficient g_v is the factor which converts the voltage of the signal sent to the motor in the force exerted on the rack. It is the product of several physical gains as shown in Equation (7).

$$f = (k_a \cdot k_t \cdot k_{mp})v(t) = g_v v(t). \tag{7}$$

Where the physical meanings are:

$$k_a = \frac{1}{R_{motor}} \approx 2\ AV^{-1} \tag{8a}$$

Electrical conductance of the motor

$$k_t \approx 0.1\ NmA^{-1} \tag{8b}$$

Motor torque constant

$$k_{mp} = \frac{1}{r_{pinion}} \approx 26.25\ m^{-1} \tag{8c}$$

Trasmission ratio of the gearing

2 System identification

2.1 Steady state analysis

From the *step response analysis* the steady state values of input and output are considered. In this analysis the ”static” coefficients can be studied, they are:

- voltage to force g_v
- springs' stiffness k_i with $i \in 1, 2, 3$

The goal of this section is to estimate g_v and to verify the ratios between the stiffnesses k_i, w.r.t the nominal values. It is not possible to verify directly the k_i values because the steady state values are 3 as many as the equations of the generated system of equations, while the variables would be 4.

2.1.1 Calculations

In order to generate the system of equations, the static gain vector g_{dc} of the system is computed. A procedure may be to apply the ”CAB” formula from the state space formulation:

$$\boldsymbol{g_{dc}} = \boldsymbol{C}\boldsymbol{A}^{-1}\boldsymbol{B} \tag{9}$$

which is the transfer function at $s = 0$. Since equation (9) implies the inverse of the 6×6 matrix $\boldsymbol{A}$, another computation is performed. Using the formulation in Equation (4), the following limits are performed:

$$\begin{cases} \lim_{t\to+\infty} \dot{x} = 0 \\ \lim_{t\to+\infty} \ddot{x} = 0 \end{cases} \tag{10}$$

The substitution (10) in (4a) yields:

$$\boldsymbol{K}x = \boldsymbol{b} \tag{11}$$

The static gain is then:

$$\boldsymbol{g_{dc}} = \boldsymbol{K^{-1}b} \tag{12}$$

$$\boldsymbol{g_{dc}} = \left[\begin{array}{ccc} g_v \dfrac{k_1k_2 + k_3k_2 + k_3k_1}{k_1k_2k_3} & g_v \dfrac{k_2 + k_3}{k_2k_3} & g_v \dfrac{1}{k_3} \end{array} \right]^\top \tag{13}$$

Comment 1. Apply (10) to equation (4a) is equivalent to apply the Laplace tranform to the equation (4a) and apply the final value theorem to it

Comment 2. Looking at the Equation (13) is evident the series connections of the springs.

The steady state value of the three output are available. Some other computations has to be made to make equation (13) suitable for the check on the stiffnesses ratios and the new estimation of g_v. The equation for the steady state values is:

$$\boldsymbol{Cx_\infty} = \boldsymbol{g_{dc}}v_\infty \tag{14}$$

Where the ∞ denotes the steady state value of the quantity ($\lim_{t\to\infty} f(t)$). The Equation (14) is now expressed in terms of stiffnesses ratio: k_3 is fixed to the nominal value and two ratio are defined as following.

$$R_{31} = \frac{k_3}{k_1} \tag{15a}$$

$$R_{32} = \frac{k_3}{k_2} \tag{15b}$$

This operations are made in order to make the system easy to solve, compute at first the factor g_v. This allows to solve the system in a cascade fashion from g_v to R_{13}. The equation (14) is now expressed in the following form:

$$k_3 \left[\begin{array}{c} x_1(\infty) \\ x_2(\infty) \\ x_3(\infty) \end{array} \right] = v(\infty) \left[\begin{array}{c} g_v + g_vR_{13} + g_vR_{23} \\ g_vR_{23} \\ g_v \end{array} \right] \tag{16}$$

2.1.2 Steady state value picking

Now the next step is to use the measured value of steady state response (input and output), take g_v as new estimated value and verify that R_{12} and R_{32} matches the nominal values. The input is set on $v(\infty) = 0.5$.
The value of steady state response ($x(\infty)$) are picked from the data directly in a point far from the transient, as shown in Figure 2. The input waveform is shown in Figure 3.

The obtained values for each output ($\boldsymbol{x_\infty}$) are 4. The mean value of the ratios (R_{31},R_{32}) is estimated by the **arithmetic average**. Given ($\boldsymbol{x_\infty}$) averaged from the data, Equation (16) is solved and the results are shown in Table 2. The error is defined as `error`$\% = 100\dfrac{data\ value}{nominal}$

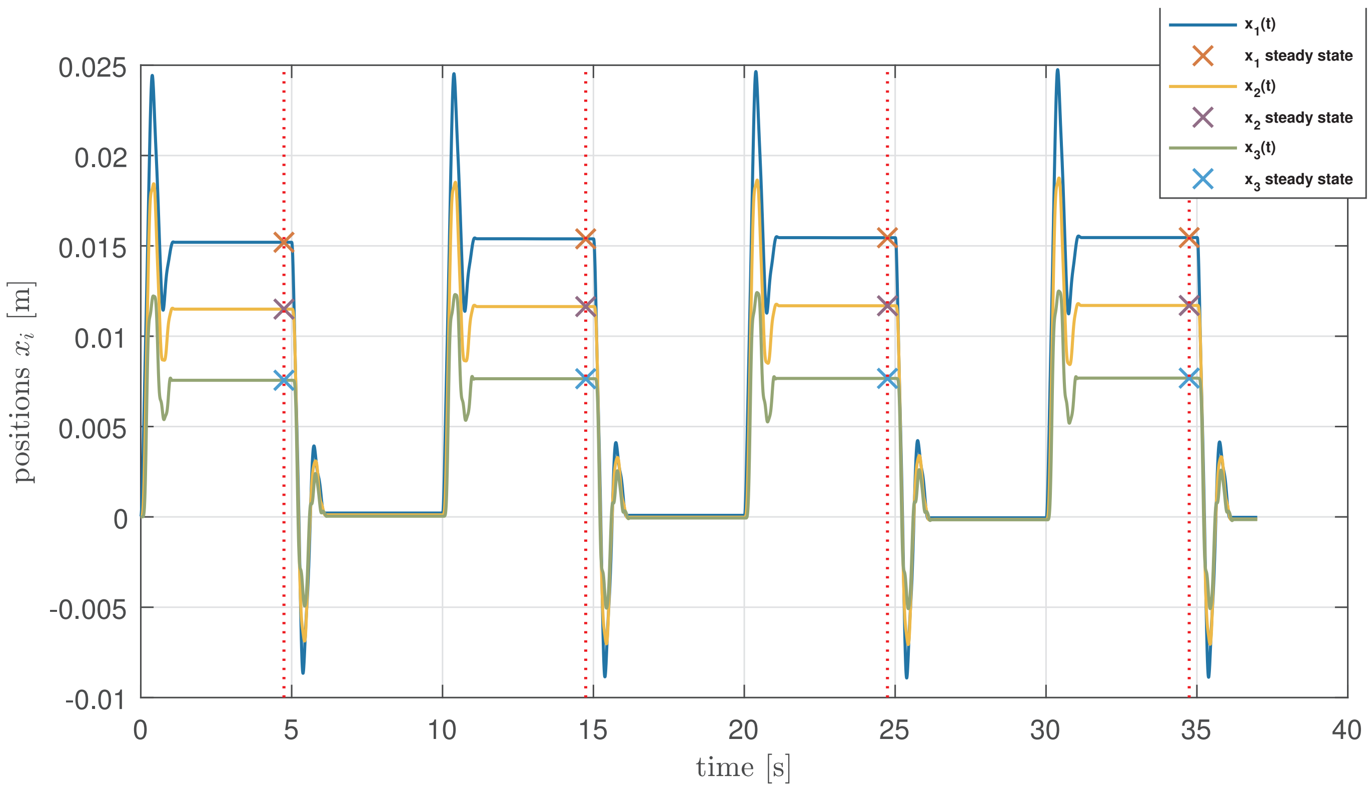


Figure 2: Steady state value picking

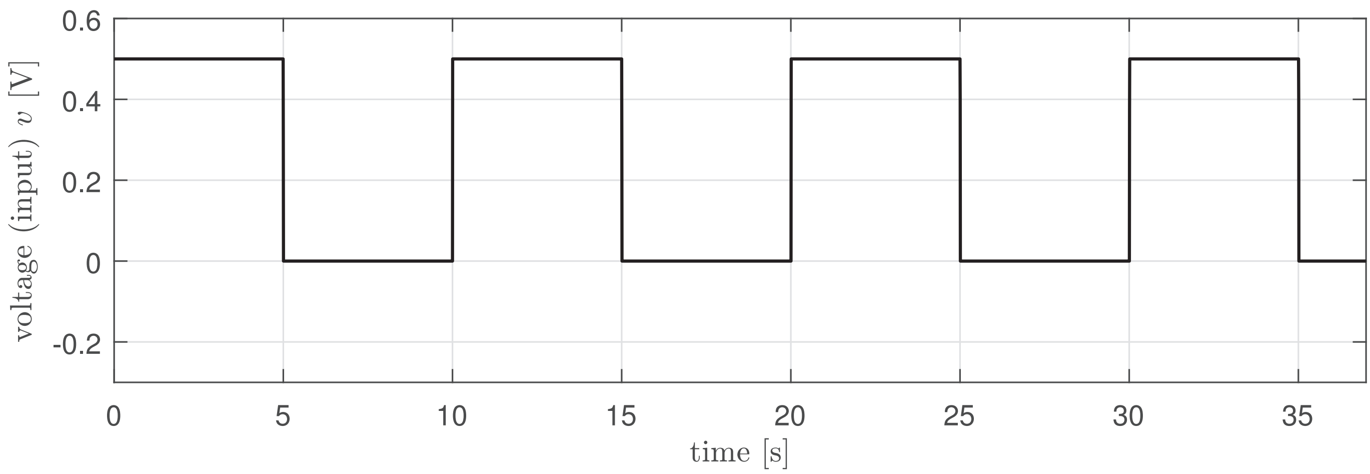


Figure 3: Step input

	from data	nominal	error %
k_3/k_2	0.5227	0.5143	6.7 %
k_3/k_1	0.4900	0.5116	4.4 %
	from data	initial	error %
g_v	6.0497	5.2500	15.2 %

