

# Trainable Fourth-order Partial Differential Equations for Image Noise Removal

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## Abstract

Decreasing noise in images is an important task in computer vision, and image processing by partial differential equations (PDEs) has been an active topic in this area. In PDE-based methods for image processing the unprocessed, original image is considered as the initial value for the PDE and the solution of the equation is the outcome of the model. Despite the advantages of using PDEs in image processing, designing and modeling different equations for various types of applications has always been a challenging and interesting problem. In this paper, we aim to tackle this problem by introducing a fourth-order equation with flexible, trainable coefficients and with the help of an optimal control problem, the coefficients are determined therefore the proposed model adapts itself to each particular application. At the final stage, the image enhancement is performed on the noisy test image and the performance of our proposed method is compared to other PDE-based models.

**Keywords:** Partial Differential Equations, Image Processing, Image De-noising, Optimal Control.

## 1 Introduction

Image de-noising is an important preliminary step in many computer vision and image processing problems which can affect further processing, and has been an open research area for a long time [28, 31, 37]. By considering the

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image as a function  $I$  in  $\mathbb{R}^2$ , the aim of de-noising an image is to extract the clear image  $I$  from the noisy image  $I_0$  which has been degraded by the model  $I_0 = I + n$ , in which  $n$  is a noise function.

Various approaches for image de-noising have been developed, such as methods based on non-local means (NLM) filters, wavelets, total variation (TV), block-matching and three-dimensional filtering (BM3D), sparse representation, Markov random field, neural network, etc [35, 10, 29, 4, 1, 6, 26, 2, 7, 8, 45, 18, 25].

In contrast to the discrete methods used in signal processing in the past, which were the basis of digital image processing, in the past decades, continuous methods based on partial differential equations (PDEs) have become an effective tool.

First efforts of using PDEs in computer vision and image processing tasks date back to the 1960s [11, 15], however, these methods did not attract much attention. In the 80s, the importance of multi-scale descriptions of images has been recognized. The importance of scale-space filtering in images was introduced by Witkin [40] and then developed by Koenderink [17]. The idea of their approach is to generate a family of scaled images  $I(x, y, t)$  by convolving the original image  $I_0(x, y)$  with a Gaussian kernel  $G(x, y, t)$  of variance  $t$ :

$$I(x, y, t) = I_0(x, y) \otimes G(x, y, t). \quad (1)$$

The time variable  $t$  is said to be the scale parameter and the higher the scale gets, it results in an image with reduced resolution, therefore, the noise vanishes.

It was pointed out by Koenderink [17] that,  $I(x, y, t)$  is the fundamental solution to the heat conduction equation as below, with the original image  $I_0(x, y)$  as the initial condition and the conductance coefficient  $c$ , a real number:

$$I_t = c(I_{xx} + I_{yy}). \quad (2)$$

Although the outcome of this equation is a less noisy image, the main disadvantage of this approach is the fact that while smoothing, it blurs important image features, in other words, the Gaussian scale-space does not respect the edges of objects in an image. Therefore, the scaled image loses important details after the process.

In 1992, a method introduced by Perona and Malik on anisotropic diffusion [29] drew great attention on PDE methods in image enhancement. In fact, they designed the following equation, known as PM second-order PDE, to achieve a good balance between noise removal and retaining the edges of objects in the image:

$$I_t = c(x, y, t)(I_{xx} + I_{yy}). \quad (3)$$

In the above PDE, the conductance coefficient function  $c(x, y, t)$ , is chosen to decrease smoothing around the edges and reduce noise in other areas. Following their model, many equations have been introduced on this basis. See for instance [16, 38, 5].

Although the PM second-order PDE and its variances were successfully applied in image de-noising problems, these methods cause blocky effects on the processed image. In [42], it is noted that the PM PDE is a second-order model and this feature guarantees its ability to reconstruct images with discontinuities but is responsible for the blocky effect [44].

In recent years, many efforts have been made to resolve this problem in the PM method. Introducing different coefficients, different equations for processing, raising the order of the equation and so on. One of the innovative efforts to tackle the problems of second-order approaches was inspired by learning-based methods in machine learning [19]. Afterward, a number of second-order learning PDEs (L-PDEs) were introduced which adopt a technique called PDE-based optimal control [14] and uses training images to learn the coefficients in the PDE, and once they are computed, then the PDE is obtained and can be applied to test images.

In learning PDEs proposed in [19, 22, 23], the coefficient functions are assumed to minimize a functional subject to some PDE constraints which are designed according to the problem.

In these classes of approaches, the structure of the problem is considered and the learned coefficients will form the PDE according to the training images. Thus, the most effort on obtaining a PDE is to prepare some input/output training image pairs.

One of the other successful strategies in enhancing the performance of second-order PDEs for image de-noising is to design higher-order equations in replace of quadratic ones [13, 21, 24]. In 2000, in a method proposed by You and Kaveh, known as YK fourth-order PDE [41], the following equation was introduced:

$$\frac{\partial I}{\partial t} = -\nabla^2 (c_1 (|\nabla^2 I|) |\nabla^2 I|), \quad (4)$$

where  $\nabla^2$  denotes Laplacian operator and conductance coefficient  $c_1(.)$  in YK fourth-order PDE, is a function of the absolute value of the Laplacian of the image intensity, that is

$$c_1 (|\nabla^2 I|) = \frac{1}{1 + (|\nabla^2 I|/k)^2}, \quad (5)$$

where  $k > 0$  is the Laplacian threshold. The YK fourth-order PDE attempts to remove noise and preserve edges by approximating an observed image with a piecewise planar image [41].

In recent years, more PDE-based models have been developed, improving the performance of the YK method. In 2016, Zhang et al [46] proposed the adaptive fourth-order PDE for noise removal

$$\frac{\partial u}{\partial t} = -\text{div}^2(h(\nabla u, D^2 u) \frac{D^2 u}{|D^2 u|}) + \mu \text{div}((1 - h(\nabla u, D^2 u)) \frac{\nabla u}{|\nabla u|}), \quad (6)$$

where  $h(\nabla u, D^2 u)$  is an edge detector chosen by

$$h(\nabla u, D^2 u) = \frac{1}{1 + k_1 |\nabla(G_\sigma \otimes u)|^2 + k_2 |D^2(G_\sigma \otimes u)|^2 + \gamma}, \quad (7)$$

where  $\gamma > 0$  is a small constant to guarantee  $h < 1$ ,  $k_1$  and  $k_2$  are positive constants and  $\mu > 0$ ,  $\lambda > 0$  are parameters balancing the contribution of the three terms in the objective function.

Later in 2018, Siddig et al [33] introduced the following equation

$$\begin{cases} \frac{\partial u}{\partial t} + D_{ij}^2 \left( \frac{\alpha(u) D_{ij}^2 u}{|D_{ij}^2 u|} \right) = 0, & (x, t) \in \Omega_T = (0, T) \times \Omega, \\ u(x, t) = 0, & (x, t) \in (0, T) \times \partial\Omega, \\ \frac{\partial u}{\partial \vec{n}} = 0, & (x, t) \in (0, T) \times \partial\Omega, \\ u(x, 0) = u_0(x), & x \in \Omega. \end{cases} \quad (8)$$

where  $\alpha(u) = \frac{1}{\sqrt{1 + |G_\sigma \otimes \nabla u|^2}}$ ,  $G_\sigma(x)$  is the Gaussian filter with  $\sigma$  as its parameter. In these models (Siddig's and Zhang's PDEs), they tend to overcome the disadvantages of YK and TV model while keeping the pros of PDE algorithms.

