Question 1: Implementation of the Line Search Algorithm

Consider the algorithm that was explained in the workshop. As mentioned in Slide 9 of the second workshop, the Line Search Algorithm is guaranteed to work. The gradient descent algorithm is not necessarily guaranteed to converge rapidly, but it ensures that the step length chosen is the largest feasible step that can be taken using the gradient descent algorithm.

You are required to implement the Line Search algorithm and then execute it on the following functions:

1. $f_1(x) = f(x_1, x_2) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2$
2. $f_2(x) = f(x_1, \dots, x_4) = (x_1 - 10x_2)^2 + 5(x_3 - x_4)^2 + (x_2 - 2x_3)^4 + 10(x_1 - x_4)^4$
   where $x_0 = (1, 2, 2, 2)^T$

Show $f(x)$ during different iterations. Additionally, compare the number of evaluations of the function $f(x_k)$ and $||x_{k+1} - x_k||$ with the results of the gradient descent algorithm using graphs. Report the results and analyze them in conjunction with the series approximation $\phi'(x)$ and $\phi(x)$ and its gradient.

Algorithm 2 Line Search

Initialization: $i = 1$, $\alpha_0 = 0$, $\alpha_{max} > 0$ and $\alpha_1 \in (0, \alpha_{max})$;
while $True$ **do**
    Evaluate $\phi(\alpha_i)$;
    if $\phi(\alpha_i) > \phi(0) + c_1\alpha_i\phi'(0)$ **then**
        $\alpha_* \leftarrow$ **ZOOM**(α_{i-1}, α_i) ;
        Break
    end if
    Evaluate $\phi'(\alpha_i)$;
    if $|\phi(\alpha_i)| \leq c_2\phi'(0)$ **then**
        $\alpha_* \leftarrow \alpha_i$;
        Break
    end if
    if $\phi'(\alpha_i) \geq 0$ **then**
        $\alpha_* \leftarrow$ **ZOOM**(α_i, α_{i-1}) ;
        Break
    end if
    Choose $\alpha_{i+1} \in (\alpha_i, \alpha_{max})$
    $i \leftarrow i + 1$;
end while

Algorithm 3 ZOOM$(\alpha_{low}, \alpha_{high})$

while $True$ **do**
    Interpolate to find a α_j between α_{low} andα_{high};
    Evaluate $\phi(\alpha_j)$;
    if $\phi(\alpha_j) > \phi(0) + c_1\alpha_j\phi'(0)$ **then**
        $\alpha_{high} \leftarrow \alpha_j$;
    else
        Evaluate $\phi'(\alpha_j)$;
        if $|\phi(\alpha_j)| \leq c_2\phi'(0)$ **then**
            $\alpha_* \leftarrow \alpha_j$;
            Break
        end if
        if $\phi'(\alpha_j)(\alpha_{high} - \alpha_{low}) \geq 0$ **then**
            $\alpha_{high} \leftarrow \alpha_{low}$;
        end if
        $\alpha_{low} \leftarrow \alpha_j$;
    end if
end while

Question 2: Implementation of the BFGS Algorithm

Implement the BFGS algorithm as presented in Slide 19 of the second workshop, and similarly execute it on functions (1) and (2) as in the previous question. Finally, report and analyze the results of the BFGS algorithm as mentioned in Question 1. It is required to use the Line Search algorithm for step length selection in the BFGS implementation.

We change the green part in code and correct them.

$minimize \; ||H - H_k||$

Algorithm 8 BFGS

Initialization: x_0, convergence tolerance $\epsilon > 0$;
Hessian approx. H_0 and $k = 0$;
while $\nabla f_k \geq \epsilon$ **do**
    $p_k = -H_k \nabla f(x_k)$;
    line search based on Wolfe condition to find α;
    $x_{k+1} = x_k + \alpha p_k$
    $s_k = \alpha p_k, \quad y_k = \nabla f(x_{k+1}) - \nabla f(x_k)$
    $\rho_k = 1/s_k^\intercal y_k$
    $H_{k+1} = (I - \rho_k s_k y_k^\intercal) H_k (I - \rho_k s_k y_k^\intercal) + \rho_k s_k s_k^\intercal$
    $k = k + 1$
end while