Table 2: Stiffnesses ratios and voltage-to-force coefficients results from steady state analysis $error = \dfrac{value - nominal}{nominal}$

The *voltage-to-force* estimation is shown in table 2.

2.2 Parameters estimation

To estimate the parameters the **impulse response** is used. The *voltage-to-force* coefficient g_v is one of the parameters to be estimate in order to include the possibility to cross-check the result with the step response results. The parameters to estimate are shown in table 1.

2.2.1 Estimation strategy

The strategy to estimate the parameters consists in use the **linear model** described in 1.2.2, use the same input as the real model, and compare the output with the real output tuning the parameters to minimize the difference in a LSQ sense.

Box 1 Estimation strategy key-points

1. Choose the linear model
2. Provide the same input as the real system (an impulse-like signal)
3. Simulate the output (time domain)
4. Compare the outputs with the real ones.
5. Computing the sum of squares of the residuals
6. Tune the parameters iteratively to minimize this quantity

The implementation of the strategy in Box 1 is implemented creating a MATLAB function which presents in input the parameters and in output the sum of the squares of the residuals. This is performed with a simulation of the system in (5) executed inside the function, the data of the real system (output and input) are available as a global variable, the inputs are the parameters to tune, the only output of the function is the sum of square to minimize. The strategy and the function to minimize (orange box) is shown in Figure 4.

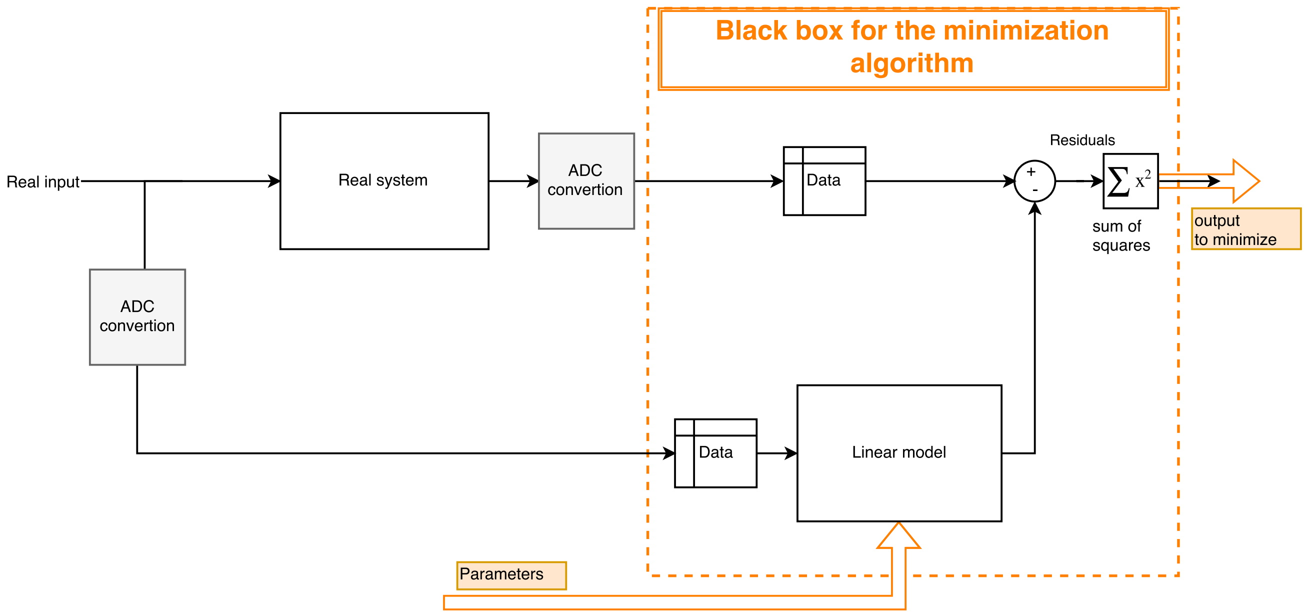


Figure 4: estimation strategy scheme, Function to minimize in orange

Comment 3. A single evaluation of the function corresponds to a simulation of the linear system, the parameters are updated by the algorithm after the evaluation of the function

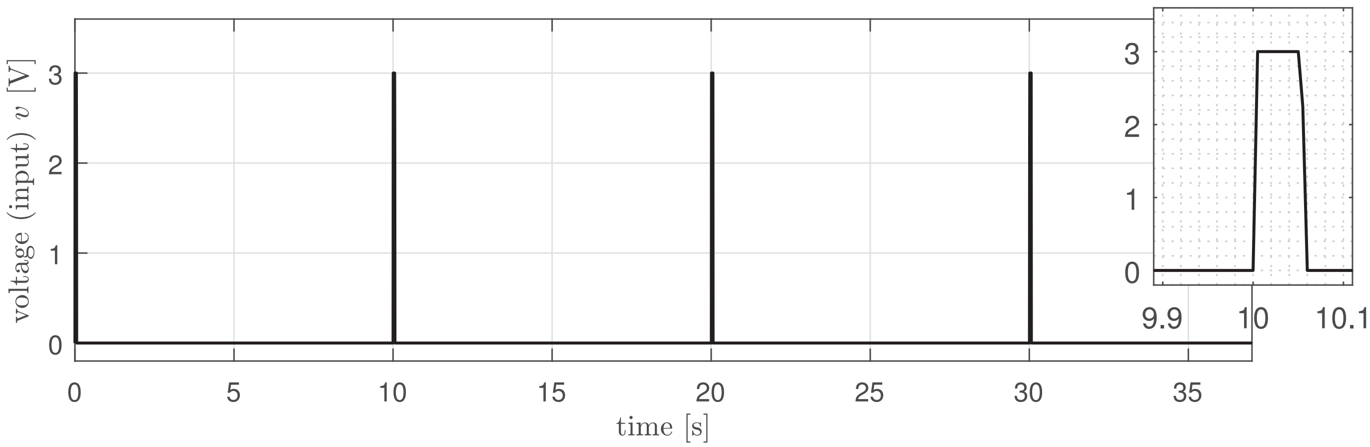


Figure 5: The pulse-like input (zoom of the single pulse on the right)

The implementation of the strategy is performed with MATLAB, the function `lsqnonlin` is used. The input used is a pulse-like waveform which excite the system four times, the signal is shown in Figure 5. The estimation will be performed with two different models: the **free damping** case (equation (5)) and the **proportional damping** case, it is explained in section 2.2.3.

2.2.2 Free damping case

The first estimation is performed using the model in Equations (5), the parameters to estimate are 9.
The results of this optimization are shown in table 3

Name	g_v	m_1	m_2	m_3	c_1	c_2	c_3	c_{12}	c_{23}
Value	6.223	1.565	1.461	1.146	2.923	1.806	2.002	0.00221	0.00001
Unit	N/V	K_g			$N s\, m_{-1}$				

Table 3: Estimation results (model with free damping)

A plot with the comparison of the models is provided in Figure 6. The normalized root means square errors are provided for each DOF in table 5

2.2.3 Proportional damping case

The procedure is exactly the same as the case of the free damping. The parameters to estimate now are 6 instead of 9. The damping now is represented by only 2 parameters since it is a linear combination of the mass matrix and the stiffness matrix, as shown in Equation (17). The interesting property of this representation is the possibility to perform the modal decomposition on the system. This property will be exploited in Section 3.3.1

$$\boldsymbol{C} = c_a \boldsymbol{M} + c_b \boldsymbol{K} \tag{17}$$

Name	g_v	m_1	m_2	m_3	c_a	c_b
Value	6.223	1.565	1.461	1.146	1.528	8.583e-05
Unit	N/V	K_g			$N s\, m_{-1}$	

Table 4: Estimation results (model with proportional damping)

Figure 6: Comparison between the response of the model and the response of the system **free damping case**