The theoretical analysis in [9, 12] shows the advantages of fourth-order equations over second-order ones. First, fourth-order diffusion reduces fluctuation faster than second-order ones. Secondly, since second-order PDE will evolve toward a piecewise constant approximation in smooth regions, unlike second-order PDE, fourth-order PDE will evolve toward and settle down to a piecewise smooth image if the image support is infinite [43].

In this paper, motivated by previous efforts on solving second-order equations' problems, we propose a fourth-order PDE with trainable coefficients for image de-noising to enhance the performance of the second-order L-PDEs. To mention two advantages of this PDE, one should notice the advantage of using fourth-order PDEs, and also the proposed PDE uses image differential invariants up to fourth-order which can detect more image features compared to those of second-order PDEs. The results obtained from our scheme are compared to state of the art previous models, such as second-order learning-based PDEs [22], YK fourth-order PDE [41] and a more recent fourth-order PDEs, Siddig [33] and Zhang's [46], on this topic. The experimental results show that, in terms of subjective and objective measures, the proposed model can outperform the pre-existing methods.

The remainder of this paper is organized as follows. Section 2 presents the trainable fourth-order PDE with a detailed description of the solution of the optimal control problem. Experimental results are presented in Section 3, and the paper is concluded in Section 4.

## 2 Proposed model

In this section, firstly the formulation of the fourth-order PDE in our model for noise reduction is presented and then an optimal control problem will be introduced to assist in completing the PDE. Finally, the optimality conditions will be investigated for the recommended model, which helps us to extract enough information to solve the problem.

The framework of this paper is based on the fourth-order noise removal PDE models and inspired by second-order learning PDEs mentioned in the previous section, we will consider flexible trainable coefficients for the proposed PDE.

In this work, the typical notations in image processing and optimal control literature are used. We denote  $f$  as the input image and  $I$  as the desired output image.  $\Omega$  is an open bounded region of  $\mathbb{R}^2$  and  $\partial\Omega$  is its boundary, the spatial variable  $(x, y) \in \Omega$  and  $t \in (0, T_f)$  is the temporal variable,  $\Omega \times (0, T_f)$  is named  $Q$  and  $\Gamma$  is its boundary. We then define the set  $\eta$  as bellow and  $|p|$  is the length of string  $p$  which  $p \in \eta$ .

$$\eta = \{0, x, y, xy, xx, yy, xxx, xxy, xyy, yyy, xxxx, xxxy, xxyy, xyxy, yyyx\}$$

To construct our PDE, we used a set of differential invariants up to fourth-order. The main reason for using these types of partial derivatives to form our equation is to create a model that can cover more image features compared to second-order ones and therefore be more sensitive to edges in the image. Furthermore, these differential invariants stay unchanged under various 2D transformations like rotation and translation [27]. Consider the fourth-order PDE below

$$\begin{cases} \frac{\partial I}{\partial t} = \mathbf{u}(t)^T \text{diff}(I), & \text{in } Q, \\ I(x, y, t) = 0, & \text{on } \Gamma, \\ I(x, y, 0) = f, & \text{in } \Omega, \end{cases} \quad (9)$$

in which  $\mathbf{u}(t) = [u_0(t) \dots u_6(t)]^T$  is the coefficient function and  $\text{diff}(\cdot)$  includes the differential parts defined as

$$\begin{aligned} \text{diff}_1(I) &= I, \\ \text{diff}_2(I) &= I_y^2 + I_x^2, \\ \text{diff}_3(I) &= I_{xx}I_{yy} - I_{xy}^2, \\ \text{diff}_4(I) &= I_{xx}I_y^2 - 2I_{xy}I_xI_y + I_{yy}I_x^2, \\ \text{diff}_5(I) &= I_{xx}I_xI_y - I_{xy}I_y^2 - I_{xy}I_x^2 + I_{yy}I_xI_y, \\ \text{diff}_6(I) &= I_{xxx}I_x^3 + 3I_{xxy}I_yI_x^2 + 3I_{xyy}I_y^2I_x + I_{yyy}I_y^3, \\ \text{diff}_7(I) &= I_{xxx}I_x^4 + 4I_{xyy}I_y^3I_x + 6I_{xxy}I_x^2I_y^2 + 4I_{xxy}I_yI_x^3 + I_{yyy}I_y^4. \end{aligned} \quad (10)$$

The initial value for this equation is  $f$ , the noisy image, and the output  $I(\cdot)$  is the desired clear image. To obtain a workable equation, we need to determine the coefficient function  $\mathbf{u}(t)$  which is used to control the evolution

of  $I$ . Our aim is to attain a flexible system that can train the coefficients, then function  $\mathbf{u}(t)$  would vary in different problems and be adapted according to the given training images.

As mentioned previously, with using differential invariants in our model, the PDE system (9) is rotationally invariant and the coefficient functions must be independent of  $(x, y)$  so that (9) is shift-invariant as well.

**Proposition 1.** *Considering the equation (9), the control functions  $\{u_i\}_{i=0}^6$  must be independent of  $(x, y)$ .*

*Proof.* Considering

$$D(I, \mathbf{u}) := \mathbf{u}(t)^T \text{diff}(I), \quad (11)$$

we then rewrite  $D(I, \mathbf{u}) = \tilde{D}(I, x, y, t)$ . Therefore, it is enough to prove the independence of  $\tilde{D}$  of  $(x, y)$ .

Due to the translation invariance of the equation (9), when we change  $f(x, y)$  to  $f(x - x', y - y')$ ,  $I(x, y)$  changes to  $I(x - x', y - y')$  and we have

$$\frac{\partial I(x - x', y - y')}{\partial t} = \tilde{D}(I(x - x', y - y'), x, y, t).$$

Then, by the conversion  $x - x' = x$  and  $y - y' = y$ , we get

$$\frac{\partial I(x, y)}{\partial t} = D(I(x, y), x + x', y + y', t).$$

On the other hand

$$\begin{aligned} \frac{\partial I(x, y)}{\partial t} &= D(I(x, y), x, y, t), \\ \Rightarrow \forall (x', y'); D(I, x + x', y + y', t) &= D(I, x, y, t). \end{aligned}$$

Hence,  $D$  is independent of  $(x, y)$ , which indicates the independence of the function  $\mathbf{u}$ .  $\square$

## 2.1 Extracting the coefficient function

Attempting to solve the problem (9), first, we need to determine the coefficients  $\mathbf{u}(t)$  which is done by an optimal control problem, therefore we recollect some necessary definitions in this field [36]. We begin by assuming  $I \in H_0^1(\Omega)$ .

**Definition 1.** *Considering the optimal control problem*

$$\begin{aligned} &\min J(I, u) \\ \text{s.t. } &T(I, u) = 0 \\ &u \in U_{ad} \end{aligned} \quad (12)$$

where  $J : H_0^1(\Omega) \times L^2(\Omega) \rightarrow \mathbb{R}$  is called the object function of the optimal control problem and  $U_{ad} \in L^2(\Omega)$  is the admissible set.

A vector  $\bar{u} \in U_{ad}$  is called an optimal control for (12), if  $J(\bar{I}, \bar{u}) \leq J(I, u)$  for each  $u \in U_{ad}$  and  $\bar{I}$  is called the optimal state associated with  $\bar{u}$ .

