Mechanical vibration - System identification and modal analysis of 3-DOF linear system

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1 Dynamical system

1.1 The linear model

The chosen model is a linear plant consisting of 3 masses, 3 springs between them, and 3 dampers between each mass and the ground. The model is shown in Figure 1.

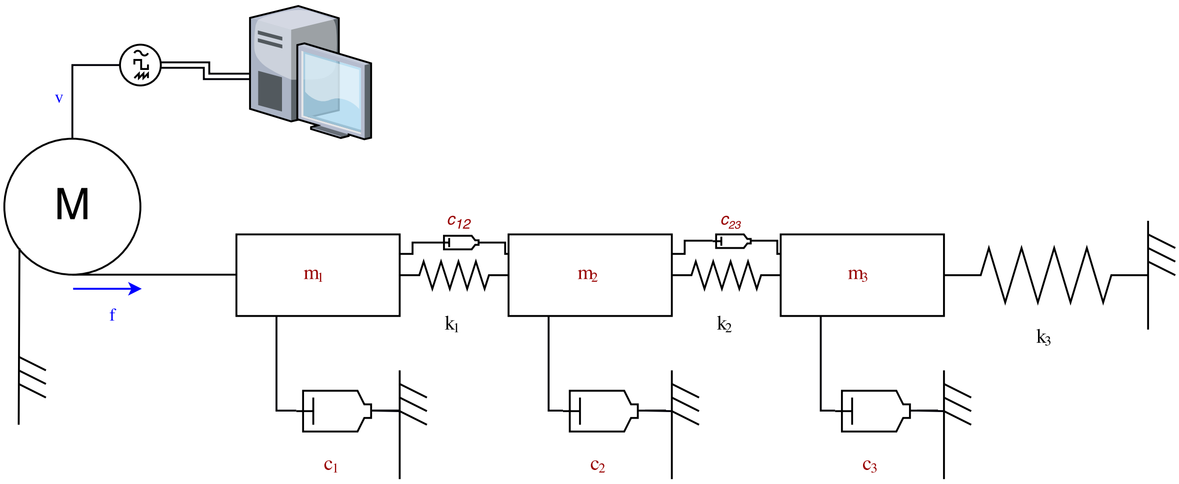

Figure 1: The chosen plant, in red the unknown parameters

1.2 equation of motion

$$\begin{cases} m_1\ddot{x}_1 = +k_1\left(x_2 - x_1\right) + c_{12}\left(\dot{x}_2 - \dot{x}_1\right) - c_1\dot{x}_1 + g_v v \\ m_2\ddot{x}_2 = +k_1\left(x_1 - x_2\right) + k_2\left(x_3 - x_2\right) + c_{12}\left(\dot{x}_1 - \dot{x}_2\right) + c_{23}\left(\dot{x}_3 - \dot{x}_2\right) - c_2\dot{x}_2 \\ m_3\ddot{x}_3 = +k_2\left(x_2 - x_3\right) + c_{23}\left(\dot{x}_2 - \dot{x}_3\right) - c_3\dot{x}_3 - k_3x_3 \end{cases} \tag{1}$$

In the classical matrix form:

$$\boldsymbol{M}\ddot{x} + \boldsymbol{C}\dot{x} + \boldsymbol{K}x = \boldsymbol{b} \tag{2a}$$

where:

$$\boldsymbol{K} = \begin{bmatrix} +k_1 & -k_1 & 0 \\ -k_1 & +k_1 + k_2 & -k_2 \\ 0 & -k_2 & +k_3 \end{bmatrix} \tag{2b}$$

$$\boldsymbol{M} = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \tag{2c}$$

$$\boldsymbol{b} = \begin{bmatrix} g_v \\ 0 \\ 0 \end{bmatrix} \tag{2d}$$

$$\boldsymbol{C} = \begin{bmatrix} +c_1 + c_{12} & -c_{12} & 0 \\ -c_{12} & +c_2 + c_{12} + c_{23} & -c_{23} \\ 0 & -c_{23} & c_3 + c_{23} \end{bmatrix} \tag{2e}$$