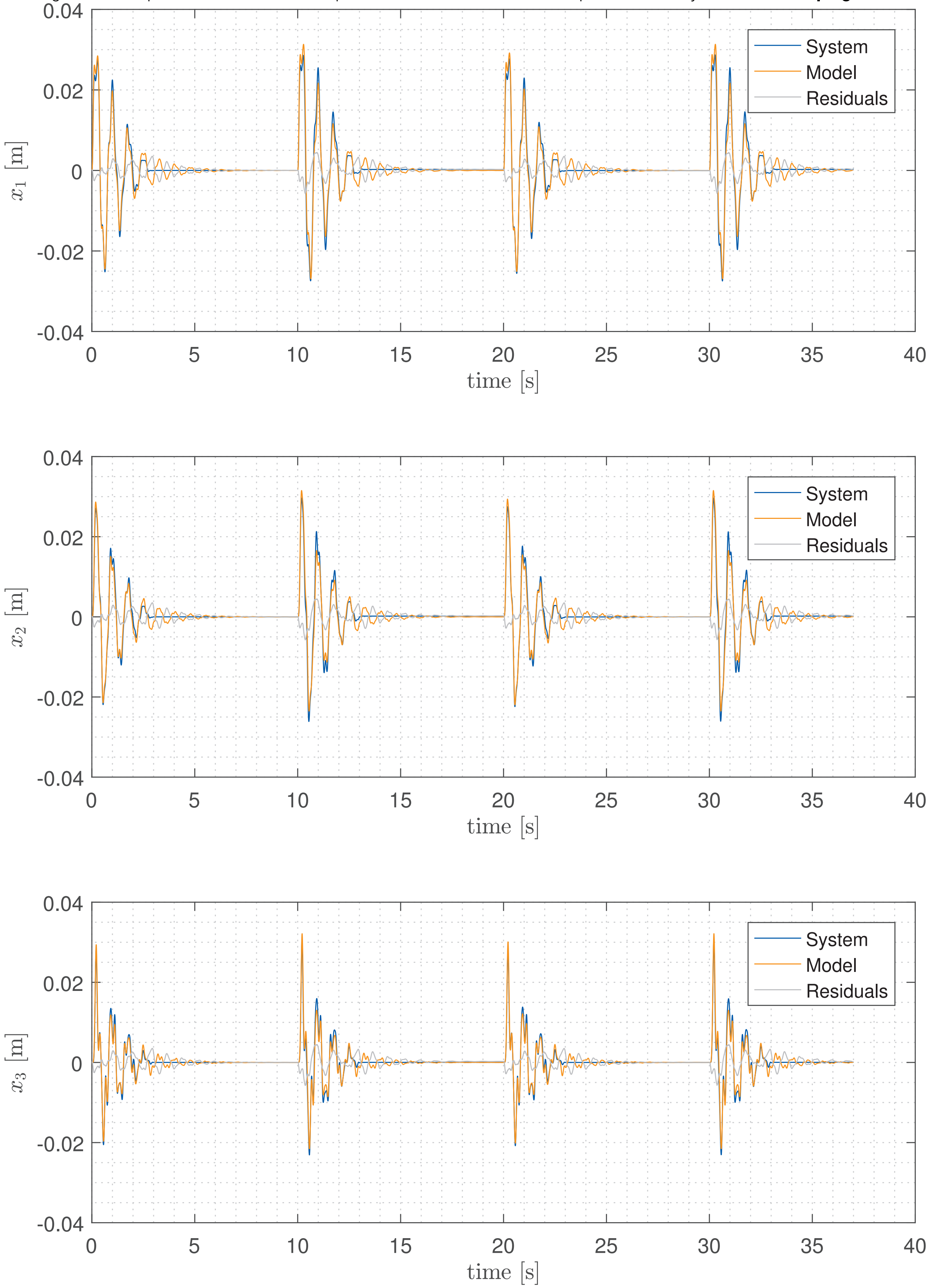
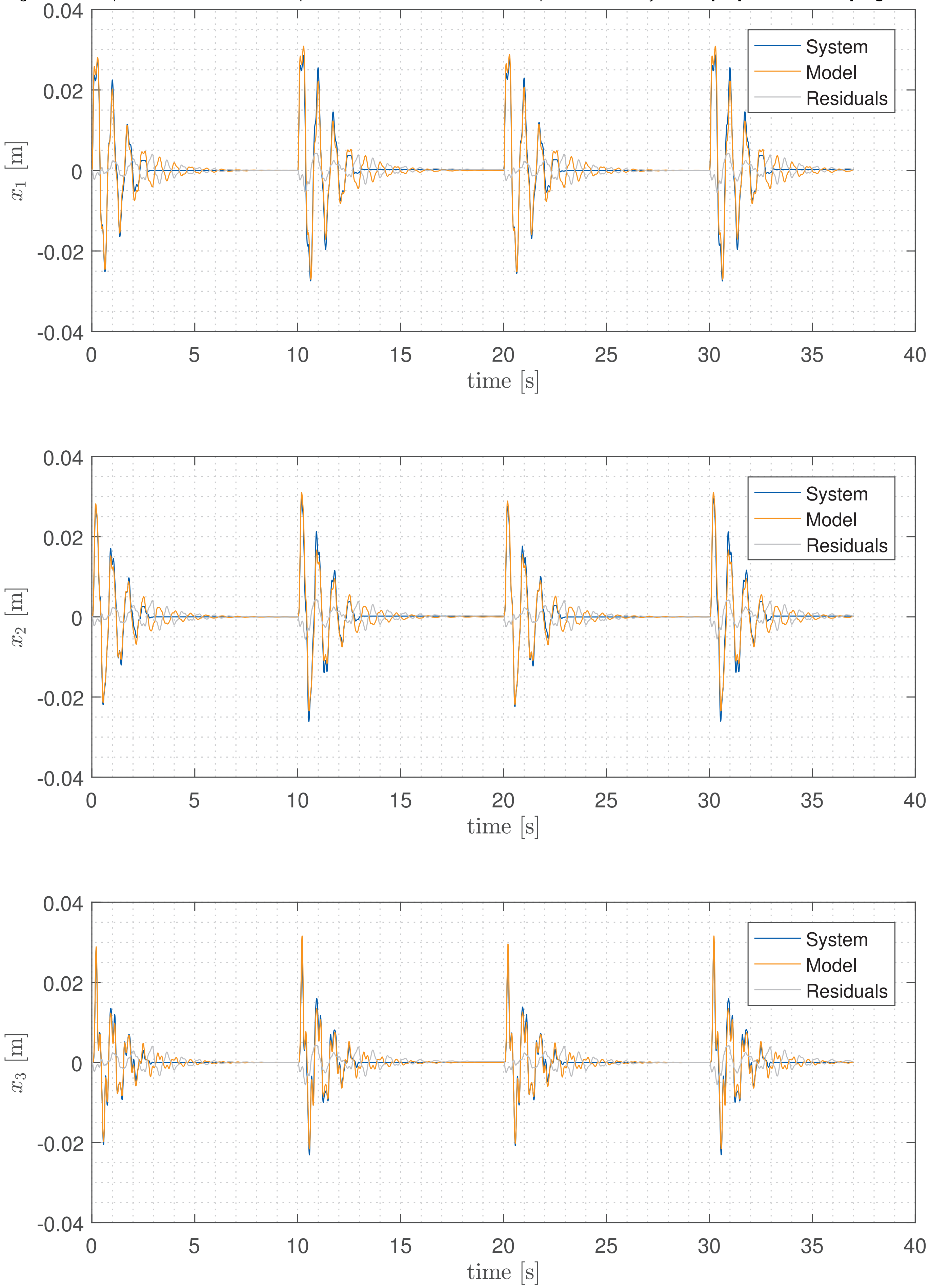


Figure 7: Comparison between the response of the model and the response of the system - **proportional damping case**



2.3 Conclusion and considerations

First guesses The procedure in box 1 requires a first guess. The bounds of the masses and the damping coefficients were known. Values within the bound were chosen as first guesses. No significant changes are detected. (10^{-4} K_g on masses for example).

Spring damping As evident in table 3, the damping between the masses seems negligible. This is also manifested in the prevalence of the diagonal term in the damping matrix:

$$\begin{bmatrix} 2.925 & -0.002 & 0.000 \\ -0.002 & 1.809 & -0.000 \\ 0.000 & -0.000 & 2.002 \end{bmatrix} \tag{18}$$

where p-pedix indicates the parameters estimated in the proportional damping case.

Proportional damping The proportional damping model provides very similar results, it means a modal decomposition will be possible witouth approximating to much the damping matrix.

Dry friction A nonlinearity in the model may be the dry friction between bodies and carriers. The static friction in particular is evident in the final transient. See Figure 8.