**Definition 2.** Let  $U$  and  $V$  be normed vector spaces and  $U_{ad} \subseteq U$ ,  $f : U_{ad} \rightarrow V$ , and consider some  $u \in U_{ad}$ . Let  $h \in U$ , if  $u + th \in U_{ad}$  for sufficiently small  $t > 0$ , and the limit

$$\delta f(u, h) = \lim_{t \rightarrow 0} \frac{f(u + th) - f(u)}{t},$$

exists in  $V$ , then  $\delta f(u, h)$  is called the directional derivative of  $F$  at  $u$  in the direction  $h$ . If the directional derivative  $\delta f(u, h)$  exists for each  $h$ , then the map

$$\delta f(u, \cdot) : U \rightarrow V, h \mapsto \delta f(u, h),$$

is called the first variation of  $f$  at  $u$ . In addition, if the first variation constitutes a bound linear operator, then it is called the Gateaux derivative of  $f$ .

**Definition 3.** Letting  $X$  and  $Y$  be Banach spaces, let  $A : X \rightarrow Y$  be a bounded linear operator. The map

$$A^* : Y^* \rightarrow X^*, A^*(f) = f \circ A,$$

is called the adjoint of  $A$ . For the optimal control problem (12), the adjoint equation is of the form

$$D_y T(I, u)^T \lambda = \nabla_y J(y, u), \quad (13)$$

and its solution  $\lambda$  is the adjoint state.

**Definition 4.** The function  $L(I, u, \lambda) : H_0^2(\Omega) \times L^2(\Omega) \times H_0^2(\Omega) \rightarrow \mathbb{R}$  defined as below is called the Lagrangian function for the problem (12).

$$L(I, u, \lambda) := J(I, u) - \langle T(I, u), \lambda \rangle \quad (14)$$

where  $\langle \cdot, \cdot \rangle$  denotes the inner product defined on  $H_0^2(\Omega)$ .

Our motivation here was the optimal heating problem which uses an optimal control problem to get the desired output from a heat equation. For instance, in a transient heating problem, the goal is to reach a certain temperature  $\bar{y}$  by controlling the heating elements [30]. Likewise, our objective in this paper is to approximate the target clean noise-free image  $\tilde{I}$  by controlling the coefficient function. Therefore, a PDE based optimal control problem will be presented. As the first step, we need to prepare some input/output training images  $(f_k, \tilde{I}_k)$ , where  $f_k$  is the noisy image and  $\tilde{I}_k$  is the clear image. The final output of the equation needs to be close to the ground truth, thus the coefficient functions must minimize the following functional:

$$\begin{aligned} J(\{I_k\}_{k=1}^K, \mathbf{u}) &= \frac{1}{2} \sum_{k=1}^K \int_{\Omega} (I_k(T_f) - \tilde{I}_k)^2 d\Omega \\ &+ \frac{1}{2} \sum_{i=0}^6 \alpha_i \int_0^{T_f} u_i^2(t) dt, \end{aligned} \quad (15)$$

where  $K$  is the number of the training images,  $I_k(T_f)$  is the output image determined from (9) at time  $t = T_f$  when the initial value is  $f_k$  and  $\alpha_i$  are positive weighting parameters<sup>1</sup>. The first term of the functional  $J$  requires the final output of our PDE to be close to the ground truth and the second term is for regularization so that this optimal control problem is well-posed and counteracts the tendency of the control to become locally bounded as  $J$  approaches its minimum [36].

We have the following optimal control problem with PDE constrains:

$$\min_{\mathbf{u}} J(\{I_k\}_{k=1}^K, \mathbf{u}); \quad s.t. \quad \begin{cases} \frac{\partial I_k}{\partial t} = \mathbf{u}(t)^T \text{diff}(I_k), & \text{in } Q, \\ I_k(x, y, t) = 0, & \text{on } \Gamma, \\ I_k(x, y, 0) = f_k, & \text{in } \Omega. \end{cases} \quad (16)$$

In addition, we consider an inequality constraint  $0 \leq \mathbf{u}(t)$  for the problem (16) to make sure that the coefficients are all positive. In the following, the necessary conditions for this problem (16) will be examined so that we would be able to solve the problem by proper numerical methods.

**Lemma 1.** Suppose  $J(I, u) = \int_Q g(u, I) dQ$  is of class  $C^1$ , where  $g$  is a smooth function. Then considering the function  $D(I, \mathbf{u})$  in (11)

$$\frac{DJ}{Du} = D_I^*(I, u)(\lambda) + g_I^*(I, u)(T_f),$$

where  $D_I^*$  and  $g_I^*$  are the adjoint operators of  $D_I$  and  $g_I$  respectively,  $\frac{DJ}{Du}$  is the Gateaux derivative of  $J$  with respect to the control  $u$  and  $\lambda$  is the adjoint state of (9).

*Proof.* We first define an operator  $\psi$ , mapping  $u$  to the solution of (9);  $I = \psi(u)$ . Now consider

$$d = \lim_{\varepsilon \rightarrow 0} \frac{\psi(u + \varepsilon \delta u) - \psi(u)}{\varepsilon},$$

where  $\delta u$  is the perturbation of  $u$ .

We have

$$\begin{aligned} \left( \frac{DJ}{Du}, \delta u \right)_Q &= \lim_{\varepsilon \rightarrow 0} \frac{J(I, u + \varepsilon \delta u) - J(I, u)}{\varepsilon} \\ &= \lim_{\varepsilon \rightarrow 0} \frac{J(\psi(u + \varepsilon \delta u), u + \varepsilon \delta u) - J(\psi(u), u)}{\varepsilon} \\ &= \lim_{\varepsilon \rightarrow 0} \frac{J(I + \varepsilon d + o(\varepsilon), u + \varepsilon \delta u) - J(I, u)}{\varepsilon} \\ &= \int_Q (g_I(u, I)(d) + g_I(u, I)(\delta u)) dQ \\ &= (g_I(u, I)(d), 1)_Q + (g_I(u, I)(\delta u), 1)_Q \\ &= (g_I^*(u, I)(1), d)_Q + (g_I^*(u, I)(1), \delta u)_Q. \end{aligned} \quad (17)$$

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<sup>1</sup>in this paper we fix  $\alpha_i = 10^{-7}$ ,  $i = 0, \dots, 6$ .



If we take  $\lambda$  as the adjoint state of (16), then from the PDE constrains of (16) we have

$$((\psi(u))_t, \lambda)_Q = (D(u, \psi(u)), \lambda)_Q, \quad (18)$$

and

$$((\psi(u + \varepsilon \delta u))_t, \lambda)_Q = (D(u + \varepsilon \delta u, \psi(u + \varepsilon \delta u)), \lambda)_Q. \quad (19)$$

Now by subtracting (19) with (18)

$$\begin{aligned} & ((\psi(u + \varepsilon \delta u) - \psi(u))_t, \lambda)_Q \\ &= (D(u + \varepsilon \delta u, \psi(u + \varepsilon \delta u)) - D(u, \psi(u)), \lambda)_Q \\ &= (\varepsilon D_u(u, I)(\delta u) + D_I(u, I)(\psi(u + \varepsilon \delta u) - \psi(u)), \lambda) + o(\varepsilon). \end{aligned} \quad (20)$$

In this step, we divide both sides with  $\varepsilon$  and let  $\varepsilon \rightarrow 0$ ,

$$(d_t, \lambda)_Q = (D_u(u, I)(\delta u), \lambda)_Q + (D_I(u, I)(d), \lambda)_Q. \quad (21)$$