1.3 state-space model

The linear model of the plant, expressed by the equation 2, is a SIMO model. A state-space form was chosen to represent this model. The matrices are the follwing:

$$A = \begin{bmatrix} \boldsymbol{Z}_{3\times3} & \boldsymbol{I}_{3\times3} \\ -\boldsymbol{M}^{-1}\boldsymbol{K} & -\boldsymbol{M}^{-1}\boldsymbol{C} \end{bmatrix} \tag{3a}$$

$$\boldsymbol{B} = \begin{bmatrix} \boldsymbol{Z}_{3\times1} \\ -\boldsymbol{M}^{-1}\boldsymbol{b} \end{bmatrix} \tag{3b}$$

$$\boldsymbol{C} = \begin{bmatrix} \boldsymbol{I}_{3\times3} & \boldsymbol{Z}_{3\times3} \end{bmatrix} \tag{3c}$$

$$\boldsymbol{D} = \begin{bmatrix} \boldsymbol{Z}_{3\times1} \end{bmatrix} \tag{3d}$$

where $\boldsymbol{I}$ is the identity matrix and $\boldsymbol{Z}$ is a matrix with all the entries equal to zero.

1.4 experimental setup

1.4.1 data processing

Few operations on data must be performed in order to use them. At first, the data on diplacements is provided in *encoder counts*. They are converted in meters with the following convertion factor (g_x):

$$g_x = \frac{\Delta x}{\Delta \texttt{counts}} = 2\pi r_e \cdot \frac{\Delta \texttt{counts}}{16000 \frac{\texttt{counts}}{\texttt{encoder revolution}}} \tag{4}$$

1.5 parameters and data available

1.6 initial hypothesis and approximations

- neglected motor electrical dynamics (instantaneous transmission of torque)
- rectilinear motion (all perfect aligned)
- inertia and damping of the motor are merged respectively into m_1 and c_1.

$$\begin{cases} m_1 = m_{block} + \frac{J_{motor}|_{zz}}{r^2} \\ c_2 = c_{block} + \frac{c_{motor}}{r^2} \end{cases} \tag{5}$$

  where r is the radius of the gear-rack coupling (gear wheel), $J_{motor}|_{zz}$ is the inertia of the motor , c_{motor} the rotational damping and "block" quantities are the ones stricly related to the physical first mass.

- The term c_3 contains the viscous friction with the ground and the one due to the spring. In the model those two contributions cannot be quantified separately.

2 System identification

2.1 step response analysis

First of all, the step response analysis can be performed. In this analysis the "static" coefficients can be estimated, they are:

- voltage to force g_v
- springs' stiffness k_i with $i \in 1, 2, 3$

The coefficient to be estimated is the *voltage-to-force* coefficient

$$f = (\boldsymbol{k_a} \cdot \boldsymbol{k_t} \cdot \boldsymbol{k_{mp}})v = g_v v \tag{6}$$

. In order to estimate the parameters the static gain vector g_{dc} of the system has to be computed. The standard procedure is to apply the "CAB" formula from the state space formulation:

$$\boldsymbol{g_{dc}} = \boldsymbol{C}\boldsymbol{A}^{-1}\boldsymbol{B} \tag{7}$$

which is the transfer function at $s = 0$. Since this computation implies the inverse of the 6×6 matrix $\boldsymbol{A}$, another computation is performed. Using the formulation in Equation 2, we can perform the following limits:

$$\begin{cases} \lim_{t\to+\infty} \dot{x} = 0 \\ \lim_{t\to+\infty} \ddot{x} = 0 \end{cases} \tag{8}$$

The substitution 8 in 2a yields:

$$\boldsymbol{K}x = \boldsymbol{b} \tag{9}$$

The static gain is then:

$$\boldsymbol{g_{dc}} = \boldsymbol{K^{-1}b} \tag{10}$$

Note that it is equivalent to apply the Laplace tranform to the equation 2a and apply the final value theorem to it.

$$\boldsymbol{g_{dc}} = \begin{bmatrix} g_v \dfrac{k_1k_2 + k_3k_2 + k_3k_1}{k_1k_2k_3} & g_v \dfrac{k_2 + k_3}{k_2k_3} & g_v \dfrac{1}{k_3} \end{bmatrix}^\top \tag{11}$$

Note that in equation 11 is evident from the expression the parallel between the stiffnesses (at steady state inertia and damping are invisible). The steady state value of the three output are available. Some other computations has to be made to make equation 11 suitable for the check on the stiffnesses ratios and the new estimation of g_v. The equation for the steady state values is:

$$\boldsymbol{Cx}_\infty = \boldsymbol{g_{dc}}u_\infty \tag{12}$$

Where the ∞ denotes the steady state value of the quantity. The Equation 12 is now expressed in terms of stiffnesses ratio: k_3 is fixed to the nominal value and two ratio are defined:

- fix k_3 on nominal value
- multiplied both sides times k_3
- replace $R_{32} = \dfrac{k_3}{k_2}$
- replace $R_{31} = \dfrac{k_3}{k_1}$

This operations are made in order to make the system easy to solve, and uncoupling the nonlinear factor g_v. This allow to make to solve the system in a cascade fashion from g_v to R_{13}. The equation 12 is now expressed in the following form:

$$k_3 \begin{bmatrix} x_1(\infty) \\ x_2(\infty) \\ x_3(\infty) \end{bmatrix} = u(\infty) \begin{bmatrix} g_v + g_vR_{13} + g_vR_{23} \\ g_vR_{23} \\ g_v \end{bmatrix} \tag{13}$$

Now the next step is to use the measured value of steady state response (input and output), take g_v as new estimated value and verify that R_{12} and R_{32} matches the nominal values. As set before $u(\infty) = 0.5$. Results on the ratios are shown in Table 2.1

	from data	nominal	error %
k_3/k_2	0.5	0.5	6.7 %
k_3/k_1	0.5	0.5	4.4 %
	from data	initial	error %
g_v	6.0497	5.2500	15.2 %

Table 1: Stiffnesses ratios and voltage-to-force coefficients results

The $voltage-to-force$ estimation is shown in table 2.1.

2.2 Parameters estimation

To estimate the parameters the impulse reponse is used. The *voltage-to-force* coefficient g_v is one of the parameters to be estimate in order to have the possibility to cross-check the result with the step response result.

2.2.1 Free damping case

The first estimation is performed using the model in Equations 3.

3 modal analysis

3.1 Eigenvalue problem

3.2 Undamped case

3.2.1 Raileight quotient

The Rayleight quotient is

$$R_q(\boldsymbol{x}) = \frac{\boldsymbol{x}^\top \boldsymbol{K}\boldsymbol{x}}{\boldsymbol{x}^\top \boldsymbol{M}\boldsymbol{x}} \tag{14}$$

where x is the vector of the positions, R_q as a function of the ladder, presents stationary points in the neigborhood of the modal shapes, and the value corresponds to the respective eigenfrequency. The first stationary point it's easy to find, but the other two are not. In order to find the other stationary points, the property of orthogonality of the modal shapes vector is used. The basic idea is to reduce a degree of freedom every frequency using this property. The procedure is the following:

- Provide an initial guess, the convergence to the first minima is quite robust to the first guess. Vector $[\ 1\ 1\ 1\ ]^\top$ is used.
- Use Matlab function `fminunc` to find the minima and the value of x, varying only two parameters of the vector ($x = [\ 1\ \alpha\ \beta\ ]^\top$)
- Check if the minimum is the lowest frequency with a contour plot or performing the next steps. Define the first modal shape vector u_{1R}.
- Compute the null space of the first modal shape vector $Ker(u_{1R}^\top)$
- find the minimum of

  $$R_{q2} = \frac{[\ 1\ \alpha\ ]B_k^\top \boldsymbol{K} B_k [\ 1\ \alpha\ ]^\top}{[\ 1\ \alpha\ ]B_k^\top \boldsymbol{M} B_k [\ 1\ \alpha\ ]^\top} \tag{15}$$

  where $img(B_k) = Ker(u_{1R}^\top)$ is a base of the Kernel of the first modal shape vector, the value of the function is another resonance frequency.
- The last modal shape vector and the last resonance frequency can be computed respectively as a base of the Kernel of the two other modal shape vectors and evaluating R_q at the last modal shape vector.

$$u_{3R} \in Ker(\begin{bmatrix} u_{1R} & u_{2R} \end{bmatrix}^\top) \tag{16}$$

The results are shown in Table **??** and the contour plot of the Raileight coefficient is shown in Figure **??**

3.3 Proportional damping case

The mode shapes of the system remains the same but the frequencies will be different and complex (BIBLIO RAO PAG 914).

given the modal shape vectors matrix U obtained from the eigenvalue problem 3.1 with the masses from the proportional damping estimation, the modal coordinates w.r.t the previous coordinates are the following:

$$\tilde{x} = U^\top x \tag{17}$$

Which is basically a projection of the coordinate on the modal shapes vector (scalar products). The matrices M, K and C becomes...