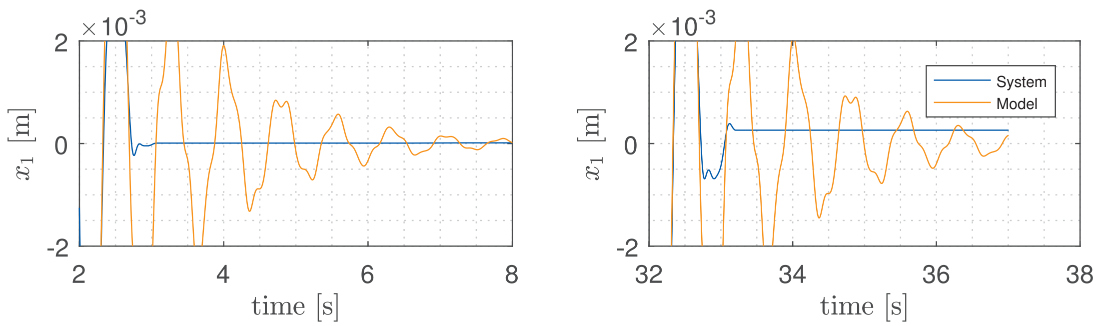


Figure 8: Sudden stop of the motion because of the static friction, a difference in equilibrium positions denotes the presence of a stick set.

2.3.1 Multiple single DOF system case

It is possible to detach the springs and to fix the masses on the equilibrium position. Then the system set-up can be disassembled and reassembled in order to estimate the parameters as a combination of analysis of single DOF systems. To obtain a suitable strategy, some considerations has to be made.
For each single DOF system, 3 parameters has to be estimated: m , k and ζ (or c). It is not possible to get all the parameters from the free response of a single DOF system. Looking at the equation (19).

$$x(t) = X_o e^{-\zeta\omega_n t} \sin\left(\omega_n \sqrt{1-\zeta^2} t + \phi_o\right) \tag{19}$$

mass *m* and *k*, appear only into $\omega_n = \sqrt{k/m}$. Even trying to rearrange the response with $\zeta\omega_n = c/2m$ the three variables appears only in two ratios. To make the mass estimation possible with the response in (19), The stiffness k must be estimated with a different experiment (or the mass if it is possible to use a weight-scale).
Once k is known, it is possible to estimate the damping ratio ζ and then the mass by means of the **logarithmic decrement**.

A possible strategy for the system identification is then summarized as following (Box 2).

Box 2 Strategy
Measure at first **all the springs stiffness separately**. It is possible applying a know **constant force** and fixing the **second mass**. With the displacement x_1 it is possible to compute the stiffness, as shown in figure 9.

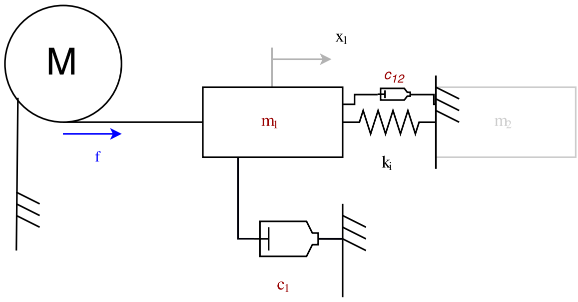


Figure 9: configuration for the estimation of the masses and the daping coefficients

For each spring k can be derived as:

$$k = \frac{f_{constant}}{x_1(\infty)} \tag{20}$$

where $x(\infty)$ is the steady state value. It can be averaged to decrease the zero-mean noise. Once the stiffness are known, the logarithmic decrement method for the estimation of m and ζ can be performed, with the configurations shown in figure 10.

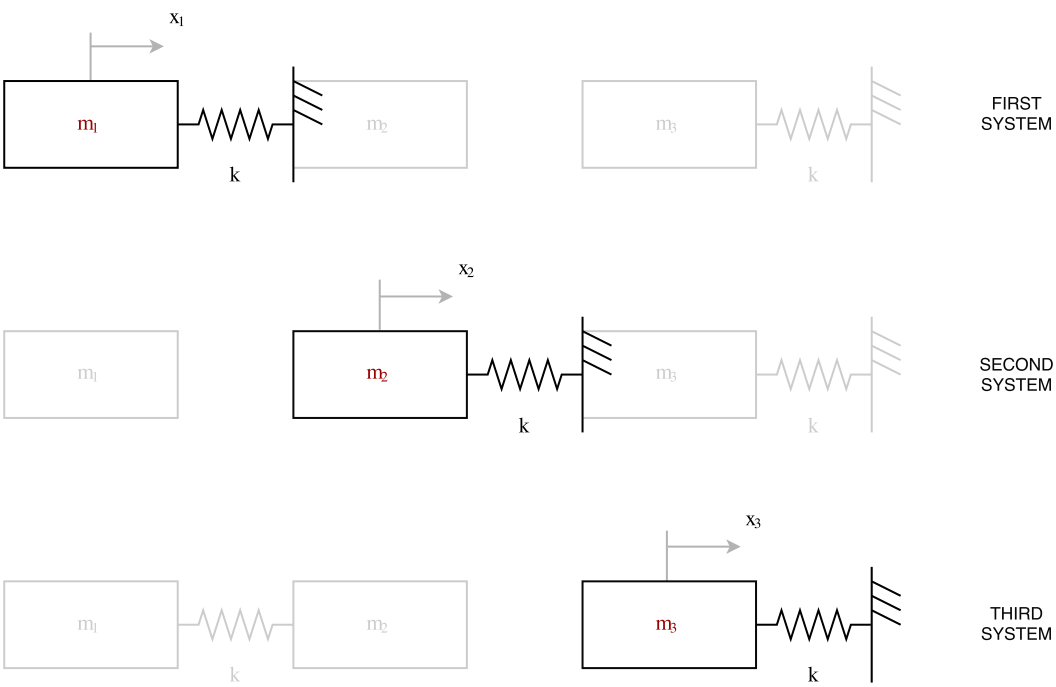


Figure 10: configuration for the estimation of the masses and damping

The masses can be moved **manually**, since the initial conditions are not required. ⚠Note that it is not possible to uncouple the springs damping and the carrier damping in one DOF systems. To separate this two contributions at least two mass are needed.

Table 5: NRMSE for the outputs

	Free damping	Proportional damping
x_1 `NRMSE`	81.52 %	81.22 %
x_2 `NRMSE`	81.35 %	81.05 %
x_3 `NRMSE`	81.87 %	81.35 %

3 Modal analysis

3.1 Eigenvalue problem

Define a system expressed by the following equation:

$$\boldsymbol{M}\ddot{x}(t) + \boldsymbol{K}x(t) = \boldsymbol{b}v(t) \tag{21}$$

this system is the **undamped** version of system (4). The resonance frequencies of the system in Equation (4) (the same as equation (21)) can be derived solving an eigenvalue problem stated as follows:

Box 3 The resonance frequencies , modal shape vectors and eigenvalue problem

Given the system expressed in (4) define a resonance frequency an $\omega > 0$ with $x_e \neq 0_v \in \Re^3$ such that :

$$\left(\boldsymbol{K} - \omega^2\boldsymbol{M}\right)x = 0_v \tag{22}$$

Equation (22) is equivalent to Equation (23).

$$Ker\left(\boldsymbol{K} - \omega^2\boldsymbol{M}\right) \neq 0_v \quad \Leftrightarrow \quad det\left(\boldsymbol{K} - \omega^2\boldsymbol{M}\right) = 0 \tag{23}$$

The quantities ω are the **natural frequencies** of the system, and the eigenvectors x_e are the **modal shape vectors** of the system. Associate a frequency ω with its square ω^2 which is an eigenvalue of (22), take the corresponding eigenvector and define the pairs:

$$(u_i, \omega_i) \quad i \in \{1, 2, 3\} \tag{24}$$

Define the **Modal shape matrix**, the matrix of column vector u_i sorted by increasing frequency.

$$\boldsymbol{U} = \left[\begin{array}{ccc} u_1 & u_2 & u_3 \end{array}\right] \tag{25}$$

Comment 1. solving the problem in the Box 3, is equivalent to find the poles of the transfer function between the input and the output of the undamped system 21. the poles are $\pm i\omega$.

To solve the problem in Box 3 MATLAB provides a function `eig`, which receives in input the two matrices ($\boldsymbol{K}$ and $\boldsymbol{M}$) and returns the eigenvalues and the eigenvectors in a matrix. Taking the square root of the ladders the ω are obtained, the Modal shape matrix is directly provided by the function. The modal shape vectors are orthogonal each other and this is a property which can be verified.

3.1.1 Results

The procedure shown in the previous Section 3.1 is the same for both the case of Free damping and proportional damping. The difference leads in the different values estimated. They are shown in table 3 and 4. The values obtained from the eigenvalue problem are shown in table 6.

Table 6: Values of frequencies and modal shapes vector obtain solving the eigenvalue problem

	Free damping			proportional damping		
ω [rad/s]	8.2808	27.4038	41.8258	8.2709	27.4019	41.8500
	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$\boldsymbol{U}$	0.8614	-0.5183	-2.5368	0.8609	-0.5271	-2.5621
	0.6099	-1.3058	2.3301	0.6093	-1.3111	2.3721

3.2 Rayleight method

The Rayleight method is a method to find simultaneously the resonance frequencies and the modal shapes vector. Ths method is based on the search of stationary point in a function, the Rayleight quotient. The Rayleight quotient is

$$R_q(\boldsymbol{x}) = \frac{\boldsymbol{x}^\top \boldsymbol{K} \boldsymbol{x}}{\boldsymbol{x}^\top \boldsymbol{M} \boldsymbol{x}}. \tag{26}$$

The Rayleight quotient manifests two properties which make it suitable to do so:

1. Given an eigenvector (modal shape vector) $R_q(U_i)$ = ω_i^2
2. $R_q(x)$ Presents a stationary point in the neighborhood of $x = U_i$

   Additional properties are:
3. The first frequency (the lowest) the first mode corresponds to a minimum of $R_q(x)$
4. The last frequency (the largest) the last mode corresponds to a maximum of $R_q(x)$.

