Note on freeway node models and optimization

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1 Freeway node models

Freeway node models are a class of node models restricted to two inputs and two outputs, and with one of the outputs having infinite supply. There are several alternatives for freeway node models.

1. fixed priority
2. demand proportional (XXX)
3. Daganzo's model [1]
4. capacity proportional (XXX)
5. directed capacity proportional [3]
6. ACTM [2]
7. max flow (XXX)

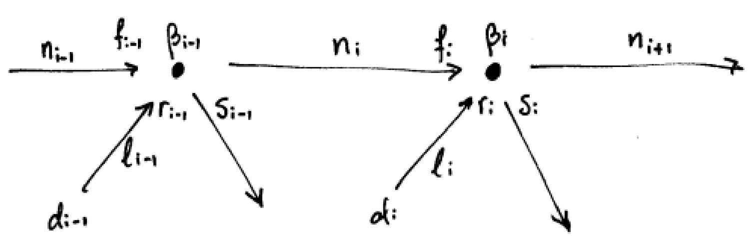

Figure 1: Network structure

All of these models solve the node modeling problem here stated. In figure 1, given the mainline and onramp states (n's and l's), find the flows through the node (f's and r's). The split ratio assumption is always made:

$$s^i = \beta^i f^i \tag{1}$$

All of the models are formulated in terms of *demand* and *supply*. These are functions of the upstream and downstream state. We will use the following notation for demand and supply:

S_f^i	mainline demand
S_r^i	onramp demand
R^i	mainline supply

1.1 Supply allocation models

Several but not all of the models listed above can be expressed in this way,

$$f_i = \min(\ S_f^i\ ,\ p_f^i R^{i+1}\ ) \tag{2}$$

$$r_i = \min(\ S_r^i\ ,\ p_r^i R^{i+1}\ ) \tag{3}$$

Drop the section indices.

$$f = \min(\ S_f\ ,\ p_f R\ ) \tag{4}$$

$$r = \min(\ S_r\ ,\ p_r R\ ) \tag{5}$$

The interpretation is that each upstream branch is allocated a portion of the downstream supply R, according to its *priority* p. The flow it sends is the minimum of its demand and its allocated downstream space. We call such models *supply allocation* models. As we shall see, all of the ones we have listed, except the Daganzo and the max flow models, are supply allocation models.

1.2 Flow maximization in supply allocation models

One criterion that has been suggested by both (Daganzo) and (Tampere) for a node model is that it should not waste space. This means that either both the upstream links should send their full demand (free flow), or the downstream supply should be fully utilized (congestion). In our notation,

$$\left\{ \begin{array}{c} f = S_f \\ r = S_r \end{array} \right\} \qquad OR \qquad \bar{\beta} f + r = R \tag{6}$$

In figure 2, it means that the solution must either be at point **A**, or on line **R**. Equation (6) on the other hand defines a solution in the set **OAB**.

For large R, equation (6) prescribes flows on point **A**, as desired. However, for small R the solution is $(f, r) = (p_f, p_r)R$. This implies that flow is maximized if and only if

$$\bar{\beta} p_f + p_r = 1 \tag{7}$$

For intermediate values of R we have,

$$(f, r) = (p_f R, S_r) \ \text{ or } (S_f, p_r R) \tag{8}$$

These imply requirements on the priorities:

$$\begin{array}{c} p_f = (R - S_r)/\bar{\beta} R \\ \text{or} \\ p_r = (R - \bar{\beta} S_f)/R \end{array}$$

Given that any of these conditions may occur at one time or another, we see that the priorities must depend on the demand if flow is to be maximized in this type of model. The three proposals for overcoming this problem are:

1. LP: Forget about allocation, solve a flow maximization lp.

2. Daganzo: Single branch is a special case. Leads to model with "mid" function.

3. Demand proportional: Eliminate the single branch case by always distributing congestion to both branches simultaneously.