Integrating by parts

$$(d, \lambda)_\Omega|_0^T - (d, \lambda_t)_Q = (\delta u, D_u^*(u, I)(\lambda))_Q + (d, D_I^*(u, I)(\lambda)). \quad (22)$$

Now, since  $\lambda$  is the adjoint equation of the problem, we have

$$(d, \lambda)_\Omega|_0^T = 0, (g_I^*(u, I)(T), d)_Q = (-\lambda_t - D_I^*(u, I)(\lambda), d), \quad (23)$$

and combining it with (21)

$$(g_I^*(u, I)(T), d)_Q = (-\lambda_t - D_I^*(u, I)(\lambda), d) = (\delta u, D_u^*(u, I)(\lambda)). \quad (24)$$

Therefore we arrive at

$$\frac{DJ}{Du} = g_I^*(u, I)(T) + D_u^*(u, I)(\lambda).$$

□

Now, we investigate the conditions that the optimal vectors  $\bar{u}$  and  $\bar{I}$  must satisfy to be then determined, using numerical methods.

**Theorem 1.** *Considering the optimal control problem (16), and suppose that*

$$U_{ad} = \{\mathbf{u} \in L_2(\Omega); \mathbf{u}_a \leq \mathbf{u}\} \quad (25)$$

*is the admissible control set, where  $U_{ad} \subseteq O \subseteq L^2(\Omega)$ ,  $O$  is an open subset of  $L^2(\Omega)$ , in our case  $\mathbf{u}_a = 0$  and  $J : H_0^1(\Omega) \times L_2(\Omega) \rightarrow \mathbb{R}$  is of class  $C^1$ . If  $\bar{\mathbf{u}} \in U_{ad}$  is an optimal control for (16) in the sense of Def.1 with corresponding state  $\bar{I}$  and adjoint state  $\bar{\lambda}$ , then  $\bar{\mathbf{u}} = (\bar{u}_i)_{0 \leq i \leq 6}$  satisfies for each  $i \in \{0, \dots, 6\}$*

$$\bar{u}_i = u_{a,i}, \quad \text{where } L_u(\bar{I}, \bar{u}, \bar{\lambda}) > 0 \quad (26)$$

In addition, by introducing the extended Lagrange function

$$L(I, u, \lambda, \mu_a) = L(I, u, \lambda) + \langle u_a - u, \mu_a \rangle, \quad (27)$$

and letting

$$\mu_a := \max\{0, L_u(I, u, \lambda)\},$$

the 4-tuples  $(\bar{I}, \bar{u}, \bar{\lambda}, \mu_a) \in (H_0^2(\Omega), L^2(\Omega), H_0^2(\Omega), L^2(\Omega))$  satisfies

$$\frac{\partial J}{\partial u_i} = \alpha_i u_i - \sum_{k=1}^K \int_{\Omega} \lambda_k \text{diff}_i(I_k) d\Omega - \mu_a; \quad i = 0, \dots, 6. \quad (28)$$

*Proof.* First note that, (25) implies that  $U_{ad}$  is convex. We have the following optimality system which can be used to determine the optimal control  $\bar{u}$  and  $\bar{I}$ :

$$\begin{aligned} \nabla_{\lambda} L(\bar{y}, \bar{u}, \bar{\lambda}) &= 0, \\ \nabla_I L(\bar{y}, \bar{u}, \bar{\lambda}) &= 0, \\ \nabla_u L(\bar{y}, \bar{u}, \bar{\lambda}) &= 0, \\ \langle L_u(\bar{I}, \bar{u}, \bar{\lambda}), u - \bar{u} \rangle &\geq 0, \text{ for each } u \in U_{ad}. \end{aligned} \quad (29)$$

Every solution to the general form of optimal control (12) must satisfy this system [36].

Slightly rearranging the fourth optimal condition, we get

$$\langle L_u(\bar{I}, \bar{u}, \bar{\lambda}), \bar{u} \rangle_{L^2(\Omega)} \leq \langle L_u(\bar{I}, \bar{u}, \bar{\lambda}), u \rangle_{L^2(\Omega)}$$

it implies that  $\bar{u}$  is the solution to the optimal problem

$$\min_{u \in U_{ad}} \langle L_u(\bar{I}, \bar{u}, \bar{\lambda}), u \rangle_{L^2(\Omega)} = \min \sum_{i=0}^6 L_u(\bar{I}, \bar{u}_i, \bar{\lambda}) u_i \quad (30)$$

the components  $u_i$  can be varied independently because of the form of  $U_{ad}$ , such that the sum in (30) attains its min if and only if each summand is minimal. Therefore,

$$L_u(\bar{I}, \bar{u}_i, \bar{\lambda}) \bar{u}_i = \min_{u \in U_{ad}} L_u(\bar{I}, \bar{u}_i, \bar{\lambda}) u_i \quad (31)$$

Now, (26) is a direct result of (31).

Moreover, according to the definition of  $L(I, u, \lambda, \mu_a)$  in (27),

$$\begin{aligned} \nabla_{\lambda} L(I, u, \lambda, \mu_a) &= \nabla_{\lambda} L(\bar{I}, \bar{u}, \bar{\lambda}), \\ \nabla_I L(I, u, \lambda, \mu_a) &= \nabla_I L(\bar{I}, \bar{u}, \bar{\lambda}), \\ \nabla_u L(I, u, \lambda, \mu_a) &= \nabla_u L(\bar{I}, \bar{u}, \bar{\lambda}) - \mu_a, \end{aligned} \quad (32)$$

therefore, we continue the proof with the old form of Lagrangian for simplicity. By Def(4) the Lagrangian function for the optimal control problem (16) is as bellow

$$L(\{I_k\}_{k=1}^K, \mathbf{u}) = J(\{I_k\}_{k=1}^K, \mathbf{u}) + \sum_{k=1}^K \int_Q \lambda_k ((I_k)_t - D(I_k, \mathbf{u})) dQ, \quad (33)$$

where the multiplier  $\lambda$  is the adjoint state. It can be seen that the first optimality condition  $\nabla_\lambda L(\bar{y}, \bar{u}, \bar{\lambda}) = 0$  is the PDE constraint in (16).

To find the adjoint state  $\lambda$ , we first perturb  $D(I_k)$  with respect to  $I$ :

$$\begin{aligned} & D(I_k + \varepsilon \delta I_k, \mathbf{u}) - D(I_k, \mathbf{u}) \\ &= \varepsilon \cdot \left( \frac{\partial D}{\partial (I_k)_x} \frac{\partial (\delta I_k)}{\partial x} + \dots + \frac{\partial D}{\partial (I_k)_{yyyy}} \frac{\partial^4 (\delta I_k)}{\partial y^4} \right) + O(\varepsilon) \\ &= \varepsilon \cdot \sum_{p \in \eta} \sigma_p(I_k) \frac{\partial^{|p|} (\delta I_k)}{\partial p} + O(\varepsilon), \end{aligned} \quad (34)$$

in which

$$\sigma_p(I_k) = \frac{\partial D(I_k, \mathbf{u})}{\partial I_k} = \mathbf{u}^T(t) \cdot \frac{\partial (\text{diff}(I_k))}{\partial (I_k)_p} = \sum_{i=0}^6 u_i \frac{\partial \text{diff}_i(I_k)}{\partial (I_k)_p}. \quad (35)$$