The Rayleight method consists in find the stationary values of $R_q(x)$.

3.2.1 Implementation

The procedure is implemented in MATLAB with the following steps.

- The Rayleight quotient is defined as a function of x, conserving only two degree of freedom of the vector x, the function to minimize is then:

$$R_q(\alpha, \beta) = \frac{\begin{bmatrix}1 & \alpha & \beta\end{bmatrix} \boldsymbol{K} \begin{bmatrix}1 & \alpha & \beta\end{bmatrix}^\top}{\begin{bmatrix}1 & \alpha & \beta\end{bmatrix} \boldsymbol{M} \begin{bmatrix}1 & \alpha & \beta\end{bmatrix}^\top} \tag{27}$$

- The gradient in α and β of $R_q(\alpha, \beta)$ is computed analytically (it is not reported here) (this is convenient since there are 3 DOF).

$$\nabla R_q = \begin{bmatrix} \frac{\partial R_q}{\partial \alpha} & \frac{\partial R_q}{\partial \beta} \end{bmatrix} \tag{28}$$

- The equation (29) (null gradient) is solved. Where the solution are α_i and β_i with $i \in 1, 2, 3$.

$$\nabla R_q = 0_{1\times 2} \tag{29}$$

- The quantities are computed as shown in Equation (30), the results are shown in Table 7.

$$\begin{cases} U_i = [\ 1\ \alpha_i\ \beta_i\ ] \\ \omega_i = R_q(\alpha_i,\ \beta_i) \end{cases} \tag{30}$$

- Since $R_q(\alpha, \beta)$ is a surface. A contour plot is provided to verify the points. Property 3 and 4 are used to check the last and the first frequencies, which have to be respectively a maximum and a minimum. The contour plot is shown in Figures 11 and 12, for the cases of free damping and proportional damping respectively.
- The orthogonality of the modal shape vectors is check performing scalar product between them. The expected result has to be zero or very small because of the numerical approximation ($\div 10^{-16}$).

Table 7: Values of frequencies and modal shapes vector obtained using the Rayleight method

	Free damping			proportional damping		
ω [rad/s]	8.2808	27.4038	41.8258	8.2709	27.4019	41.8500
$\boldsymbol{U}$	1.0000 0.8614 0.6099	1.0000 -0.5183 -1.3058	1.0000 -2.5368 2.3301	1.0000 0.8609 0.6093	1.0000 -0.5271 -1.3111	1.0000 -2.5621 2.3721

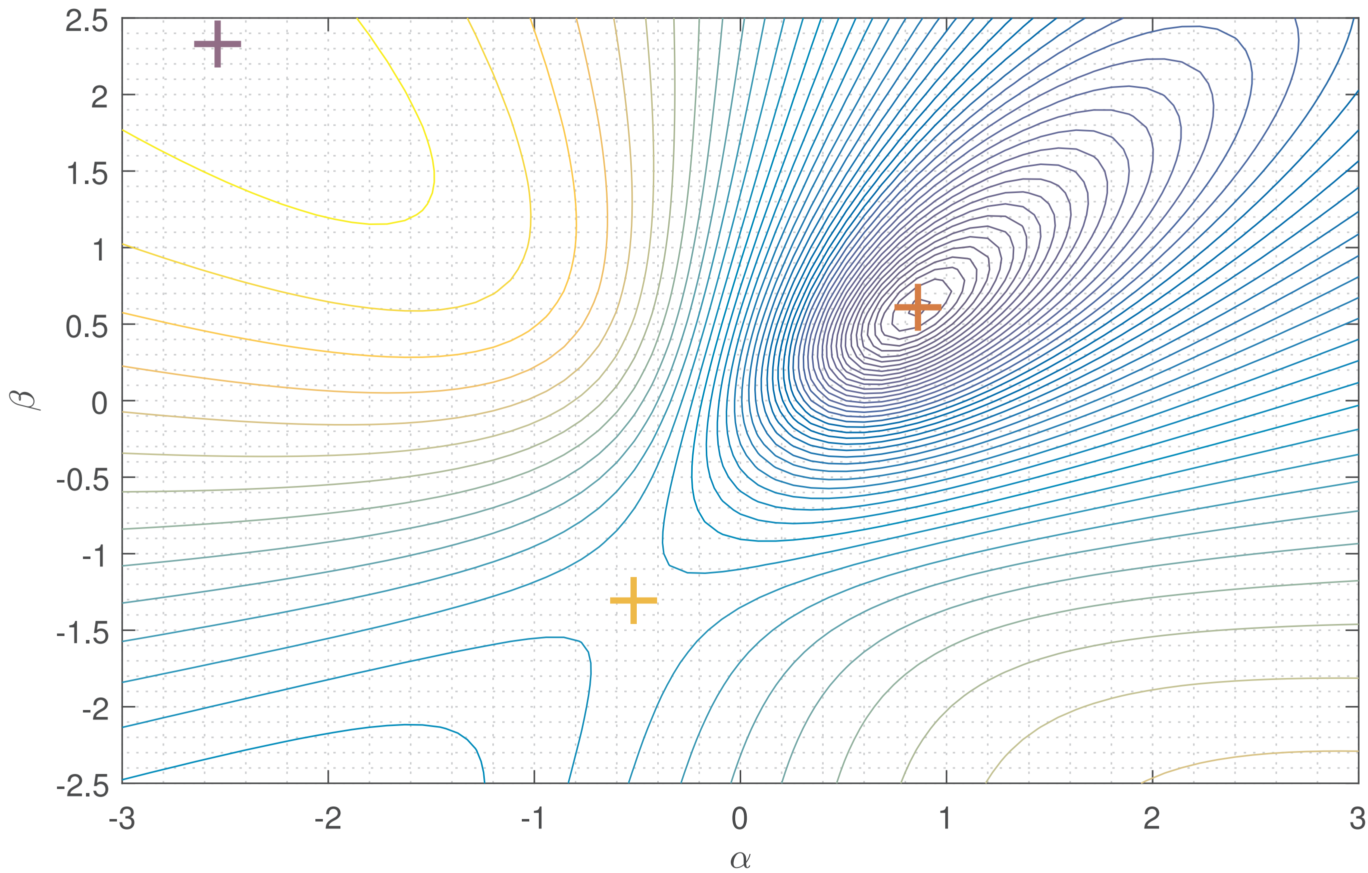


Figure 11: Controur plot of the Rayleight quotient in case of free damping, with stationary points