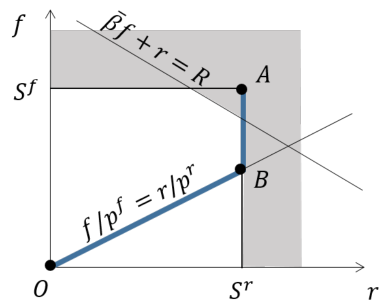

Figure 2: Supply allocation model

1.3 Daganzo

Daganzo uses constant priorities, but departs from the pure supply allocation scheme in order to maximize flow. The priorities are selected so that the flow maximization condition for the fully congested case is satisfied: $\bar{\beta}p_f + p_r = 1$. Then, in the partial congestion case, the solution chosen is the intersection of the **AB** segment with **R**. That is, we do not use the value returned by the minimum formula, $p_f R$, because it is too small. Rather, we fill in all of the remaining space $R - S_r$.

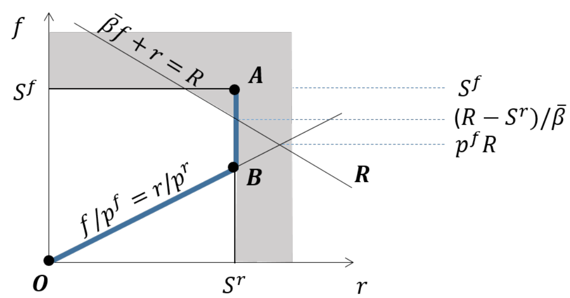

Figure 3: Flow maximizing node model

The formulas are:

$$f = \begin{cases} S_f & \text{if } R > \bar{\beta}S_f + S_r \\ \text{mid}(S_f, R - S_r, p_f R) & \text{otherwise} \end{cases} \tag{9}$$

$$r = \begin{cases} S_r & \text{if } R > \bar{\beta}S_f + S_r \\ \text{mid}(S_r, R - S_f, p_r R) & \text{otherwise} \end{cases} \tag{10}$$

1.4 LP

Another was of ensuring that supply is fully utilized is to explicitly maximize the flow into the downstream link. The flows are computed as the solutions to the following LP,

$$\text{maximize } \bar{\beta}f + r \tag{11}$$

$$\text{subject to } f \in [0, S_f] \tag{12}$$

$$r \in [0, S_r] \tag{13}$$

$$\bar{\beta}f + r \leq R \tag{14}$$

This problem clearly has non-unique solution whenever $\bar{\beta}S_f + S_r \geq R$, and a means of resolving this problem is needed. In fact, the Daganzo method is one such possibility. The main advantage of the LP approach is that it can be easily extended to nodes with larger numbers of inputs and outputs.

1.5 Demand proportional

Each upstream link is prioritized according to its demand,

$$p_f = \frac{S_f}{\bar{\beta}_i S_f + S_r} \tag{15}$$

$$p_r = \frac{S_r}{\bar{\beta}_i S_f + S_r} \tag{16}$$

These priorities satisfy equation (7), and therefore maximize flow in the fully congested case. The priority line goes through point **A**, and thus the partial congestion case is eliminated.

1.6 ACTM

The ACTM was created for the purpose of optimizing ramp metering rates. The equations of the ACTM are,
conservation

$$n_i \leftarrow n_i + f_{i-1} + r_i - \frac{1}{\bar{\beta}_i} f_i \tag{17}$$

$$l_i \leftarrow l_i + d_i - r_i \tag{18}$$

flows

$$f_i = \min(v_i(n_i + \gamma r_i), w_{i+1}(\bar{n}_{i+1} - n_{i+1} - \gamma r_{i+1}), F_i) \tag{19}$$

$$r_i = \min(l_i + d_i, \xi_i(\bar{n}_i - n_i), c_i) \tag{20}$$

To map this onto a supply allocation model, we make the following substitutions:

$$\gamma = 0 \tag{21}$$

$$S_f = \min(v_i n_i, F_i) \tag{22}$$

$$R = \bar{n}_{i+1} - n_{i+1} \tag{23}$$

$$S_r = \min(l_i + d_i, c_i) \tag{24}$$

$$pf = w_{i+1} \tag{25}$$

$$p_r = \xi \tag{26}$$

The ACTM is therefore simply a supply allocation model with constant priority, and with a built-in weaving model supplied by γ (this is the asymmetric part). For flow maximization in the fully congested regime, the ACTM should have:

$$\xi = w\beta \tag{27}$$

(...with the appropriate section indices). But again, flow maximization is not generally possible with the ACTM because it is a constant-priority supply allocation model.