Besides, for having  $L_{I_k}(I, u, \lambda)$ , we perturb  $I_k$  in  $L$ :

$$\begin{aligned} L_{I_k}(I, u, \lambda) &= \lim_{\varepsilon \rightarrow 0} \frac{L(\dots, I_k + \varepsilon \delta I_k, \dots) - L(\dots, I_k, \dots)}{\varepsilon} \\ &= \lim_{\varepsilon \rightarrow 0} \left( \frac{1}{2\varepsilon} \int_{\Omega} ((I_k + \varepsilon \delta I_k)(x, y, T_f) - \tilde{I}_k(x, y))^2 d\Omega - \frac{1}{2} \int_{\Omega} (I_k(x, y, T_f) - \tilde{I}_k(x, y))^2 d\Omega \right. \\ &\quad \left. + \int_Q \lambda_k [((I_k + \varepsilon \delta I_k)_t - D(I_k + \varepsilon \delta I_k)) - ((I_k)_t - D(I_k))] dQ \right) \\ &= \int_{\Omega} (I_k(x, y, T_f) - \tilde{I}_k(x, y)) \delta I_k(x, y, T_f) d\Omega + \int_Q \lambda_k (\delta I_k(x, y, T_f))_t dQ \\ &\quad - \int_Q \lambda_k \sum_{p \in \eta} \sigma_p \frac{\partial^{|p|} (\delta I_k)}{\partial p} dQ + O(\varepsilon), \end{aligned} \quad (36)$$

and  $\delta I_k$  should satisfy

$$\begin{aligned} \delta I_k &= 0 \quad \text{on } \Gamma, \\ \delta I_k &= 0 \quad \text{in } \Omega. \end{aligned}$$

Due to the boundary and initial conditions of  $I_k$ , if we assume that  $\lambda_k = 0$  on  $\Gamma$ , then upon integration by parts (with respect to  $t$  in  $(I_k)_t$  and spatial variables in  $\partial^{|p|}(\delta I_k)$ ), we have

$$\begin{aligned}
L_{I_k}(I, u, \lambda) &= \int_{\Omega} (I_k(x, y, T_f) - \tilde{I}_k(x, y)) \delta I_k(x, y, T_f) d\Omega \\
&\quad + \int_{\Omega} (\lambda_k \cdot \delta I_k)(x, y, T_f) d\Omega - \int_Q (\lambda_k)_t \delta I_k dQ \\
&\quad - \int_Q \sum_{p \in \eta} (-1)^{|p|} \frac{\partial^{|p|}(\sigma_p(I_k) \lambda_k)}{\partial p} \cdot \delta I_k dQ \\
&= \int_Q [(\lambda_k + I_k - \tilde{I}_k) \delta(t - T_f) \\
&\quad - (\lambda_k)_t - \sum_{p \in \eta} (-1)^{|p|} \frac{\partial^{|p|}(\sigma_p \lambda_k)}{\partial p}] \delta I_k dQ + O(\varepsilon),
\end{aligned} \tag{37}$$

where  $\delta(\cdot)$  is the Dirac function. Finally, by considering the second optimality condition  $L_u(I, u, \lambda) = 0$ , our argument yields the following system as the adjoint equation for  $\lambda_k$ :

$$\begin{cases} \frac{\partial \lambda_k}{\partial t} + \sum_{p \in \eta} (-1)^{|p|} (\sigma_p \lambda_k)_p = 0, & \text{in } Q, \\ \lambda_k = 0, & \text{on } \Gamma, \\ \lambda_k(x, y, T_f) = \tilde{I}_k - I_k(T_f), & \text{in } \Omega, \end{cases} \tag{38}$$

for  $k = 1, \dots, K$ .

Now by perturbing  $u_i$  in  $L$  we get  $L_{u_i}(I, u, \lambda)$

$$\begin{aligned}
L_{u_i}(I, u, \lambda) &= \lim_{\varepsilon \rightarrow 0} \frac{L(\dots, u_i + \varepsilon \cdot \delta u_i, \dots) - L(\dots, u_i, \dots)}{\varepsilon} \\
&= \lim_{\varepsilon \rightarrow 0} \left( \frac{1}{2\varepsilon} \alpha_i \int_0^T (u_i + \varepsilon \cdot \delta u_i)^2(t) dt - \frac{1}{2} \alpha_i \int_0^T u_i^2(t) dt \right. \\
&\quad \left. - \sum_{k=1}^K \int_Q \lambda_k((u_i + \varepsilon \cdot \delta u_i)(t) - u_i^t(t)) \operatorname{diff}(I) dQ \right) \\
&= \int_0^T (\alpha_i u_i \cdot \delta u_i)(t) dt - \int_0^T \left( \sum_{k=1}^K \int_{\Omega} \lambda_k \operatorname{diff}(I_k) d\Omega \right) \delta u_i dt.
\end{aligned} \tag{39}$$

Therefore, we get

$$L_{u_i}(I, u, \lambda) = \alpha_i u_i - \sum_{k=1}^K \int_{\Omega} \lambda_k \operatorname{diff}_i(I_k) d\Omega; \quad i = 0, \dots, 6, \tag{40}$$

where the function  $\lambda_k$  is the solution to (38). Now we get back to the Lagrangian (27) of our model, we have

$$L_{u_i}(I, u, \lambda, \mu_a) = \alpha_i u_i - \sum_{k=1}^K \int_{\Omega} \lambda_k \text{diff}_i(I_k) d\Omega - \mu_{a,i}; \quad i = 0, \dots, 6, \quad (41)$$

From Lemma(1), we have  $\frac{DJ}{Du} = L_u(I, u, \lambda)$ . Therefore we arrive at

$$\frac{DJ}{Du_i} = \alpha_i u_i - \sum_{k=1}^K \int_{\Omega} \lambda_k \text{diff}_i(I_k) d\Omega - \mu_{a,i}; \quad i = 0, \dots, 6. \quad (42)$$

□

In summary, we have the Gateaux derivative of the object function  $J$  with respect to the control  $u$ ,

$$\frac{DJ}{Du_i} = \alpha_i u_i - \sum_{k=1}^K \int_{\Omega} \lambda_k \text{diff}_i(I_k) d\Omega - \mu_{a,i}; \quad i = 0, \dots, 6, \quad (43)$$

where the adjoint state  $\lambda$  is the solution to the PDE bellow

$$\begin{cases} \frac{\partial \lambda_k}{\partial t} + \sum_{p \in \eta} (-1)^{|p|} (\sigma_p \lambda_k)_p = 0, & \text{in } Q, \\ \lambda_k = 0, & \text{on } \Gamma, \\ \lambda_k(x, y, T_f) = \tilde{I}_k - I_k(T_f), & \text{in } \Omega, \end{cases} \quad (44)$$

For more details and a more mathematically precise explanation, the interested reader is referred to [36, 20]. Now that we have extracted enough information from the optimality conditions, we can use numerical methods to determine  $\bar{\mathbf{u}}$  for (16).