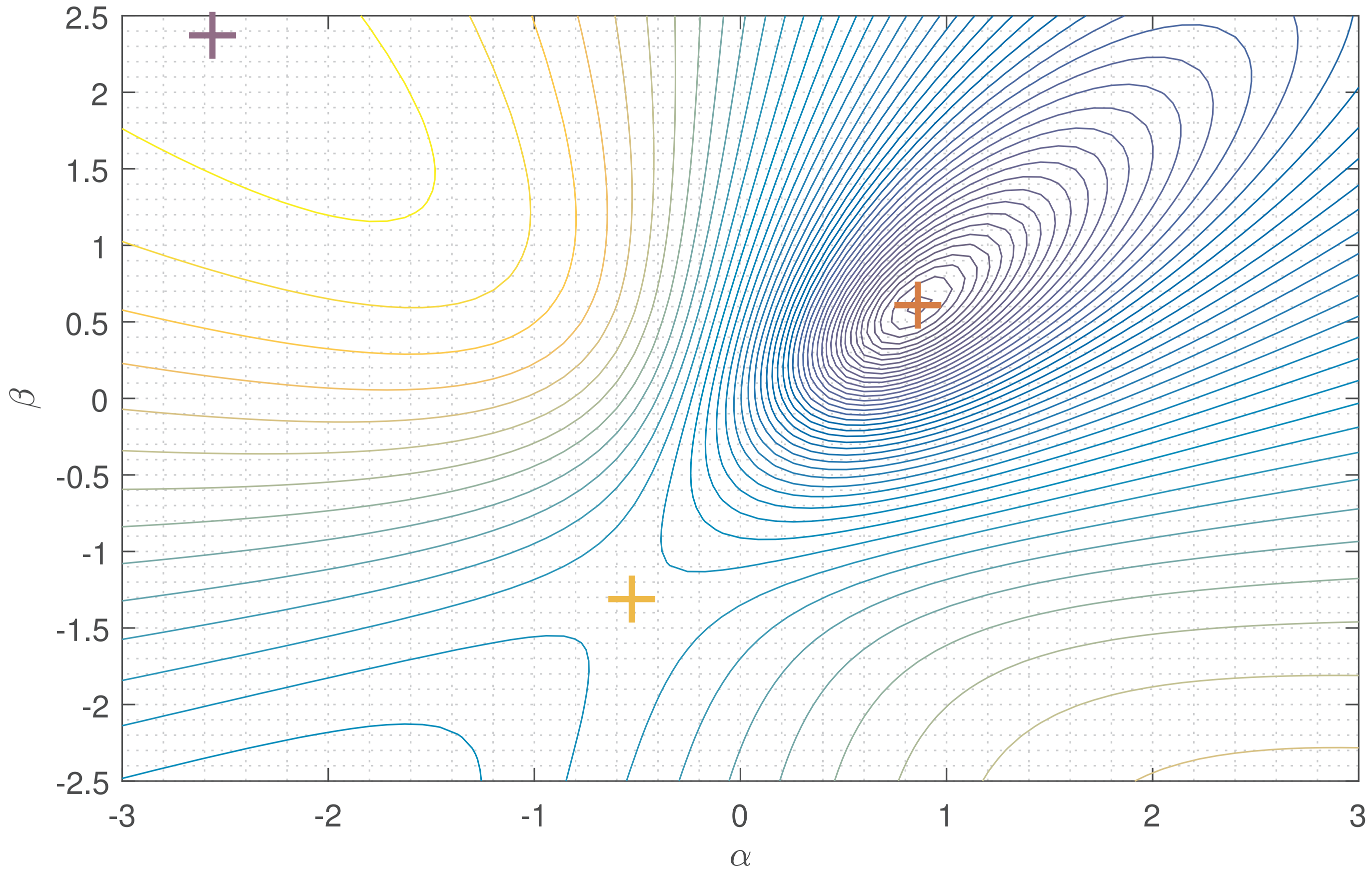


Figure 12: Controur plot of the Rayleight quotient in case of proportional damping, with stationary points

3.3 Matrix Iteration Method

Another method to find the eigenpairs is the Matrix Iteration Method. This method, as the name suggest, is an iterative method. Define the dynamic matrix:

$$\boldsymbol{D} = \boldsymbol{K}^{-1}\boldsymbol{M} \tag{31}$$

The algorithm

1. Define the first matrix $\boldsymbol{D}$ from Equation (31).
2. Define a first guess of the vector x.
3. Premultiply times the matrix D the vector x obtaining a new vector x^+.
$$x^+ = \boldsymbol{D}x \tag{32}$$
4. Normalize the vector x on the forst element.
$$x^+ = \frac{x}{x(1)} \tag{33}$$
5. Repeat steps 3. and 4. on the new x.
6. Repeat steps 3. 4. 5. N times. The greater the number of iteration N the greater the accuracy. In this case N=15 is chosen since the computation is not heavy.
7. After the N iterations, the vector x converge to the first eigenvector of $\boldsymbol{D}$, the one with the highest eigenvalue (lowest frequency of the undamped system).
8. The frequency is computed according to equation (34), the values of the frequency and the eigenvector are stored. The frequency is the square root of the reciprocal of the eigenvalue. The modal shape vectors are the eigenvectors (see comment 2) Actually this step is redundant, because the eigenvalue can be picked also from $x(1)$ before the last step 4.
$$\begin{cases} \omega_i = \left(\dfrac{x^\top D_i x}{x^\top x}\right)^{-\dfrac{1}{2}} \\ u_i = x \end{cases} \tag{34}$$

Matrix deflation: Since the method can detect only the first eigenvector, to find the ther ones somehow an elimination of the first eigenvalue from the image of the matrix has to be performed. A manipulation can be used to achieve this effect: the matrix deflation. This procedure update the Matrix D without affecting the eigenvectors, but reducing to zero a chosen eigenvalue. (Numerically it means reduce it to small values like 10^{-18})

9. The matrix D is updated performing the matrix deflation with the last eigenvalue.

$$\boldsymbol{D}^{+} = \boldsymbol{D} - \frac{u_i u_i^\top \boldsymbol{M}}{u_i^\top \boldsymbol{M} u_i} \tag{35}$$

10. The procedure from steps 3 to 9 is repeated for each additional eigenpair to find.

Comment 2. Equation (34) comes from the standard eigenvalue problem in Box 3:

$$\left(\boldsymbol{K} + \omega^2 \boldsymbol{M}\right) x = 0 \Leftrightarrow \frac{x}{\omega^2} = \boldsymbol{K}^{-1} \boldsymbol{M} x \Leftrightarrow \frac{x}{\omega^2} = \boldsymbol{D} x \tag{36}$$

From Equations (36) is evident that the eigenvalues of $\boldsymbol{D}$ are the reciprocals of the frequencies squared and the eigenvectors are the same as the modal shape vectors.

The whole procedure is performed for the case of free damping and the case of proportional damping. The results are shown in table 8. A block diagram which resumes the implemented method is shown in Figure 13. Again a orthogonality check is performed on the modal shape vectors.

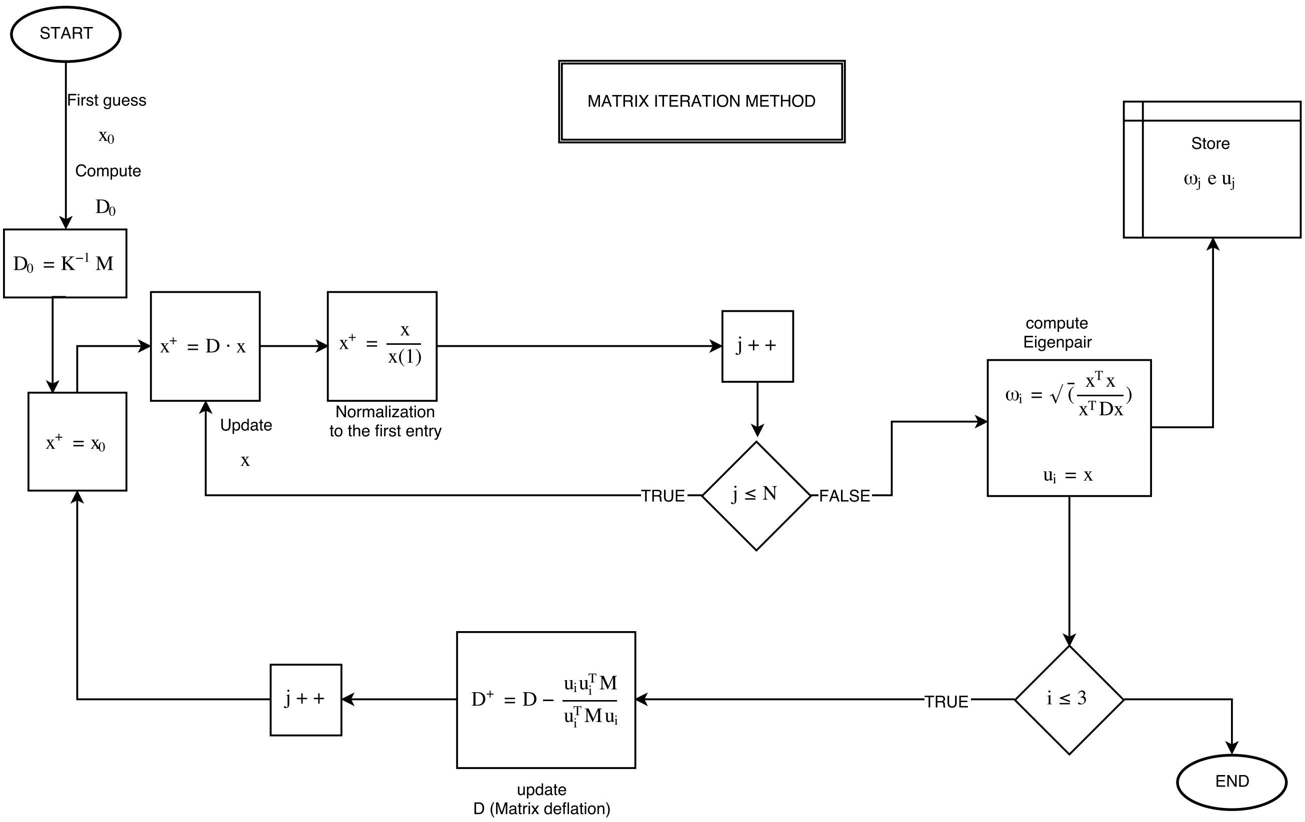


Figure 13: the Matrix Iteration Method

Table 8: Values of frequencies and modal shapes vector obtained using the Matrix Iteration Method

	Free damping			proportional damping		
ω [rad/s]	8.2808	27.4038	41.8258	8.2709	27.4019	41.8500
$\boldsymbol{U}$	1.0000 0.8614 0.6099	1.0000 -0.5183 -1.3058	1.0000 -2.5368 2.3301	1.0000 0.8609 0.6093	1.0000 -0.5271 -1.3111	1.0000 -2.5621 2.3721

3.3.1 Modal decomposition

In the proportional damping case the modal decomposition can be performed. Using the modal shape matrix U, definitively the one from the proprtional damping parameters. The new coordinates are:

$$\tilde{x} = \boldsymbol{U}^{\top} x \tag{37a}$$

Also the vector of the inputs has to be projected:

$$\tilde{b} = \boldsymbol{U}^{\top} b \tag{37b}$$

The matrices becomes diagonal and they can be computed as shown below:

$$\tilde{\boldsymbol{M}} = \boldsymbol{U}^{\top}\boldsymbol{M}\boldsymbol{U} \quad (37c) \qquad \tilde{\boldsymbol{K}} = \boldsymbol{U}^{\top}\boldsymbol{K}\boldsymbol{U} \quad (37d) \qquad \tilde{\boldsymbol{C}} = \boldsymbol{U}^{\top}\boldsymbol{C}\boldsymbol{U} \quad (37e)$$

Comment 3. In Equation (37a) it is possible to use the inverse of U instead of the transpose to get the coordinates. This is due to the orthogonality property of U.