2 Optimal ramp metering with the ACTM

2.1 Original problem

Find $\{n^*, l^*, f^*, r^*, c^*\}$ that solve,

$$\text{minimize } \sum_i \sum_k \left(n_{(i,k)} + l_{(i,k)} - \eta\left(f_{(i,k)} + r_{(i,k)}\right)\right) \tag{28}$$

subject to :

conservation

$$n_{(i,k+1)} = n_{(i,k)} + f_{(i-1,k)} + r_{(i,k)} - f_{(i,k)}/\bar{\beta}_i \tag{29}$$

$$l_{(i,k+1)} = l_{(i,k)} + d_{(i,k)} - r_{(i,k)} \tag{30}$$

mainline flows

$$f_{(i,k)} = \min\left(\bar{\beta}_i v_i(n_{(i,k)} + \gamma r_{(i,k)}), w_{i+1}(\bar{n}_{i+1} - n_{(i+1,k)} - \gamma r_{(i+1,k)}), F_i\right) \tag{31}$$

onramp flows

$$r_{(i,k)} = \min\left(l_{(i,k)} + d_{(i,k)}, \xi_i(\bar{n}_i - n_{(i,k)}), c_{(i,k)}\right) \tag{32}$$

onramp bounds

$$c_{(i,k)} \leq \bar{c}_i \tag{33}$$

$$l_{(i,k)} \leq \bar{l}_i \tag{34}$$

non-negativity

$$c_{(i,k)} \geq 0 \tag{35}$$

2.2 Relaxed problem

Find $\{n^*, l^*, f^*, r^*\}$ that solve,

$$\text{minimize} \sum_i \sum_k \left(n_{(i,k)} + l_{(i,k)} - \eta \left(f_{(i,k)} + r_{(i,k)}\right)\right) \tag{36}$$

subject to :

conservation

$$n_{(i,k+1)} = n_{(i,k)} + f_{(i-1,k)} + r_{(i,k)} - f_{(i,k)}/\bar{\beta}_i \tag{37}$$

$$l_{(i,k+1)} = l_{(i,k)} + d_{(i,k)} - r_{(i,k)} \tag{38}$$

mainline flows

$$f_{(i,k)} \leq \bar{\beta}_i v_i (n_{(i,k)} + \gamma r_{(i,k)}) \tag{39}$$

$$f_{(i,k)} \leq w_{(i+1,k)} (\bar{n}_{i+1} - n_{(i+1,k)} - \gamma r_{(i+1,k)}) \tag{40}$$

$$f_{(i,k)} \leq F_i \tag{41}$$

onramp flows

$$r_{(i,k)} = d_{(i,k)} \qquad \text{if uncontrolled} \tag{42}$$

$$r_{(i,k)} \leq l_{(i,k)} + d_{(i,k)} \qquad \text{if controlled} \tag{43}$$

onramp bounds

$$r_{(i,k)} \leq \bar{c}_i \tag{44}$$

$$l_{(i,k)} \leq \bar{l}_i \tag{45}$$

non-negativity

$$r_{(i,k)} \geq 0 \tag{46}$$

2.3 Solution algorithm

Set $c^* = r^*$ to get an optimal solution for the original problem.