Now, we need to find the initial  $\mathbf{u}(t)$  as the starter of the numerical method. At each time step,  $\frac{\partial I_k(t)}{\partial t}$  is expected to be  $\frac{\tilde{I}_k - I_k(t)}{T_f - t}$  so that  $I_k$  tends to the expected output  $\tilde{I}_k$ . On the other hand, we have  $\frac{\partial I_k(t)}{\partial t} = L(I_k, \mathbf{u})$ . Next, we aim to find  $\mathbf{u}(t)$  to minimize:

$$\sum_{k=1}^K \int_{\Omega} \left( L(I_k, \mathbf{u}) - \frac{\partial I_k(t)}{\partial t} \right)^2 d\Omega = \sum_{k=1}^K \int_{\Omega} [p_k(t)^T \mathbf{u}(t) - d_k(t)]^2 d\Omega.$$

Therefore, the initial  $\mathbf{u}(t)$  can be obtained by solving the following system:

$$P(t)\mathbf{u}(t) = \mathbf{d}(t), \quad (45)$$

where we have  $P(t) = \sum_{k=1}^K \int_{\Omega} p_k(t) p_k(t)^T d\Omega$  and  $\mathbf{d}(t) = \sum_{k=1}^K \int_{\Omega} p_k(t) d_k(t)^T d\Omega$ ,

in which  $p_k(t) = \text{diff}(I_k)$  and  $d_k(t) = \frac{\tilde{I}_k - I_k(t)}{T_f - t}$ .

Finally, with the help of the above  $\mathbf{u}(t)$ , and by considering the Gateaux derivative (43), we can perform Conjugate Gradient (CG) method [32, 34] to solve the optimal control problem (16).

Our PDE framework is summarized in Algorithm (1). Once the coefficients are learned, the equation (9) is completed and can be applied on test images where the noisy image  $f$  is the input as the initial value and the output  $I(T_f)$  is the desired de-noised image.

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**Algorithm 1** The framework to learn PDEs for image restoration

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**Input:** Training image pairs  $(f_k, \tilde{I}_k)$ ,  $k = 1, \dots, K$ ,  $T_f$ .

- 1: Initialize  $\mathbf{u}(t)$ ,  $t \in [0, T_f]$ , by solving (45).
  - 2: **while** not converged **do**
  - 3:     Compute  $\frac{\partial J}{\partial u_i}$ ,  $i = 0, \dots, 6$  using (43).
  - 4:     Perform conjugate gradient method using  $\frac{\partial J}{\partial u_i}$  [32, 34].
  - 5: **end while**
  - 6: **return** Coefficient functions  $\mathbf{u}(t)$ ,  $t \in [0, T_f]$ .
  - 7: Solve (9) using the result from previous step.
- 

### 3 Experimental Results

In this section, the application of our proposed PDE for image de-noising is demonstrated. The experiments are performed on gray-scale images. We compare the results obtained from our PDE with the evolutionary previous works, PM second-order PDE [29], YK fourth-order PDE [41], second-order L-PDE [22], Zhang [46] and Siddig's [33] fourth-order PDE approaches for image de-noising. For selecting the parameters in PM and YK method, we chose the values which produce the most appealing results and the best result is chosen as the output of the method. For all experiments, the grid size  $h = 1$  and  $T_f = 5$  are considered. All our experiments are performed in Matlab R2018b on an Intel(R) Core(TM) i5, 2.40 GHz, 8 GB RAM laptop.

For a quantitative comparison between the previous methods and our scheme, the peak signal-to-noise ratio (PSNR) and structural similarity index (SSIM) of the processed images are considered. It should be noted that PSNR is most easily defined via the mean squared error (MSE). Given a noise-free  $m \times n$  monochrome image  $I$  and its noisy approximation  $f$ , MSE is defined as [43]:

$$MSE = \frac{1}{mn} \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} (I(i, j) - f(i, j))^2,$$

then PSNR is obtained as follows:

$$\begin{aligned} PSNR &= 10 \cdot \log_{10} \left( \frac{\max^2 I}{\sqrt{MSE}} \right) \\ &= 20 \cdot \log_{10}(\max I) - 10 \cdot \log_{10}(MSE). \end{aligned} \tag{46}$$

Here,  $\max I$  is the maximum possible pixel value of the image. It can easily be seen that, the closer an image is to its ground truth, the higher its PSNR is. Given similar assumptions, SSIM is defined as

$$SSIM = \frac{(2\bar{I}\bar{f} + C_1)(2\sigma_{If} + C_2)}{(\mu_I^2 + \mu_f^2 + C_1)(\sigma_I^2 + \sigma_f^2 + C_2)}$$

in which  $\bar{I}$  and  $\bar{f}$  are the mean values of  $I$  and  $f$ ,  $\sigma_I$  and  $\sigma_f$  are the standard deviation of them and  $\sigma_I^2$  and  $\sigma_f^2$  are their variances and  $\sigma_{If}$  is the variance of  $I$  and  $f$ . For stabilizing the division with the weak denominator,  $C_1 = (K_1L)^2$  and  $C_2 = (K_2L)^2$  are two variables and  $L$  is the dynamic range of pixel values,  $K_1 = 0.01$  and  $K_2 = 0$ . Like PSNR, the closer an image is to its ground truth, the closer the SSIM will be to one.

For choosing images in our experiment, firstly, we used a group of images containing 50 clear images and their noisy version, with additive white Gaussian noise, in various sizes, categorized in ten different classes including animals, sea, personal picture, city view, nature images, etc. These images are gathered from SIPI image database [39], Berkeley image database [3] and Matlab's image library. Four pictures in each class were used for training the PDE, and one was used as the test image and the same test image was used as the initial image for the other de-noising methods.

In figures, 1, 2, the performance of our proposed method and the other five de-noising models are shown. Firstly, "Lena" picture with size  $512 \times 512$ , 1(a) original clean image, 1(b) noisy version of it with additive white noise followed by 1(c-g) showing the de-noised images by PM, YK, 2nd L-PDE, Zhang's's PDE and Siddig's model respectively and 1(h) is the de-noised image with our proposed method. It can be seen that compared to the other five methods, our fourth-order PDE demonstrates better results in preserving edges while decreasing the noise of the image. In Figure 2, the "City" image with size  $850 \times 480$  is displayed followed by 2(b), its noisy version and on 2(c-g), the enhanced form of it by the Pm, YK, 2nd order L-PDE, Zhang's model and Siddig's approach are showed and in 2(h), the result of our de-noising model is illustrated. It is vivid that the recommended PDE reduces noise without blurring the image features or leaving any speckles.

The PSNR and SSIM values collected from de-noising ten images with additive white Gaussian noise with all the mentioned models are presented in Table 1. Comparing the values obtained from discussed methods in each image shows that our proposed scheme has the best performance among the other methods, which is indicated by the higher PSNR and SSIM values.

In addition, to better investigate the performance of our suggested method on real images with unknown noises, the second group consists of 50 images, in five separate classes with real unknown noise such as aerial and texture images are chosen. This group of images is also gathered from SIPI and Berkeley image databases and pictures taken by a Canon EOS 750D camera. Table 2 represents the results of performing all the methods on our five image classes which can be seen that our suggested model results in better outcomes.