The matrices are shown below.

$$\tilde{\boldsymbol{M}} = \begin{pmatrix} m_1\,\mathrm{U}_{11}{}^2 + m_2\,\mathrm{U}_{12}{}^2 + m_3\,\mathrm{U}_{13}{}^2 & 0 & 0 \\ 0 & m_1\,\mathrm{U}_{21}{}^2 + m_2\,\mathrm{U}_{22}{}^2 + m_3\,\mathrm{U}_{23}{}^2 & 0 \\ 0 & 0 & m_1\,\mathrm{U}_{31}{}^2 + m_2\,\mathrm{U}_{32}{}^2 + m_3\,\mathrm{U}_{33}{}^2 \end{pmatrix} \tag{38}$$

$$\tilde{\boldsymbol{K}} = \left[\, \tilde{K_1}\; \tilde{K_2}\; \tilde{K_3} \,\right] \tag{39a}$$

$$\tilde{K_1} = \begin{pmatrix} \mathrm{U}_{11}\,(\mathrm{U}_{11}\,k_1 - \mathrm{U}_{12}\,k_1) - \mathrm{U}_{12}\,(\mathrm{U}_{11}\,k_1 + \mathrm{U}_{13}\,k_2 - \mathrm{U}_{12}\,(k_1+k_2)) - \mathrm{U}_{13}\,(\mathrm{U}_{12}\,k_2 - \mathrm{U}_{13}\,(k_2+k_3)) \\ 0 \\ 0 \end{pmatrix} \tag{39b}$$

$$\tilde{K_2} = \begin{pmatrix} 0 \\ \mathrm{U}_{21}\,(\mathrm{U}_{21}\,k_1 - \mathrm{U}_{22}\,k_1) - \mathrm{U}_{22}\,(\mathrm{U}_{21}\,k_1 + \mathrm{U}_{23}\,k_2 - \mathrm{U}_{22}\,(k_1+k_2)) - \mathrm{U}_{23}\,(\mathrm{U}_{22}\,k_2 - \mathrm{U}_{23}\,(k_2+k_3)) \\ 0 \end{pmatrix} \tag{39c}$$

$$\tilde{K_3} = \begin{pmatrix} 0 \\ 0 \\ \mathrm{U}_{31}\,(\mathrm{U}_{31}\,k_1 - \mathrm{U}_{32}\,k_1) - \mathrm{U}_{32}\,(\mathrm{U}_{31}\,k_1 + \mathrm{U}_{33}\,k_2 - \mathrm{U}_{32}\,(k_1+k_2)) - \mathrm{U}_{33}\,(\mathrm{U}_{32}\,k_2 - \mathrm{U}_{33}\,(k_2+k_3)) \end{pmatrix} \tag{39d}$$

$$\tilde{\boldsymbol{C}} = c_a\tilde{\boldsymbol{M}} + c_b\tilde{\boldsymbol{K}} \tag{40}$$

Comment 4. Equation (40) comes from the fact that c_b and c_a are scalars.

This decomposition uncouples the degree of freedom. It means each row of the matrix equation:

$$\tilde{\boldsymbol{M}}\ddot{\tilde{x}} + \tilde{\boldsymbol{C}}\dot{\tilde{x}} + \tilde{\boldsymbol{K}}\tilde{x} = \tilde{b}v \tag{41}$$

can be treated as a single degree of freedom system.

3.3.2 Modal damping

Since the system is now uncoupled, it can be expresse in the form of 3 independent equations.

$$\tilde{\boldsymbol{M}}_{ii}\ddot{\tilde{x}}_i + \tilde{\boldsymbol{C}}_{ii}\dot{\tilde{x}}_i + \tilde{\boldsymbol{K}}_{ii}\tilde{x}_i = \tilde{b}_i v \quad i \in 1,2,3 \tag{42}$$

Natural frequencies can be derived. ⚠They are not the resonance frequencies, but the frequencies at which the **damped** system will oscillate in the impulse response. They are shown in (43) and the values in table 9

$$\zeta_i = \frac{c_a + c_b\omega_i^2}{2\omega_i} \tag{43a}$$

$$\omega_{ni} = \omega_i\sqrt{1-\zeta_i} \tag{43b}$$

Where the equation of the i^{th} mode can be written as

$$\ddot{\tilde{x}}_i + 2\zeta_i\omega_{ni}\dot{\tilde{x}}_i + \omega_{ni}^2\tilde{x} = \frac{\tilde{b}_i}{\boldsymbol{M}_{ii}}v \tag{43c}$$

modal coordinate	ζ_i	ω_{ni} [rad/s]
$\tilde{x}_1$	0.093	8.235
$\tilde{x}_2$	0.029	27.390
$\tilde{x}_3$	0.020	41.842

Table 9: Damping quantities (frequency and damping coefficient)

4 Transfer functions plots

The transfer functions are converted from the state space form (Equation (5)) through the formula (44). The input is the voltage v.

$$\frac{X(s)}{V(s)} = C(sI-A)^{-1}B. \tag{44}$$

To compute the transfer function between the force and the positions, the voltage-to-foce coefficient gain has to be removed. Given $F(s)$ the Laplace transform of the force signal.

$$\frac{X(s)}{F(s)} = \frac{1}{g_v}\frac{X(s)}{V(s)}. \tag{45}$$

The transfer function is a column vector with 3 entries, one for each output, they have the same denominator. The bode diagram are shown in Figure 14. Both the cases are plotted: the free damping and and the proportional damping case.

The two transfer function are barely distinguishable. The resonance peaks are clearly visible (different from the ones of the undamped version of the system).

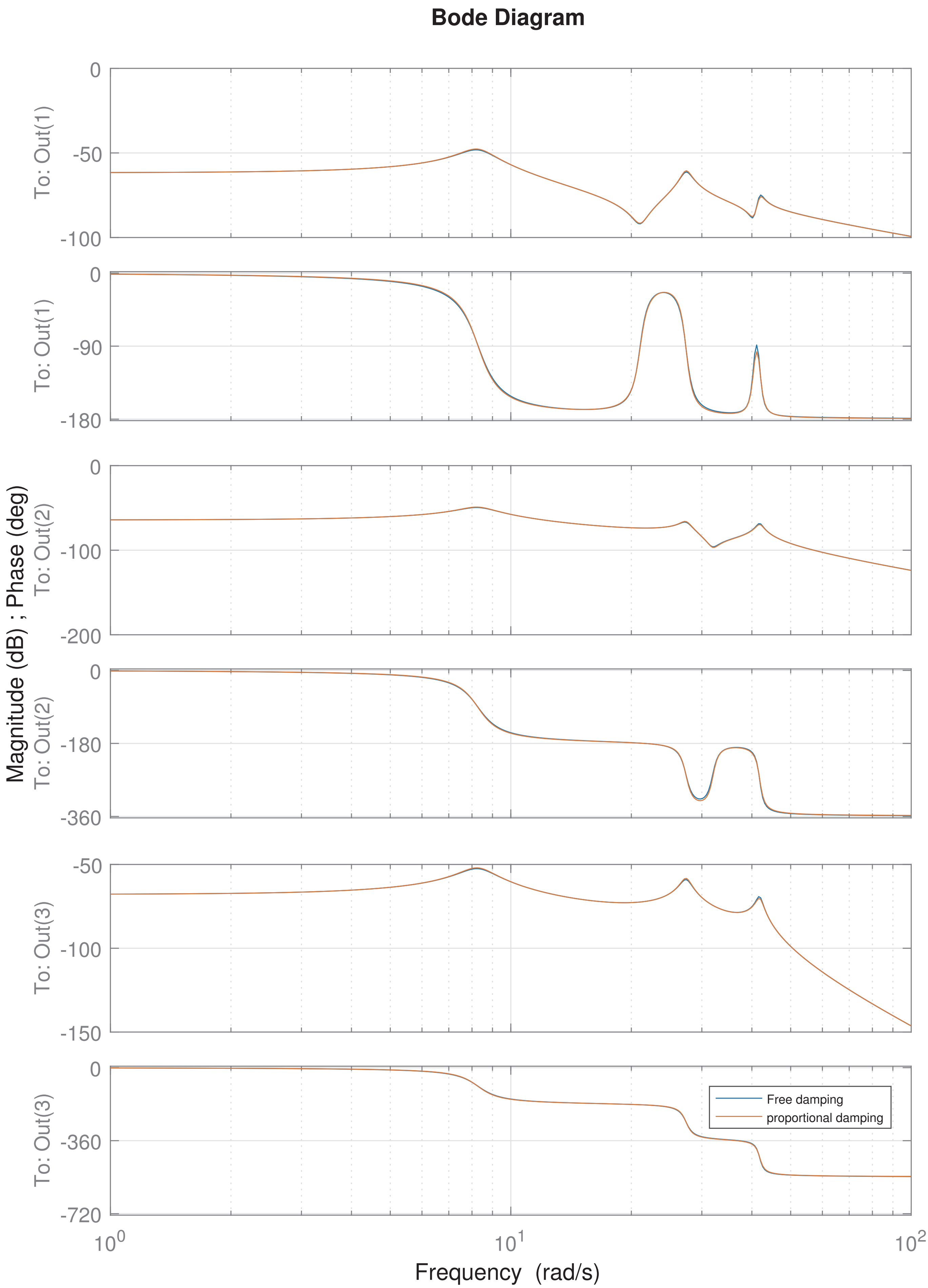


Figure 14: Bode diagram of the tranfer functions $\frac{X(s)}{F(s)}$

5 Sine sweep analysis

The inputs provided are two sine sweep signal with different sample frequencies. $5ms$ and $10ms$. The signal is the same, the only different is the sampling frequency. To visualize the spectra of the signals, the fft is performed on both the signal. The spectrum is plotted with the real frequencies in 15.