3 Optimal ramp metering and VSL with the demand-proportional model

3.1 Original problem

Find $\{n^*, l^*, f^*, r^*, c^*\}$ that solve,

$$\text{minimize } \sum_i \sum_k \left(n_{(i,k)} + l_{(i,k)}\right) \tag{47}$$

subject to :

conservation

$$n_{(i,k+1)} = n_{(i,k)} + \bar{\beta}_{i-1} f_{(i-1,k)} + r_{(i-1,k)} - f_{(i,k)} \tag{48}$$

$$l_{(i,k+1)} = l_{(i,k)} + d_{(i,k)} - r_{(i,k)} \tag{49}$$

flows

$$f_{(i,k)} = S_{f(i,k)} \min\left(1, R_{(i+1,k)}/\bar{S}_{(i,k)}\right) \tag{50}$$

$$r_{(i,k)} = S_{r(i,k)} \min\left(1, R_{(i+1,k)}/\bar{S}_{(i,k)}\right) \tag{51}$$

$$R_{(i+1,k)} = \min\left(w_{i+1}(\bar{n}_{i+1} - n_{(i+1,k)}), F_{i+1}\right) \tag{52}$$

$$S_{f(i,k)} = \min\left(v_i n_i, F_i\right) \tag{53}$$

$$S_{r(i,k)} = \min\left(l_i, c_i\right) \tag{54}$$

$$\bar{S}_{(i,k)} = \bar{\beta}_i S_{f(i,k)} + S_{r(i,k)} \tag{55}$$

onramp bounds

$$c_{(i,k)} \in [0, \bar{c}_i] \tag{56}$$

$$l_{(i,k)} \leq \bar{l}_i \tag{57}$$

Non-negativity

$$n, l, f, r \tag{58}$$

3.2 Original modified problem

Find $\{n^*, l^*, f^*, r^*, v^*, S_r^*\}$ that solve,

$$\text{minimize} \sum_i \sum_k \left(n_{(i,k)} + l_{(i,k)}\right) \tag{59}$$

subject to :

conservation

$$n_{(i,k+1)} = n_{(i,k)} + \bar{\beta}_{i-1} f_{(i-1,k)} + r_{(i-1,k)} - f_{(i,k)} \tag{60}$$

$$l_{(i,k+1)} = l_{(i,k)} + d_{(i,k)} - r_{(i,k)} \tag{61}$$

flows

$$f_{(i,k)} = S_{f(i,k)} \min\left(1, R_{(i+1,k)}/\bar{S}_{(i,k)}\right) \tag{62}$$

$$r_{(i,k)} = S_{r(i,k)} \min\left(1, R_{(i+1,k)}/\bar{S}_{(i,k)}\right) \tag{63}$$

$$R_{(i+1,k)} = \min\left(w_{i+1}(\bar{n}_{i+1} - n_{(i+1,k)}), F_{i+1}\right) \tag{64}$$

$$S_{f(i,k)} = \min\left(v_i n_i, F_i\right) \tag{65}$$

$$\bar{S}_{(i,k)} = \bar{\beta}_i S_{f(i,k)} + S_{r(i,k)} \tag{66}$$

Bounds

$$v_{(i,k)} \in [0, V_i] \tag{67}$$

$$S_{r(i,k)} \in [0, \min(l_{(i,k)}, \bar{c}_i)] \tag{68}$$

$$l_{(i,k)} \leq \bar{l}_i \tag{69}$$

Non-negativity

$$n, l, f, r \tag{70}$$

3.3 Relaxed problem

Find $\{n^*, l^*, f^*, r^*\}$ that solve,

$$\text{minimize} \sum_i \sum_k \left(n_{(i,k)} + l_{(i,k)}\right) \tag{71}$$

subject to :

conservation

$$n_{(i,k+1)} = n_{(i,k)} + \bar{\beta}_{i-1} f_{(i-1,k)} + r_{(i-1,k)} - f_{(i,k)} \tag{72}$$

$$l_{(i,k+1)} = l_{(i,k)} + d_{(i,k)} - r_{(i,k)} \tag{73}$$

mainline flows

$$f_{(i,k)} \leq v_i n_{(i,k)} \tag{74}$$

$$f_{(i,k)} \leq F_i \tag{75}$$

$$f_{(i,k)} \bar{\beta}_i + r_{(i,k)} \leq w_{i+1}(\bar{n}_{i+1} - n_{(i+1,k)}) \tag{76}$$