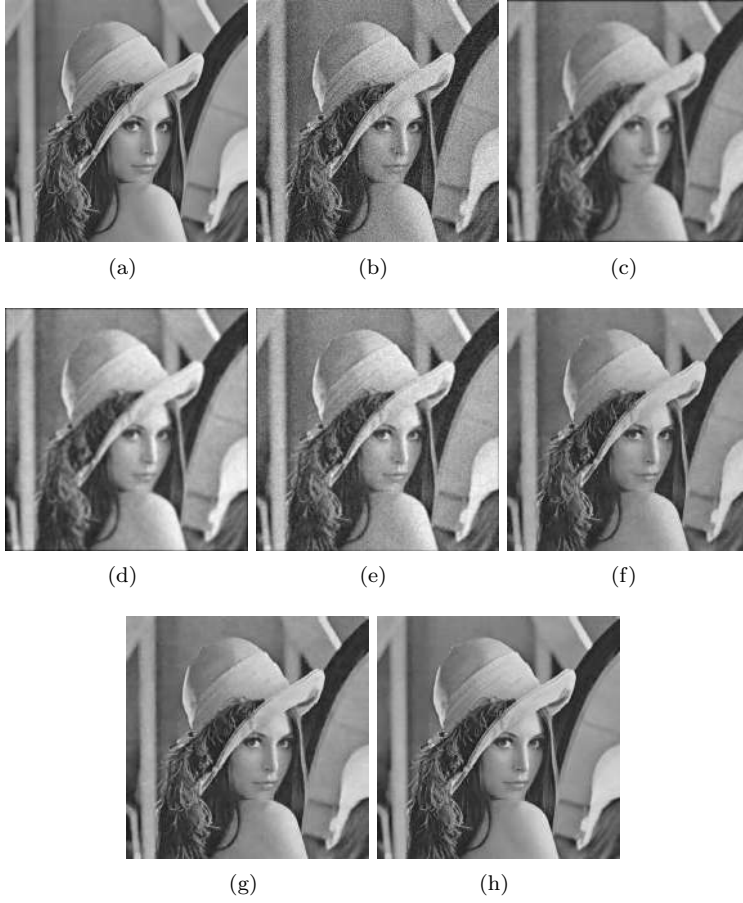


Figure 1: Comparison of proposed scheme with five methods on “Lena” image with added white Gaussian noise. (a) Original image; (b) Noisy image; (c) PM second-order PDE; (d) YK fourth-order PDE; (e) second-order L-PDE; (f) Zhang’s fourth-order PDE; (g) Siddig’s model; (h) proposed fourth-order PDE

In training the L-PDEs in [22] and our model for de-noising the images with unknown noise, we used more training images to learn the coefficients due to the more complex noise in each image. In figures 3 and 4, the noisy test image “Aerial” and “Tile” are shown with their de-noised versions and it is observed that our model is superior in decreasing noise while preserving the image features.

In Figure 3, a noisy aerial image with unknown noise is showed and its de-noised forms are 3(b-g). Even with the complex unknown noise, the perfor-



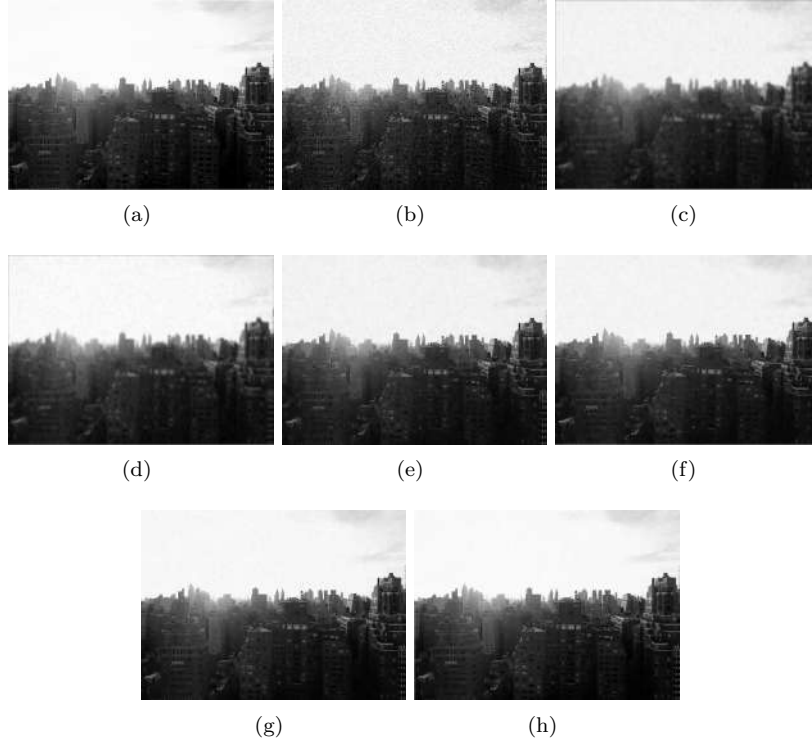


Figure 2: Comparison of proposed scheme with five methods on “City” image with added white Gaussian noise. (a) Original image; (b) Noisy image; (c) PM second-order PDE; (d) YK fourth-order PDE; (e) second-order L-PDE; (f) Zhang’s fourth-order PDE; (g) Siddig’s model; (h) proposed fourth-order PDE

mance of our proposed quadratic model is better than the other competitors. In Figure 4, a noisy tile image is presented. As it was expected from the PSNR and SSIM values in Table 2, the recommended learning-based PDE still yields better results in comparison to the other contestants.

Finally, we compare the average PSNR values resulted from all the models, in Figure 5. We can see that by comparing the average values obtained from each method on both image groups with Gaussian and unknown noise, that the average values gained from our proposed PDE is larger than the other approaches on both types of noises which indicates that our scheme outperforms other models in both image categories.

Table 1: PSNR and SSIM results for ten test images with additive white Gaussian noise.

	PM	YK	2nd L-PDE	Zhang	Siddig	<b>Proposed</b>
Lena	19.4755	20.8096	23.8938	27.7531	28.5738	<b>30.4873</b>
	0.7527	0.8031	0.8093	0.8069	0.8421	<b>0.9184</b>
City	20.3853	20.7052	24.2705	28.2597	29.6089	<b>29.1105</b>
	0.6908	0.7288	0.8041	0.8472	0.8133	<b>0.8563</b>
Text	18.0646	21.9834	23.9050	28.0671	28.5636	<b>31.2598</b>
	0.7093	0.7313	0.8527	0.8530	0.8600	<b>0.8697</b>
Woman	19.8594	20.7346	23.0504	29.4682	30.1833	<b>30.4666</b>
	0.6551	0.6995	0.8211	0.8873	0.8237	<b>0.9030</b>
Eiffel	19.1024	21.4389	24.1924	28.0046	30.1453	<b>30.2713</b>
	0.6694	0.6890	0.7806	0.8074	0.8497	<b>0.8901</b>
Sea	20.2217	21.0962	23.9878	27.2673	27.5078	<b>31.0694</b>
	0.7589	0.7854	0.8030	0.8364	0.7890	<b>0.8751</b>
Nightcity	19.8372	21.9698	25.2569	28.5490	28.6472	<b>29.3441</b>
	0.7048	0.7589	0.8497	0.7573	0.8628	<b>0.8638</b>
Nature	20.2255	22.3523	24.6548	29.7069	29.6567	<b>30.4155</b>
	0.6508	0.7001	0.7945	0.7886	0.8349	<b>0.9153</b>
Flowers	19.3536	20.9230	25.2612	28.9910	29.0531	<b>29.9513</b>
	0.6461	0.6934	0.7914	0.8199	0.8276	<b>0.8785</b>
Trees	20.3819	22.1760	25.8949	28.6375	28.8983	<b>30.5744</b>
	0.6835	0.7412	0.7364	0.8536	0.8619	<b>0.9143</b>

Table 2: PSNR and SSIM results for five test images with real unknown noise.