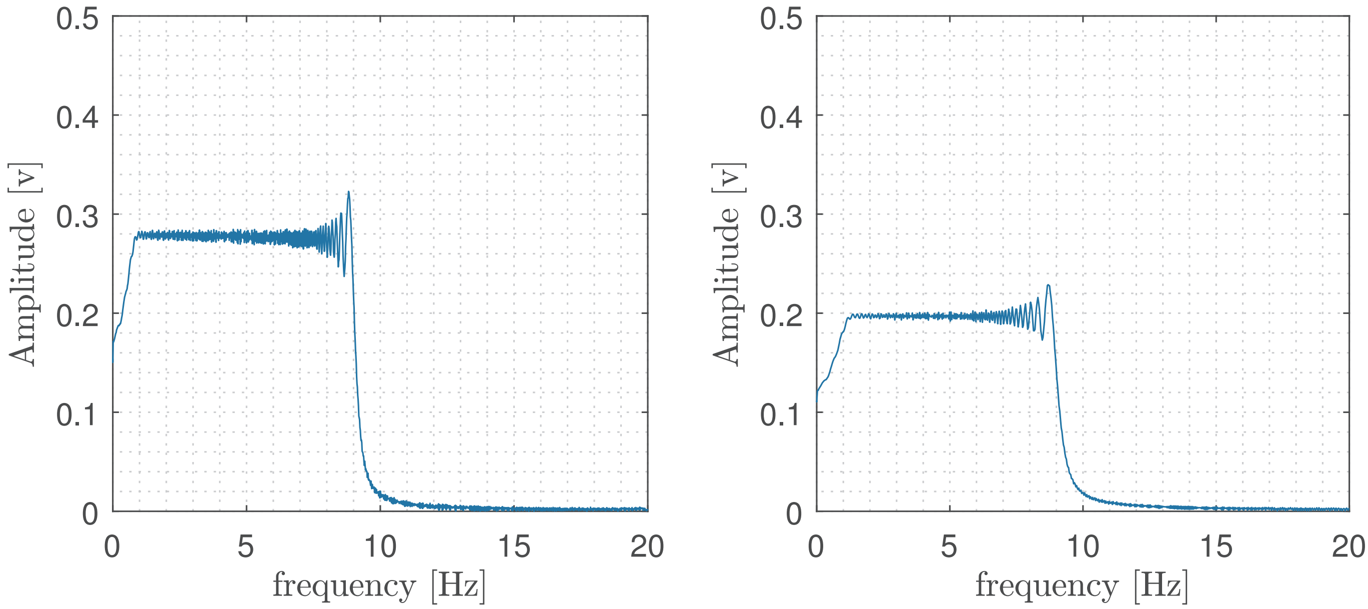


Figure 15: Sine sweep spectra, (slow on the left and fast on the right)

The two signal has the same frequency content. The difference is only the sampling time. The spectra cover all the resonance peaks of the transfer function. Despite the spectrum, the sine sweep excites each frequency for a limited amount of time (in fact a spectrogram is a more specific tool for the varying-in-time spectrum). From the time plot 16, it is possible to notice that the third mode is not clearly visible .

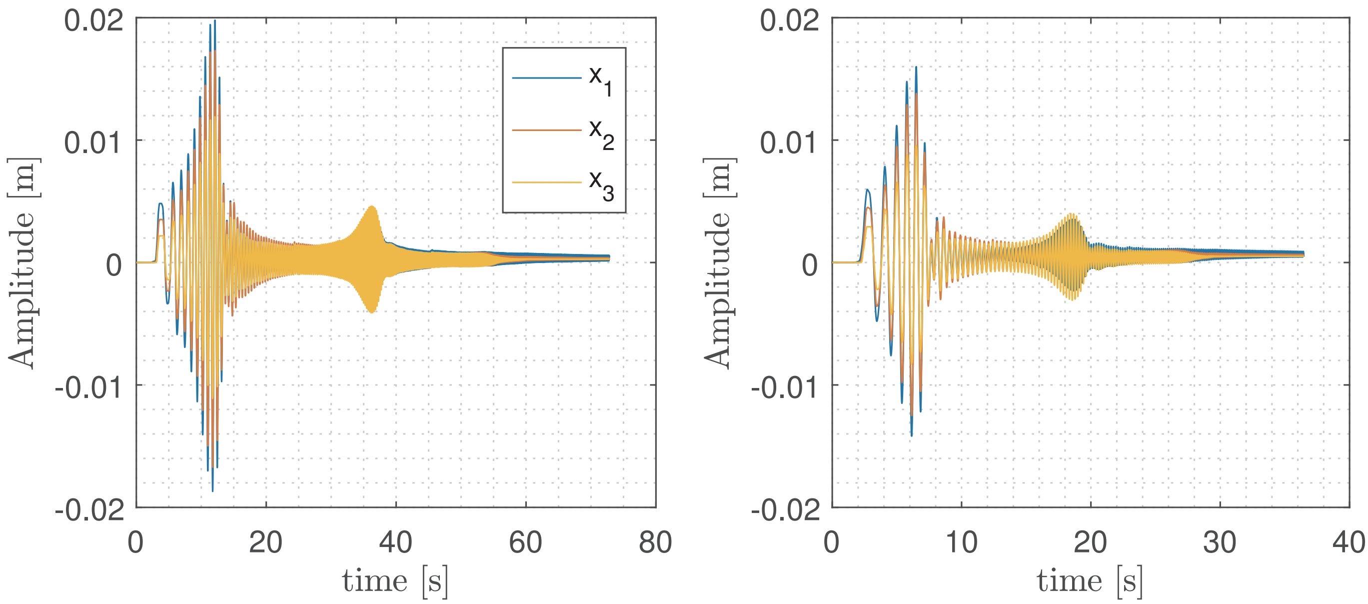


Figure 16: Sine sweep outputs of the system, (slow on the left and fast on the right)

To estimate the transfer function from this data MATLAB provides several solution, the chosen one is `tfest`. It is oriented to LSQ instead of some other method like `tfestimate` which use the averaged periodogram method, and the signal is not periodic.

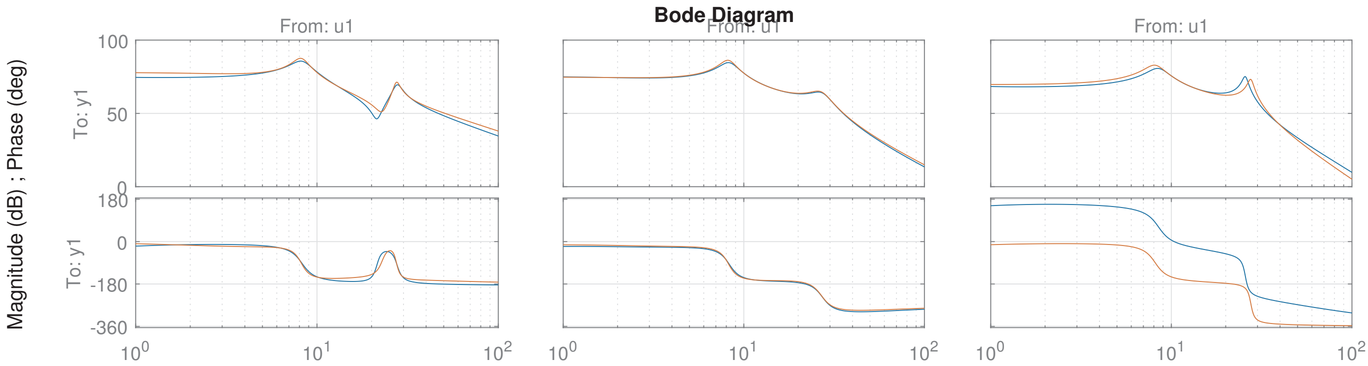


Figure 17: Sine sweep outputs of the system, (slow on the left and fast on the right)

As expected, the first two modes are detected but not the third. To estimate a transfer function a signal with a wider spectrum would be better, or a pseudo random binary sequence.