$$f_{(i,k)} \bar{\beta}_i + r_{(i,k)} \leq F_{i+1} \tag{77}$$

onramp flows

$$r_{(i,k)} = d_{(i,k)} \qquad \text{if uncontrolled} \tag{78}$$

$$r_{(i,k)} \leq l_{(i,k)} \qquad \text{if controlled} \tag{79}$$

onramp bounds

$$r_{(i,k)} \leq \bar{c}_i \tag{80}$$

$$l_{(i,k)} \leq \bar{l}_i \tag{81}$$

Non-negativity

$$n, l, f, r \tag{82}$$

3.4 Conversion algorithm

From the optimal solution to the relaxed problem, find a solution to the modified original problem. That is, find speed v_i and onramp demand S_r that that satisfy the demand-proportional node model.

The relaxed problem defines constraints on the mainline and onramp flows as represented in Figure 4. The demand-proportional model says that these flows must be on the corner of the box (if the supply line does not intersect the box), or on a straight line joining the origin with the far corner of the box. We divide the problem into 4 cases, indicated in the figure.

Case A is a freeflow case, where the mainline travels at the maximum free-flow speed. There is in this case no speed limit regulation, and we can set $S_r = r^*$, while keeping $v_i = V_i$.

Case B covers the

4 Replacements that make them identical

- In ACTM, set $\gamma = 0$.
- Change of variables: LNCTM's $(\bar{\beta} f) \leftrightarrow$ ACTM's (f)
- Onramp freeflow constraint in ACTM includes demand. LNCTM doesn't.
- LNCTM includes non-negativity constraints.

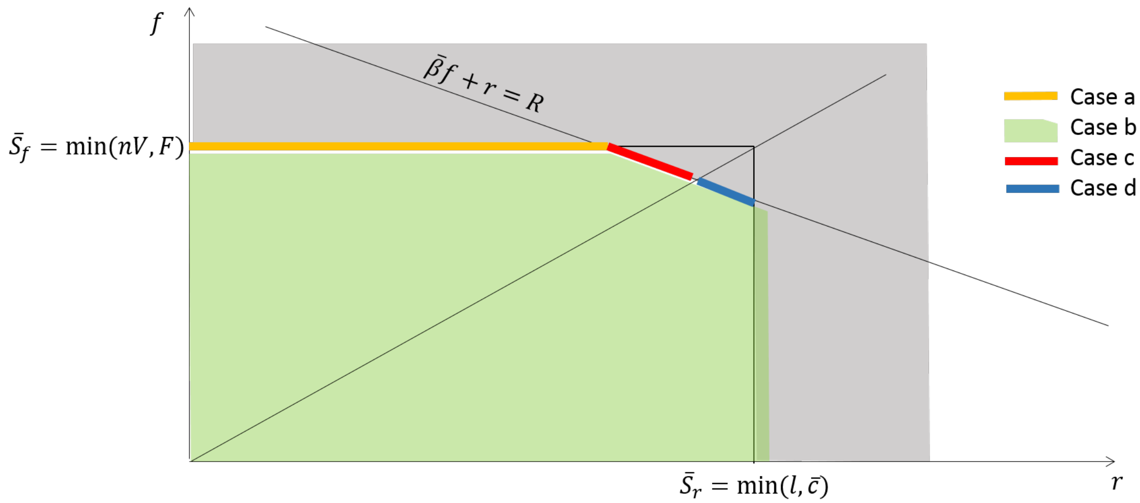

Figure 4: Conversion algorithm

References

[1] C. Daganzo. The cell transmission model, part ii: network traffic. *Transportation Research Part B*, 29:79–93, 1995.

[2] G. Gomes and R. Horowitz. Optimal freeway ramp metering using the asymmetric cel transmission model. *Transportation Research Part C*, 14:244–262, 2006.

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