	PM	YK	2nd L-PDE	Zhang	Siddig	<b>Proposed</b>
Truck	16.1163	17.2961	18.6548	19.9679	21.0791	<b>22.5288</b>
	0.7001	0.7058	0.7413	0.7538	0.8103	<b>0.8526</b>
Aerial	15.9888	18.1222	18.9908	19.5223	20.9317	<b>22.9611</b>
	0.6846	0.7140	0.7851	0.7987	0.8159	<b>0.8794</b>
Personal	16.6055	18.3664	19.0258	20.6046	21.6548	<b>23.7669</b>
	0.6422	0.6988	0.7423	0.7642	0.7941	<b>0.8682</b>
Texture	15.4097	17.0267	18.9516	19.5583	20.9513	<b>22.0304</b>
	0.7085	0.7631	0.7970	0.8155	0.8289	<b>0.8437</b>
Tile	16.2849	17.9806	18.9823	20.7140	21.1675	<b>23.6656</b>
	0.6827	0.7125	0.7681	0.8155	0.8468	<b>0.8549</b>

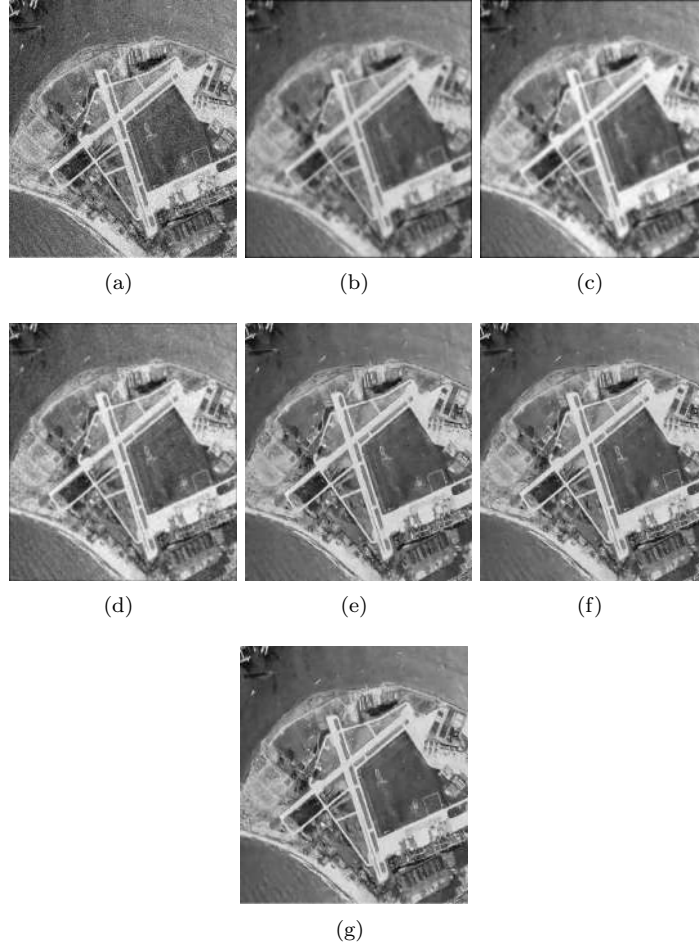


Figure 3: Comparison of proposed scheme with five methods on “Aerial” image with real unknown noise. (a) Noisy image; (b) PM second-order PDE; (c) YK fourth-order PDE; (d) second-order L-PDE; (e) Zhang’s fourth-order PDE; (f) Siddig’s model; (g) proposed fourth-order PDE

## 4 Conclusion

In this paper, we proposed a fourth-order PDE for image de-noising with trainable coefficients using optimal control. The experimental results show improvements in performance in comparison to previous approaches. Compared to previous methods, our scheme results in better outcomes. For future studies, we want to strengthen our approach in the following areas. First, design a more general equation to adapt better to different problems.

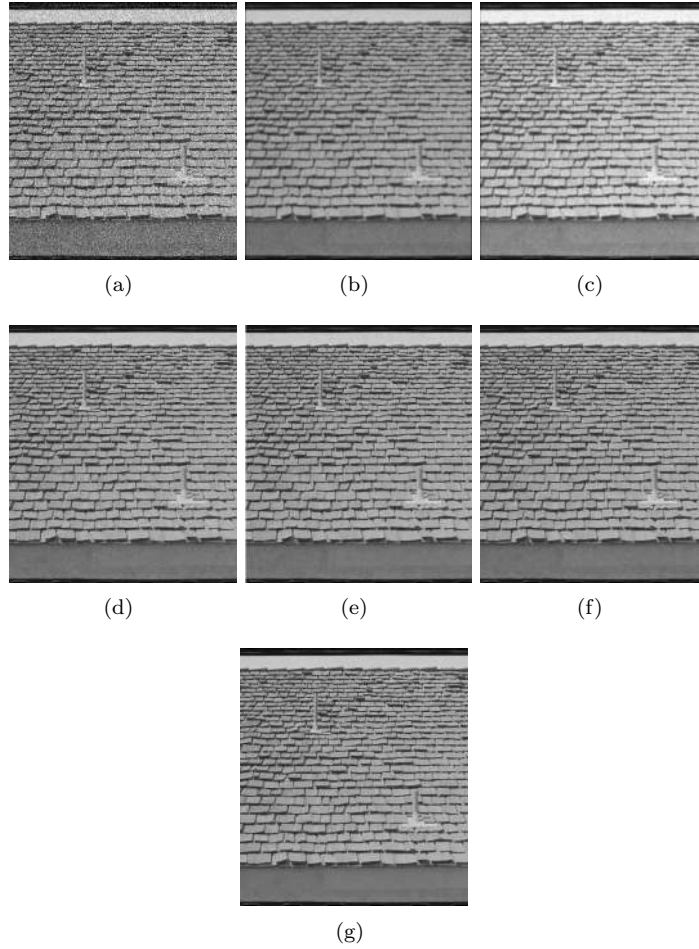


Figure 4: Comparison of proposed scheme with five methods on “Tile” image with real unknown noise. (a) Noisy image; (b) PM second-order PDE; (c) YK fourth-order PDE; (d) second-order L-PDE; (e) Zhang’s fourth-order PDE; (f) Siddig’s model; (g) proposed fourth-order PDE

Second, to introduce a region-based PDE with trainable coefficients. And finally, Finding a more suitable numerical method for solving the optimal control problem.

## Conflict of interest

The authors declare that they have no conflict of interest.

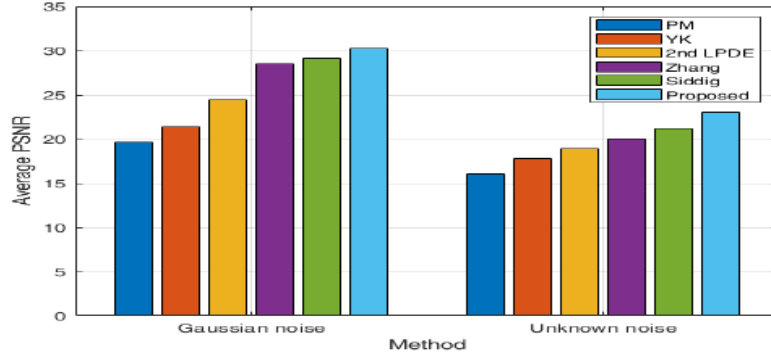


Figure 5: comparison of average PSNR values in 4 methods

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